Multiperiod Multiproduct Advertising Budgeting: Stochastic Optimization Modeling

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Abstract

We propose a stochastic optimization model for the Multiperiod Multiproduct Advertising Budgeting problem, so that the expected profit of the advertising investment is maximized. The model is a convex optimization problem that can readily be solved by plain use of standard optimization software. It has been tested for planning a realistic advertising campaign. In our case study, the expected profit of the stochastic approach has been favorably compared with the expected profit of the deterministic approach. This provides a quantitative argument in favor of the stochastic approach for managerial decision making in a data-driven framework.

Key words: Marketing, advertising budgeting, deterministic optimization, stochastic optimization, convex optimization.

1 Introduction

In this paper we address the Multiperiod Multiproduct Advertising (MAB) Budgeting problem (notice that we drop one M for short). More specifically, we wish to simultaneously optimize the advertising campaigns of different products [8], considering a multiperiod planning horizon and having the *cumulative advertising effect* [20] as the inter-period linking variables.

This work can be considered as the second part of [3] where: First, the relevance of the MAB problem and its state-of-the-art were discussed. Second, a deterministic MAB optimization model was introduced. Third, its theoretical properties were studied (convexity, separability and optimality conditions). The purpose of this second part is to introduce a stochastic MAB optimization model and compare it with its deterministic counterpart introduced in [3]. In the remaining of the paper, we will replace the term 'optimization model' by 'model' for short. In deterministic models, each unknown parameter is substituted by an estimate, that is, all the problem parameters are considered to be known. However, in real situations many of the problem parameters are unknown at the moment of deciding the advertising budget. In this paper we will enhance our previous MAB model by explicitly dealing with this uncertainty: Each unknown parameter will be incorporated into the MAB model as a random variable instead of as a single estimate. Thus, the new stochastic MAB model proposes an optimal advertising budget adapted not only to the estimated parameter value, but to a set of representative values of the uncertain parameters.

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Stochastic models of the advertising budgeting problem under uncertainty have been proposed in the literature from different perspectives. In [24] a stochastic game theory approach is used to analyze the optimal advertising spending in a duopolistic market where the share of each firm depends on its own and its competitors advertising decisions. In [2] a Bayesian dynamic linear model is used for studying the wear out effects of different themes of advertising (for example, price advertisements versus product advertisements) in order to improve the effectiveness of advertising budget allocation across different themes. In [10] a Markov decision process approach is used for modeling the multistage advertising budgeting problem. Other approaches that have been proposed to solve the advertising budgeting problem under uncertainty are based on multicriteria fuzzy optimization [22], chance constraint goal optimization [4] and robust optimization [1]. Other works that consider the stochastic side of the advertising budgeting problem in a multiperiod environment are [2, 24], among others. A recent survey about dynamic advertising (deterministic and stochastic approaches) can be found in [17].

As we have mentioned, different stochastic aspects of the advertising budgeting problem have been considered in literature. But, as far as we know, a stochastic version of the MAB problem has not yet been addressed. Thus, for example, [9] considers a multiproduct model but for a single period and with a deterministic approach, [27] considers a multiperiod model but for a single product and also with a deterministic approach, [10] considers a stochastic multiperiod model but for a single product. However, as we will illustrate in our case study, a relevant expected profit improvement can be achieved if the parameter uncertainty in MAB models is taken into account (stochastic models) compared to ignore it (deterministic models). In this paper we propose and analyze a stochastic optimization version of the MAB problem, that improves two previous advertising budgeting models presented in [3] and [9], respectively. The proposed model has appealing properties: it is a convex optimization model and it is a probability based optimization model (it takes into account the likelihood of the representative values of the uncertain parameters). There are other choices to deal with the uncertainty in the MAB problem. Thus for example, the chance constraint approach allows for incorporating probabilistic constraints which are interesting from a modelling point of view, but, in general, the resulting deterministic equivalent optimization problem is non-convex. Another possible approach is robust optimization which deals with uncertainty by considering uncertainty sets for the model parameters, but this approach, in contrast with the approach we use, does not take into account the likelihood of each parameter value given by its probability distribution. See [18].

As it has been stated above, the aim of the stochastic MAB model here presented consists of obtaining the optimal expected advertising budget and its related allocation, by considering that the probability distribution of the uncertain parameters is known. Our model aims to addressing the following questions: a) Which is the optimal multiproduct advertising budget for the whole planning horizon? b) Given an advertising budget, how it can be optimally allocated along the planning horizon? c) Is it advisable to consider stochastic MAB models? or on the contrary, is it enough to consider deterministic ones?

The MAB model introduced in [3] is simple but realistic enough to be used in the advertising industry. From a mathematical point of view, it corresponds to a convex optimization problem which is numerically tractable and allows for computing the global optimal solutions with moderate computational effort. We will analyze under which conditions these desirable properties are inherited by the stochastic MAB model.

The remainder of the paper is organized as follows. Section 2 summarizes some basic but useful properties for formulating and solving single-stage stochastic optimization problems with deterministic feasible set. In Section 3 a stochastic version of the MAB problem is introduced and analyzed. A realistic case study is introduced in Section 4 to illustrate the effectiveness of the stochastic model as well as the theoretical concepts of Section 3. Section 5 concludes and outlines future research.

2 Single-stage stochastic optimization with deterministic feasible set

Stochastic optimization literature most frequently considers an optimization horizon with two or more decision stages (two-stage and multistage models, respectively). In contrast, single-stage models consider only one decision stage, that is, the decision is to be made 'here and now' and the model does not account for any corrective (recourse) actions as in the two-stage and multistage cases, see [5, 18], among others. Note: In case of the recourse actions are performed, they are made at the stages of the planning horizon at which the realization of the random variables in the model becomes known. Single-stage models have received much less attention than two-stage or multistage ones, for example, they are not considered in the books [5] and [26]. However, their relevance in emphasized in [18]. In the field of advertising budgeting we will show that for a slightly higher conceptual and computational effort, the single-stage stochastic approach may significantly improve the single-stage deterministic one in two aspects: expected profit and accuracy (see Section 4).

In this section we deal with the single-stage stochastic optimization problem with deterministic feasible set, that is, with deterministic parameters in the constraints. Let us consider the decision vector $x \in \mathcal{D} \subset \mathbb{R}^n$ and the random vector $\xi \in \Xi \subset \mathbb{R}^m$, where $\mathcal{D}$ is called the feasible set and Ξ is called the support of ξ. Given the cost function $F(x, \xi)$ we define the stochastic problem P_S as

$$\min_{x \in \mathcal{D}} F_S(x),$$

where $F_S(x) := \mathbb{E}[F(x, \xi)]$. Notice that the uncertain parameters are only in the cost function and not in $\mathcal{D}$. In some situations problem P_S is approximated by a deterministic model named the expected value problem P_{EV} that can be expressed as

$$\min_{x \in \mathcal{D}} F_{EV}(x),$$

where $F_{EV}(x) := F(x, \bar{\xi})$ and $\bar{\xi} = \mathbb{E}[\xi]$. In this context let $x^*_{(\cdot)}$ and $F^*_{(\cdot)}$ denote the optimal solution and cost of problem $\mathrm{P}_{(\cdot)}$, respectively. The deterministic and stochastic MAB models, to be presented in Section 3, have the structure of problems P_{EV} and P_S, respectively, and for this reason we review some basic but useful properties to formulate and solve those two problems (see Appendix, Proposition 1).

In the context of two-stage stochastic linear optimization let EV, EEV and RP denote the optimal cost of the Expected Value problem, the Expected result of using the EV solution and the optimal cost of the Recurse Problem, respectively; see e.g. [5, 18]. Given that the problem of concern in this section does not consider a recourse (corrective action), let us use the term SP (optimal cost of the Stochastic Problem) instead of RP. These concepts can be adapted in our context as follows,

$$\mathrm{EV} = F^*_{EV}, \qquad \mathrm{EEV} = F_S(x^*_{EV}), \qquad \mathrm{SP} = F^*_S.$$

With this notation, Proposition 1 recovers the well known result in stochastic linear optimization EV $\leq$ SP $\leq$ EEV. Another useful concept is VSS (Value of the Stochastic Solution), such that VSS = EEV $-$ SP. It measures the cost of ignoring uncertainty in choosing a decision; see again e.g. [5, 18].

Example 1 *Let us illustrate by example that the expected value function F_{EV} may be a bad approximation to the stochastic function F_S for optimization purposes. Consider the cost function*

$$F(x, \xi) = k_1 + k_2\, e^{\xi_1 x} - \xi_2\, x,$$

where $k_1 \in \mathbb{R}$ and $k_2 > 0$ are constants, and ξ_1 and ξ_2 are normal random variables $N(\bar{\xi}_1, \sigma_1^2)$ and $N(\bar{\xi}_2, \sigma_2^2)$, respectively. It is easy to prove that $F(x, \xi)$ is convex in ξ, then by Proposition 1 it results

$$k_1 + k_2\, e^{\bar{\xi}_1 x} - \bar{\xi}_2\, x \quad \leq \quad \mathbb{E}\left[k_1 + k_2\, e^{\xi_1 x} - \xi_2\, x\right] \quad \forall x \in \mathbb{R}.$$

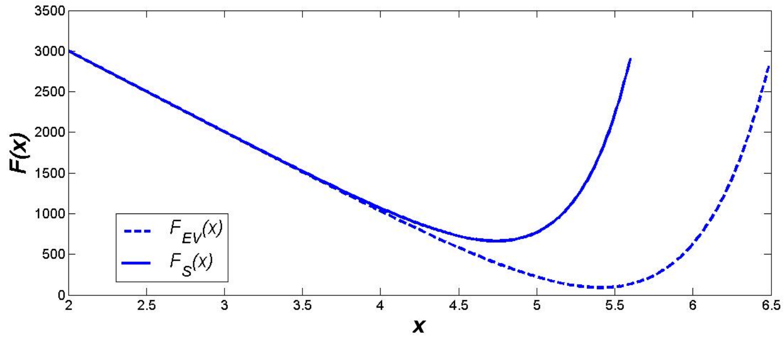

Figure 1: In Example 1, $F_{EV}(x)$ is a bad approximation to $F_S(x)$ for optimization purposes.

Consider problems P_S and P_{EV} associated to $F(x, \xi)$ and $\mathcal{D} = \mathbb{R}$. In this case we have:

$$\begin{aligned} F_S(x) &= \mathbb{E}\left[k_1 + k_2\, e^{\xi_1 x} - \xi_2\, x\right] \\ &= k_1 + k_2\, \mathbb{E}\left[e^{\xi_1 x}\right] - \bar{\xi}_2\, x \\ &= k_1 + k_2\, e^{\bar{\xi}_1 x + 0.5\sigma_1^2 x^2} - \bar{\xi}_2\, x, \end{aligned}$$

where we have used that $\mathbb{E}\left[e^{\xi_1 x}\right]$ corresponds to the moment-generating function of the normal random variable ξ_1 and therefore it is equal to $e^{\bar{\xi}_1 x + 0.5\sigma_1^2 x^2}$ (see [19] for details).

However, in the expected value problem, the function $F_S(x)$ is approximated by

$$F_{EV}(x) = k_1 + k_2\, e^{\bar{\xi}_1 x} - \bar{\xi}_2\, x.$$

Thus, the approximation

$$\mathbb{E}\left[e^{\xi_1 x}\right] = e^{\bar{\xi}_1 x + 0.5\sigma_1^2 x^2} \approx e^{\bar{\xi}_1 x}$$

may produce poor optimization results as it is depicted in Figure 1 for the instance with the following parameters: $k_1 = 5000, k_2 = 0.01, \bar{\xi}_1 = 2, \sigma_1^2 = 0.1, \bar{\xi}_2 = 1000, \sigma_2^2 = 100$. After solving P_{EV} and P_S it results

$$\begin{aligned} (x_{EV}^*, F_{EV}^*) &= (5.4,\ 90), \\ (x_S^*, F_S^*) &= (4.7,\ 663), \\ F_S(x_{EV}^*) &= 1750. \end{aligned}$$

Since EV = 90, SP=663 and EEV = 1750, the chain of inequalities EV $\leq$ SP $\leq$ EEV holds. Furthermore, although the deterministic and the stochastic optimal solutions are similar (5.4 versus 4.7), the corresponding expected costs are very different (1750 versus 663), giving a VSS equal to 1087. Of course, this is only a toy example, but it illustrates the well known fact that the expected value function F_{EV} may be a bad approximation to the stochastic function F_S for optimization purposes.

3 Multiperiod multiproduct advertising budgeting under uncertainty

We introduced in [3] a deterministic model for the multiperiod multiproduct advertising budgeting (MAB) problem, as well as, a brief review of some key concepts in the advertising industry such as sales driver (promotions, advertising copy, media selection and timing, etc.), reach, frequency, exposure, GRP (Gross Rating Point)... (see [3], Section 2: 'Problem formulation'). Further details can be found in [2, 6, 7, 15], among others. A more realistic approach to this problem consists of allowing randomness of the main parameters such as product sales, advertising saturation levels, advertising diminishing returns and white noise of the profit function. With this objective in mind, in this section the stochastic optimization version of the MAB problem is introduced, directly derived from the deterministic model in [3]. Prior to define the MAB profit function, we list the notation.

3.1 Notation

	Indexes:	
t	Index for periods, $t \in \mathcal{T} = \{1, \ldots, T\}$	
j	Index for products, $j \in \mathcal{J} = \{1, \ldots, J\}$	
i	(Auxiliary) index for products, $i \in \mathcal{I} = \mathcal{J}$	
k	Index for sales drivers, $k \in \mathcal{K} = \{1, \ldots, K\}$	$j \in \mathcal{J}$
$\mathcal{JK}$	stands for $\mathcal{J} \times \mathcal{K}$	$j \in \mathcal{J}$
$\mathcal{TIJK}$	stands for $\mathcal{T} \times \mathcal{I} \times \mathcal{J} \times \mathcal{K}$	$j \in \mathcal{J}$
$\mathcal{TJ}$	stands for $\mathcal{T} \times \mathcal{J}$	
$\mathcal{TJK}$	stands for $\mathcal{T} \times \mathcal{J} \times \mathcal{K}$	$j \in \mathcal{J}$
	Deterministic parameters:	
c_{tjk}	Cost of sales driver jk in period $t \in \{1, \ldots, T+1\}$, such that $c_{tjk} > 0$ (note that, for simplicity, to refer to the investment in the k-th driver to advertise product j, we use the expression 'driver jk')	$jk \in \mathcal{JK}$
δ_{jk}	Retention rate of the advertising effect from period to period for driver jk, $\delta_{jk} \in\ ]0, 1[$	$jk \in \mathcal{JK}$
p_{tj}	Profit per unit of product j in period t, $p_{tj} > 0$	$tj \in \mathcal{TJ}$
$\widetilde{x}_{0jk}$	Advertising effect of driver jk previous to the first period, $\widetilde{x}_{0jk} \geq 0$	$jk \in \mathcal{JK}$
	Stochastic parameters (random variables):	
γ_{tijk}	Sales of product i in period t induced by one unit of driver jk, $\gamma_{tijk} \in \mathbb{R}$ (note that, for simplicity, to refer to the k-th driver of product j, we use the expression 'driver jk')	$tijk \in \mathcal{TIJK}$
α_{tjk}	Advertising saturation level in period t of driver jk, $\alpha_{tjk} > 0$	$tjk \in \mathcal{TJK}$
β_{tjk}	Advertising diminishing return to scale in period t of driver jk, $\beta_{tjk} > 0$	$tjk \in \mathcal{TJK}$
ε	Stochastic error of the profit function such that $\mathbb{E}[\varepsilon] = 0$	
ξ	Random vector that accounts for all the random parameters ($\alpha_{tjk}, \beta_{tjk}\gamma_{tijk}$ and ε) of the profit function $P(g, \xi)$	

	Functions:	
S_{tjk}	Sales of product j in period t due to driver jk	$tjk \in \mathcal{TJK}$
C_{tijk}	Cross product effect: sales of product $i \neq j$ in period t due to driver jk	$tijk \in \mathcal{TIJK}$
P_{tjk}	Profit in period t due to driver jk	$tjk \in \mathcal{TJK}$.
P	Profit function	
x_{tjk}	Cumulated advertising effect of driver jk in period t ('adstock')	$tjk \in \mathcal{TJK}$
	Decision variables:	
g_{tjk}	Investment in GRPs of driver jk in period t (GRP stands for 'Gross Rating Points')	$tjk \in \mathcal{TJK}$

3.2 Profit function

In this section a stochastic MAB profit function is introduced directly derived from the deterministic one in [3], where further modeling details can be found (Section 2.2: 'The multiproduct sales response function'). We define the stochastic MAB profit function $P(g, \xi)$ as follows:

$$P(g, \xi) \quad = \sum_{tjk \in \mathcal{TJK}} P_{tjk}(g) + \varepsilon \tag{1}$$

where

$$\begin{aligned}
g &= (g_{tjk})_{tjk \in \mathcal{TJK}} \\
\xi &= \big((\alpha_{tjk})_{tjk \in \mathcal{TJK}},\ (\beta_{tjk})_{tjk \in \mathcal{TJK}},\ (\gamma_{tijk})_{tijk \in \mathcal{TIJK}},\ \varepsilon\big) && && (2)\\
P_{tjk}(g) &= p_{tj}\, S_{tjk}(g) + \sum_{i \in \mathcal{I}} p_{ti}\, C_{tijk}(g) -\ c_{tjk}\, g_{tjk} + V_{tjk}(g) && tjk \in \mathcal{TJK}, \\
S_{tjk}(g) &= \alpha_{tjk}(1 - e^{-\beta_{tjk}\, x_{tjk}(g)}) && tjk \in \mathcal{TJK}, && (3)\\
C_{tijk}(g) &= \gamma_{tijk}\, x_{tjk}(g) && tijk \in \mathcal{TIJK} && (4)\\
x_{tjk}(g) &= \delta_{jk}\, x_{t-1,jk}(g) + g_{tjk} && tjk \in \mathcal{TJK}, && (5)\\
x_{0jk} &= \widetilde{x}_{0jk} && jk \in \mathcal{JK}, && (6)\\
V_{1jk}(g) &= -c_{1jk}\, \delta_{jk} \widetilde{x}_{0jk} && jk \in \mathcal{JK}, && (7)\\
V_{tjk}(g) &= 0, && 1 < t < T, jk \in \mathcal{JK}, && (8)\\
V_{Tjk}(g) &= c_{T+1,jk}\, \delta_{jk}\, x_{Tjk}(g) && jk \in \mathcal{JK}. && (9)
\end{aligned}$$

The following comments for model (1)-(9) are in order:

(1) The stochastic error ε of the profit function is required both by (possibly) omitted variables in the model and by truly random disturbances, as pointed out in [15]. Observe that considering the random parameters that define $P_{tjk}(g)$ we should write $P_{tjk}(g, \alpha, \beta, \gamma)$ instead, but we drop them to simplify the notation.

(2) Also for simplifying, coefficients γ_{tijk} with $i = j$ are assumed both to exist and to be 0.

(3) Sales of product j induced by driver jk in period t.

(4) Cross product effect: sales on product $i \neq j$ due to driver jk in period t.

(5) Dynamic behavior of variable adstock for driver jk.

(6) Initial level of the adstock variable for driver jk.

(7) Accounting cost of the initial adstock level for driver jk.

(8) For notational convenience we use these dummy terms and set them equal to 0.

(9) Accounting value of the final adstock level for driver jk.

Notice that (5) is the discrete time version of the Nerlove-Arrow continuous time model for the adstock variable [20]. Finally, being ξ a random vector, the profit function $P(g, \xi)$ is a random variable. In the literature it is common to formulate the stochastic optimization problems in terms of scenarios. In the MAB context, a scenario is a realization of the random vector ξ. Let $\widetilde{\xi}^\omega$ denote the realization of the random parameters for scenario $\omega \in \Omega$, where Ω is the index set of the scenarios that are considered, such that for each scenario it is not required that at each period there are scenario groups in the typical scenario tree where those scenarios have the same values of the uncertain parameters up to the given period. That is the scenarios here considered are independent one from the other.

In scenario based notation, the profit function (1) would be written as

$$P(g, \widetilde{\xi}^\omega) = \sum_{tjk \in \mathcal{T}\mathcal{J}\mathcal{K}} P_{tjk}(g) + \widetilde{\varepsilon}^\omega \qquad \omega \in \Omega. \tag{10}$$

Usually the cardinality of Ω is considered to be finite, therefore in (10) it is implicitly assumed that ξ is a discrete random vector. However, in the MAB problem ξ could be either discrete or continuous and, for this reason, the notation $P(g, \xi)$ is used instead of $P(g, \widetilde{\xi}^\omega)$.

3.3 The stochastic and deterministic optimization models

3.3.1 Stochastic optimization model

Let us see that the Stochastic Multiperiod Multiproduct Advertising Budgeting problem MAB_S can be defined as a *single-stage* stochastic optimization problem with deterministic feasible set. Notice that in a different way as traditionally it is done in stochastic optimization for multiperiod problems (see for example [5, 18]), all (multiperiod) decisions are taken at the beginning of the planning horizon as it is a practice in the advertising sector and, then, the decision vector g is unique independently of the scenario to occur. That is, it is a non-recourse problem. In order to follow the notation defined in Section 2, the cost function $F(g, \xi)$ is defined as the opposite value of the profit function:

$$F(g, \xi) := -P(g, \xi).$$

Then, problem MAB_S can be defined as follows

$$\min_{g \in \mathcal{D}} \quad F_S(g), \tag{11}$$

where $F_S(g) = \mathbb{E}[F(g, \xi)] = -\mathbb{E}[P(g, \xi)]$. In Proposition 2 of the Appendix we show that under mild assumptions this cost function can be stated as follows

$$\begin{aligned} F_S(g) \quad = \quad - \sum_{tjk \in \mathcal{T}\mathcal{J}\mathcal{K}} \Big\{ & p_{tj} \, \bar{\alpha}_{tjk} \, \left(1 - \mathbb{E}\left[e^{-\beta_{tjk} \, x_{tjk}(g)} \right] \right) + \sum_{i \in \mathcal{I}} p_{ti} \, \bar{\gamma}_{tijk} \, x_{tjk}(g) \\ & -c_{tjk} \, g_{tjk} + V_{tjk}(g) \Big\}. \end{aligned} \tag{12}$$

Notice that the stochastic parameters only appear in the cost function since the feasible set $\mathcal{D}$ is assumed to be deterministic and convex. A typical convex feasible set only considers linear constraints and can be defined as

$$\mathcal{D} = \left\{ g \in \mathbb{R}^n \quad | \quad Ag \leq b, \qquad \underline{g} \leq g \leq \overline{g} \right\}$$

where A is the constraint matrix, b is the right-hand-side and $\underline{g}$ and $\overline{g}$ are lower and upper bounds, respectively. See the case study presented in Section 4.

Notice that problem MAB$_S$ has the same structure as problem P$_S$ considered in Section 2. Also observe that, although problem MAB$_S$ considers T periods (i.e., it is a *multiperiod* problem) the value of the decision variables does not depend on each realization of the random vector ξ. Therefore, it is a *single-stage* stochastic optimization problem where, as we stated above, the value of each decision variable is unique. In spite of the problem being a single-stage one, it takes into account all the realizations of the random vector ξ, in contrast with the deterministic approach, that simply takes into account the expected value of the parameters.

In order to calculate the expected profit $\mathbb{E}\big[P(g,\xi)\big]$, notice that regarding the random parameters $\alpha_{tjk}, \gamma_{tijk}$ and the random error ε it is equivalent to consider their full probability distribution or to consider their expected value. The reason is that $P(x,\xi)$ is linear in these parameters. In contrast, $P(x,\xi)$ is a non-linear function in the random parameters β_{tjk} because of the non-linear term

$$\mathbb{E}\left[e^{-\beta_{tjk}\; x_{tjk}(g)}\right].$$

This term is considered in the stochastic approach, whereas the deterministic approach approximates it by

$$e^{-\mathbb{E}[\beta_{tjk}]\; x_{tjk}(g)}.$$

While in some (rare) situations this approximation might be appropriate, in general it is a very crude approach: the whole probability distribution of β_{tjk} is collapsed into a one-point distribution, as pointed out in [18]. Therefore, the stochastic MAB model presented in this paper improves the deterministic one presented in [3]. At a low computational cost, the stochastic MAB model computes the true expected cost whereas the deterministic MAB model uses a potentially crude approximation, as we show in Example 1 and in the case study of Section 4. Furthermore, Problem MAB$_S$ is a convex optimization problem, which is a very convenient feature for optimization and computational aspects (see Appendix, Proposition 3).

3.3.2 Deterministic optimization model

The stochastic parameters of problem MAB$_S$ are replaced with their expected value to obtain the so called Expected Value problem, that by definition is a deterministic one. So, the Expected Value Multiperiod Multiproduct Advertising Budgeting problem MAB$_{EV}$ can be expressed as

$$\min_{g\in\mathcal{D}} \quad F_{EV}(g), \tag{13}$$

where $F_{EV}(g) = F(g,\mathbb{E}[\xi]) = -P(g,\mathbb{E}[\xi])$ and can be stated as follows

$$\begin{aligned} F_{EV}(g) \quad = \quad - \sum_{tjk\in\mathcal{T}\mathcal{J}\mathcal{K}} & \Big\{ p_{tj}\; \bar{\alpha}_{tjk}\; \left(1 - e^{-\bar{\beta}_{tjk}\; x_{tjk}(g)}\right) + \sum_{i\in\mathcal{I}} p_{ti}\; \bar{\gamma}_{tijk}\; x_{tjk}(g) \\ & -c_{tjk}\; g_{tjk} + V_{tjk}(g) \Big\}. \end{aligned} \tag{14}$$

On the other hand, problem MAB$_{EV}$ has the same feasible set $\mathcal{D}$ as problem MAB$_S$ (see the previous section). This model corresponds to the one introduced in [3]. Notice that problem MAB$_{EV}$ has the same structure as problem P$_{EV}$ analyzed in Section 2. Also notice that, in this deterministic version the random vector ξ is replaced by its expectation $\mathbb{E}[\xi]$, so we minimize $-P(g,\mathbb{E}[\xi])$ (equivalent to maximize $P(g,\mathbb{E}[\xi])$). In contrast, in the stochastic MAB model we maximize $\mathbb{E}[P(x,\xi)]$ which is, in general, different from $P(g, \mathbb{E}[\xi])$. Thus, as we have already pointed out, the improvement of problem MAB$_S$ is to consider the true expected profit $\mathbb{E}[P(x, \xi)]$ instead of its approximation $P(g, \mathbb{E}[\xi])$ used in problem MAB$_{EV}$. Furthermore, problem MAB$_{EV}$ is also a convex optimization problem (see Appendix, Proposition 4). Finally it is important to mention that under mild assumptions we have that

$$F^*_{EV} \leq F^*_S \leq F_S(g^*_{EV})$$

(see Appendix, Proposition 5).

3.4 Computing $\mathbb{E}\left[\, e^{-\beta x} \,\right]$

In order to compute $F_S(x)$ in problem MAB$_S$, the expected value $\mathbb{E}[\, e^{-\beta x} \,]$ is required to be calculated (see the Appendix, Proposition 2). It can be done by considering different alternatives depending on the existence or not of a closed formula for obtaining the moment generating function of the random variable β (see Example 1). We favor (see the case study presented in Section 4) the scenario based approach based on the big data that use to be available. In this case, the computation is as follows,

$$\mathbb{E}\left[e^{-\beta x}\right] = \sum_{\omega \in \Omega} w^\omega e^{-\widetilde{\beta}^\omega \, x}, \tag{15}$$

where the index set Ω is assumed to be finite, $\{\widetilde{\beta}^\omega\}_{\omega \in \Omega}$ is the support of β and w^ω is the probability or weight assigned to the occurrence of $\widetilde{\beta}^\omega$. Each ω corresponds to a scenario and formulation (11) written in terms of expression (15) corresponds to the Deterministic Equivalent Model [5].

4 Case study

4.1 Case description

An experimental case is analyzed for showing the improvement that the single-stage stochastic model can bring to the deterministic one of the multiperiod multiproduct advertising budgeting (MAB) problem under parameter uncertainty. Notice that the level of improvement depends on the instance considered. In any case, regarding the expected profit, the stochastic approach will always be at least as good as the deterministic one (Proposition 5). The objective of the current case study is to show by example that in the MAB problem, for a slightly higher conceptual and computational effort, the stochastic approach may significantly improve the deterministic approach in two aspects: expected profit and accuracy. As it was pointed out in the introduction, considering that for some MAB parameters only its probability distribution is known, the aim of this work is to answer the following questions: a) Which is the optimal multiproduct advertising budget for the whole planning horizon? b) Given an advertising budget, how can we optimally allocate it along the planning horizon? c) Is it important to consider stochastic models? or on the contrary, is it enough to consider deterministic ones?

The computations have been conducted on a laptop under Windows XP, with a processor Intel Core Duo 2.40 GHz and with 3.48 GB of RAM. The implementation has been written in Matlab (R2008b) and the constrained convex minimization problems MAB$_S$ and MAB$_{EV}$, have been solved by function `fmincon` from the Matlab Optimization Toolbox (V4.1) with default parameters. In Tables 6 and 8 we report the number of SQP iterations and CPU time needed to solve our MAB instances (12 periods, 2 products and 2 drivers).

The version of problem MAB$_S$ solved in this case study corresponds to

$$\min_{g \in \mathcal{D}} \quad F_S(g), \tag{16}$$

Table 1: Problem dimensions and profit per unit of product.

Parameter	Value	Units
T	12	month
I	2	product
K	2	driver
p_{t1}	1.75	euro/unit of product P1
p_{t2}	1.40	euro/unit of product P2

Table 2: Advertising retention rate δ_{jk}, initial adstock $\widetilde{x}_{0jk}$ and advertising diminishing return $\bar{\beta}_{tjk}$.

j	δ_{j1}	δ_{j2}	$\widetilde{x}_{0j1}$	$\widetilde{x}_{0j2}$	$\bar{\beta}_{tj1}$	$\bar{\beta}_{tj2}$
1	0.660	0.552	300	300	0.010	0.010
2	0.588	0.552	50	50	0.005	0.005

Table 3: Advertising saturation level $\bar{\alpha}_{tjk}$.

t	$\bar{\alpha}_{t11}$	$\bar{\alpha}_{t12}$	$\bar{\alpha}_{t21}$	$\bar{\alpha}_{t22}$
1	345,000	270,000	86,400	105,600
2	389,850	305,100	83,700	102,300
3	493,350	386,100	113,400	138,600
4	510,600	399,600	137,700	168,300
5	731,400	572,400	199,800	244,200
6	838,350	656,100	251,100	306,900
7	897,000	702,000	278,100	339,900
8	969,450	758,700	259,200	316,800
9	734,850	575,100	224,100	273,900
10	386,400	302,400	191,700	234,300
11	427,800	334,800	189,000	231,000
12	407,100	318,600	218,700	267,300

where a finite set of scenarios Ω is considered such that

$$\begin{aligned} F_S(g) \quad = \quad & -\sum_{tjk \in \mathcal{TJK}} \Bigg\{ p_{tj}\, \bar{\alpha}_{tjk} \left(1 - \sum_{\omega \in \Omega} w^{\omega} e^{-\widetilde{\beta}^{\omega}_{tjk}\, x_{tjk}(g)} \right) + \sum_{i \in \mathcal{I}} p_{ti}\, \bar{\gamma}_{tijk}\, x_{tjk}(g) \\ & -c_{tjk}\, g_{tjk} + V_{tjk}(g) \Bigg\}. \end{aligned} \tag{17}$$

The feasible set accounts for the budged constraint, i.e.,

$$\mathcal{D} = \left\{ g \in \mathbb{R}^n \quad | \quad \sum_{tjk \in \mathcal{TJK}} c_{tjk}\, g_{tjk} \leq b, \qquad \underline{g} \leq g \leq \overline{g} \right\}, \tag{18}$$

where the advertising budget b is allocated among all the drivers and along all the periods. Of course, many other kind of constraints could be considered: the company could be interested in limiting the advertising budget within each time period, it could be interested in imposing a threshold for the

Table 4: Cost of driver c_{tjk}.

t	c_{t11}	c_{t12}	c_{t21}	c_{t22}
1	480	528	432	475
2	480	528	432	475
3	640	704	576	634
4	640	704	576	634
5	480	528	432	475
6	480	528	432	475
7	640	704	576	634
8	640	704	576	634
9	480	528	432	475
10	480	528	432	475
11	640	704	576	634
12	640	704	576	634

adstock variables at the end of the planning horizon, it could impose a budget for each driver, etc. On the other hand, the problem MAB$_{EV}$ here solved corresponds to equations (13)-(14) (notice that problems MAB$_S$ and MAB$_{EV}$ have the same feasible set $\mathcal{D}$).

The MAB$_S$ and MAB$_{EV}$ instances that are considered in this computational experiment were derived from a real-life case addressed at the consulting company Bayes Forecast[1] to plan the advertising campaign for a leading fast moving consumer goods company. The instance we present here, considers a twelve months planning horizon (T = 12), two products that for confidentiality we denote by P1 and P2 (I = 2) and two sales drivers (K = 2). The first driver corresponds to TV advertising and the second driver corresponds to in-store promotions. Table 1 shows the problem dimensions as well as the unit profit per product, being the same for all periods in the planning horizon. Table 2 gives the advertising retention rate δ_{jk} (no units), the initial adstock $\widetilde{x}_{0jk}$ (GRPs) and the advertising diminishing return $\bar{\beta}_{tjk}$ (GRPs^{-1}) related to function (19), being the same for all periods in the planning horizon. Notice that only the expected result is given, since the corresponding discrete estimated distributions consider 300 realization for each β_{tjk} and therefore are too large to be included in the paper. Nevertheless, they can be downloaded from Section 'Publications' at `http://bayes.cct.urjc.es/∼cbeltran/CV/`. Table 3 shows the advertising saturation level α_{tjk} (units of product j) related to function (19). Table 4 gives the cost (euro/GRP) of drivers in the periods along the planning horizon. Notice that these costs define alternating low and high price periods of two months. This cost structure allow us to observe the reaction of the two optimization approaches in front of dynamic GRP prices.

In the MAB model the single product sales response function denominated 'modified exponential' [15] is used, where sales due to advertising are modeled as a function of the cumulated advertising effect $x(g)$ as follows,

$$S(g) = \alpha \left(1 - e^{-\beta\, x(g)}\right). \tag{19}$$

Note: Function S_{tjk} was introduced in Section 3, and here the indexes tjk are dropped for simplicity of exposition. The positive parameter α corresponds to the advertising saturation level. This means that no matter how much marketing effort is expended, the sales due to advertising will not be higher than α. The positive parameter β regulates the advertising diminishing return to scale. On the other hand, γ, the cross product sales effect between products P1 and P2, is due to substitution, i.e., advertising

[1] Bayes Forecast S.L., Madrid (Spain), www.bayesforecast.com

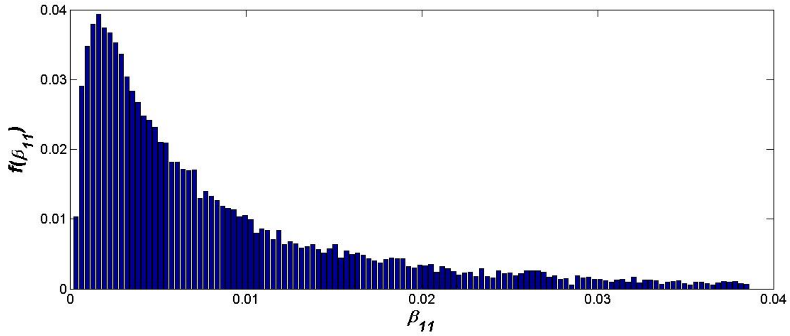

Figure 2: Discrete estimation of $f_{\beta_{11}}$. The probability of each possible realization $\widetilde{\beta}_{11}$ is given by $f_{\beta_{11}}(\widetilde{\beta}_{11})$.

on, say P1, will increase P1 sales but will reduce P2 sales and viceversa. This effect is known as *cannibalization* [12]. Under cannibalization, the cross product effect parameter γ is negative (see Table 5).

As we saw in Section 3.2, the sales functions $S_{tjk}(g)$ and $C_{tijk}(g)$ are based on the unknown parameters α, β and γ. For the deterministic model MAB$_{EV}$ it is enough to estimate their expected values, i.e., $\bar{\alpha}, \bar{\beta}$ and $\bar{\gamma}$. On the other hand, the stochastic model MAB$_S$, is based on the expected values $\bar{\alpha}$ and $\bar{\gamma}$ and on the probability distribution of β, needed to compute $\mathbb{E}[e^{-\beta x}]$. At this point several approaches could be possible. One approach would consist in estimating the probability density function of β, say f_β, by fitting a theoretical distribution (e.g. log-normal) and then to compute $\mathbb{E}[e^{-\beta x}]$ either analytically, if possible, or by discretizing the estimation of f_β in order to use formula (15). As an alternative, in this case study, we have followed a Bayesian approach which, in our opinion is more effective, since it directly obtains a discrete estimation of f_β. More specifically, by using Bayesian inference combined with Markov Chain Monte Carlo (MCMC) methods [21] one can obtain a sample of the so-called posterior probability distribution of β, which corresponds to a discrete estimation of f_β. For example, Figure 2 depicts the discrete estimation of $f_{\beta_{11}}$ which gives its possible realizations and their associated probability. Note: Since in this case study, β_{tjk} does not depend on the specific time period t, index t can be dropped to just write β_{jk}. The full description of the Bayesian approach for obtaining a discrete estimation of f_β and the verification of its validity is out of the scope of this work (see for example [21] for details).

The expected value of function (19) in problem MAB$_S$ is exactly computed as

$$\mathbb{E}\left[\alpha\left(1-e^{-\beta x}\right)\right] = \bar{\alpha}\left(1-\mathbb{E}\left[e^{-\beta x}\right]\right),$$

where the right-hand-side expectation is calculated by means of formula (15). Observe that, in fact, it is only needed the probability distribution of β. For α, it is enough to consider its expected value $\bar{\alpha}$. The same happens with the cross product effect parameter γ such that $\bar{\gamma}_{tijk}$ is given in Table 5 (in units of product i per unit of driver jk).

Table 5: Cross product effect parameter $\bar{\gamma}_{tijk}$.

i	$\bar{\gamma}_{ti11}$	$\bar{\gamma}_{ti21}$	$\bar{\gamma}_{ti12}$	$\bar{\gamma}_{ti22}$
1	0	-0.00010	0	-0.00015
2	-0.00010	0	-0.00015	0

It is worthy to mention that problem MAB$_S$ implicitly considers a huge number of scenarios. As indicated above, a scenario is a realization of the random vector ξ, so $\widetilde{\xi}^{\omega}$ denotes the realization of the random parameters for scenario $\omega \in \Omega$, where Ω is the set of scenarios that are considered. In the MAB$_S$ instance here solved, there are in total 145 random parameters: four triples (α, β, γ) per each of the 12 periods plus one ε. However, to compute the MAB expected profit it is enough to consider the random variables β_{tjk}, and the expected values $\bar{\alpha}_{tjk}, \bar{\gamma}_{tijk}$ and $\bar{\varepsilon}$. Thus the total number of random parameters explicitly considered is 48: four β's per each of the 12 periods. For the remaining 97 random parameters it is enough to consider their expected values. All in all, taking into account that the discrete estimation distribution of each random variable β_{tjk}, for $tjk \in \mathcal{TJK}$, considers 300 possible realizations, the total number of scenarios implicitly considered in the MAB$_S$ instance is $300^{48} \approx 10^{119}$. This has been possible since the structure of the MAB$_S$ problem allows for applying Proposition 2, which dramatically simplifies the computation of the expected profit.

Problem MAB$_{EV}$ simplifies problem MAB$_S$ by just considering the expected values $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ and $\bar{\varepsilon}$. Thus, for example, the expected value of function (19) is approximated by

$$\mathbb{E}\left[\alpha\left(1 - e^{-\beta x}\right)\right] \approx \bar{\alpha}\left(1 - e^{-\bar{\beta} x}\right).$$

Notice that the two problems, MAB$_{EV}$ and MAB$_S$, are built from almost the same data (Table 1 to Table 5). The difference is that problem MAB$_{EV}$ is based on the expected values $\bar{\beta}_{tjk}$ in Table 2, whereas problem MAB$_S$ considers a discrete estimation of the probability density function of each random variable β_{tjk} which has 300 possible realizations.

4.2 Determining the optimal budget and its allocation

As considered in Section 3, when solving the MAB problem one can compute the optimal budget taking into account the parameter uncertainty with a single-stage stochastic optimization model (i.e., MAB$_S$ problem) or, alternatively, one can compute the optimal budget by considering that all parameters are known with a single-stage deterministic optimization model (i.e., MAB$_{EV}$ problem). The deterministic problem MAB$_{EV}$ usually gives a solution which is good but suboptimal for problem MAB$_S$. One of the aims of our case study is to show by example that relevant losses can be incurred by implementing the deterministic optimal allocation.

In the example it is assumed that in the feasible set (18) there is not a finite limit in the budget (i.e. $b = +\infty$.). Furthermore, the lower bound vector $\underline{g}$ and the upper bound vector $\overline{g}$ are 0 and $+\infty$, respectively. Table 6 shows the main results of the MAB$_S$ and MAB$_{EV}$ models, in particular the optimal budget and profit as well as the related CPU time and SQP iterations. As an example, the optimal budget allocation corresponding to driver 1 of product P1, is depicted in Figure 3, where it can be observed that the stochastic approach is more sensitive to price changes than the deterministic approach. The former allocates more GRPs in low price periods (months 1-2, 5-6 and 9-10) and less GRPs in high price periods (months 3-4, 7-8 and 11-12).

The Expected profit of using the Expected Value solution (EEV) is computed by fixing the g decision variables in the MAB$_S$ problem to the optimal solution values of the MAB$_{EV}$ problem and computing the profit function (1) for each scenario $\omega \in \Omega$ by considering the related realization $\widetilde{\xi}^{\omega}$, such that EEV is the expected value of the profit function over all the scenarios. Table 6 shows that the advertising budged proposed by the stochastic model is 13.93% (531,705 euros) higher than the advertising budged proposed by the deterministic model. It also shows that the stochastic approach produces an expected profit 4.38% (738,146 euros) higher than the expected profit of the deterministic approach given by the EEV. Of course, the stochastic improvement in expected profit observed in this case study does not guarantee this level of improvement for all the MAB instances. Finally, notice that in this case

Table 6: Optimal budget and optimal profit.

	MAB_{EV}	EEV	MAB_{S}	Variation
Budget	3,818,334	-	4,350,039	+13.93%
Deterministic profit	23,276,709	-	-	-
Expected profit	-	16,870,731	17,608,877	+4.38%
CPU time (s)	2	-	54	-
SQP iterations	43	-	35	-

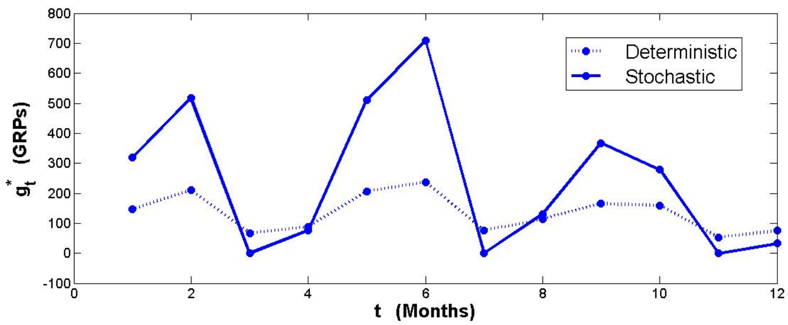

Figure 3: Optimal budget allocation g_t^* for driver 1 of product P1: Deterministic approach versus stochastic approach.

study the deterministic approach is inaccurate, thus misleading for managerial purposes, since EEV (16,870,731 euros) has dropped more than 25% compared to EV (23,276,709 euros).

So far we have compared the stochastic versus the deterministic (expected value) approaches by comparing the corresponding expected profits $\mathbb{E}[P(g_S^*, \xi)]$ and $\mathbb{E}[P(g_{EV}^*, \xi)]$, as numerical values. It can also be useful to compare the corresponding profits $P(g_S^*, \xi)$ and $P(g_{EV}^*, \xi)$, as random variables, whose sample probability distribution can be seen in Figures 4 and 5, respectively. These distributions have been obtained by computing the profits associated to a sample of size 20000 of the random vector ξ. That is, for each sample vector $\widetilde{\xi}^k$, we have computed the corresponding sample profits $P(g_S^*, \widetilde{\xi}^k)$ and $P(g_{EV}^*, \widetilde{\xi}^k)$ whose histogram can be seen in Figures 4 and 5, respectively. The sample expected profits for the deterministic and stochastic approaches thus computed are 16,855,996 and 17,619,252 euros, respectively which are not far from the model expected profits in Table 6, 16,870,731 and 17,608,877 euros, respectively. Other relevant sample statistical parameters can be found in Table 7.

For a more precise comparison of the previous random profits, their Cumulative Distribution Function (CDF) can be used (see Figure 6). Since the deterministic approach CDF is above the stochastic

Table 7: Sample statistical parameters for the deterministic and stochastic approaches (in euros).

	Deterministic	Stochastic
Min	11,814,360	12,483,785
Max	21,071,046	21,242,252
Mean	16,855,996	17,619,252
Median	16,895,003	17,678,295
Standard deviation	1,305,596	1,093,411

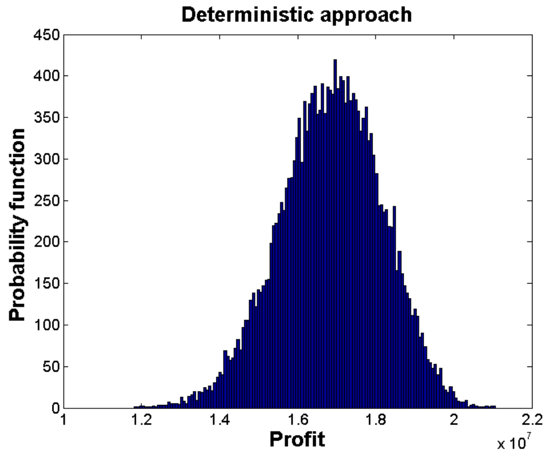

Figure 4: Sample probability distribution of the profit (euros) that may be obtained with the deterministic approach (sample expected profit 16,855,996 euros).

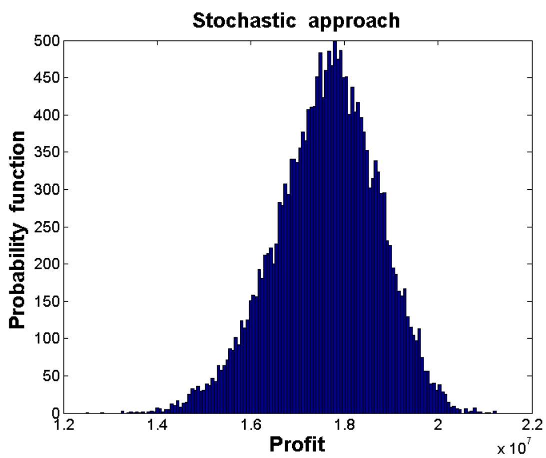

Figure 5: Sample probability distribution of the profit (euros) that may be obtained with the stochastic approach (sample expected profit 17,619,252 euros).

approach CDF, we can conclude that decision g_S^* has first-order stochastic dominance over decision g_{EV}^* [14]. Therefore decision g_S^* can be ranked as superior to decision g_{EV}^* from a probabilistic point of view (on top of a better expected profit). Notice that the these CDFs give useful further information as for example, the percentage of scenarios with shortfall (i.e., scenarios whose profit is below a given threshold).

4.3 Determining the optimal allocation for a given budget

To compute the optimal budget one assumes that there is no limit on the available budget, as in the previous section. However, it is usual to allocate a reduced advertising budget. For example, if we were limited to 50% of the optimal budget as computed in Section 4.2, model MAB$_S$ should be solved with $b = 2,175,020$ euros in (18). Table 8 shows the results obtained by the two approaches: optimal budget versus reduced budget. Observe that by reducing 50% (2,175,020 euros) the optimal budget for the 12 months, the optimal expected profit is reduced by 5.86% (1,032,054 euros). As an example,

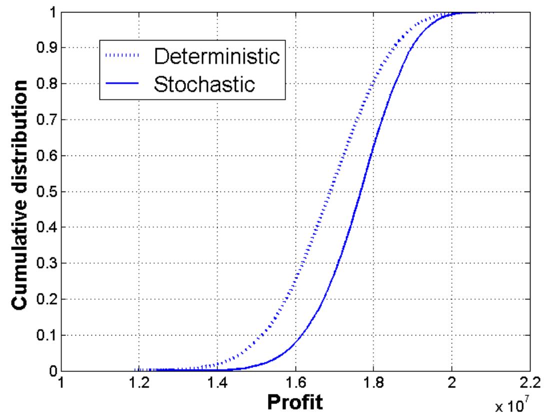

Figure 6: It is more likely to obtain a low profit with the deterministic approach.

Table 8: Expected profit for the optimal budged and a reduced budget (stochastic approach).

	Optimal budget	Reduced budget	Variation
Budget	4,350,039	2,175,020	-50.00%
Expected profit	17,608,877	16,576,823	-5.86%
CPU time (s)	54	40	
SQP iterations	35	26	

Figure 7 depicts the budget allocation corresponding to driver 1 of product P1 obtained by the two approaches.

5 Concluding remarks

The main contribution of this paper is to introduce a stochastic model for the Multiperiod Multiproduct Advertising Budgeting (MAB) problem. We call it MAB_S and it is intended to solve the MAB problem under uncertainty, that is, by taking into account the randomness of the problem parameters. As far as

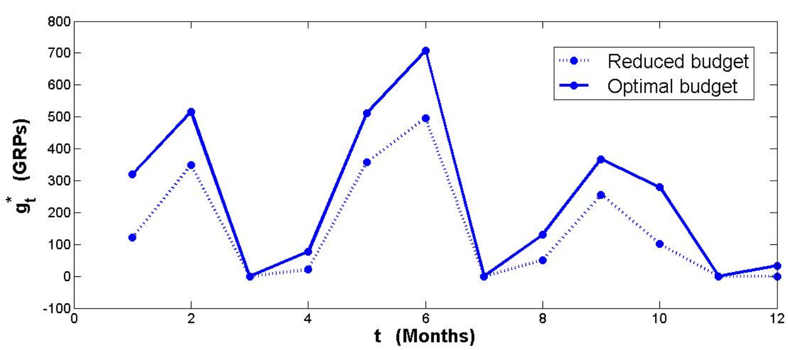

Figure 7: Optimal budget allocation g_t^* for driver 1 of product P1 considering the optimal or the reduced budget, respectively.

we know, a *stochastic* version of the MAB problem has not been considered in literature.

The MAB$_S$ problem addressed in this work corresponds to a *single-stage* stochastic optimization problem with deterministic feasible set. From a theoretical point of view, it has been shown that MAB$_S$ is a convex optimization problem, which is numerically tractable and produces global optimal solutions for the expected profit (i.e., in a risk neutral environment). We have also proven that the optimal expected profit given by the stochastic approach is at least as good as the expected profit given by the deterministic approach.

From a practical point of view, first, it has been proven that the stochastic model, in comparison with its deterministic counterpart, allows for a better allocation of the advertising investment along the planning horizon. A MAB$_S$ instance derived from a realistic advertising campaign has been used as a pilot case in the computational experiment, where the stochastic model has improved by 4.38% the optimal expected profit computed by the deterministic one. Second, it has been assessed that the deterministic approach may be inaccurate, thus misleading for managerial purposes. In the case study, the expected profit of using the expected value solution (EEV) has dropped more than 25% compared to the optimal profit of the expected value problem (EV). Third, it has been shown by example that the MAB$_S$ model can be used not only to compute the optimal budget, but for allocating (suboptimal) limited budgets combined with other managerial constraints. For example, it has been observed that, by reducing 50% the optimal budget, the advertising expected profit drops 5.86%.

Therefore, we can conclude that it is important for advertising budgeting to consider effective stochastic models as the one presented in this work. In this case, for a slightly higher conceptual and computational effort, the stochastic model MAB_S may significantly improve the deterministic model MAB_{EV} in two aspects: expected profit and accuracy.

Finally, we would like to point out some limitations of the MAB$_S$ model here presented. The first limitation is that this stochastic model is risk neutral as we stated above. That is, it is based on the expected profit and it does not incorporate any risk measure to cope with the cases of high variability of the profit over the scenarios as for example conditional Value-at-Risk [23, 25, 26] and stochastic dominance [11, 13, 14] among others. The second limitation is that the model does not take the product price as a sales driver, that is, as a decision variable (notice that prices are input data in the current version of the model). The third limitation is that the model does not take into account the competitive aspects of advertising budgeting. As a sequel of this paper, we are planning to improve this version of the MAB model regarding these three aspects. Furthermore, we will study the advantage of using a (restricted) multistage approach to the problem by clustering consecutive periods into stages and taking stage-based decisions allowing then recourse actions.

6 Appendix. Theoretical results

In Proposition 1 we review some classical results which are basic but useful in this context (notice that we use some notation defined in Section 2). The other propositions concern the MAB model introduced in this paper.

Proposition 1

1. *(Jensen's inequality [16]) Let ξ be a random vector such that $\mathbb{E}\left[\xi\right] = \bar{\xi}$ and $G(\xi)$ be a convex function. Then*

$$G(\bar{\xi}) \leq \mathbb{E}[G(\xi)].$$

2. *Let ξ be a random vector and $F(x,\xi)$ be a convex function in ξ. Then*

$$\min_{x\in\mathcal{D}} F(x,\bar{\xi}) \leq \min_{x\in\mathcal{D}} \mathbb{E}\big[F(x,\xi)\big].$$

3. *If $F(x,\xi)$ is a convex function in ξ then:*

$$F_{EV}^* \leq F_S^* \leq F_S(x_{EV}^*).$$

4. *If $F(x,\bar{\xi})$ is a convex function in x, then P_{EV} is a convex optimization problem (i.e., minimization of a convex cost function and convex feasible set).*

5. *If $F(x,\xi)$ is a convex function in x for all $\xi \in \Xi$, then P_S is a convex optimization problem.*

Proposition 2 *If each $(\alpha_{tjk},\beta_{tjk})$, for all $tjk \in \mathcal{TJK}$, is a pair of independent random variables, then $\mathbb{E}\big[P(g,\xi)\big]$ can be computed as follows*

$$\begin{aligned}\mathbb{E}\big[P(g,\xi)\big] \quad = \quad & \sum_{tjk\in\mathcal{TJK}} \Big\{ p_{tj}\ \bar{\alpha}_{tjk}\ \left(1-\mathbb{E}\left[e^{-\beta_{tjk}\ x_{tjk}(g)}\right]\right) + \sum_{i\in\mathcal{I}} p_{ti}\ \bar{\gamma}_{tijk}\ x_{tjk}(g) \\ & -c_{tjk}\ g_{tjk} + V_{tjk}(g) \Big\},\end{aligned}$$

where $\bar{\alpha}_{tjk}$ and $\bar{\gamma}_{tjk}$ represent expected values.

Proof: It follows by considering two basic results in probability. The first one is that $\mathbb{E}[\cdot]$ is a linear operator such that $\mathbb{E}[k_1\xi_1 + k_2\xi_2] = k_1\mathbb{E}[\xi_1] + k_2\mathbb{E}[\xi_2]$ for any pair $(k_1,\ k_2)$ of constants and any pair $(\xi_1,\ \xi_2)$ of random variables. The second one is that $\mathbb{E}[\xi_1\xi_2] = \mathbb{E}[\xi_1]\mathbb{E}[\xi_2]$ for any pair $(\xi_1,\ \xi_2)$ of independent random variables. ∎

Proposition 3 *Problem MAB$_S$ is a convex optimization problem.*

Proof: For a fixed ξ, the cost function $F(g,\xi)$ is convex in g if α_{tjk} and β_{tjk}, for all $tjk \in \mathcal{TJK}$, are positive (see [3]). This condition is satisfied since, α_{tjk} and β_{tjk}, for all $tjk \in \mathcal{TJK}$, are positive random variables in problem MAB$_S$. Then, $F(g,\xi)$ is a convex function in g for all $\xi \in \Xi$ and by Proposition 1, problem MAB$_S$ is a convex optimization problem (the convexity of $\mathcal{D}$ is assumed by hypothesis). ∎

Proposition 4 *Problem MAB$_{EV}$ is a convex optimization problem.*

Proof: Analogous to the proof of Proposition 3.

Proposition 5 *If each $(\alpha_{tjk},\beta_{tjk})$, for all $tjk \in \mathcal{TJK}$, is a pair of independent random variables then for problems MAB$_{EV}$ and MAB$_S$ it results*

$$F_{EV}^* \leq F_S^* \leq F_S(g_{EV}^*).$$

Proof: Let us consider the random vector $\beta = (\beta_{tjk})_{tjk \in \mathcal{TJK}}$ and the function

$$\begin{aligned} Q(g,\beta) \quad = \quad & \sum_{tjk \in \mathcal{TJK}} \Big\{ p_{tj}\ \bar{\alpha}_{tjk} \ \left(1 - e^{-\beta_{tjk}\ x_{tjk}(g)}\right) + \sum_{i \in \mathcal{I}} p_{ti}\ \bar{\gamma}_{tijk}\ x_{tjk}(g) \\ & -c_{tjk}\ g_{tjk} + V_{tjk}(g) \Big\}. \end{aligned}$$

It is clear that $\mathbb{E}[P(g,\xi)] = \mathbb{E}[Q(g,\beta)]$ for all g. Therefore $F_S(g) = -\mathbb{E}[Q(g,\beta)]$. Similarly, $P(g,\mathbb{E}[\xi]) = Q(g,\mathbb{E}[\beta])$ for all g. Therefore $F_{EV}(g) = -Q(g,\mathbb{E}[\beta])$. This implies that Q could be used instead of P to define problems MAB_S and MAB_{EV}. On the other hand, it is easy to see that $-Q(g,\beta)$ is a convex function in β. The proof can be completed by applying Proposition 1. ■

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