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Corresponding Author:	Kerim Yunt, Dr. sc. techn. Zurich, Zurich SWITZERLAND
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	
Corresponding Author's Secondary Institution:	
First Author:	Kerim Yunt, Dr. sc. techn.
First Author Secondary Information:	
Order of Authors:	Kerim Yunt, Dr. sc. techn.
Order of Authors Secondary Information:	
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The Principle of Hamilton for Mechanical Systems with Impacts and Unilateral Constraints

Kerim Yunt
Senior Engineer Mechanical Development
MAN Diesel and Turbo Schweiz AG
Zurich, Switzerland 8005
Email: kerim.yunt@man.eu

ABSTRACT

An action integral is presented for Hamiltonian mechanics in canonical form with unilateral constraints and/or impacts. The transition conditions on generalized impulses and the energy are presented as variational inequalities, which are obtained by the application of discontinuous transversality conditions. The energetical behavior for elastic, plastic and blocking type impacts are analyzed. A general impact equation is obtained by the stationarity conditions, which is compatible with the most general impact laws and is applicable to various impactive processes straightforwardly. The crux in achieving energetical behaviour which conforms with the physics of the impactive process, is shown to be the consistency conditions on the impact time variations.

1 Introduction

Out of historical circumstances, energy conservation played an emminent role in the maturation of classical mechanics and traditionally the principles of mechanics are basically stated for conservative evolutionary processes, and are extended then by technical formalism to dissipative processes. The overwhelming amount of scientific publications which deal with the conservation properties of various mechanical principles in comparison to works which mainly deal with dissipation, are testifying this philosophical approach of the classical mechanicists traditionally, though dissipation is the reflection of a fundamental principle in physics to mechanics. This is the principle of entropy increase in physics, which is also known as the second law of thermodynamics, to which all principles in physics must yield.

Hamilton postulated in 1835 in his seminal work [1] [2], that if a Lagrangian system occupies certain positions at fixed times t_0 and t_f, then it should move between these two positions along those admissible arcs $q(t) \in C^1_n[t_0, t_f]$, which make the action integral

$$J(q) = \int_{t_0}^{t_f} L(q(s), \dot{q}(s))\, ds \tag{1}$$

stationary. The integrand $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called the Lagrangian and is defined as $L = T(q, \dot{q}) - V(q)$, where $V(q)$ and $T(q, \dot{q})$ represent the potential and kinetic energy, respectively. The original form of the Hamilton's principle deals with conservative systems with equality constraints, which are perfect bilateral constraints. Though the Hamiltonian mechanics has been born later than Lagrangian mechanics, the domains of physics in which its formalism is used even reach to more modern branches of physics, such as relativistic/quantum mechanics [3] and in control theory in optimal control [4]. Therefore it is important to formulate results of Lagrangian formalism in Hamiltonian formalism. Given the generalized coordinates the following Legendre transformation on the generalized velocities:

$$p = \partial_{\dot{q}} L \tag{2}$$

where p is called the conjugate generalized momentum, the Hamiltonian canonical equations are obtained as the stationarity condition:

$$\dot{p} = -\partial_q H, \quad \dot{q} = \partial_p H. \tag{3}$$

in smooth conservative motion. The Legendre transformation, which yields the Hamiltonian is achieved by the following supremum operation:

$$H(q,p) = \sup_{\dot{q}} \langle p, \dot{q} \rangle - L(q,\dot{q}). \tag{4}$$

If condition (2) holds, then the supremum in (4) is attained. The class of mechanical systems for which the total mechanical energy is given by the Hamiltonian $H(q,p)$ are scleronomic (not explicitly time dependent) systems with solely general position dependent potentials as discussed in [5].

The missing link in analytical mechanics which shows, that general impactive processes are obtained by extremizing some sort of action integral for which momentum and energy are not necessarily conserved is recently shown in [6] for elastic contact impacts in the Lagrangian formalism. In [6] the conditions, under which general non-conserving impacts become a part of an extremizing solution for mechanical systems, which are scleronomic and holonomic, are investigated. The general momentum balance and the total energy change over a collisional impact for a mechanical scleronomic holonomic finite-dimensional Lagrangian system are obtained in the form of stationarity conditions of a modified action integral. The reference [6] has been preceded/succeeded by many works such as [7–14], which were not able to present impulsive action integrals for impacts without energy conservation. Recently in [15] it is shown, that if the principle of Jourdain is extended to cover dissipative force laws, then it receives a meaning as the maximization of irreversibility and thereby becomes the second-law of thermodynamics form for temperature-independent mechanics. In [16] blocking as a dissipative impactive process is analyzed by the technique in [6] and an impulsive action integral in the Lagrangian formalism is presented. In this work, by making use of the results obtained in [6] and [16] the impactive principle of stationary action for impactive processes are obtained by maximizing, for which momentum and energy are not necessarily conserved over the impact in the Hamiltonian formalism. A general introduction to impacts is given by [17] and [18]. The standard reference for inequality problems in mechanics is authored by Panagiotopoulos [19]. The Hamiltonian inclusion in the evolution of differential inclusions in variational calculus and optimal control is presented in [20]. The basics of nonsmooth variational analysis is presented in the classical work of [21] by Clarke. Mathematical fundamentals of nonsmooth mechanics are provided in the classical works of Moreau given in the references [22,23]. There is a vast literature on a multiplicity of methods which deal with the principle of stationary/least action for dissipative processes without impacts such as [24–27].

In this work, a smooth Riemannian configuration manifold $\mathcal{M}$, for which q denotes the n-tuple of generalized local coordinates is considered. The kinetic metric associated with $\mathcal{M}$ is given by $M(q)$ at each q. The generalized velocity of the system $\dot{q}$ lives in the tangent space of the manifold $T_{\mathcal{M}}(q)$. If the motion of the system is constrained to a submanifold of $\mathcal{M}$ denoted by the admissible set $\mathcal{C}$, then the tangent space $T_{\mathcal{M}}(q)$ is subdivided into a pair of cones $T_{\mathcal{C}}(q)$ and $T^{\perp}_{\mathcal{C}}(q)$, which are orthogonal to each other in the kinetic metric. The cotangent space is denoted by $N_{\mathcal{C}}(q)$ and the cones $T_{\mathcal{C}}(q)$ and $T^{\perp}_{\mathcal{C}}(q)$ are subspaces of $T_{\mathcal{M}}(q)$. It is assumed, that the constraint structure may differ in the pre-impact and post-impact phases. There one can distinguish among the following cases:

1. The impact may be induced by an abrupt change in the constraints such as blocking [16].
2. Nondifferentiability or nonsmoothness in the constraint (C^0 constraints) equations like in the example studied in [28].
3. The impactive process may supply enough work on the constraint structure, such that by exceeding a certain treshhold the holonomic constraints may change or disappear.

The total energy of the scleronomic holonomic Lagrangian system is given by its total mechanical energy:

$$H(q,p) = T(q,p) + V(q).$$

The differential measure of the total mechanical energy is given by:

$$dH(q,p) = \partial_t H(q,p)\,dt + \partial_\sigma H(q,p)\,d\sigma.$$

The absolutely continuous part of the measure dH is denoted by $\frac{dH}{dt}$. The singular part of dH, is represented as $\frac{dH}{d\sigma}$, where $d\sigma$ a regular Borel measure, and $\frac{dH}{d\sigma}$ is the Radon-Nikodym derivative of dH with respect to $d\sigma$. The Lebesgue-Stieltjes integration of the differential measure of the total mechanical energy over the impact time yields:

$$\int_{\{t_s\}} dH = H^+ - H^- = T^+ - T^- = L^+ - L^-. \tag{5}$$

The latter equality in (5) is due to the fact, that the potential energy V remains unaltered, since it only depends on the generalized positions, which remain constant over an impact. The Borel measurable part of the total mechanical energy E is related to the jump in the kinetic energy. The jump in the kinetic energy is:

$$\begin{aligned} T^+ - T^- &= \frac{1}{2}\left\langle p(t_s^+) + p(t_s^-), M^{-1}(q(t_s))\,\left(p(t_s^+) - p(t_s^-)\right)\right\rangle \\ &= \frac{1}{2}\left\langle \dot{q}(t_s^+) + \dot{q}(t_s^-), M(q(t_s))\,\left(\dot{q}(t_s^+) - \dot{q}(t_s^-)\right)\right\rangle . \end{aligned} \tag{6}$$

The existence of the generalized velocities at pre-impact and post-impact instants in the limiting sense and their nonexistence during impacts renders the kinetic energy undefined at an instant of an impact.

The evolution of a Lagrangian conservative system without discontinuities in the velocity/generalized conjugate momenta and under the influence of smooth forces subject to smooth constraints is generally given by the well-known framework:

$$M(q)\,\ddot{q} - N(q\,,\dot{q}\,) = \partial V(\,q\,) + \lambda_{\mathcal{C}}, \tag{7}$$
$$q \in \mathcal{C}, \quad \dot{q} \in T_{\mathcal{C}}(q), \tag{8}$$

where $\mathcal{C}$ is the abstract constraint set. Here $\lambda_{\mathcal{C}}$ denotes the constraint forces and the vector $N(\,q\,,\dot{q}\,)$ comprises corriolis and gyroscopic forces. If instantaneous jumps of acceleration, velocity and/or conjugate momenta is considered, one distinguishes between the right-continuous and left-continuous dynamics:

$$M(q)\,\ddot{q}^+ - N(q\,,\dot{q}^+\,) = \partial V(\,q\,) + \lambda_{\mathcal{C}^+}, \tag{9}$$
$$q \in \mathcal{C}, \quad \dot{q}^+ \in T_{\mathcal{C}^+}(q), \tag{10}$$

and

$$M(q)\,\ddot{q}^- - N(q\,,\dot{q}^-\,) = \partial V(\,q\,) + \lambda_{\mathcal{C}^-}, \tag{11}$$
$$q \in \mathcal{C}, \quad \dot{q}^- \in T_{\mathcal{C}^-}(q), \tag{12}$$

respectively, which correspond to forward and backward evolutions in time. The forward and backward evolutions coincide in the case of smooth motion. If discontinuities on acceleration and/or velocity level arise and the pre-impact and post-impact transition sets $\mathcal{C}^-$ and $\mathcal{C}^+$ differ, then in the variational analysis of the process, the consequences of the distinction among the forward and backward evolutions in time, must be considered. Out of these considerations, the author, focused on internal boundary variations, which lead via variational inequalities to the concept of discontinuous transversality conditions. This approach has been applied to derive necessary conditions in the impulsive optimal control of Lagrangian systems [29, 30] and in analyzing the principle of stationary action in the Lagrangian framwork such as [6, 16, 31, 32].

The regularity of the pre-impact and post-impact transition sets at the instant and position of impact is an assumption of local convexity. The irregularity of the constraint set at an instant of impact is visualized in mechanics in the form of inward/re-entrant corners as discussed in [33] and [34]. If the at the location of impactive transition the regularity is not present either at pre-transition and/or post-transition state, in the sense that the cotingent cone does not overlap with the tangent cone, then the obtained stationarity conditions are weakened to substationarity conditions, and the variational inequalities are termed as hemivariational or quasivariational inequalities [19].

The following main theorem is proven in this work for generalized positions from the space of absolutely continuous functions AC and for conjugate momenta from the space of locally bounded variation functions LBV:

Main Theorem: If there exist arcs $\tilde{q} \in AC_n[t_0, t_f]$ and $\tilde{p} \in LBV_n[t_0, t_f]$, impact position $\tilde{q}(\tilde{t}_s)$, pre-impact and post-impact conjugate momenta $\tilde{p}(\tilde{t}_s^-)$ and $\tilde{p}(\tilde{t}_s^+)$ at an impact time $\tilde{t}_s$ because of a impactive process at multiple contacts/locations, which induces the system, which moves in $\mathcal{C}^-$, to evolve on the constraint $\mathcal{C}^+$ and if these arcs provide for the action integral in (43) a maximizer, then the following conditions hold:

1. The Hamilton canonical equations on $[t_0, t_f]$ in the almost everywhere sense

$$\dot{\tilde{p}}_j = -\partial_{q_j} H(\tilde{q}, \tilde{p}), \quad \dot{\tilde{q}}_j = \partial_{p_j} H(\tilde{q}, \tilde{p}), \quad j = 1, 2, \ldots, n. \tag{13}$$

2. The conjugate momentum balance:

$$\tilde{p}^+ - \tilde{p}^- = D^+ \left(\tilde{q}(\tilde{t}_s)\right) \tilde{\Lambda}^+ + D^- \left(\tilde{q}(\tilde{t}_s)\right) \tilde{\Lambda}^- . \tag{14}$$

Here the matrices D^- and D^+, are the pre-impact and post-impact generalized impulse direction. The vectors $\tilde{\Lambda}^-$ and $\tilde{\Lambda}^+$ are Lagrange multipliers/impulses for which $\tilde{\Lambda}^+ \in N_{C^+}(q)$ and $\tilde{\Lambda}^- \in N_{C^-}(q)$ hold.

3. Bounded amount of energy is removed or added to the system.

The theory of subgradients and variational inequalities has found in the recent decades many fields of application. The concept of variational inequalities plays an important role in applied mathematics since its introduction by Hartman and Stampacchia [35]. For the systems which are considered in this work, it is assumed that the mechanical systems are well-posed, in the sense that the arcs do not possess accumulation points of impacts, like in the case of the jumping rigid ball, which is exposed to infinitely many impacts in finite time. An important well-posedness result states, that if the generalized velocities are of bounded variation (BV), then the impact times are countable as discussed in the references [36–41].

2 Internal Boundary Variations (IBV) and Discontinuous Transversality Conditions (DTC)

The Fermat rule for polynomials (or stationary principle) discovered in 1636, according to which gradients of differentiable functions must vanish at points of local minima and maxima, has provided a major means to find optima. If the classical Fermat rule is applied to the action integral, then the Hamilton's canonical necessary conditions are obtained. Non-smooth analysis provides more general criteria than the Fermat principle, because it enables the characterization of optima at which the functional is not differentiable in the classical sense. The extremality conditions at the boundaries of a dynamical process are covered by transversality conditions. The essential idea is to consider every point of the domain, where continuity ceases to exist, as a boundary of the problem. The discontinuous transversality conditions follow straightforwardly by the evaluation of the corresponding variational inequalities to the internal boundary variations. The boundary constitutes an upper boundary for one segment of the interval whereas for the other segment a lower boundary in the time domain at time t_s. The pre-transition and post-transition variations are interrelated by the transition conditions. Several families of variational curves, which are parameterized by a nonnegative ε, are introduced in order to generate the variations:

$$\begin{aligned}
p(t,\varepsilon) &= p(t) + \varepsilon\hat{p}(t) = p(t) + \delta p(t),\\
q(t,\varepsilon) &= q(t) + \varepsilon\hat{q}(t) = q(t) + \delta q(t),\\
q(t_s^+,\varepsilon) &= q(t_s^+) + \varepsilon\hat{q}(t_s^+) = q(t_s^+) + \delta q(t_s^+),\\
q(t_s^-,\varepsilon) &= q(t_s^-) + \varepsilon\hat{q}(t_s^-) = q(t_s^-) + \delta q(t_s^-),\\
t_s(\varepsilon) &= t_s + \varepsilon\hat{t}_s = t_s + \delta t_s.
\end{aligned}$$

The variations of the pre-, and post-transition positions at fixed time $\hat{q}(t_s^+)$, $\hat{q}(t_s^-)$ are the following Gâteaux derivatives:

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{p(t,\varepsilon)-p(t,0)}{\varepsilon} = \hat{p}, \quad \text{with}\, p(t,0) = q,\\
&\lim_{\varepsilon\downarrow 0}\frac{q(t,\varepsilon)-q(t,0)}{\varepsilon} = \hat{q}, \quad \text{with}\, q(t,0) = q,\\
&\lim_{\varepsilon\downarrow 0}\frac{q(t_s^+,\varepsilon)-q(t_s^+,0)}{\varepsilon} = \hat{q}(t_s^+), \quad \text{with}\, q(t_s^+,0) = q(t_s^+),\\
&\lim_{\varepsilon\downarrow 0}\frac{q(t_s^-,\varepsilon)-q(t_s^-,0)}{\varepsilon} = \hat{q}(t_s^-), \quad \text{with}\, q(t_s^-,0) = q(t_s^-),\\
&\lim_{\varepsilon\downarrow 0}\frac{t_s(\varepsilon)-t_s(0)}{\varepsilon} = \hat{t}_s, \quad \text{with}\, t_s(0) = t_s,
\end{aligned}$$

and are related with the total variations in these entities $\hat{q}_s^+$, $\hat{q}_s^-$ at the internal boundary by the following affine relations:

$$\hat{q}(t_s^+) = \hat{q}_s^+ - \dot{q}(t_s^+)\,\hat{t}_s, \tag{15}$$
$$\hat{q}(t_s^-) = \hat{q}_s^- - \dot{q}(t_s^-)\,\hat{t}_s. \tag{16}$$

The variations of the post-transition and pre-transition positions are given by:

$$\delta q_s^+ = \delta q\left(t_s^+\right) + \dot{q}\left(t_s^+\right)\,\delta t_s, \tag{17}$$
$$\delta q_s^- = \delta q\left(t_s^-\right) + \dot{q}\left(t_s^-\right)\,\delta t_s. \tag{18}$$

The equations (17) and (18) show, that the total variations in the pre-, and post-transition positions consist of two parts, namely, the variations at fixed time $\delta q\left(t_s^+\right)$ and $\delta q\left(t_s^-\right)$; and the variations due to the transition time t_s. Since the generalized position does not change, their respective pre-transition and post-transition variations equal each other:

$$\delta q_s^+ = \delta q_s^-. \tag{19}$$

If the decomposition of the boundary variations is considered, the equality in (19) means, that:

$$\delta q(t_s^+) + \dot{q}(t_s^+)\delta t_s = \delta q(t_s^-) + \dot{q}(t_s^-)\delta t_s. \tag{20}$$

The tangent cone to a set $\mathcal{C}$ at a given point x in the regular case is:

$$T_{\mathcal{C}}(x) = \left\{ y \in K \,|\, \lim_{\tau \downarrow 0} \frac{d_{\mathcal{C}}(x+\tau y)}{\tau} = 0 \right\}. \tag{21}$$

Here $d_{\mathcal{C}}(x)$ denotes, the distance function of the point x to the set $\mathcal{C}$, where it takes the value zero, if and only if $x \in \mathcal{C}$. The tangent cone $T_{\mathcal{C}}(x)$ to a set $\mathcal{C} \subset K$ is polar to a certain nonempty convex cone $N_{\mathcal{C}}(x)$ in the dual space K^*:

$$N_{\mathcal{C}}(x) = \{ z \in K^* \,|\, \langle y, z \rangle \leq 0, \quad y \in T_{\mathcal{C}}(x) \}. \tag{22}$$

The allowable pre-impact and post-impact position variations are limited to:

$$\delta q_s^+ \in T_{\mathcal{C}^+}\left(q(t_s^+)\right), \quad \delta q_s^- \in T_{\mathcal{C}^-}\left(q(t_s^-)\right). \tag{23}$$

The continuity of the positions requires the equality of the pre-impact and post-impact position variations:

$$\delta q_s^+ = \delta q_s^- = \delta q_s. \tag{24}$$

According to (24), δq_s fulfills both the pre-impact and post-impact conditions:

$$\delta q_s \in T_{\mathcal{C}^+} \wedge \delta q_s \in T_{\mathcal{C}^-} \Rightarrow \delta q_s \in T_{\mathcal{C}^+}(q_s) \cap T_{\mathcal{C}^-}(q_s). \tag{25}$$

The following set relations hold:

$$T_{\mathcal{C}^+}(q_s) \cap T_{\mathcal{C}^-}(q_s) \equiv T_{\mathcal{C}^+ \cap \mathcal{C}^-}(q_s), \tag{26}$$
$$N_{\mathcal{C}^+ \cap \mathcal{C}^-}(q_s) \equiv N_{\mathcal{C}^+}(q_s) \oplus N_{\mathcal{C}^-}(q_s). \tag{27}$$

Here $\oplus$ denotes the set addition. The equality (24) means, that:

$$\delta q\left(t_s^+\right) - \delta q\left(t_s^-\right) = \left(\dot{q}\left(t_s^+\right) - \dot{q}\left(t_s^-\right)\right)\delta t_s \tag{28}$$

must hold in general. Consider the approximation $q(t+\varepsilon) \approx q(t) + \varepsilon\hat{q}(t)$ of the position q at a $t+\varepsilon$ with $\varepsilon \geq 0$, which should fulfill $q(t+\varepsilon) \in \mathcal{C}$ or $d_{\mathcal{C}}(q(t+\varepsilon)) = 0$. The condition $d_{\mathcal{C}}(q(t+\varepsilon)) = 0$ holds, if and only if the infinitesimal arc, that connects $q(t)$ and $q(t+\varepsilon)$ fulfills:

$$\lim_{\varepsilon \downarrow 0} \frac{d_{\mathcal{C}}(q(t)+\varepsilon\hat{q}(t)) - d_{\mathcal{C}}(q(t))}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{d_{\mathcal{C}}(q(t)+\varepsilon\hat{q}(t))}{\varepsilon} = 0, \tag{29}$$

for $d_{\mathcal{C}}(q(t)) = 0$, which requires:

$$\delta q \in T_{\mathcal{C}}(q). \tag{30}$$

By a similar argument the following condition is valid in phases of motion, where the generalized velocities are continuous:

$$\dot{q} \in T_{\mathcal{C}}(q). \tag{31}$$

At an instant of velocity jump, which may be accompanied by alteration in the pre-impact and post-impact constraint sets, it is necessary to distinguish among forward and backward dynamics:

$$\dot{q}^+ \in T_{\mathcal{C}^+}\left(q(t_s^+)\right), \quad \dot{q}^- \in T_{\mathcal{C}^-}\left(q(t_s^-)\right). \tag{32}$$

The allowable pre-impact and post-impact position variations are expressed as:

$$\delta q_s^+ \in T_{\mathcal{C}^+}\left(q(t_s^+)\right), \quad \delta q_s^- \in T_{\mathcal{C}^-}\left(q(t_s^-)\right). \tag{33}$$

The equality $\delta q_s^+ = \delta q_s^- = \delta q_s$ requires, that:

$$\delta q_s \in T_{\mathcal{C}^+ \cap \mathcal{C}^-}\left(q(t_s^+)\right). \tag{34}$$

The allowable pre-impact and post-impact velocity variations are expressed as:

$$\delta \dot{q}_s^+ \in T_{T_{\mathcal{C}^+}\left(q(t_s^+)\right)}\left(\dot{q}(t_s^+)\right), \quad \delta \dot{q}_s^- \in T_{T_{\mathcal{C}^-}\left(q(t_s^-)\right)}\left(\dot{q}(t_s^-)\right). \tag{35}$$

Analogously, the spaces of the variations of the pre-impact and post-impact conjugate momenta read:

$$\delta p^+ \in N^{\perp}_{T_{\mathcal{C}^+}\left(q(t_s^+)\right)}\left(p(t_s^+)\right), \quad \delta p^- \in N^{\perp}_{T_{\mathcal{C}^-}\left(q(t_s^-)\right)}\left(p(t_s^-)\right). \tag{36}$$

The set in $T_{\mathcal{C}^+ \cap \mathcal{C}^-}(q(t_s^+))$ should cover all possible candidate directions for the internal boundary variations, so that the obtained extremizing arc is the extremizer over all comparison curves. This property is guaranteed by the tangential regularity of the transition sets.

Lemma [21]: Let $\mathcal{C} = \{x | f_i(x) \leq 0, i = 1, \ldots, n.\}$ and let x be such that $f_i(x) = 0, i = 1, \ldots, n$. Then, if each f_i is strictly differentiable at x, and if all $\nabla_x f_i$ are positively linearly independent, it follows that $\mathcal{C}$ is tangentially regular at x, and one has:

$$N_{\mathcal{C}}(x) = \left\{ \sum_{i=1}^{n} \lambda_i \nabla_x f_i(x) \,|\, \lambda_i \geq 0, \quad i = 1, \ldots, n \right\}. \tag{37}$$

It is assumed, that the constraint sets $\mathcal{C}^+$ and $\mathcal{C}^-$ are regular, and the impactive process does not impair thereby the regularity of the post-impact set.

3 Stationarity of the Impulsive Action Integral

The functional $f : X \to \mathbb{R}$, which is defined on arbitrary Banach spaces, is said to be Lipschitz of rank K near a given point $x \in X$, if for some $\varepsilon > 0$, there exists a neighborhood $\mathbb{B}(x, \varepsilon)$, such that

$$|f(y) - f(z)| \leq K \parallel y - z \parallel, \quad \forall y, z \in \mathbb{B}(x, \varepsilon), \tag{38}$$

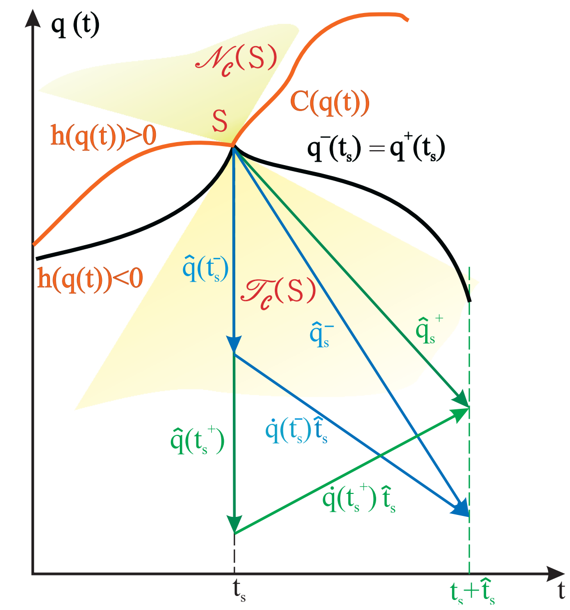

Fig. 1. General decomposition of boundary variations at an internal boundary with a constraint for the re-entrant (nonregular) case in an elastic impact.

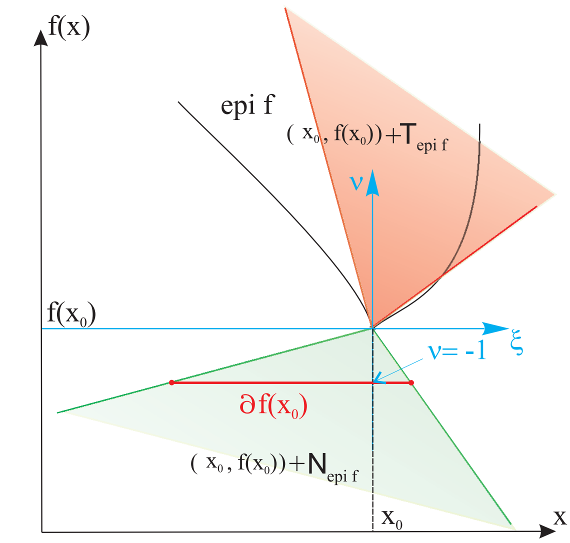

Fig. 2. The subdifferential of a Lipschitz function f at its minimum.

holds, where $\mathbb{B}(x, \varepsilon)$ is the ball with radius ε and center x. The functional $f : X \to \mathbb{R}$ is convex, provided that, for all y and z in X and $\gamma \in [0,1]$, one has:

$$f(\gamma y + (1-\gamma)z) \leq \gamma f(y) + (1-\gamma)f(z). \tag{39}$$

There is an equivalent definition of the subdifferential through an inequality. The directional derivative of $f(x)$, if $f(x)$ is Lipschitz around x is defined by:

$$f^0(x;y) = \limsup_{\substack{x' \to x \\ \varepsilon \downarrow 0}} \frac{f(x' + \varepsilon y) - f(x')}{\varepsilon}. \tag{40}$$

The subdifferential is, then defined alternatively by:

$$\partial f(x) := \left\{ z \in Z \,|\, f^0(x;y) \geq \langle z, y \rangle, \forall y \in X \right\}. \tag{41}$$

If f is continuously differentiable, then $\partial f(x)$ consists of a single element, namely, $\partial f(x) = \{\nabla_x f(x)\}$. The function $y \to f^0(x;y)$ is finite, positively homogeneous, and subadditive on X, and satisfies $|f^0(x;y)| \leq K \| y \|$. The directional derivative $f^0(x;y)$ is as a function of y Lipschitz of rank K on X. The following relation holds: $-f^0(x;y) = f^0(x;-y)$. [21]. If f attains a local mininum or maximum at x then the zero vector is an element of the subdifferential: $0 \in \partial f(x)$. For every y in X, the directional majorizes the following expression:

$$f^0(x;y) = \max \{ \langle \xi, y \rangle \, \forall \xi \in \partial f(x) \}. \tag{42}$$

The condition of a maximum (minimum) requires: $f^0(x;y) \leq 0 \quad (f^0(x;y) \geq 0)$. The above framework is applicable to integral functionals [21]. Having set the stage, the impulsive action integral becomes:

$$\begin{aligned} J(q,p,t_s) = &\int_{t_0}^{t_s^-} \langle p, \dot{q} \rangle - H(q,p)\, ds \\ &+ \int_{t_s^+}^{t_f} \langle p, \dot{q} \rangle - H(q,p)\, ds = J_1(q,t_s) + J_2(q,t_s). \end{aligned} \tag{43}$$

The regularity of the functional in (43) is guaranteed by the regularity of the integrands and of the transition sets $\mathcal{C}^+$ and $\mathcal{C}^-$.

3.1 Proof of the Main Theorem

Let $q(t) \in AC^1[t_0,t_f]$ be an arc and t_s be an transition time for which the action integral $J(q(t), p(t), t_s)$ is well-defined and finite. The arc $\tilde{q}(t) \in AC^1[t_0,t_f]$ and $\tilde{t}_s \in \mathbb{R}$ is a weak local maximum for (43), if there exist $\varepsilon > 0$ and $\varepsilon_t > 0$ such that every $\hat{q}(t) \in AC^1[t_0,t_f]$ with $\|\hat{q}(t)\|_\infty + \|\hat{p}(t)\|_\infty < \varepsilon$ and $\|\hat{t}\| < \varepsilon_t$ gives rise to a well-defined objective value $J(\tilde{q}(t) + \varepsilon\hat{q}(t), \tilde{p}(t) + \varepsilon\hat{p}(t), \tilde{t}_s + \varepsilon_t\hat{t}_s)$, which satisfies

$$J(\tilde{q}(t) + \varepsilon\hat{q}(t), \tilde{p}(t) + \varepsilon\hat{p}(t), \tilde{t}_s + \varepsilon_t\hat{t}_s) \leq J(\tilde{q}(t), \tilde{p}(t), \tilde{t}_s). \tag{44}$$

If there exist arcs $\tilde{q}$ and $\tilde{p}$, transition position $\tilde{q}(\tilde{t}_s)$, pre-transition and post-transition conjugate momenta $\tilde{p}(\tilde{t}_s^-)$ and $\tilde{p}(\tilde{t}_s^+)$ at a transition time $\tilde{t}_s$, which all together maximize the functional in (43), such that the value functional assumes the finite value $\tilde{J}(\varepsilon = 0) = J(\tilde{q}, \tilde{t}_s)$, then the following variational inequality is fulfilled:

$$\sum_{\forall \hat{\psi}_j} \tilde{J}^0(\cdot; \varepsilon\,\hat{\psi}_j) \leq 0, \forall \hat{\psi}_j \in \{\hat{q}(t_s), \hat{t}_s\} \bigcup \{\hat{q}, \hat{p}\}, \tag{45}$$

since J is subdifferentially regular at any extremal solution. The regularity of the integrands is due to their strict differentiability with respect to their arguments. Here the internal boundary variations are given by $\hat{q}(t_s)$ and $\hat{t}_s$. The following one parameter functionals are defined:

$$G_i(\varepsilon) = \langle p + \varepsilon\hat{p}, \dot{q} + \varepsilon\dot{\hat{q}} \rangle - H_i(q + \varepsilon\hat{q}, p + \varepsilon\hat{p}), \quad i = 1,2, \tag{46}$$

in order to investigate the validity of the following stationarity condition:

$$\limsup_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\left[\int_{t_0}^{t_s^- + \varepsilon\hat{t}} G_1(\varepsilon)\,ds - \int_{t_0}^{t_s^-} G_1(0)\,ds\right] \tag{47}$$

$$\begin{aligned}
&+ \frac{1}{\varepsilon}\left[\int_{t_s^+ + \varepsilon\hat{t}}^{t_f} G_2(\varepsilon)\,ds - \int_{t_s^+}^{t_f} G_2(0)\,ds\right] = \\
&\limsup_{\varepsilon \to 0^+} \int_{t_0}^{t_s^-} \frac{G_1(\varepsilon) - G_1(0)}{\varepsilon}\,ds + \int_{t_s^+}^{t_f} \frac{G_2(\varepsilon) - G_2(0)}{\varepsilon}\,ds \\
&+ \int_{t_s^-}^{t_s^- + \varepsilon\hat{t}} \frac{G_1(\varepsilon)}{\varepsilon}\,ds + \int_{t_s^+ + \varepsilon\hat{t}}^{t_s^+} \frac{G_2(\varepsilon)}{\varepsilon}\,ds = \\
&\int_{t_0}^{t_s^-} \limsup_{\varepsilon \to 0^+} \frac{G_1(\varepsilon) - G_1(0)}{\varepsilon}\,ds + \int_{t_s^-}^{t_s^- + \varepsilon\hat{t}} \limsup_{\varepsilon \to 0^+} \frac{G_1(\varepsilon)}{\varepsilon}\,ds + \\
&\int_{t_s^+}^{t_f} \limsup_{\varepsilon \to 0^+} \frac{G_2(\varepsilon) - G_2(0)}{\varepsilon}\,ds + \int_{t_s^+ + \varepsilon\hat{t}}^{t_s^+} \limsup_{\varepsilon \to 0^+} \frac{G_2(\varepsilon)}{\varepsilon}\,ds \\
&= \int_{t_0}^{t_s^-} \langle \hat{p}, \dot{q} \rangle + \langle p\,\dot{\hat{q}} \rangle - \left\langle \frac{\partial H_1}{\partial q}, \hat{q} \right\rangle - \left\langle \frac{\partial H_1}{\partial p}, \hat{p} \right\rangle ds \\
&+ \int_{t_s^+}^{t_f} \langle \hat{p}, \dot{q} \rangle + \langle p, \dot{\hat{q}} \rangle - \left\langle \frac{\partial H_2}{\partial q}, \hat{q} \right\rangle + \left\langle \frac{\partial H_2}{\partial p}, \hat{p} \right\rangle ds \\
&+ \langle p(t_s^-)\,\dot{q}(t_s^-) \rangle\,\hat{t} - \langle p(t_s^+), \dot{q}(t_s^+) \rangle\,\hat{t} + H_1(q(t_s^-), p(t_s^-))\,\hat{t} \\
&- H_2(q(t_s^+), p(t_s^+))\,\hat{t} \leq 0.
\end{aligned} \tag{48}$$

The lemma of Raymond-Dubois [42] states:

$$\langle p(s), \hat{q}(s) \rangle \big|_a^b = \int_a^b \langle \dot{p}(s), \hat{q}(s) \rangle + \langle p(s), \dot{\hat{q}}(s) \rangle\,ds. \tag{49}$$

By inserting (49) in (47) the following is obtained:

$$\begin{aligned}
&\int_{t_0}^{t_s^-} \left\langle \dot q(s) - \partial_p H_1 \,, \hat p(s)\right\rangle - \left\langle \partial_q H_1 + \dot p(s) \,, \hat q(s)\right\rangle \mathrm{d}s \\
&+ \int_{t_s^+}^{t_f} \left\langle \dot q(s) - \partial_p H_2 \,, \hat p(s)\right\rangle - \left\langle \partial_q H_2 + \dot p(s) \,, \hat q(s)\right\rangle \mathrm{d}s \\
&+ \langle p(s), \hat q(s)\rangle \big|_{t_s^+}^{t_f} + \langle p(s), \hat q(s)\rangle \big|_{t_0}^{t_s^-} \\
&\left\langle p(t_s^-) \,, \dot q(t_s^-)\right\rangle \hat t - \left\langle p(t_s^+) \,, \dot q(t_s^+)\right\rangle \hat t \\
&+ H_1(q(t_s^-), p(t_s^-)) - H_2(q(t_s^+), p(t_s^+))\,\hat t \leq 0.
\end{aligned} \tag{50}$$

The insertion of (17) and (18) into (50) yields:

$$\begin{aligned}
&\int_{t_0}^{t_s^-} \left\langle \dot q(s) - \partial_p H_1 \,, \hat p(s)\right\rangle - \left\langle \partial_q H_1 + \dot p(s) \,, \hat q(s)\right\rangle \mathrm{d}s \qquad (51)\\
&+ \int_{t_s^+}^{t_f} \left\langle \dot q(s) - \partial_p H_2 \,, \hat p(s)\right\rangle - \left\langle \partial_q H_2 + \dot p(s) \,, \hat q(s)\right\rangle \mathrm{d}s \\
&\left[\left\langle p(t_s^-) - p(t_s^+) \,, \hat q_s\right\rangle + H_1^- - H_2^+\right] \hat t \leq 0.
\end{aligned}$$

in the limit. By the application of Fatou's Lemma on the integral in (51) the Hamilton's canonical equations are obtained in the almost everywhere sense as given in (13). The directional derivative of J_2 in the direction δq_s^+ is:

$$J_2^0\left(\cdot; \varepsilon \hat q_s^+\right) = \left\langle -p(t_s^+), \delta q_s^+\right\rangle.$$

The directional derivative of J_1 in the direction δq_s^- is:

$$J_1^0\left(\cdot; \varepsilon \hat q_s^-\right) = \left\langle p(t_s^-), \delta q_s^-\right\rangle.$$

The validity of (24) relates their directional derivatives under regularity of J_2 and J_1 at $(q(t_s), t_s)$:

$$J^0\left(\cdot; \varepsilon \hat q_s\right) = J_1^0\left(\cdot; \varepsilon \hat q_s^-\right) + J_2^0\left(\cdot; \varepsilon \hat q_s^+\right). \tag{52}$$

The condition for a maximum is:

$$J^0\left(\cdot; \varepsilon \hat q_s\right) \leq 0, \quad \forall \hat q_s \in T_{(\mathcal{C}^+ \cap \mathcal{C}^-)}(\tilde q). \tag{53}$$

By making use of definition (22), this optimality condition is fulfilled for:

$$\tilde p(t_s^-) - \tilde p(t_s^+) \in N_{(\mathcal{C}^+ \cap \mathcal{C}^-)}(\tilde q). \tag{54}$$

The inclusion (54) is equivalently expressed as:

$$\tilde p(t_s^+) - \tilde p(t_s^-) = \tilde D^+ \tilde \Lambda^+ + \tilde D^- \tilde \Lambda^-. \tag{55}$$

The directional derivative of J_2 in the direction δt_s is given by:

$$J_2^0\left(\cdot; \varepsilon \hat t_s\right) = H^+ \delta t_s \tag{56}$$

where H^+ is the post-impact Hamiltonian. The directional derivative of J_1 in the direction δt_s is given by:

$$J_1^0(\cdot;\varepsilon\hat{t}_s) = -H^-\,\delta t_s, \tag{57}$$

where H^- is the pre-impact Hamiltonian. Combining (56) and (57) yields:

$$J^0(\cdot;\varepsilon\hat{t}_s) = J_1^0(\cdot;\varepsilon\hat{t}_s) + J_2^0(\cdot;\varepsilon\hat{t}_s) = \left(H^+ - H^-\right)\delta t_s. \tag{58}$$

The variational inequality pertaining to the impact time variation yields the optimality condition:

$$J^0(\cdot;\varepsilon\hat{t}_s) = \left(H^+ - H^-\right)\delta t_s \leq 0, \tag{59}$$

3.2 The Consistency Conditions on the Time Variation in Different Scenarios

By the property, that boundary variations of the generalized positions consist of two components, which are independent of each other as in (17) and (18), it is assumed that:

$$\delta q(t_s^+) \in T_{\mathcal{C}^+}\left(q(t_s^+)\right) \;\wedge\; \dot{q}(t_s^+)\,\delta t_s \in T_{\mathcal{C}^+}\left(q(t_s^+)\right), \tag{60}$$

$$\delta q(t_s^-) \in T_{\mathcal{C}^-}\left(q(t_s^-)\right) \;\wedge\; \dot{q}(t_s^-)\,\delta t_s \in T_{\mathcal{C}^-}\left(q(t_s^-)\right). \tag{61}$$

4 Elastic Rigid Body Collisions

This is the most studied case in impact mechanics. A given vector-valued function $g(q)$ ($h(q) = -g(q)$) represents the shortest distances between the rigid bodies in the system and these distances are always nonnegative (nonpositive) due to the impenetrability assumption. If elements of g become zero, then contact among rigid bodies occurs and the mechanical system reaches the boundary of the admissible set $\mathcal{C}$. The signed distance vector $g(q)$ is differentiable in the classical sense. At the instant of a multi-impact at m contacts, at which $g(q) = 0$ is valid, the pre-impact and the post-impact normal relative velocities are given by:

$$v^- = {}^-D^T(q)\,\dot{q}^- \quad \text{and} \quad v^+ = {}^+D^T(q)\,\dot{q}^+, \tag{62}$$

respectively. The linear operator $D(q) \in \mathbb{R}^{n\times m}$ is defined as $D(q) = \nabla_q\, g(q)$. The linear operator $D(q)$ is assumed to have full rank with $m \leq n$.

In rigid body collisions one has a nonpositive pre-impact relative velocities in the approach phase, and nonnegative post-impact relative velocities due to the rigidity or impenetrability condition, which are stated in (62). If the impactive transition sets $\mathcal{C}^+$ and $\mathcal{C}^-$ are defined by:

$$\mathcal{C}^+ = \left\{q(t_s^+) \,|\, h^+(q(t_s)) \leq 0\right\}, \tag{63}$$

$$\mathcal{C}^- = \left\{q(t_s^-) \,|\, h^-(q(t_s)) \leq 0\right\}, \tag{64}$$

respectively, then the position variations at pre-impact and post-impact instants, are in the respective tangent cones:

$$T_{\mathcal{C}}\left(q\left(t_s^+\right)\right) = T_{\mathcal{C}^+} = \left\{\xi \,|\, {}^+D^T(q(t_s))\,\xi \geq 0\right\}, \tag{65}$$

$$T_{\mathcal{C}}\left(q\left(t_s^-\right)\right) = T_{\mathcal{C}^-} = \left\{\xi \,|\, {}^-D^T(q(t_s))\,\xi \geq 0\right\}, \tag{66}$$

respectively, if $h(q(t_s^+)) = 0$ and/or $h(q(t_s^-)) = 0$. The normal cones of the sets $\mathcal{C}^+$ and $\mathcal{C}^-$ become:

$$N_{\mathcal{C}^+}(q(t_s)) = \left\{-{}^+D\left(q\left(t_s^+\right)\right)\Lambda^+ \,|\, \Lambda^+ \geq 0\right\},$$

$$N_{\mathcal{C}^-}(q(t_s)) = \left\{-{}^-D\left(q\left(t_s^-\right)\right)\Lambda^- \,|\, \Lambda^- \geq 0\right\},$$

respectively.

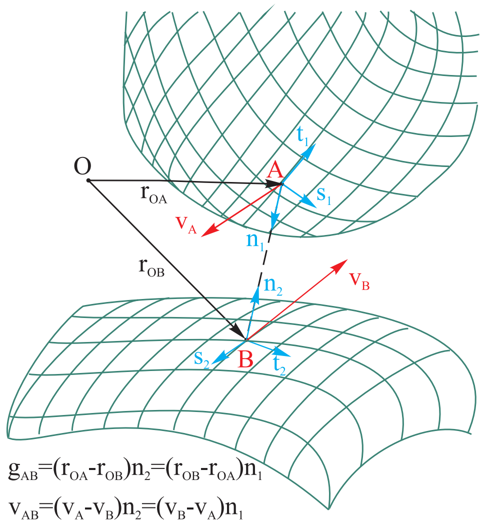

Fig. 3. General contact kinematics.

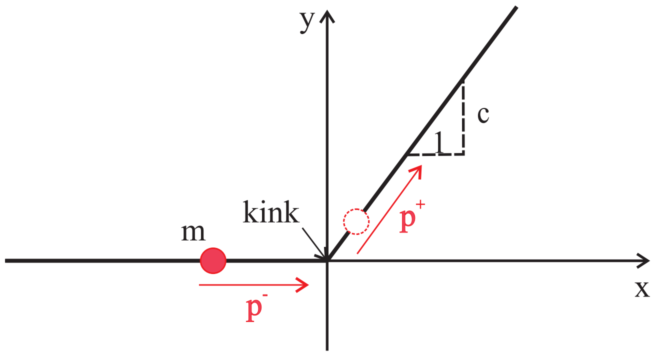

Fig. 4. Point mass moving along a nonsmooth constraint.

For the variational parts pertaining to impact time variations, the inclusions in (60) and (61) mean, that:

$$ {}^{+}D^{T}\left(q\left(t_s^{+}\right)\right)\dot{q}_s^{+}\delta t_s \geq 0, \quad {}^{-}D^{T}\left(q\left(t_s^{-}\right)\right)\dot{q}_s^{-}\delta t_s \geq 0. \tag{67} $$

These two conditions impose further restrictions on impact time variations:

$$ {}^{+}D^{T}\left(q(t_s^{+})\right)\dot{q}_s^{+} \geq 0 \wedge \dot{q}(t_s^{+})\delta t_s \in T_{C^{+}} \Rightarrow \hat{t}_s \geq 0, \tag{68} $$

$$ {}^{-}D^{T}\left(q(t_s^{-})\right)\dot{q}_s^{-} \leq 0 \wedge \dot{q}(t_s^{-})\delta t_s \in T_{C^{-}} \Rightarrow \hat{t}_s \leq 0. \tag{69} $$

The conditions in (68) and (69) clearly require, that in case of collisional impacts the impact time variations must vanish $\hat{t}_s = 0$, in order to maintain the consistency of the internal boundary variations. In the case of collisional impacts we further have in addition to (24):

$$ \delta q(t_s^{+}) = \delta q(t_s^{-}). \tag{70} $$

The equality of position variations at fixed impact time is due to the vanishing impact time variations. which is fulfilled for arbitrary energy jumps $H^{+} - H^{-}$, because the consistency of the internal boundary variations require $\delta t_s = 0$. The unboundedness of the energy change $H^{+} - H^{-}$ would contradict the assumption, that the generalized velocities belong to the class of bounded variation functions, in view of the fact, that the jump in the total energy and the magnitude of the jump of the pre-impact and post-impact generalized velocities are related by (6). Due to the rigidity assumption the pre-impact and post-impact transition sets are identical, so that for the generalized impulse directions $D^{-} = D^{+}$ holds, and the impact equation takes the well-known form:

$$ \tilde{p}^{+} - \tilde{p}^{-} = D\left(\tilde{q}(\tilde{t}_s)\right)\left(\tilde{\Lambda}^{+} + \tilde{\Lambda}^{-}\right), \tag{71} $$

5 Impactive processes arising Due to Blocking and Nonsmooth Constraints

The case is investigated by the author in ([16]) by making use of DTC and its similarity to fully inelastic impacts is stated. At an instant at of blocking, the directions characterized by D^{+}, are after the transition time, abruptly closed for evolution, which requires $D^{+}\dot{q} = 0$ to hold. In this case the consistency conditions on the time variations (60) and (61) translate into:

$$ {}^{+}D^{T}\left(q(t_s^{+})\right)\dot{q}_s^{+} = 0 \wedge \dot{q}(t_s^{+})\delta t_s \in T_{C_2^{+}} \Rightarrow \hat{t}_s \,\text{free}, \tag{72} $$

$$ {}^{+}D^{T}\left(q(t_s^{-})\right)\dot{q}_s^{-} \leq 0 \wedge \dot{q}(t_s^{-})\delta t_s \in T_{C_1^{+}} \Rightarrow \hat{t}_s \leq 0. \tag{73} $$

The conditions (72) and (73) require that $\hat{t}_s \leq 0$. If this is considered, together with the stationarity condition (59), then an decrease in the total mechanical energy is required: $H^- \geq H^+$. The energy decrease is in accordance with the physics of the process [16]. The post-impact tangent cone is given by:

$$T_{C_2}\left(q\left(t_s^+\right)\right) = T_{C_2^+} = \left\{\xi | {}^+D^T\left(q\left(t_s\right)\right)\xi = 0\right\}, \tag{74}$$
$$T_{C_1}\left(q\left(t_s^+\right)\right) = T_{C_1^+} = \left\{\xi | {}^+D^T\left(q\left(t_s\right)\right)\xi \geq 0\right\}. \tag{75}$$

Consider the motion of a mass particle without friction and gravity along an ideal holonomic constraint with a kink at the origin as shown in figure (4). The particle is supposed to move along the line $g_1(x,y) : y = 0$ until the origin, and to follow the line $g_2(x,y) : y - cx = 0$ beginning at the origin. It has a pre-impact velocity of $\dot{x}^-$ to the right. The impactive process is modelled by a release of the constraint $g_1(x,y)$ and blocking of contraint $g_2(x,y)$ at the kink. The pre-impact and post-impact generalized directions become:

$$D^- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad D^+ = \begin{pmatrix} -c \\ 1 \end{pmatrix}, \tag{76}$$

In [16] it is shown, that the change in the conjugate momentum and the total mechanical energy for a releasing-blocking type impactive action is given by:

$$p^+ - p^- = -D^+ G_{bb}^{-1} v^- = D^+ \Lambda^+ + D^- \Lambda^-. \tag{77}$$

and

$$H^+ - H^- = -\frac{1}{2}\left\langle v^-, G_{bb}^{-1} v^-\right\rangle, \tag{78}$$

respectively, where the Delassus' operator G_{bb} is given by: $G_{bb} = {}^+D^T M^{-1} D^+$ and v^- denotes the pre-impact relative velocity with respect to the blocking constraint: $v^- = D^+ \dot{q}^-$. Substituting relevant entities to the example in the relations (78) and (77) the following is obtained:

$$H^+ - H^- = -\frac{mc^2}{2(1+c^2)}(\dot{x}^-)^2, \quad p^+ - p^- = \begin{pmatrix} -\frac{c^2}{(1+c^2)} \\ \frac{c}{(1+c^2)} \end{pmatrix} m\dot{x}^- \tag{79}$$

such that the conjugate momenta after the impact become:

$$p_x^+ = \frac{m}{(1+c^2)}\dot{x}^-, \quad p_y^+ = \frac{mc^2}{(1+c^2)}\dot{x}^- \tag{80}$$

and the impulses are given by:

$$\Lambda^- = 0, \quad \Lambda^+ = \frac{mc\dot{x}^-}{(1+c^2)}. \tag{81}$$

The decrease in energy is correctly characterized by the discontinuous transversality conditions, and the right-hand side of the impact equation describes an impactive process for which the change the constraint structure induces an impact. In the case of nonsmooth constraints and blocking action, the impact equation is driven by the post-impact impulses.

6 Impacts accompanied by Alteration in the Mass Structure

Consider for example a stellar or atomic scale collision process in which one of the participating objects disappear through energy emittance and/or the compressive phase impulses exceeds a certain treshhold such that one "bursts". Due to the nonexistence of a post-impact constraint structure $\mathcal{C}^+$, one is only left with the condition (61), which translates in this case into:

$$^{-}\mathrm{D}^{\mathrm{T}}\left(\mathrm{q}(\mathrm{t}_s^-)\right)\dot{\mathrm{q}}_s^- \leq 0 \wedge \dot{\mathrm{q}}(\mathrm{t}_s^-)\,\delta\mathrm{t}_s \in \mathrm{T}_{\mathcal{C}^-} \Rightarrow \hat{\mathrm{t}}_s \leq 0. \tag{82}$$

The condition (82) requires that $\hat{\mathrm{t}}_s \leq 0$. If this is considered, together with the stationarity condition (59), then an decrease in the total mechanical energy is required: $\mathrm{H}^- \geq \mathrm{H}^+$. The energy decrease is here interpretable, as the activation energy required to dissolve the other particle in the form of compressive work in the sense of a Poisson type impact process and the conversion to other types of energies from mechanical energy. The impact equation here involves only the compression/pre-impact part:

$$\tilde{\mathrm{p}}^+ - \tilde{\mathrm{p}}^- = {}^{-}\mathrm{D}\left(\tilde{\mathrm{q}}(\tilde{\mathrm{t}}_s)\right)\tilde{\Lambda}^-. \tag{83}$$

7 Discussion and Conclusions

There are impactive processes in which the mass structure may not be preserved, which is among others the case if atomic particles collide. The discontinuity in the conjugate momenta arises due to the sudden change in the mass distributions, which is accompanied by absorption or emittance of energy. Another generalization of the impact law is achieved by considering the change in the constraint structure. The sudden change in the constraint structure may induce impacts as studied in [16] and [28]. As a consequence the pre-impact and post-impact generalized impulsive force directions D^+ and D^- differ from each other. The introduced Hamiltonian framework is capable of providing an impact equation which able to cope with impactive processes, for which:

1. The energy is not conserved, and dissipative impacts become eligible.
2. The mass distribution may change abruptly.
3. The constraint structure due to addition and removal of constraints is altered suddenly.

The impact equation which is obtained by the proposed variational formulation is compatible with the most common impact equations such as Newton's or Poisson's impact law. The Poisson's impact law, which requires to distinguish between compression phase and decompression phase impulsive forces Λ^- and Λ^+, which are interrelated by a restitution coefficient of in the form $\Lambda^+ = \varepsilon_p \Lambda^-$, is directly structurally adaptable to in the impactive process due to the clear distinction between the pre-impact and post-impact phases. Different then in many preceeding studies, which were limited to energy conserving impactive processes, the variational framework for the Hamiltonian formulation enables the implementation of arbitrary restitution coefficients with Newton's and Poisson's impact laws.

The stationarity conditions of a nonsmooth action integral are obtained in the form of maximizing conditions also in [10] and [11], where the vanishing on the variations of impact time is recognized, as the main mechanism for nonconserving impacts but the mechanism that renders the time variations at impact zero, has been not known, until in [6,32] the consistency conditions on the time variations related the time variations directly to the physics of the process. The equation (28) is in comparison to the following assumptions

$$\delta\mathrm{q}\left(\mathrm{t}_s^+\right) = \dot{\mathrm{q}}\left(\mathrm{t}_s^+\right)\delta\mathrm{t}_s, \quad \delta\mathrm{q}\left(\mathrm{t}_s^-\right) = \dot{\mathrm{q}}\left(\mathrm{t}_s^-\right)\delta\mathrm{t}_s, \tag{84}$$

which are used in the works [13] and [8], an improvement, because it enables nonconservative impactive processes to become extremizing arcs. Given the central role of the consistency conditions on the time variations in adapting the energetical requirements to the physics of the impactive process, the assumption (15) is a restriction to only conservative impacts.

The Weierstrass-Erdmann [43] conditions and the discontinuous transversality conditions differ in their philosophical approach to corner points fundamentally. The Weierstrass-Erdmann conditions are internalizing the corner points into the domain, whereas the discontinuous transversality conditions are externalizing the corner conditions by assigning them to boundaries and utilizing corresponding tools of the variational calculus. The thorough evaluation of the internal boundary variations for collisional impacts of this type showed that the variation of impact time vanishes, which leaves the energetical behaviour over an impact unconstrained except, that it must be bounded. It is straightforward to show, that the conditions of the main theorem are valid also in the strong norm. A detailed account and a comparison between the Weierstrass-Erdmann conditions and Discontinuous Transversality conditions for impactive processes of finite-dimensional Lagrangian systems is given in [6]. A historical review on the treatment of discontinuous solutions in the calculus of variations is found in [44].

Irrespective whether the set $\mathcal{C}^+ \cap \mathcal{C}^-$ is tangentially regular at $\mathrm{q}(\mathrm{t}_s)$ or not, the generalized impulse exists in $\mathrm{N}_{\mathcal{C}^+\cap\mathcal{C}^-}\left(\mathrm{q}(\mathrm{t}_s)\right)$.

The vast majority of the literature on the stationarity principle focuses on determining the stationarity conditions based on the classical rule of Fermat which requires: $\delta\mathrm{J} = 0$ for all admissible variations. If the functional J is smooth and $\partial\mathrm{J}$

reduces to a singleton, then the stationarity conditions are obtained, which may correspond to the stationarity conditions of a maximum or a minimum. If the functional J is merely Lipschitz, then the stationarity conditions for a minimum and a maximum need to be investigated separately, which require $\delta J \geq 0$ or $\delta J \leq 0$, respectively. If the stationarity conditions for a nonsmooth minimum is investigated in the presented case, then the impulse and energy balance equations becomes physically incorrect and by sign reversal.

The generalized directional derivative of Clarke does not require the existence of any limit in the vicinity of the point of interest, and involves only the behaviour of the functional near the stationary point in the sense of an upper derivative, which is in the sense of the analysis presented here, because of the discontinuity of the generalized conjugate momenta, the integrand does not exist at an instant of impact and the stationarity conditions are to be understood in the limiting sense.

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