

The Quadratic Assignment Problem is Easy for Robinsonian Matrices

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Abstract

We present a new polynomially solvable case of the Quadratic Assignment Problem in Koopmans-Beckman form $\text{QAP}(A, B)$, by showing that the identity permutation is optimal when A and B are respectively a Robinson similarity and dissimilarity matrix and one of A or B is a Toeplitz matrix. A Robinson (dis)similarity matrix is a symmetric matrix whose entries (increase) decrease monotonically along rows and columns when moving away from the diagonal, and such matrices arise in the classical seriation problem.

Keywords: *Quadratic Assignment Problem; Robinson (dis)similarity; seriation; well solvable special case*

1 Introduction

In this paper we consider two problems over permutations. Our main problem of interest is the *Quadratic Assignment Problem* (QAP), a well studied hard combinatorial optimization problem, introduced by Koopmans and Beckman [16] in 1957 as a mathematical model for the location of indivisible economic activities. In the problem $\text{QAP}(A, B)$, we are given n facilities, n locations, a *flow matrix* A whose entry A_{ij} represents the flow of activity between two facilities i and j , and a *distance matrix* B whose entry B_{ij} represents the distance between the locations i and j . Then the objective is to find an assignment of the facilities to the locations, i.e., a permutation π of the set $[n] = \{1, \dots, n\}$, minimizing the total cost of the assignment. That is, solve the following optimization problem over all permutations π of $[n]$:

$$\text{QAP}(A, B) \quad \min_{\pi} \sum_{i, j=1}^n A_{ij} B_{\pi(i)\pi(j)}, \quad (1)$$

where $A_{ij} B_{\pi(i)\pi(j)}$ is the cost inferred by assigning facility i to location $\pi(i)$ and facility j to location $\pi(j)$. QAP has been extensively studied in the past decades, in particular due to its many real world wide applications. We refer e.g. to [4, 6] and references therein for an exhaustive survey. QAP is an NP-hard problem and it cannot even be approximated within a constant factor [21]. However, there exist many special cases which are solvable in polynomial time by exploiting the structure of the data. Specifically, we are interested in those “easy cases” where an optimal

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solution is known in explicit form and is represented by a *fixed* permutation. These cases have a practical importance in designing heuristics, approximation and enumeration algorithms and they occur when the matrices A and B have an specific ordered structure, like being Monge, Toeplitz or monotone matrices (see [5, 3, 9], [6, §8.4] for a survey and the recent work [7]). For instance, if A is monotonically nondecreasing (i.e., if both its rows and column are nondecreasing) and B is monotonically nonincreasing (i.e., if both its rows and column are nonincreasing), then it is known that the identity permutation is an optimal solution to QAP(A, B) [6, Proposition 8.23].

The main contribution of this paper is to provide a new class of QAP instances that admit a closed form optimal solution. These instances arise when both matrices A and B have a special ordered structure related to the so-called seriation problem, which is our second problem of interest and is described below.

The *seriation problem* asks (roughly) to linearly order a set of objects in such a way that similar objects are close to each other and dissimilar objects are far apart. This classical problem was introduced by Robinson [20] in 1951 for reconstructing the chronology of archeological graves from information about their similarities. Beside archeological dating [20, 15] the seriation problem has applications in many other areas including biology [18], machine learning [11], scheduling [13] and sparse matrix reordering [2]. A more exhaustive list of applications can be found in [17].

Specifically, an $n \times n$ symmetric matrix A is called a *Robinson similarity matrix* if its entries decrease monotonically in the rows and the columns when moving away from the main diagonal, i.e., if

$$A_{ik} \leq \min\{A_{ij}, A_{jk}\} \quad \text{for all } 1 \leq i \leq j \leq k \leq n. \quad (2)$$

Given a symmetric matrix $A = (A_{ij})$, the seriation problem asks to find a simultaneous reordering of its rows and columns, i.e., a permutation π of $[n]$, so that the permuted matrix $A_\pi = (A_{\pi(i)\pi(j)})$ is a Robinson similarity. If such a permutation exists then A is said to be a *Robinsonian similarity*. Analogously a symmetric matrix B is called a *Robinson dissimilarity matrix* if its entries increase monotonically in the rows and columns when moving away from the main diagonal, i.e., if $-B$ is a Robinson similarity; B is then a *Robinsonian dissimilarity* if $-B$ is a Robinsonian similarity. Note that the 0/1 Robinsonian similarity matrices are exactly the symmetric matrices with the well known *consecutive ones property*.

Recall that a matrix $A \in \mathbb{R}^{n \times n}$ is a Toeplitz matrix if it has constant entries along its diagonals, i.e., if $A_{ij} = A_{i+1, j+1}$ for all $1 \leq i, j \leq n-1$. Our main results in this paper can then be formulated as follows. Assume that A is a Robinson similarity, B is a Robinson dissimilarity and that one of the two matrices A or B is a Toeplitz matrix. Then the identity permutation is an optimal solution for problem QAP(A, B) (Theorem 4). From this we can derive a more general result when both matrices are Robinsonian (Theorem 5).

Our result can be seen as an analogue for symmetric matrices of the above mentioned result about QAP for monotone matrices, where we replace the monotonicity property by the Robinson property (which implies unimodal rows and columns). This result uncover an interesting connection between QAP and the seriation problem. Since there exist polynomial time algorithms for recognizing Robinsonian (dis)similarity matrices, this permit us to introduce a new class of QAP instances which is solvable in polynomial time.

The rest of the paper is organized as it follows. In Section 2 we introduce the seriation problem and discuss some known algorithms to solve it in polynomial time, and we also discuss links to the 2-SUM problem and other instances of QAP. Section 3 contains the main result

of the paper and in Section 4 we present some applications of it leading to new instances of polynomially solvable QAP's.

Notation and some preliminaries

Throughout \mathcal{S}^n denotes the set of symmetric $n \times n$ matrices and $J \in \mathcal{S}^n$ denotes the all-ones matrix. For two matrices A, B , $\langle A, B \rangle = \text{Tr}(A^\top B) = \sum_{i,j=1}^n A_{ij}B_{ij}$ denotes the usual trace inner product on $\mathbb{R}^{n \times n}$. Let \mathcal{P}_n denote the set of permutations of $[n]$. For $\pi \in \mathcal{P}_n$ and $A \in \mathcal{S}^n$, set $A_\pi = (A_{\pi(i)\pi(j)})_{i,j=1}^n \in \mathcal{S}^n$. We will use the following properties: given $A, B \in \mathcal{S}^n$ and $\pi, \tau \in \mathcal{P}_n$,

$$(A_\pi)_\tau = A_{\pi\tau} \quad \text{and} \quad \langle A, B_\tau \rangle = \langle A_{\tau^{-1}}, B \rangle. \quad (3)$$

Recall that $A \in \mathcal{S}^n$ is a Robinson similarity if it satisfies the linear inequalities (2) and A is a Robinson dissimilarity if $-A$ is a Robinson similarity. We also say that A is a Robinson matrix if A or $-A$ is a Robinson similarity matrix. Clearly, the all-ones matrix J is both a similarity and a dissimilarity Robinson matrix (as it satisfies (2) at equality). Hence, translating by a multiple of J preserves the Robinson property. Therefore, we may assume without loss of generality that A is entrywise nonnegative. Moreover, for Robinson similarities, as the (easy to check) next lemma shows, we may assume without loss of generality that all diagonal entries are equal to some given constant.

Lemma 1. *Let $A \in \mathcal{S}^n$, set $\alpha := \max_{i,j=1}^n A_{ij}$ and define the matrix $A' \in \mathcal{S}_n$ with $A'_{ii} = \alpha$ for $i \in [n]$ and $A'_{ij} = A_{ij}$ for $i \neq j \in [n]$. Then, A is a Robinsonian similarity if and only if $A_{ii} \geq \max_{j \in [n]} A_{ij}$ for all $i \in [n]$ and A' is a Robinsonian similarity.*

Hence, in the rest of the paper, we may and will thus assume that (dis)similarity matrices are entrywise nonnegative and that Robinson similarities have the highest (constant) value on the diagonal entries. In turn, this implies that when dealing with Robinson dissimilarities, we may assume without loss of generality that all diagonal entries are equal to 0 (which is common practice in the literature when dealing with dissimilarities).

2 Seriation and 2-SUM

Given a set of n objects to order and a similarity matrix $A = (A_{ij})$ which represents the pairwise correlations between the objects, a *consistent ordering* of the objects is a permutation π of $[n]$ for which the permuted matrix A_π is a Robinson similarity, i.e., satisfies the linear constraints (2). The seriation problem consists in finding (if they exist) all possible consistent orderings of the objects.

It is straightforward to verify whether a given matrix $A \in \mathcal{S}^n$ is a Robinson similarity. Moreover one can decide in polynomial time whether A is Robinsonian and if so construct all consistent reorderings. Mirkin and Rodin first introduced in 1984 an $O(n^4)$ algorithm to recognize Robinsonian similarities. Later, Chepoi and Fichet [8] and Seston [22] improved the algorithm to respectively $O(n^3)$ and $O(n^2 \log n)$ and, very recently, Prea and Fortin [19] presented an $O(n^2)$ algorithm. These algorithms apply to recognize Robinsonian dissimilarities and thus also Robinson similarities as discussed above. They are based on a characterization of Robinsonian matrices in terms of interval hypergraphs [13].

A completely different approach to recognize Robinsonian similarities was used by Atkins et al. [1], who introduced an interesting spectral sequencing algorithm. Given a matrix $A \in \mathcal{S}^n$, its *Laplacian matrix* is the matrix $L_A = \text{diag}(Ae) - A \in \mathcal{S}^n$, where e is the all-ones vector and

$\text{diag}(Ae)$ is the diagonal matrix whose diagonal is given by the vector Ae . If $A \geq 0$ then L_A is positive semidefinite and thus its smallest eigenvalue is $\lambda_1(L_A) = 0$. Moreover, the second smallest eigenvalue $\lambda_2(L_A)$ is positive if A is irreducible. The *Fiedler vector* $y_F \in \mathbb{R}^n$ of A is the eigenvector corresponding to the second smallest eigenvalue $\lambda_2(L_A)$ of L_A , which is also known as the *Fiedler value*. The spectral algorithm of Atkins et al. [1] relies on the following properties of the Fiedler vector of a Robinson similarity matrix.

Theorem 1. [1] *If $A \in \mathcal{S}^n$ is a Robinson similarity then it has a monotone Fiedler vector y_F , i.e., satisfying: $y_F(1) \leq \dots \leq y_F(n)$ or $y_F(n) \leq \dots \leq y_F(1)$.*

Theorem 2. [1] *Assume that $A \in \mathcal{S}^n$ is a Robinsonian similarity, that its Fiedler value is simple and that the Fiedler vector y_F has no repeated entries. Let π be the permutation induced by sorting monotonically the values of y_F (in increasing or decreasing order). Then the matrix A_π is a Robinson similarity matrix.*

In other words, the above results show that sorting monotonically the Fiedler vector y_F reorders A as a Robinson similarity. The complexity of the algorithm of [1] in the general case is given by $O(n(T(n) + n \log n))$, where $T(n)$ is the complexity of computing (approximately) eigenvalues of an $n \times n$ symmetric matrix.

Barnard et al. [2] had earlier used the same spectral algorithm to solve the 2-SUM problem. Given a matrix $A \in \mathcal{S}^n$, the 2-SUM problem is the special instance QAP(A, B) of QAP, where the distance matrix is of the form $B = ((i - j)^2)_{i,j=1}^n$. That is,

$$\text{2-SUM} \quad \min_{\pi \in \mathcal{P}_n} \sum_{i,j=1}^n A_{\pi(i)\pi(j)} (i - j)^2. \quad (4)$$

The authors of [14] show that 2-SUM is an NP-complete problem and they use the above spectral method of reordering the Fiedler vector of A as a heuristic to produce a feasible solution for problem (4). This in turn permits to bound important matrix structural parameters, like envelope-size and bandwidth. However, no assumption is made on the structure of the matrix A . As observed in [14, 12], the following fact can be used to motivate the spectral approach for 2-SUM. If we define the vectors $x = (1, \dots, n)^\top, x_\pi = (\pi(1), \dots, \pi(n))^\top \in \mathbb{R}^n$ for $\pi \in \mathcal{P}_n$, then $\sum_{i,j=1}^n A_{ij} (\pi(i) - \pi(j))^2 = (x_\pi)^\top L_A x_\pi$ and thus computing the Fiedler value arises as a natural continuous relaxation for the 2-SUM problem (4).

Fogel et al. [12] point out an interesting connection between the seriation and 2-SUM problems. They consider a special class of Robinson similarity matrices for which they can show that the identity permutation is optimal for the 2-SUM problem (4). Namely they consider the following *cut matrices*: given two integers $1 \leq u \leq v \leq n$, the *cut matrix* $\text{CUT}(u, v)$ is the symmetric $n \times n$ matrix with (i, j) -th entry 1 if $u \leq i, j \leq v$ and 0 otherwise. Clearly each cut matrix is a Robinson similarity and thus conic combinations of cut matrices as well. The following result is shown in [12], for which we give a short proof.

Proposition 1. [12] *If $A \in \mathcal{S}^n$ can be written as a conic combination of cut matrices, then the identity permutation is optimal for the 2-SUM problem (4). More generally if, for some $\pi \in \mathcal{P}_n$, A_π can be written as a conic combination of cut matrices, then π is optimal for the 2-SUM problem (4).*

Proof. First we show the result when A is a cut matrix. Say, $A = \text{CUT}(u, v)$ and set $t = v - u + 1$. Then, we need to show that $\sum_{i,j=1}^n A_{ij} (\pi(i) - \pi(j))^2 \geq \sum_{i,j=1}^n A_{ij} (i - j)^2$ for any permutation

$\pi \in \mathcal{P}_n$. Equivalently, we need to show that

$$\sum_{1 \leq r < s \leq t} (i_r - i_s)^2 \geq \sum_{1 \leq r < s \leq t} (r - s)^2$$

for any integers $1 \leq i_1 < \dots < i_t \leq n$. Now the latter easily follows from the fact that $|i_r - i_s| \geq |r - s|$ for all r, s . Hence we have shown that the identity permutation is an optimal solution of problem (4) when A is a cut matrix and this easily implies that this also holds when A is a conic combination of cut matrices. The second statement follows as a direct consequence. \square

In other words, the above result shows that for similarity matrices as in Proposition 1, any permutation reordering A as a Robinson matrix also solves (4). As not every Robinson similarity is a conic combination of cut matrices, this raises the question whether the above result extends to the case when A is an arbitrary Robinson similarity matrix.

There is a second possible way in which one may want to generalize the result of Proposition 1. Indeed, the 2-SUM problem (4) is the instance of QAP(A, B), where the distance matrix is $B = ((i - j)^2)_{i,j=1}^n$, which turns out to be a Toeplitz dissimilarity Robinson matrix. In fact there are many other interesting classes of QAP whose distance matrix B is a Toeplitz Robinson dissimilarity matrix. For instance, QAP(A, B) models the minimum linear arrangement (aka 1-SUM) problem when $B = (|i - j|)_{i,j=1}^n$ and, more generally, the p -SUM problem when $B = (|i - j|^p)_{i,j=1}^n$ (for $p \geq 1$), and QAP(A, B) models the minimum bandwidth problem when A is the adjacency matrix of a graph and B is of the form B_n^Δ , as defined in relation (5) below. For more details on these and other graph (matrix) layout problems with practical impact we refer to the survey [10] and references therein.

This thus raises the further question whether the result of Proposition 1 extends to instances of QAP(A, B), where B is an arbitrary Toeplitz dissimilarity matrix. This is precisely what we do in this paper. We remove both assumptions on A and B and show that the identity permutation is optimal for QAP(A, B) when A is any Robinson similarity and B is any Robinson dissimilarity, assuming that B (or A) has a Toeplitz structure.

3 The main result

Let A be a Robinson similarity matrix and let B be a Toeplitz Robinson dissimilarity matrix. The first key (easy) observation is that we can decompose B as a conic combination of 0/1 Toeplitz Robinson dissimilarities. Given an integer $\Delta \in [n]$, we define the symmetric matrix $B_n^\Delta \in \mathcal{S}^n$ with entries

$$(B_n^\Delta)_{ij} = \begin{cases} 1 & \text{if } |i - j| \geq n - \Delta \\ 0 & \text{else} \end{cases} \quad \text{for } i, j = 1, \dots, n. \quad (5)$$

Note that for $\Delta = n$, we have that $B_n^\Delta = J$. Clearly, each matrix B_n^Δ is a Toeplitz matrix and a Robinson dissimilarity. In fact all Toeplitz Robinson dissimilarities can be decomposed in terms of these matrices B_n^Δ .

Lemma 2. *Let $B \in \mathcal{S}^n$ be a Toeplitz matrix and let $\beta_0, \dots, \beta_{n-1} \in \mathbb{R}$ such that $B(i, j) = \beta_k$ for all $i, j \in [n]$ with $|i - j| = k$ for $0 \leq k \leq n$. Then,*

$$B = \beta_0 J + \sum_{k=1}^{n-1} (\beta_k - \beta_{k+1}) B_n^{n-k}. \quad (6)$$

If moreover B is a Robinson dissimilarity, i.e., if $\beta_0 = 0 \leq \beta_1 \leq \dots \leq \beta_{n-1}$, then B is a conic combination of the matrices B_n^Δ (for $\Delta = 1, \dots, n - 1$).

Proof. Direct verification. □

Our main result, which we show in this section, is that the identity permutation is optimal for $\text{QAP}(A, B_n^\Delta)$ for any integer $1 \leq \Delta \leq n - 1$. As we will see in Section 3, this result implies that the identity permutation is optimal for $\text{QAP}(A, B)$.

Theorem 3. *Let $A \in \mathcal{S}^n$ be a Robinson similarity matrix and let $\Delta \in [n - 1]$. Then, for any permutation π of $[n]$, we have:*

$$\langle A_\pi, B_n^\Delta \rangle \geq \langle A, B_n^\Delta \rangle. \quad (7)$$

Proof. Let E_n^Δ denote the support of the matrix B_n^Δ , i.e., the set of upper triangular positions where B_n^Δ has a nonzero entry. That is:

$$E_n^\Delta = \{\{i, n - \Delta + j\} : 1 \leq i \leq j \leq \Delta\}.$$

Then we can reformulate the inner products in (7) as

$$\begin{aligned} \langle A_\pi, B_n^\Delta \rangle &= \sum_{i,j=1}^n A_{\pi(i),\pi(j)} (B_n^\Delta)_{i,j} = 2 \sum_{\{i,j\} \in E_n^\Delta} A_{\pi(i),\pi(j)}, \\ \langle A, B_n^\Delta \rangle &= \sum_{i,j=1}^n A_{i,j} (B_n^\Delta)_{i,j} = 2 \sum_{\{i,j\} \in E_n^\Delta} A_{i,j}, \end{aligned}$$

and (7) is equivalent to the following inequality:

$$\sum_{\{i,j\} \in E_n^\Delta} A_{\pi(i),\pi(j)} \geq \sum_{\{i,j\} \in E_n^\Delta} A_{i,j}. \quad (8)$$

The proof is by induction on $n \geq 2$. The base case $n = 2$ is trivial, since then $\Delta = 1$ and both summations in (8) are identical. We now assume that the result holds for $n - 1$ and we show that it also holds for n . For the remaining of the proof, we fix a Robinson similarity matrix $A \in \mathcal{S}^n$, an integer $\Delta \in [n - 1]$ and a permutation π of $[n]$. Moreover we let $k \in [n]$ denote the index such that $n = \pi(k)$.

The key idea in the proof is to show that there exist a subset $F \subseteq E_n^\Delta$ and a permutation τ of $[n]$ satisfying the following properties:

(C1) $|F| = \Delta$,

(C2) the indices $\min\{\pi(i), \pi(j)\}$ for the pairs $\{i, j\} \in F$ are pairwise distinct,

(C3) $\tau(n) = k$ and the set $R := E_n^\Delta \setminus F$ satisfies

$$R = \tau(E_{n-1}^{\Delta-1}) := \{\{\tau(i), \tau(j)\} : \{i, j\} \in E_{n-1}^{\Delta-1}\}.$$

where the set $E_{n-1}^{\Delta-1}$ is the support of the matrix $B_{n-1}^{\Delta-1}$, defined in the same fashion as E_n^Δ . Indeed, as we now show, if such set F and permutation τ exist, then we can conclude the proof using the induction assumption.

Claim 1. *Assume that there exist a set $F \subseteq E_n^\Delta$ and a permutation τ of $[n]$ satisfying (C1)-(C3), then the inequality (8) holds.*

Proof. (of Claim 1). Let us decompose the summation:

$$\Sigma_{\pi}(A) := \sum_{\{i,j\} \in E_n^{\Delta}} A_{\pi(i),\pi(j)}$$

as the sum of the two terms:

$$\Sigma_{\pi}(A) = \Sigma_{\pi,F}(A) + \Sigma_{\pi,R}(A), \quad (9)$$

where we set:

$$\Sigma_{\pi,F}(A) := \sum_{\{i,j\} \in F} A_{\pi(i),\pi(j)}, \quad \Sigma_{\pi,R}(A) := \sum_{\{i,j\} \in R} A_{\pi(i),\pi(j)}.$$

We now bound each term separately. First we consider the term $\Sigma_{\pi,F}(A)$. As A is a Robinson similarity matrix, it follows that for all indices $i, j \in [n]$:

$$A_{ij} \geq A_{n,\min\{i,j\}}.$$

Hence, we can deduce:

$$\Sigma_{\pi,F}(A) = \sum_{\{i,j\} \in F} A_{\pi(i),\pi(j)} \geq \sum_{\{i,j\} \in F} A_{n,\min\{\pi(i),\pi(j)\}} \geq \sum_{i=1}^{\Delta} A_{n,i}, \quad (10)$$

where for the right most inequality we have used the conditions (C1),(C2) combined with the fact A is a Robinson similarity.

We now consider the second term $\Sigma_{\pi,R}(A)$. Define the permutation $\sigma = \pi\tau$. Then, by (C3), we have that $\sigma(n) = \pi(k) = n$ and thus $\sigma(E_{n-1}^{\Delta-1}) = \pi(\tau(E_{n-1}^{\Delta-1})) = \pi(R)$. We can then write:

$$\Sigma_{\pi,R}(A) = \sum_{\{i,j\} \in E_{n-1}^{\Delta-1}} A_{\sigma(i),\sigma(j)}. \quad (11)$$

As $\sigma(n) = n$, the permutation σ of $[n]$ induces a permutation σ' of $[n-1]$. We let $A', B' \in \mathcal{S}^{n-1}$ denote the principal submatrices obtained by deleting the row and column indexed by n in A and B_n^{Δ} , respectively. Then A' is again a Robinson similarity matrix (now of size $n-1$) and $B' = B_{n-1}^{\Delta-1}$ is supported by the set $E_{n-1}^{\Delta-1}$. Then, using the induction assumption applied to A' , $\Delta-1$ and σ' , we obtain:

$$\Sigma_{\sigma'}(A') := \sum_{\{i,j\} \in E_{n-1}^{\Delta-1}} A'_{\sigma(i),\sigma(j)} \geq \Sigma_{\text{id}}(A') := \sum_{\{i,j\} \in E_{n-1}^{\Delta-1}} A'_{i,j}. \quad (12)$$

Using (11), we get:

$$\Sigma_{\pi,R}(A) = \Sigma_{\sigma'}(A') \geq \Sigma_{\text{id}}(A') = \sum_{\{i,j\} \in E_{n-1}^{\Delta-1}} A_{i,j}. \quad (13)$$

Finally, combining (9),(10) and (13), we get the desired inequality:

$$\Sigma_{\pi}(A) \geq \sum_{i=1}^{\Delta} A_{i,n} + \sum_{\{i,j\} \in E_{n-1}^{\Delta-1}} A_{ij} = \sum_{\{i,j\} \in E_n^{\Delta}} A_{ij},$$

which concludes the proof of the claim. \square

Hence, in order to conclude the proof of Theorem 3, it suffices to show that we can find a set $F \subseteq E_n^\Delta$ and a permutation τ of $[n]$ for which conditions (C1)-(C3) hold. In a first step, we formulate two additional conditions (C4) and (C5) which together with (C1),(C2) imply (C3). Nevertheless, they will turn simpler to check than (C3).

Claim 2. *Assume that the sets $F \subseteq E_n^\Delta$ and $R := E_n^\Delta \setminus F$ satisfy the conditions (C1),(C2) and the following two conditions:*

(C4) *no pair in the set R contains the element k ,*

(C5) *no pair $\{i, n - \Delta + i\}$ with $1 \leq i \leq \Delta$ and $k + 1 - n + \Delta \leq i \leq k - 1$ belongs to the set R .*

Define the cyclic permutation $\tau = (k, k + 1, \dots, n)$. Then, we have $R = \tau(E_{n-1}^{\Delta-1})$.

Proof. (of Claim 2). By (C1), the set R has the same cardinality as the set $\tau(E_{n-1}^{\Delta-1})$ and therefore it suffices to show the inclusion $R \subseteq \tau(E_{n-1}^{\Delta-1})$. For this consider a pair $\{i, n - \Delta + j\}$ in R , where $1 \leq i \leq j \leq \Delta$. We show that $i = \tau(r)$ and $j = \tau(s)$ for some $1 \leq r \leq s \leq \Delta - 1$. In view of (C4), we shall define r, s depending whether i lies before or after k , getting the following cases:

- 1) $i \leq k - 1$, then $r = i$ and $i = \tau(i)$,
- 2) $i \geq k + 1$, then $r = i - 1$ and $i = \tau(i - 1)$.

We do the same for index j , getting the following cases:

- a) $n - \Delta + j \leq k - 1$, then $s = j$ and $n - \Delta + j = \tau(n - \Delta + j)$,
- b) $n - \Delta + j \geq k + 1$, then $s = j - 1$ and $n - \Delta + j = \tau(n - \Delta + j - 1)$.

It remains only to check that $1 \leq r \leq s \leq \Delta - 1$ holds. For this, we now discuss all possible combinations for indices i and j according to the above cases:

- 1a) Then, $r = i$ and $s = j$. Since $1 \leq i \leq j \leq \Delta$, we only have to check that $j \leq \Delta - 1$. Indeed, if $j = \Delta$, then from a) we get $n \leq k - 1$, which is impossible.
- 1b) Then, $r = i$ and $s = j - 1$. It suffices to check that $r \leq s$, i.e., $i \neq j$. Indeed, assume that $i = j$. Then, the pair $\{i, n - \Delta + i\}$ belongs to R with $i \leq k - 1$ (as we are in case 1) for index i) and $i \geq k + 1 - n + \Delta$ (as we are in case b) for index j), which contradicts the condition (C5).
- 2a) Then, $r = i - 1$ and $s = j$. It suffices to check that $r \geq 1$ and $s \leq \Delta - 1$. The first one holds since $i \geq 2$ as we are in case 2) for index i . The second one is also true, since we are in case a) for index j and $k \leq n$.
- 2b) Then, $r = i - 1$ and $s = j - 1$. It suffices to check that $r \geq 1$, which holds since we are in case 2) for index i and thus $i \geq 2$.

Thus we have shown that $R \subseteq \tau(E_{n-1}^{\Delta-1})$, which concludes the proof of the claim. \square

In the rest of the proof, we define the permutation τ of $[n]$ to be the cyclic permutation $\tau = (k, k + 1, \dots, n)$ (so that $\tau(n) = k$) and we show how to construct the sets F and $R := E_n^\Delta \setminus F$ so that they satisfy the conditions (C1),(C2),(C4) and (C5). If we can do this, then we know that (C3) is satisfied too and thus the proof is complete (in view of Claims 1 and 2) .

The following terminology will be useful, regarding the pairs $\{i, n - \Delta + j\}$ (for $1 \leq i \leq j \leq \Delta$) in the set E_n^Δ . We refer to the pairs $\{i, n - \Delta + i\}$ (with $1 \leq i \leq \Delta$) as the *diagonal pairs*. Furthermore, for each $1 \leq i_0 \leq \Delta$, we refer to the pairs $\{i_0, n - \Delta + j\}$ (with $i_0 + 1 \leq j \leq \Delta$) as the *horizontal pairs* on row i_0 , meaning the pairs of E_n^Δ in the row indexed by i_0 . Finally, for each $k_0 = n - \Delta + j_0$ such that $1 \leq j_0 \leq \Delta$, we refer to the pairs $\{i, n - \Delta + j_0\}$ (with $1 \leq i \leq j_0 - 1$) as the *vertical pairs* on column k_0 , meaning the pairs of E_n^Δ in the column indexed by k_0 . Note that, in both horizontal and vertical pairs, the diagonal pair $\{i, n - \Delta + i\}$ is not included. As an illustration see Figure 1.

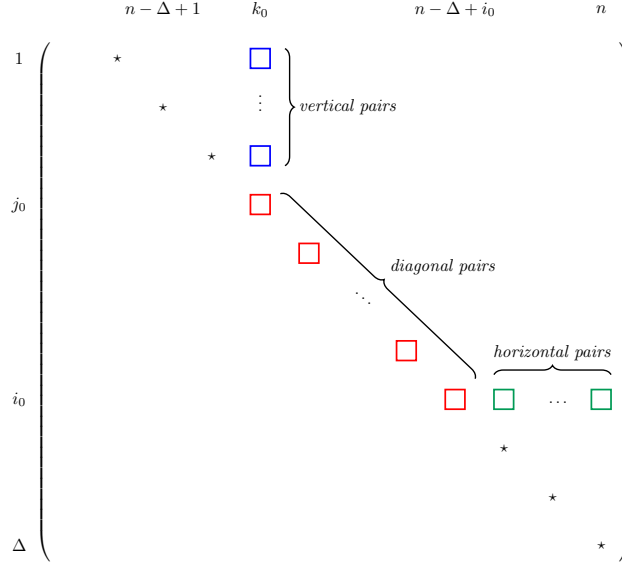


Figure 1: Vertical, diagonal and horizontal pairs in the set E_n^Δ

Moreover, we denote by:

$$U = U_R \cup U_C, \text{ where } U_R = \{1, \dots, \Delta\}, \quad U_C = \{n - \Delta + 1, \dots, n\}$$

the set consisting of the row and column indices for the nonzero entries of the matrix B_n^Δ .

In the rest of the proof we discuss depending whether the index k belongs to U_R and/or U_C . Namely we consider the following four cases: (1) $k \notin U_R \cup U_C$, (2) $k \in U_R \setminus U_C$, (3) $k \in U_C \setminus U_R$, and (4) $k \in U_R \cap U_C$. In each of these cases we define the set F and in fact it will follow easily from its definition that the conditions (C1), (C4) and (C5) hold (so we give no further details about this). Hence it will remain only to show that condition (C2) holds in each of the four cases and this is what we do below.

Case (1): $k \notin U_R \cup U_C$.

Then, $\Delta + 1 \leq k \leq n - \Delta$, which implies $\Delta \leq (n - 1)/2$. In this case we define F as the set of all diagonal pairs (see Figure 2a at page 10), namely:

$$F = \{\{i, n - \Delta + i\} : 1 \leq i \leq \Delta\}$$

To see that (C2) holds, let $r \neq s \in [\Delta]$ and we have to show that $\min\{\pi(r), \pi(n - \Delta + r)\} \neq \min\{\pi(s), \pi(n - \Delta + s)\}$.

This is clear since the four indices $\pi(r)$, $\pi(n - \Delta + r)$, $\pi(s)$, and $\pi(n - \Delta + s)$ are pairwise distinct. Indeed, if equality $\pi(r) = \pi(n - \Delta + s)$ would hold, this would imply the inequalities: $n - \Delta + 1 \leq r = n - \Delta + s \leq \Delta$ and thus $\Delta \geq (n + 1)/2$, a contradiction.

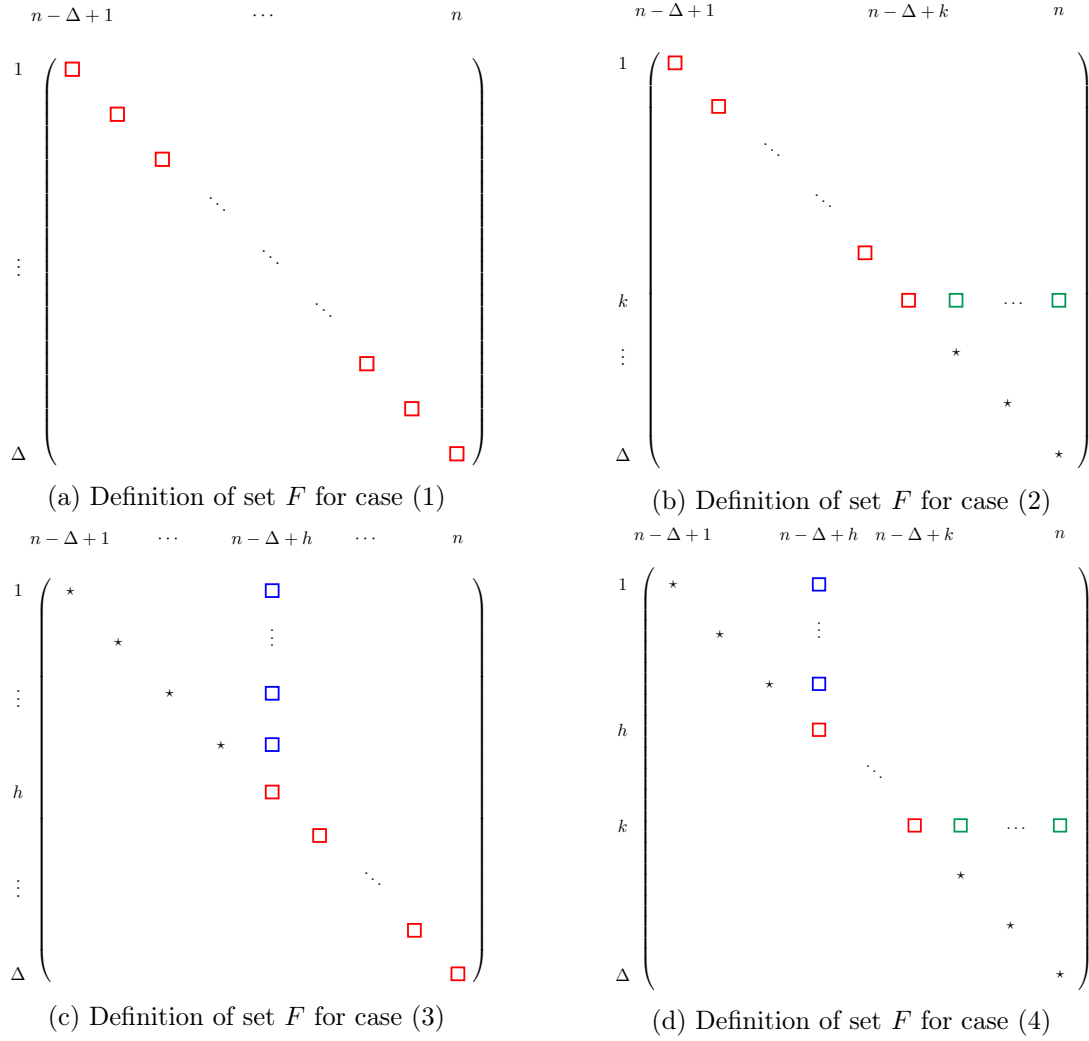


Figure 2: Different definition of set F for all 4 cases

Case (2): $k \in U_R \setminus U_C$.

Then, $1 \leq k \leq \Delta$ and $k \leq n - \Delta$. In this case we let F consist of the diagonal pairs till position $(k, n - \Delta + k)$ and then of the horizontal pairs on the k -th row (see Figure 2b at page 10), namely:

$$F = \{\{i, n - \Delta + i\} : 1 \leq i \leq k\} \cup \{\{k, n - \Delta + i\} : k + 1 \leq i \leq \Delta\}.$$

In order to check that (C2) holds, we consider the following three cases:

- For $r \neq s \in [k]$, we get $\min\{\pi(r), \pi(n - \Delta + r)\} \neq \min\{\pi(s), \pi(n - \Delta + s)\}$, since the four indices $\pi(r), \pi(n - \Delta + r), \pi(s), \pi(n - \Delta + s)$ are pairwise distinct (using the fact that $k \leq n - \Delta$).
- For $r \neq s \in \{k + 1, \dots, \Delta\}$, using the fact that $\pi(k) = n$, $\min\{\pi(k), \pi(n - \Delta + r)\} = \pi(n - \Delta + r) \neq \min\{\pi(k), \pi(n - \Delta + s)\} = \pi(n - \Delta + s)$.
- Finally, for $r \in [k]$ and $s \in \{k + 1, \dots, \Delta\}$, we have that $\min\{\pi(r), \pi(n - \Delta + r)\} \neq \min\{\pi(k), \pi(n - \Delta + s)\} = \pi(n - \Delta + s)$. Indeed, $\pi(r) \neq \pi(n - \Delta + s)$ (since otherwise this

would imply that $n - \Delta + k + 1 \leq r = n - \Delta + s \leq k$ and thus $n + 1 \leq \Delta$, a contradiction) and it is clear that $\pi(n - \Delta + r) \neq \pi(n - \Delta + s)$. The case for $s \in [k]$ and $r \in \{k + 1, \dots, \Delta\}$ is symmetric.

Case (3): $k \in U_C \setminus U_R$.

This case corresponds to $k = n - \Delta + h$, where $1 \leq h \leq \Delta$. Then, since k belongs to the column indices, we have that $n - \Delta + h = k \geq \Delta + 1$, which implies $\Delta \leq (n + h - 1)/2$.

In this case we let F consists of the vertical pairs on the k -th column and of the diagonal pairs from position $(k, n - \Delta + k)$ (see Figure 2c at page 10), namely:

$$F = \{\{i, k\} : 1 \leq i \leq h - 1\} \cup \{\{i, n - \Delta + i\} : h \leq i \leq \Delta\}.$$

To see that (C2) holds, we consider the following three cases.

- For $r \neq s \in [h - 1]$, $\pi(r) = \min\{\pi(r), \pi(k)\} \neq \min\{\pi(s), \pi(k)\} = \pi(s)$.
- For $r \neq s \in \{h, \dots, \Delta\}$, $\min\{\pi(r), \pi(n - \Delta + r)\} \neq \min\{\pi(s), \pi(n - \Delta + s)\}$, since the four indices $\pi(r), \pi(n - \Delta + r), \pi(s), \pi(n - \Delta + s)$ are pairwise distinct. Indeed, $\pi(r) = \pi(n - \Delta + s)$ would imply: $n - \Delta + h \leq r = n - \Delta + s \leq \Delta$ and thus $\Delta \geq (n + h)/2$, a contradiction.
- For $r \in [h - 1]$ and $s \in \{h, \dots, \Delta\}$, $\min\{\pi(r), \pi(k)\} = \pi(r) \neq \min\{\pi(s), \pi(n - \Delta + s)\}$, since the indices $\pi(r), \pi(s), \pi(n - \Delta + s)$ are pairwise distinct. Indeed, equality $\pi(r) = \pi(n - \Delta + s)$ would imply that $n - \Delta + h \leq r = n - \Delta + s \leq h - 1$ and thus $\Delta \geq n + 1$, a contradiction.

Case (4): $k \in U_R \cap U_C = \{n - \Delta + 1, \dots, \Delta\}$.

This case corresponds to $k = n - \Delta + h$, where $1 \leq h \leq 2\Delta - n$. Moreover, we have $\Delta \geq (n + 1)/2$.

Then we let F consist of the vertical pairs on the k -th column, of the diagonal pairs from position $(h, n - \Delta + h)$ to position $(k, n - \Delta + k)$, and of the horizontal pairs on the k -th row (see Figure 2d at page 10), namely:

$$F = \{\{i, k\} : 1 \leq i \leq h\} \cup \{\{i, n - \Delta + i\} : h + 1 \leq i \leq k\} \\ \cup \{\{k, n - \Delta + i\} : k + 1 \leq i \leq \Delta\}.$$

In order to check condition (C2), as F consists of the union of three subsets we need to consider the following six cases.

- For $r \neq s \in [h]$, $\min\{\pi(r), \pi(k)\} = \pi(r) \neq \min\{\pi(s), \pi(k)\} = \pi(s)$.
- For $r \neq s \in \{h + 1, \dots, k\}$, $\min\{\pi(r), \pi(n - \Delta + r)\} \neq \min\{\pi(s), \pi(n - \Delta + s)\}$, since $\pi(r), \pi(n - \Delta + r), \pi(s)$ and $\pi(n - \Delta + s)$ are pairwise distinct. Indeed, equality $\pi(r) = \pi(n - \Delta + s)$ would imply $r = n - \Delta + s$ and thus $k + 1 = n - \Delta + h + 1 \leq n - \Delta + s = r \leq k$, yielding a contradiction.
- For $r \neq s \in \{k + 1, \dots, \Delta\}$, it holds $\min\{\pi(k), \pi(n - \Delta + r)\} = \pi(n - \Delta + r) \neq \min\{\pi(k), \pi(n - \Delta + s)\} = \pi(n - \Delta + s)$.
- For $r \in [h]$ and $s \in \{h + 1, \dots, k\}$, $\min\{\pi(r), \pi(k)\} = \pi(r) \neq \min\{\pi(s), \pi(n - \Delta + s)\}$, since the indices $\pi(r), \pi(s), \pi(n - \Delta + s)$ are pairwise distinct. Indeed, $\pi(r) = \pi(n - \Delta + s)$ would imply that $n - \Delta + h + 1 \leq n - \Delta + s = r \leq h$ and thus $\Delta \geq n + 1$, a contradiction.

- For $r \in [h]$ and $s \in \{k+1, \dots, \Delta\}$, then we have that $\min\{\pi(r), \pi(k)\} = \pi(r) \neq \min\{\pi(k), \pi(n-\Delta+s)\} = \pi(n-\Delta+s)$, since the indices $\pi(r), \pi(n-\Delta+s)$ are distinct. Indeed, equality $\pi(r) = \pi(n-\Delta+s)$ would imply that $r = n-\Delta+s$ and thus $2(n-\Delta) + h + 1 = n-\Delta+k+1 \leq n-\Delta+s = r \leq h$, which implies $\Delta \geq n+1$, a contradiction.
- For $r \in \{h+1, \dots, k\}$ and $s \in \{k+1, \dots, \Delta\}$, then we have that $\min\{\pi(r), \pi(n-\Delta+r)\} \neq \min\{\pi(k), \pi(n-\Delta+s)\} = \pi(n-\Delta+s)$. Indeed, $r = n-\Delta+s$ would imply that $n-\Delta+k+1 \leq n-\Delta+s = r \leq k$, which implies $\Delta \geq n+1$, a contradiction.

Thus (C2) holds, which concludes the proof in case (4) and thus the proof of the theorem. \square

4 Applications of the main result

We now formulate several applications of our main result in Theorem 3. As a first direct consequence, we can show that the identity matrix is an optimal solution to the problem $\text{QAP}(A, B)$ whenever A is a Robinson similarity matrix, B is a Robinson dissimilarity matrix, and at least one of A or B is a Toeplitz matrix.

Theorem 4. *Let $A, B \in \mathcal{S}^n$ and assume that A is a Robinson similarity matrix, B is a Robinson dissimilarity matrix and moreover A or B is a Toeplitz matrix. Then the identity permutation is an optimal solution to the problem $\text{QAP}(A, B)$.*

Proof. Assume first that B is a Toeplitz matrix. Then, by Lemma 2, B is a conic combination of the matrices B_n^Δ , i.e., $B = \sum_{\Delta=1}^{n-1} \lambda_\Delta B_n^\Delta$ for some scalars $\lambda_\Delta \geq 0$. Applying the inequality (7) in Theorem 3, we obtain that

$$\langle A_\pi, B \rangle = \sum_{\Delta=1}^{n-1} \lambda_\Delta \langle A_\pi, B_n^\Delta \rangle \geq \sum_{\Delta=1}^{n-1} \lambda_\Delta \langle A, B_n^\Delta \rangle = \langle A, B \rangle,$$

which shows that the identity permutation is optimal for $\text{QAP}(A, B)$.

Assume now that A is a Toeplitz Robinson similarity (and B is a Robinson dissimilarity). Then we simply exchange the roles of A and B after a simple modification. Namely, let α and β denote the maximum value of the entries of A and B , respectively. According to Lemma 1, let define the matrices $B' = \alpha J - A$ and $A' = \beta J - B$. Then A' is a Robinson similarity matrix and B' is a Toeplitz Robinson dissimilarity matrix. For any permutation π , by the previous result applied to $\text{QAP}(A', B')$, $\langle (A')_\pi, (B') \rangle \geq \langle A', B' \rangle$. If we compute the inner product of both sides, we obtain $\langle (A')_\pi, B' \rangle = \alpha\beta \langle J_\pi, J \rangle - \alpha \langle J_\pi, A \rangle - \beta \langle B_\pi, J \rangle + \langle B_\pi, A \rangle$ and $\langle A', B' \rangle = \alpha\beta \langle J, J \rangle - \alpha \langle J, A \rangle - \beta \langle B, J \rangle + \langle B, A \rangle$, from which we can easily conclude that $\langle A_{\pi^{-1}}, B \rangle = \langle A, B_\pi \rangle \geq \langle A, B \rangle$. Hence, this shows that the identity permutation is optimal for $\text{QAP}(A, B)$ and it concludes the proof. \square

The result from Theorem 4 extends easily to the case when the matrices A and B are Robinsonian.

Theorem 5. *Let $A, B \in \mathcal{S}^n$. Assume that A is a Robinsonian similarity matrix, B is a Robinsonian dissimilarity matrix, and let π, τ be permutations that reorder A and B as Robinson similarity and dissimilarity matrices, respectively. Assume furthermore that one of the matrices A_π or B_τ is a Toeplitz matrix. Then the permutation $\tau^{-1}\pi$ is optimal for the problem $\text{QAP}(A, B)$.*

Proof. By Theorem 4, we know that $\langle A_\pi, B_\tau \rangle$ is the optimal value of QAP(A_π, B_τ), i.e., $\langle A_\pi, B_\tau \rangle \leq \langle (A_\pi)_\sigma, B_\tau \rangle$ holds for each $\sigma \in \mathcal{P}_n$. Using (3), this implies $\langle A_{\tau^{-1}\pi}, B \rangle \leq \langle A_{\tau^{-1}\pi\sigma}, B \rangle$ for all $\sigma \in \mathcal{P}_n$ and thus the permutation $\tau^{-1}\pi$ is optimal for QAP(A, B). \square

Finally we observe that the assumption that either A or B has a Toeplitz structure cannot be omitted in Theorem 3. Indeed, consider the following matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

so that A (resp., B) is a Robinson similarity (resp., dissimilarity). In this case, the identity permutation gives a solution of value $\langle A, B \rangle = 8$. Consider now the permutation $\pi = (4, 5, 1, 2, 3)$ which reorders A as follows:

$$A_\pi = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

This gives a solution of value $\langle A_\pi, B \rangle = 4$. Hence, in this case the identity permutation is not optimal, and thus the Toeplitz assumption cannot be removed in Theorem 3.

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