

BLOCK STOCHASTIC GRADIENT ITERATION FOR CONVEX AND NONCONVEX OPTIMIZATION

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Abstract. The stochastic gradient (SG) method can minimize an objective function composed of a large number of differentiable functions or solve a stochastic optimization problem, very quickly to a moderate accuracy. The block coordinate descent/update (BCD) method, on the other hand, handles problems with multiple blocks of variables by updating them one at a time; when the blocks of variables are (much) easier to update individually than together, BCD has a (much) lower per-iteration cost. This paper introduces a method that combines the great features of SG and BCD for problems with many components in the objective and with multiple (blocks of) variables.

Specifically, a block stochastic gradient (BSG) method is proposed for both convex and nonconvex programs. At each iteration, BSG approximates the gradient of the differentiable part of the objective by randomly sampling a small set of data or sampling a few functions in the objective, and then, using the approximate gradient, it updates all the blocks of variables in either a deterministic or a randomly shuffled order. Its convergence for convex and nonconvex cases is established in different senses. In the convex case, the proposed method has the same order of convergence rate as the SG method. In the nonconvex case, its convergence is established in terms of the expected violation of a first-order optimality condition.

The proposed method was numerically tested on problems including *stochastic least squares* and *logistic regression*, which are convex, as well as *low-rank tensor recovery* and *bilinear logistic regression*, which are nonconvex. On the convex problems, it performed as well as, and often significantly better, than the SG method. On the nonconvex problems, the proposed method BSG significantly outperformed the deterministic BCD method because the latter tends to slow down or stagnate near bad local minimizers. Overall, BSG inherits the benefits of both stochastic gradient approximation and block-coordinate updates.

Key words. stochastic gradient, block coordinate update, convex optimization, nonconvex optimization, nonsmooth optimization.

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1. Introduction. In many engineering and machine learning problems, we are facing optimization problems that involve a huge amount of data. It is often very expensive to use such a huge amount of data for every update of the problem variables, and a more efficient way is to sample a small amount from the collected data for each renewal of the variables.

Keeping this in mind, in this paper, we consider the stochastic program

$$\min_{\mathbf{x}} \Phi(\mathbf{x}) = \mathbb{E}_{\xi} f(\mathbf{x}; \xi) + \sum_{i=1}^s r_i(\mathbf{x}_i), \quad \text{s.t. } \mathbf{x}_i \in \mathcal{X}_i, \quad i = 1, \dots, s, \quad (1.1)$$

where $\mathcal{X}_i \subset \mathbb{R}^{n_i}, \forall i$, are convex sets, the variable $\mathbf{x} \in \mathbb{R}^n$ is partitioned into disjoint blocks $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_s)$ with total dimension $n = \sum_{i=1}^s n_i$, ξ is a random variable, $\mathbb{E}_{\xi} f(\mathbf{x}; \xi)$ is continuously differentiable, and r_i 's are certain regularization terms (possibly non-smooth) to promote solution structures such as ℓ_1 -norm $\|\mathbf{x}_i\|_1$ or ℓ_0 seminorm $\|\mathbf{x}_i\|_0$ to promote sparsity structure of the solution. Throughout the paper, we let

$$F(\mathbf{x}) = \mathbb{E}_{\xi} f(\mathbf{x}; \xi), \quad R(\mathbf{x}) = \sum_{i=1}^s r_i(\mathbf{x}_i),$$

and for simplicity, we omit the subscript ξ in the expectation operator without causing confusion.

Note that by assuming $\text{Prob}\{\xi = \ell\} = \frac{1}{N}, \ell = 1, \dots, N$, (1.1) includes as a special case the following deterministic program

$$\min_{\mathbf{x}} \frac{1}{N} \sum_{\ell=1}^N f_{\ell}(\mathbf{x}) + \sum_{i=1}^s r_i(\mathbf{x}_i), \quad \text{s.t. } \mathbf{x}_i \in \mathcal{X}_i, \quad \forall i, \quad (1.2)$$

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where N is often very large. Many problems in applications can be written in the form of (1.1) or (1.2) such as LASSO [54], sparse logistic regression [52], bilinear logistic regression [10, 53], sparse dictionary learning [31], low-rank matrix completion problem [6], and so on.

We allow F and r_i 's to be nonconvex. But when they are convex, we have sublinear convergence of the proposed method (see Algorithm 1) in terms of objective value. Without convexity, we show its global convergence in terms of the expected violation of a first-order optimality condition. In addition, numerical experiments demonstrate that our algorithm can perform very well on both convex and nonconvex problems.

1.1. Algorithms. One difficulty to solve (1.1) is that it may be impossible or very expensive to accurately calculate the expectation to evaluate the objective and its gradient or subgradient. One approach is the *stochastic average approximation* (SAA) method [24], which generates a set of samples and then solves the empirical risk minimization problem by a certain optimization method.

Another approach to solve (1.1) is the *stochastic gradient* (SG) method (see [33, 40, 44] and the references therein), which assumes that a stochastic gradient $\mathbf{g}(\mathbf{x}; \xi)$ of F can be obtained by some oracle and then iteratively performs the update

$$\mathbf{x}^{k+1} = \arg \min_{\mathbf{x} \in \mathcal{X}} \langle \mathbf{g}(\mathbf{x}^k + \tilde{\nabla} R(\mathbf{x}^k); \xi_k), \mathbf{x} - \mathbf{x}^k \rangle + \frac{1}{2\alpha_k} \|\mathbf{x} - \mathbf{x}^k\|^2, \quad (1.3)$$

where $\mathcal{X} = \mathcal{X}_1 \times \dots \times \mathcal{X}_s$, and $\tilde{\nabla} R(\mathbf{x}^k)$ is a subgradient of R at \mathbf{x}^k . In (1.3), ξ_k is a realization of ξ at the k th iteration, and α_k is a stepsize that is typically required to asymptotically reduce to *zero* for convergence. The work [33] compares SAA and SG and demonstrates that the latter can be competitive and sometimes significantly outperforms the former for solving a certain class of problems including the stochastic utility problem and stochastic max-flow problem. The SG method has also been popularly used (e.g., [14, 42, 50, 51, 59]) to solve deterministic programming in the form of (1.2) and exhibits advantages over the deterministic gradient method when N is large and high solution accuracy is not required.

To solve deterministic problems with separable nonsmooth terms as in (1.2), the block coordinate descent (BCD) method (see [20, 29, 55–58] and the references therein) has been widely used. At each iteration, BCD updates only one block of variables and thus can have a much lower per-iteration complexity than methods updating all the variables together. Although simple, BCD has been found efficient solving many large-scale problems (see [7, 37, 39, 43, 57] for example).

Considering the block separable structure of (1.1) and maintaining the benefit of BCD, we generalize the SG update in (1.3) and propose a *block stochastic gradient* (BSG) method to solve (1.1), which is summarized in Algorithm 1.

In the algorithm, we assume that samples of ξ are randomly generated. We let $\tilde{\mathbf{g}}_i^k$ be a stochastic approximate of $\nabla_{\mathbf{x}_{\pi_i^k}} F(\mathbf{x}_{\pi_{<i}^k}^{k+1}, \mathbf{x}_{\pi_{\geq i}^k}^k)$, where $\mathbf{x}_{\pi_{<i}^k}$ is short for $(\mathbf{x}_{\pi_1}, \dots, \mathbf{x}_{\pi_{i-1}})$. In (1.5), $\tilde{\nabla} r_j(\mathbf{x}_j^k)$ is a subgradient of r_j at \mathbf{x}_j^k , and we assume it exists for all j and k . We assume that both (1.4) and (1.5) are easy to solve. We perform two different updates. When $\mathcal{X}_i = \mathbb{R}^{n_i}$, we prefer (1.4) over (1.5) since proximal gradient iteration is typically faster than proximal subgradient iteration (see [5, 19, 36, 38] and the references therein); when $\mathcal{X}_i \neq \mathbb{R}^{n_i}$, we use (1.5), which takes the subgradient of r_i , since minimizing the nonsmooth function r_i subject to constraints is generally difficult.

One can certainly take $m_k = 1, \forall k$, or use larger m_k 's. In general, a larger m_k leads to a lower sample variance and incurs more computation of $\tilde{\mathbf{g}}_i^k$. Note that at the beginning of each cycle, we allow a reshuffle of the blocks, which often leads to better overall numerical performance. For the convenience of our discussion and easy notation, we assume $\pi_i^k = i, \forall i, k$, throughout our analysis, i.e., all iterations of the algorithm update the blocks in the same order. However, our analysis still goes through if the order is shuffled at the beginning of each cycle, and some of our numerical experiments use reshuffling.

Algorithm 1: Block stochastic gradient for solving (1.1)

Input: starting point \mathbf{x}^1 , step sizes $\{\alpha_i^k : i = 1, \dots, s\}_{k=1}^\infty$, and positive integers $\{m_k\}_{k=1}^\infty$.

for $k = 1, 2, \dots$ **do**

 Generate $\Xi_k = \{\xi_{k,1}, \xi_{k,2}, \dots, \xi_{k,m_k}\}$.

 Let $\pi_i^k = i, \forall i$, or randomly shuffle $\{1, 2, \dots, s\}$ to be $\{\pi_1^k, \pi_2^k, \dots, \pi_s^k\}$.

for $i = 1, \dots, s$ **do**

 Let

$$\tilde{\mathbf{g}}_i^k = \frac{1}{m_k} \sum_{\ell=1}^{m_k} \nabla_{\mathbf{x}_{\pi_i^k}^k} f(\mathbf{x}_{\pi_i^k}^{k+1}, \mathbf{x}_{\pi_i^k}^k; \xi_{k,\ell}).$$

if $\mathcal{X}_{\pi_i^k} = \mathbb{R}^{n_{\pi_i^k}}$ (the i th block is unconstrained) **then**

$$\mathbf{x}_{\pi_i^k}^{k+1} = \arg \min_{\mathbf{x}_{\pi_i^k}^k} \langle \tilde{\mathbf{g}}_i^k, \mathbf{x}_{\pi_i^k} - \mathbf{x}_{\pi_i^k}^k \rangle + \frac{1}{2\alpha_i^k} \|\mathbf{x}_{\pi_i^k} - \mathbf{x}_{\pi_i^k}^k\|^2 + r_{\pi_i^k}(\mathbf{x}_{\pi_i^k}). \quad (1.4)$$

else

$$\mathbf{x}_{\pi_i^k}^{k+1} = \arg \min_{\mathbf{x}_{\pi_i^k}^k \in \mathcal{X}_{\pi_i^k}} \langle \tilde{\mathbf{g}}_i^k + \tilde{\nabla} r_{\pi_i^k}(\mathbf{x}_{\pi_i^k}^k), \mathbf{x}_{\pi_i^k} - \mathbf{x}_{\pi_i^k}^k \rangle + \frac{1}{2\alpha_i^k} \|\mathbf{x}_{\pi_i^k} - \mathbf{x}_{\pi_i^k}^k\|^2. \quad (1.5)$$

TABLE 1.1

List of related methods

Abbreviation	Method
BSG (this paper)	block coordinate update using stochastic gradient
SG [25, 33]	stochastic gradient
BCD [29, 55]	block coordinate (minimization) descent
BCGD [56]	block coordinate gradient descent
SBGD [37]	stochastic block coordinate descent
SBMD [9]	stochastic block mirror descent

1.2. Related work. The BCD and SG methods are special cases of the proposed BSG method. In Algorithm 1, if $s = 1$, i.e., there is only one block of variables, the update in (1.5) becomes the SG update (1.3), and if $\pi_i^k = i$, and $\tilde{\mathbf{g}}_i^k = \nabla_{\mathbf{x}_i} F(\mathbf{x}_{<i}^{k+1}, \mathbf{x}_{\geq i}^k), \forall i, k$, it becomes the BCD update in [56, 58]. For solving problem (1.2), a special case of (1.1), the deterministic BCD method in [56, 58] requires the partial gradients of all of the component functions for every block update while BSG uses only one or several of them, and thus the BSG update is much cheaper. On the other hand, BSG updates the variables in a Gauss-Seidel-like manner while SG does it in a Jacobi-like manner. Hence, BSG often takes few iterations than SG, and our numerical experiments demonstrate such an advantage of BSG over SG. For the reader's convenience, we list the related methods in Table 1.1.

The BCD method has a long history dating back to the 1950s [22], which considers strongly concave quadratic programming. Its original format is block coordinate minimization (BCM), which cyclically updates all the blocks by minimizing the objective with respect to one block at a time while fixing all the others at their most recent values. The convergence of BCM has been extensively analyzed in the literature for both convex and nonconvex cases; see [21, 29, 41, 55, 58] for example. The work [1] combines the proximal point method [11, 45] with BCM and proposes a proximal block-coordinate update scheme. It was shown in [32] that such a scheme can perform better than the original BCM scheme for solving the tensor decomposition problem. Although BCD is straightforward to understand, there is no convergence rate known

for general convex programming¹ until [37], which proposes a stochastic block-coordinate descent (SBCD) method. At each iteration, the SBCD method randomly chooses one block of variables and updates it by performing one proximal gradient update. SBCD is analyzed in [37] for the smooth convex case, and the analysis is generalized to non-smooth convex case in [27, 43]. It is shown that SBCD has the same order of convergence rate for the non-smooth case as it does in the smooth case. For general convex case, SBCD has sublinear convergence in terms of expected objective value, and for strongly convex case, it converges linearly. SBCD has also been analyzed in [28] for non-smooth nonconvex case. Recently, [3, 48] showed that cyclic BCD can have the same order of convergence rate as SBCD for smooth convex programming, and [23] then extended the work of [3] to non-smooth convex case.

The SG method also has a long history and dates back to the pioneering work [44]. Since then, the SG method becomes very popular in stochastic programming. The classic analysis (e.g., in [8, 47]) requires second-order differentiability and also strong convexity, and under these assumptions, the method exhibits an asymptotically optimal convergence rate $\mathcal{O}(1/k)$ in terms of expected objective value. A great improvement was made in [34, 40], which propose a robust SG method applicable to general convex problems and obtain a non-asymptotic convergence rate $\mathcal{O}(1/\sqrt{k})$ by averaging all the iterates. These results were revisited in [33], which, in addition, proposes a mirror descent stochastic approximation method and achieves the same order of convergence rate with a better constant. Furthermore, the mirror descent method is accelerated in [16, 25], where the convergence results are strengthened for composite convex stochastic optimization. Recently, the SG method has been extended in [15, 17, 18] to handle nonconvex problems, for which the convergence results are established in terms of the violation of first-order optimality conditions.

A very relevant work to ours is [9], which proposes a *stochastic block mirror descent* (SBMD) method by combining SBCD with the mirror descent stochastic approximation method. The main difference between SBMD and our proposed BSG is that at each iteration SBMD randomly chooses one block of variables to update while BSG cyclically updates all the blocks of variables. We will demonstrate that BSG is competitive and often performs significantly better than SBMD when using the same number of samples. The practical advantage of BSG over SBMD should be intuitive and natural. For solving (1.1), both BSG and SBMD need samples or experimental observations of ξ . If the samples come sequentially, the computing processor can be idle between arrivals of two samples by updating only one block of variables other than all blocks, and thus it wastes the computing resource. Technically, SBMD has unbiased stochastic block partial gradient by choosing one block randomly to update while cyclic block update of BSG results in biased partial gradient, and thus the analysis of BSG would be more complicated than that of SBMD. To get the same order of convergence rate for BSG as that of SBMD, we will require stronger assumptions. Specifically, the analysis of SBMD in [9] does not assume boundedness of the iterates while our analysis of BSG requires such boundedness in expectation. SBMD also appeared in [26] for solving linear systems. Although the analysis in [26] assumes that one coordinate is updated at each iteration, its implementation updates all coordinates corresponding to the nonzeros of a randomly sampled row vector, and thus it explains somehow the benefit of updating all the blocks of variables instead of just one at a time.

1.3. Contributions. We summarize our contributions as follows.

- We propose a BSG method for solving both convex and nonconvex optimization problems. The update order of the blocks of variables at each iteration is arbitrary and independent of other iterations; it can be fixed or shuffled. The new method inherits the benefits of both SG and BCD methods. It is applicable to stochastic programs in the form of (1.1), and it applies to deterministic problems in the form of (1.2) with a huge amount of training data. It applies to problems with

¹The earlier work [29] establishes the linear convergence of BCM by assuming strong convexity on the objective.

many variables. It allows both N and n to be large.

- We analyze the BSG method for both convex and nonconvex problems. For convex problems, we show that it has the same order of convergence rate as that of the SG method, and for nonconvex problems, we establish its global convergence in terms of the expectation violation of first-order optimality conditions.
- We demonstrate applying the BSG method to two convex and two nonconvex problems with both synthetic and real-world data, and we compared it to the stochastic methods SG and SBMD, and to the deterministic method BCD. The numerical results demonstrate that it is comparable to, and can be significantly better, than SG and SBMD on convex problems, and it significantly outperforms the deterministic BCD on nonconvex problems on which SG and SBMD methods either ran out of memory or are incompetent.

Notation. Throughout the paper, we restrict our points in \mathbb{R}^n and use $\|\cdot\|$ for the Euclidean norm, but it is not difficult to generalize the analysis to any finite Hilbert space with a pair of primal and dual norms. We use $\mathbf{x}_{<i}$ for $(\mathbf{x}_1, \dots, \mathbf{x}_{i-1})$, $\nabla_{\mathbf{x}_i} F(\mathbf{x})$ for the partial gradient of F at \mathbf{x} with respect to \mathbf{x}_i , and $\tilde{\nabla} r_i(\mathbf{x}_i)$ as a subgradient in $\partial r(\mathbf{x}_i)$, which is the limiting subdifferential (see [46]). Scalars $\alpha_i^k, \alpha_k, \dots$ are reserved for stepsizes and L, L_{r_i}, \dots for Lipschitz constants. We let Ξ_k denote the random set of samples generated at the k th iteration and $\Xi_{[k]} = (\Xi_1, \dots, \Xi_k)$ as the history of random sets from the 1st through k th iterations. $\mathbb{E}[X|Y]$ denotes the expectation of X conditional on Y . In addition, we partition the block set $\{1, 2, \dots, s\}$ to \mathcal{I}_1 and \mathcal{I}_2 , where

$$\mathcal{I}_1 = \{i : \mathbf{x}_i \text{ is updated by (1.4)}\} \quad \text{and} \quad \mathcal{I}_2 = \{i : \mathbf{x}_i \text{ is updated by (1.5)}\}.$$

Given any set \mathcal{X} , we let

$$\iota_{\mathcal{X}}(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{x} \in \mathcal{X}, \\ +\infty, & \text{otherwise,} \end{cases}$$

be the indicator function of \mathcal{X} . For any function r and any convex set \mathcal{X} , we let

$$\mathbf{prox}_{\alpha r}(\mathbf{y}) = \arg \min_{\mathbf{x}} r(\mathbf{x}) + \frac{1}{2\alpha} \|\mathbf{x} - \mathbf{y}\|^2$$

be the proximal mapping, and

$$\mathcal{P}_{\mathcal{X}}(\mathbf{y}) = \arg \min_{\mathbf{x} \in \mathcal{X}} \|\mathbf{x} - \mathbf{y}\|^2$$

be the projection onto \mathcal{X} . When r is convex, $\mathbf{prox}_{\alpha r}$ is nonexpansive for any $\alpha > 0$, i.e.,

$$\|\mathbf{prox}_{\alpha r}(\mathbf{x}) - \mathbf{prox}_{\alpha r}(\mathbf{y})\| \leq \|\mathbf{x} - \mathbf{y}\|, \quad \forall \mathbf{x}, \mathbf{y}. \quad (1.6)$$

Note that $\mathcal{P}_{\mathcal{X}} = \mathbf{prox}_{\iota_{\mathcal{X}}}$, and thus $\mathcal{P}_{\mathcal{X}}$ is nonexpansive for any convex set \mathcal{X} . Other notation will be specified when they appear.

2. Convergence analysis. In this section, we analyze the convergence of Algorithm 1 under different settings. Without loss of generality, we assume a fixed update order in Algorithm 1:

$$\pi_i^k = i, \quad \forall i, k,$$

since the analysis still holds otherwise. Hence, we have

$$\tilde{\mathbf{g}}_i^k = \frac{1}{m_k} \sum_{\ell=1}^{m_k} \nabla_{\mathbf{x}_i} f(\mathbf{x}_{<i}^{k+1}, \mathbf{x}_{\geq i}^k; \xi_{k,\ell}).$$

Define

$$\begin{aligned}\mathbf{g}_i^k &= \nabla_{\mathbf{x}_i} F(\mathbf{x}_{<i}^{k+1}, \mathbf{x}_{\geq i}^k), \\ \boldsymbol{\delta}_i^k &= \tilde{\mathbf{g}}_i^k - \mathbf{g}_i^k.\end{aligned}$$

Note that although \mathbf{g}_i^k is defined using the full function F and does not explicitly depend on the random sample set Ξ_k , it in fact does depend on Ξ_k since $\mathbf{x}_j^{k+1}, \forall j < i$, depend on Ξ_k . This is a big difference between BSG and SG and makes our analysis more challenging. In our analysis below, some of the following assumptions will be made for each result.

ASSUMPTION 1. *There exist a constant A and a sequence $\{\sigma_k\}$ such that for any i and k ,*

$$\|\mathbb{E}[\boldsymbol{\delta}_i^k | \Xi_{[k-1]}]\| \leq A \cdot \max_j \alpha_j^k, \quad (2.1a)$$

$$\mathbb{E}\|\boldsymbol{\delta}_i^k\|^2 \leq \sigma_k^2. \quad (2.1b)$$

ASSUMPTION 2. *The objective function is lower bounded, i.e., $\Phi(\mathbf{x}) > -\infty$. There is a uniform Lipschitz constant $L > 0$ such that*

$$\|\nabla_{\mathbf{x}_i} F(\mathbf{x}) - \nabla_{\mathbf{x}_i} F(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|, \quad \forall \mathbf{x}, \mathbf{y}, \forall i.$$

ASSUMPTION 3. *There exists a constant ρ such that $\mathbb{E}\|\mathbf{x}^k\|^2 \leq \rho^2$ for all k .*

ASSUMPTION 4. *Every function r_i is Lipschitz continuous, namely, there is a constant L_{r_i} such that*

$$\|r_i(\mathbf{x}_i) - r_i(\mathbf{y}_i)\| \leq L_{r_i}\|\mathbf{x}_i - \mathbf{y}_i\|, \quad \forall \mathbf{x}_i, \mathbf{y}_i.$$

We let

$$L_{\max} = \max_i L_{r_i}$$

be the dominant Lipschitz constant.

REMARK 1. *In Assumption 1, we assume a bounded $\mathbb{E}[\boldsymbol{\delta}_i^k | \Xi_{[k-1]}]$ in (2.1a) since the common assumption $\mathbb{E}[\boldsymbol{\delta}_i^k | \Xi_{[k-1]}] = \mathbf{0}$ fails to hold in our algorithm. Since the block updates in Algorithm 1 are Gauss-Seidel, the gradient error $\boldsymbol{\delta}_i^k$ typically has a nonlinear dependence on $\mathbf{x}_j^{k+1}, \forall j < i$, which depends on Ξ_k . On the other hand, the boundedness assumption (2.1a) holds under proper conditions. For example, in (1.1), let $f(\mathbf{x}; \xi) = f_\xi(\mathbf{x})$ with $\text{Prob}\{\xi = \ell\} = \frac{1}{N}, \ell = 1, \dots, N$. Then $F(\mathbf{x}) = \frac{1}{N} \sum_{\ell=1}^N f_\ell(\mathbf{x})$. Assume that r_i 's are convex and $\mathcal{X}_i = \mathbb{R}^{n_i}, \forall i$, and that f_ℓ has Lipschitz continuous partial gradient with a uniform constant L , namely,*

$$\|\nabla_{\mathbf{x}_i} f_\ell(\mathbf{x}) - \nabla_{\mathbf{x}_i} f_\ell(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|, \quad \forall i, \ell, \forall \mathbf{x}, \mathbf{y}.$$

In addition, assume that each Ξ_k is a singleton, i.e., $\Xi_k = \{\xi_k\}$ with ξ_k uniformly selected from $\{1, \dots, N\}$.

Then

$$\mathbb{E}[\tilde{\mathbf{g}}_i^k | \Xi_{[k-1]}] = \sum_{\ell=1}^N \text{Prob}\{\xi_k = \ell\} \nabla_{\mathbf{x}_i} f_\ell(\mathbf{y}_{<i}^{\ell, k+1}, \mathbf{x}_{\geq i}^k) = \frac{1}{N} \sum_{\ell=1}^N \nabla_{\mathbf{x}_i} f_\ell(\mathbf{y}_{<i}^{\ell, k+1}, \mathbf{x}_{\geq i}^k),$$

where $\mathbf{y}_1^{\ell,k+1} = \mathbf{prox}_{\alpha_1^k r_1}(\mathbf{x}_1^k - \alpha_1^k \nabla_{\mathbf{x}_1} f_\ell(\mathbf{x}^k))$ and $\mathbf{y}_p^{\ell,k+1} = \mathbf{prox}_{\alpha_p^k r_p}(\mathbf{x}_p^k - \alpha_p^k \nabla_{\mathbf{x}_p} f_\ell(\mathbf{y}_{<p}^{\ell,k+1}, \mathbf{x}_{\geq p}^k))$ for $p \geq 2$, and

$$\begin{aligned} \mathbb{E}[\mathbf{g}_i^k | \Xi_{[k-1]}] &= \sum_{m=1}^N \text{Prob}\{\xi_k = m\} \nabla_{\mathbf{x}_i} F(\mathbf{y}_{<i}^{m,k+1}, \mathbf{x}_{\geq i}^k) \\ &= \frac{1}{N^2} \sum_{m=1}^N \sum_{\ell=1}^N \nabla_{\mathbf{x}_i} f_\ell(\mathbf{y}_{<i}^{m,k+1}, \mathbf{x}_{\geq i}^k) \\ &= \frac{1}{N^2} \sum_{\ell=1}^N \sum_{m=1}^N \nabla_{\mathbf{x}_i} f_\ell(\mathbf{y}_{<i}^{m,k+1}, \mathbf{x}_{\geq i}^k). \end{aligned}$$

Hence,

$$\begin{aligned} \|\mathbb{E}[\delta_i^k | \Xi_{[k-1]}]\| &= \|\mathbb{E}[\hat{\mathbf{g}}_i^k - \mathbf{g}_i^k | \Xi_{[k-1]}]\| \\ &\leq \frac{1}{N^2} \sum_{\ell=1}^N \sum_{m=1}^N \|\nabla_{\mathbf{x}_i} f_\ell(\mathbf{y}_{<i}^{\ell,k+1}, \mathbf{x}_{\geq i}^k) - \nabla_{\mathbf{x}_i} f_\ell(\mathbf{y}_{<i}^{m,k+1}, \mathbf{x}_{\geq i}^k)\| \\ &\leq \frac{L}{N^2} \sum_{\ell=1}^N \sum_{m=1}^N \|\mathbf{y}_{<i}^{\ell,k+1} - \mathbf{y}_{<i}^{m,k+1}\| \\ &\leq \frac{L}{N^2} \sum_{\ell=1}^N \sum_{m=1}^N \sum_{j<i} \|\mathbf{y}_j^{\ell,k+1} - \mathbf{y}_j^{m,k+1}\| \\ &\leq \frac{L}{N^2} \sum_{\ell=1}^N \sum_{m=1}^N \sum_{j<i} \alpha_j^k \|\nabla_{\mathbf{x}_j} f_\ell(\mathbf{y}_{<j}^{\ell,k+1}, \mathbf{x}_{\geq j}^k) - \nabla_{\mathbf{x}_j} f_m(\mathbf{y}_{<j}^{m,k+1}, \mathbf{x}_{\geq j}^k)\|, \end{aligned}$$

where the last inequality is from the nonexpansiveness of the proximal mapping in (1.6). Therefore, if $\|\nabla_{\mathbf{x}_i} f_\ell(\mathbf{x})\| \leq M, \forall i, \ell, \forall \mathbf{x}$, then we have from the above inequality that

$$\|\mathbb{E}[\delta_i^k | \Xi_{[k-1]}]\| \leq 2LsM \max_j \alpha_j^k.$$

The second condition (2.1b) in Assumption 1 is standard in the literature of stochastic gradient method. The variance bound σ_k^2 is allowed to vary with the iteration k .

REMARK 2. Assumption 3 is relatively weaker than the assumption made in the literature of stochastic gradient method; see [25, 33] for example, where they require $\mathbf{x} \in \mathcal{X}$ for a bounded set \mathcal{X} . Our analysis below for nonconvex case will remain valid if the assumption is weakened from the boundedness of $\{\mathbb{E}\|\mathbf{x}^k\|^2\}$ to that of $\{\mathbb{E}\|\nabla F(\mathbf{x}^k)\|^2\}$.

Note that Assumption 3 together with the partial gradient Lipschitz continuity of F in Assumption 2 implies the boundedness of $\{\mathbb{E}\|\nabla F(\mathbf{x}^k)\|^2\}$ by the following argument

$$\|\nabla_{\mathbf{x}_i} F(\mathbf{x}^k)\|^2 \leq 2\|\nabla_{\mathbf{x}_i} F(\mathbf{x}^k) - \nabla_{\mathbf{x}_i} F(\mathbf{0})\|^2 + 2\|\nabla_{\mathbf{x}_i} F(\mathbf{0})\|^2 \leq 2L^2\|\mathbf{x}^k\|^2 + 2\|\nabla_{\mathbf{x}_i} F(\mathbf{0})\|^2.$$

Throughout the paper, we let

$$M_\rho = \sqrt{4L^2\rho^2 + 2 \max_i \|\nabla_{\mathbf{x}_i} F(\mathbf{0})\|^2}, \quad (2.2)$$

and thus we have $\mathbb{E}\|\nabla_{\mathbf{x}_i} F(\mathbf{x}_{<i}^{k+1}, \mathbf{x}_{\geq i}^k)\|^2 \leq M_\rho^2, \forall i, k$. In addition, by Jensen's inequality, we have $\mathbb{E}\|\mathbf{x}^k\| \leq \rho$, and $\mathbb{E}\|\nabla_{\mathbf{x}_i} F(\mathbf{x}_{<i}^{k+1}, \mathbf{x}_{\geq i}^k)\| \leq M_\rho, \forall i, k$.

REMARK 3. Assumption 4 is the same as that in [25]. One example is $r_i(\mathbf{x}_i) = \lambda_i \|\mathbf{x}_i\|_1$, which satisfies (4) with $L_{r_i} = \lambda_i \sqrt{n_i}$. Note that (4) implies that $\|\mathbf{d}_i\| \leq L_{r_i}$ for any $\mathbf{d}_i \in \partial r_i(\mathbf{x}_i)$.

We first establish some lemmas, whose proofs are given in Appendix A. They are not difficult to prove and are useful in our convergence analysis.

LEMMA 2.1. *Let \mathbf{u}^k be a random vector depending on $\Xi_{[k-1]}$. Under Assumption 1, if \mathbf{u}^k is independent of δ_i^k conditional on $\Xi_{[k-1]}$, then*

$$\mathbb{E}\langle \mathbf{u}^k, \delta_i^k \rangle \leq A \left(\max_j \alpha_j^k \right) \mathbb{E} \|\mathbf{u}^k\|. \quad (2.3)$$

LEMMA 2.2. *Let*

$$\tilde{\mathbf{h}}_i^k = \frac{1}{\alpha_i^k} (\mathbf{x}_i^k - \mathbf{x}_i^{k+1}) \quad (2.4)$$

be the stochastic gradient mapping for the i th block at the k th iteration. Under Assumption 4, we have

$$\mathbb{E} \|\tilde{\mathbf{h}}_i^k\|^2 \leq 2\mathbb{E} \|\tilde{\mathbf{g}}_i^k\|^2 + 2L_{r_i}^2 \leq 4M_\rho^2 + 4\sigma_k^2 + 2L_{r_i}^2.$$

LEMMA 2.3. *If a nonnegative scalar sequence $\{A_k\}_{k \geq 1}$ satisfies*

$$A_{k+1} \leq \left(1 - \frac{a}{k}\right) A_k + \frac{b}{k^2},$$

where $a, b > 0$, then it obeys

$$A_k \leq \frac{c}{k}, \quad \forall k \geq [a] + 1, \quad (2.5)$$

with

$$c = \begin{cases} 2b/(a-1), & \text{if } a > 1, \\ \max\{b/a, A_1\}, & \text{if } a \leq 1, \end{cases}$$

where $[a]$ denotes the largest integer that is no greater than a .

Next we present the convergence results of Algorithm 1 for both convex and nonconvex cases. For the convex case, where functions F and r_i 's are convex, we establish a sublinear convergence rate in the same order of the SG method. For the nonconvex case, we establish the convergence result in terms of the expected violation of first-order optimality conditions.

2.1. Convex case. In this section, we first analyze Algorithm 1 for general convex problems and then for strongly convex problems to obtain stronger results.

THEOREM 2.4 (Ergodic convergence for non-smooth convex case). *Let $\{\mathbf{x}^k\}$ be generated from Algorithm 1 with $\alpha_i^k = \alpha_k = \frac{\theta}{\sqrt{k}} < \frac{1}{L}, \forall i, k$, for some positive constant $\theta < \frac{1}{L}$. Under Assumptions 1 through 4, if F and r_i 's are all convex, \mathbf{x}^* is a solution of (1.1) and $\sigma = \sup_k \sigma_k < \infty$, then*

$$\mathbb{E}[\Phi(\tilde{\mathbf{x}}^K) - \Phi(\mathbf{x}^*)] \leq D\theta \frac{1 + \log K}{\sqrt{1+K}} + \frac{\|\mathbf{x}^* - \mathbf{x}^1\|^2}{2\theta\sqrt{1+K}}, \quad (2.6)$$

where $\tilde{\mathbf{x}}^K = \frac{\sum_{k=1}^K \alpha_k \mathbf{x}^{k+1}}{\sum_{k=1}^K \alpha_k}$ and

$$D = \frac{s(\sigma^2 + 4L_{\max}^2)}{1 - L\theta} + \sqrt{s}(\|\mathbf{x}^*\| + \rho) \left(A + L \sqrt{\sum_{j=1}^s (4M_\rho^2 + 4\sigma_k^2 + 2L_{r_j}^2)} \right). \quad (2.7)$$

Furthermore, if the maximum number of iterations K is predetermined, then taking $\alpha_k = \frac{\theta}{\sqrt{K}}, \forall k$, we have

$$\mathbb{E}[\Phi(\tilde{\mathbf{x}}^K) - \Phi(\mathbf{x}^*)] \leq \frac{D\theta}{\sqrt{K}} + \frac{\|\mathbf{x}^* - \mathbf{x}^1\|^2}{2\theta\sqrt{K}}.$$

REMARK 4. The value of θ usually plays a vital role on the actual speed of the algorithm. In practice, we may not know the exact value of L , and thus it is difficult to choose an appropriate θ . Even if we know or can estimate L by a certain way, different θ 's can make the algorithm perform very differently. This phenomenon has also been observed for the SG method; see numerical tests in section 3 and also the discussion on page 6 of [33]. The work [49] and its references study adaptive learning rates, which is beyond the discussion of this paper.

Proof. From the Lipschitz continuity of $\nabla_{\mathbf{x}_i} F(\mathbf{x}_{<i}^{k+1}, \mathbf{x}_i, \mathbf{x}_{>i}^k)$ about \mathbf{x}_i , it holds for any $i \in \mathcal{I}_1$ that (see [35] for example)

$$\begin{aligned} & \Phi(\mathbf{x}_{\leq i}^{k+1}, \mathbf{x}_{>i}^k) - \Phi(\mathbf{x}_{<i}^{k+1}, \mathbf{x}_{\geq i}^k) \\ & \leq \langle \tilde{\mathbf{g}}_i^k, \mathbf{x}_i^{k+1} - \mathbf{x}_i^k \rangle + \frac{L}{2} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\|^2 + r_i(\mathbf{x}_i^{k+1}) - r_i(\mathbf{x}_i^k). \end{aligned} \quad (2.8)$$

In addition, from Lemma 2 of [25], it holds for $i \in \mathcal{I}_2$ that

$$\begin{aligned} & \Phi(\mathbf{x}_{\leq i}^{k+1}, \mathbf{x}_{>i}^k) - \Phi(\mathbf{x}_{<i}^{k+1}, \mathbf{x}_{\geq i}^k) \\ & \leq \langle \tilde{\mathbf{g}}_i^k + \tilde{\nabla} r_i(\mathbf{x}_i^k), \mathbf{x}_i^{k+1} - \mathbf{x}_i^k \rangle + \frac{L}{2} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\|^2 + 2L_{r_i} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\|. \end{aligned} \quad (2.9)$$

By Lemma 2 of [2], the updates in (1.4) and (1.5) indicate that for any $\mathbf{x}_i \in \mathcal{X}_i$, if $i \in \mathcal{I}_1$, then

$$\begin{aligned} & \langle \tilde{\mathbf{g}}_i^k, \mathbf{x}_i^{k+1} - \mathbf{x}_i^k \rangle + r_i(\mathbf{x}_i^{k+1}) + \frac{1}{2\alpha_k} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\|^2 \\ & \leq \langle \tilde{\mathbf{g}}_i^k, \mathbf{x}_i - \mathbf{x}_i^k \rangle + r_i(\mathbf{x}_i) + \frac{1}{2\alpha_k} (\|\mathbf{x}_i - \mathbf{x}_i^k\|^2 - \|\mathbf{x}_i - \mathbf{x}_i^{k+1}\|^2), \end{aligned} \quad (2.10)$$

and if $i \in \mathcal{I}_2$, then

$$\begin{aligned} & \langle \tilde{\mathbf{g}}_i^k + \tilde{\nabla} r_i(\mathbf{x}_i^k), \mathbf{x}_i^{k+1} - \mathbf{x}_i^k \rangle + \frac{1}{2\alpha_k} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\|^2 \\ & \leq \langle \tilde{\mathbf{g}}_i^k + \tilde{\nabla} r_i(\mathbf{x}_i^k), \mathbf{x}_i - \mathbf{x}_i^k \rangle + \frac{1}{2\alpha_k} (\|\mathbf{x}_i - \mathbf{x}_i^k\|^2 - \|\mathbf{x}_i - \mathbf{x}_i^{k+1}\|^2), \end{aligned} \quad (2.11)$$

Summing up (2.8) through (2.11) over i and arranging terms, we have

$$\begin{aligned} \Phi(\mathbf{x}^{k+1}) - \Phi(\mathbf{x}^k) & \leq \sum_{i=1}^s \langle \tilde{\mathbf{g}}_i^k - \tilde{\mathbf{g}}_i^k, \mathbf{x}_i^{k+1} - \mathbf{x}_i^k \rangle - \left(\frac{1}{2\alpha_k} - \frac{L}{2} \right) \|\mathbf{x}^{k+1} - \mathbf{x}^k\|^2 \\ & \quad + \frac{1}{2\alpha_k} (\|\mathbf{x} - \mathbf{x}^k\|^2 - \|\mathbf{x} - \mathbf{x}^{k+1}\|^2) + \sum_{i \in \mathcal{I}_1} (\langle \tilde{\mathbf{g}}_i^k, \mathbf{x}_i - \mathbf{x}_i^k \rangle + r_i(\mathbf{x}_i) - r_i(\mathbf{x}_i^k)) \\ & \quad + \sum_{i \in \mathcal{I}_2} \left(\langle \tilde{\mathbf{g}}_i^k + \tilde{\nabla} r_i(\mathbf{x}_i^k), \mathbf{x}_i - \mathbf{x}_i^k \rangle + 2L_{r_i} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\| \right). \end{aligned} \quad (2.12)$$

It follows from the convexity of F and r_i 's that

$$\Phi(\mathbf{x}^k) - \sum_{i \in \mathcal{I}_1} r_i(\mathbf{x}_i^k) \leq \Phi(\mathbf{x}) - \sum_{i \in \mathcal{I}_1} r_i(\mathbf{x}_i) + \langle \nabla F(\mathbf{x}^k), \mathbf{x}^k - \mathbf{x} \rangle + \sum_{i \in \mathcal{I}_2} \langle \tilde{\nabla} r_i(\mathbf{x}_i^k), \mathbf{x}_i^k - \mathbf{x}_i \rangle,$$

which together with (2.12) gives

$$\begin{aligned}
\Phi(\mathbf{x}^{k+1}) - \Phi(\mathbf{x}) &\leq \sum_{i=1}^s \langle \mathbf{g}_i^k - \tilde{\mathbf{g}}_i^k, \mathbf{x}_i^{k+1} - \mathbf{x}_i^k \rangle - \left(\frac{1}{2\alpha_k} - \frac{L}{2} \right) \|\mathbf{x}^{k+1} - \mathbf{x}^k\|^2 + \sum_{i \in \mathcal{I}_2} 2L_{r_i} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\| \\
&\quad + \sum_{i=1}^s \langle \tilde{\mathbf{g}}_i^k - \nabla_{\mathbf{x}_i} F(\mathbf{x}^k), \mathbf{x}_i - \mathbf{x}_i^k \rangle + \frac{1}{2\alpha_k} (\|\mathbf{x} - \mathbf{x}^k\|^2 - \|\mathbf{x} - \mathbf{x}^{k+1}\|^2) \\
&= \sum_{i=1}^s \langle -\delta_i^k, \mathbf{x}_i^{k+1} - \mathbf{x}_i^k \rangle - \left(\frac{1}{2\alpha_k} - \frac{L}{2} \right) \|\mathbf{x}^{k+1} - \mathbf{x}^k\|^2 + \sum_{i \in \mathcal{I}_2} 2L_{r_i} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\| \\
&\quad + \sum_{i=1}^s \langle \delta_i^k + \mathbf{g}_i^k - \nabla_{\mathbf{x}_i} F(\mathbf{x}^k), \mathbf{x}_i - \mathbf{x}_i^k \rangle + \frac{1}{2\alpha_k} (\|\mathbf{x} - \mathbf{x}^k\|^2 - \|\mathbf{x} - \mathbf{x}^{k+1}\|^2).
\end{aligned} \tag{2.13}$$

Noting $\frac{1}{\alpha_k} - L > 0$ and using Young's inequality $ab < ta^2 + \frac{1}{4t}b^2$, we have

$$\begin{aligned}
&\sum_{i=1}^s \langle -\delta_i^k, \mathbf{x}_i^{k+1} - \mathbf{x}_i^k \rangle - \left(\frac{1}{2\alpha_k} - \frac{L}{2} \right) \|\mathbf{x}^{k+1} - \mathbf{x}^k\|^2 + \sum_{i \in \mathcal{I}_2} 2L_{r_i} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\| \\
&\leq \sum_{i=1}^s \left(\langle -\delta_i^k, \mathbf{x}_i^{k+1} - \mathbf{x}_i^k \rangle - \frac{1}{4} \left(\frac{1}{\alpha_k} - L \right) \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\|^2 \right) \\
&\quad + \sum_{i \in \mathcal{I}_2} \left(2L_{r_i} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\| - \frac{1}{4} \left(\frac{1}{\alpha_k} - L \right) \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\|^2 \right) \\
&\leq \sum_{i=1}^s \frac{\alpha_k}{1 - L\alpha_k} \|\delta_i^k\|^2 + \sum_{i \in \mathcal{I}_2} \frac{4L_{r_i}^2 \alpha_k}{1 - L\alpha_k}.
\end{aligned} \tag{2.14}$$

In addition, letting $\mathbf{u}^k = \mathbf{x}_i - \mathbf{x}_i^k$ in Lemma 2.1, we have

$$\mathbb{E} \langle \delta_i^k, \mathbf{x}_i - \mathbf{x}_i^k \rangle \leq A\alpha_k \mathbb{E} \|\mathbf{x}_i - \mathbf{x}_i^k\|, \tag{2.15}$$

and also by Hölder's inequality, it holds that

$$\begin{aligned}
\mathbb{E} \langle \mathbf{g}_i^k - \nabla_{\mathbf{x}_i} F(\mathbf{x}^k), \mathbf{x}_i - \mathbf{x}_i^k \rangle &\leq \sqrt{\mathbb{E} \|\mathbf{g}_i^k - \nabla_{\mathbf{x}_i} F(\mathbf{x}^k)\|^2} \cdot \sqrt{\mathbb{E} \|\mathbf{x}_i - \mathbf{x}_i^k\|^2} \\
&\leq L\alpha_k \sqrt{\mathbb{E} \sum_{j=1}^s \|\tilde{\mathbf{h}}_j^k\|^2} \cdot \sqrt{\mathbb{E} \|\mathbf{x}_i - \mathbf{x}_i^k\|^2} \\
&\leq L\alpha_k \sqrt{\sum_{j=1}^s (4M_\rho^2 + 4\sigma_k^2 + 2L_{r_j}^2)} \cdot \sqrt{\mathbb{E} \|\mathbf{x}_i - \mathbf{x}_i^k\|^2},
\end{aligned} \tag{2.16}$$

where we have used Lemma 2.2 in the last inequality.

Taking expectation over both sides of (2.13), substituting (2.1b), (2.14), (2.15), and (2.16) into it, and noting

$$\sum_{i=1}^s \sqrt{\mathbb{E} \|\mathbf{x}_i - \mathbf{x}_i^k\|^2} \leq \sqrt{s} \sqrt{\mathbb{E} \|\mathbf{x} - \mathbf{x}^k\|^2} \leq \sqrt{s} (\|\mathbf{x}\| + \rho),$$

we have after some arrangement that

$$\begin{aligned}
&\alpha_k \mathbb{E} [\Phi(\mathbf{x}^{k+1}) - \Phi(\mathbf{x})] \\
&\leq \left(\frac{s(\sigma^2 + 4L_{\max}^2)}{1 - L\alpha_k} + \sqrt{s} (\|\mathbf{x}\| + \rho) \left(A + L \sqrt{\sum_{j=1}^s (4M_\rho^2 + 4\sigma_k^2 + 2L_{r_j}^2)} \right) \right) \alpha_k^2 \\
&\quad + \frac{1}{2} \mathbb{E} (\|\mathbf{x} - \mathbf{x}^k\|^2 - \|\mathbf{x} - \mathbf{x}^{k+1}\|^2).
\end{aligned} \tag{2.17}$$

Letting $\mathbf{x} = \mathbf{x}^*$ in the above inequality and taking its sum over k , we have

$$\sum_{k=1}^K \alpha_k \mathbb{E}[\Phi(\mathbf{x}^{k+1}) - \Phi(\mathbf{x}^*)] \leq D \sum_{k=1}^K \alpha_k^2 + \frac{1}{2} \|\mathbf{x}^* - \mathbf{x}^1\|^2. \quad (2.18)$$

Now use the convexity of Φ and $\alpha_k = \frac{\theta}{\sqrt{k}}$ in (2.18) to get

$$\begin{aligned} \mathbb{E}[\Phi(\tilde{\mathbf{x}}^K) - \Phi(\mathbf{x}^*)] &\leq D \frac{\sum_{k=1}^K \alpha_k^2}{\sum_{k=1}^K \alpha_k} + \frac{\|\mathbf{x}^* - \mathbf{x}^1\|^2}{2 \sum_{k=1}^K \alpha_k} \\ &\leq D\theta \frac{1 + \log K}{\sqrt{1+K}} + \frac{\|\mathbf{x}^* - \mathbf{x}^1\|^2}{2\theta\sqrt{1+K}}. \end{aligned} \quad (2.19)$$

If the maximum number of iterations K is predetermined, set $\alpha_k \equiv \frac{\theta}{\sqrt{K}}$ in (2.19) to obtain

$$\mathbb{E}[\Phi(\tilde{\mathbf{x}}^K) - \Phi(\mathbf{x}^*)] \leq \frac{D\theta}{\sqrt{K}} + \frac{\|\mathbf{x}^* - \mathbf{x}^1\|^2}{2\theta\sqrt{K}},$$

which completes the proof. \square

Under the general convex setting, the rate $O(1/\sqrt{k})$ is optimal for the SG method, and for strongly convex case, the rate $O(1/k)$ is optimal; see [25, 33, 34]. In the following theorem, we assume $\Phi(\mathbf{x})$ to be strongly convex and establish the rate $O(1/k)$. Hence, our algorithm has the same orders of convergence rates as that of the SG method.

THEOREM 2.5 (Non-smooth strongly convex case). *Let $\{\mathbf{x}^k\}$ be generated from Algorithm 1 with $\alpha_i^k = \alpha_k = \frac{\theta}{k} < \frac{1}{L}, \forall i, k$. Under Assumptions 1 through 4, if F and r_i 's are convex, Φ is strongly convex with modulus $\mu > 0$, namely,*

$$\Phi(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda\Phi(\mathbf{x}) + (1-\lambda)\Phi(\mathbf{y}) - \frac{\mu}{2}\lambda(1-\lambda)\|\mathbf{x} - \mathbf{y}\|^2, \quad \forall \mathbf{x}, \mathbf{y}, \forall \lambda \in [0, 1],$$

and $\sigma = \sup_k \sigma_k < \infty$, then

$$\mathbb{E}[\|\mathbf{x}^k - \mathbf{x}^*\|^2] \leq \frac{1}{k} \max \left\{ \frac{2D\theta(1+\mu\theta)}{\mu}, \|\mathbf{x}^1 - \mathbf{x}^*\|^2 \right\}, \quad (2.20)$$

where D is defined in (2.7).

Proof. When Φ is strongly convex with modulus $\mu > 0$, it holds that

$$\Phi(\mathbf{x}^{k+1}) - \Phi(\mathbf{x}^*) \geq \frac{\mu}{2} \|\mathbf{x}^{k+1} - \mathbf{x}^*\|^2,$$

which together with (2.17) implies

$$\begin{aligned} \|\mathbf{x}^{k+1} - \mathbf{x}^*\|^2 &\leq \frac{\|\mathbf{x}^k - \mathbf{x}^*\|^2}{1 + \mu\alpha_k} + 2D \frac{\alpha_k^2}{1 + \mu\alpha_k} \\ &= \frac{\|\mathbf{x}^k - \mathbf{x}^*\|^2}{1 + \mu\theta/k} + \frac{2D\theta^2}{k^2(1 + \mu\theta/k)} \\ &\leq \left(1 - \frac{\mu\theta}{k(1 + \mu\theta)}\right) \|\mathbf{x}^k - \mathbf{x}^*\|^2 + \frac{2D\theta^2}{k^2}. \end{aligned} \quad (2.21)$$

Using Lemma 2.3, we immediately get the desired result and complete the proof. \square

If the error σ_k decreases fast, we can show an almost linear convergence result. We assume that either $\mathcal{X}_i = \mathbb{R}^{n_i}$ or $r_i = 0$ for all i since it is impossible in general to get linear convergence for subgradient method (see [5, 19, 38] and the references therein). For the convenience of our discussion, we let

$$\hat{r}_i(\mathbf{x}_i) = \begin{cases} r_i(\mathbf{x}_i), & i \in \mathcal{I}_1, \\ \iota_{\mathcal{X}_i}(\mathbf{x}_i), & i \in \mathcal{I}_2. \end{cases} \quad (2.22)$$

This way, we can write the updates (1.4) and (1.5) uniformly into

$$\mathbf{x}_i^{k+1} = \arg \min_{\mathbf{x}_i} \langle \tilde{\mathbf{g}}_i^k, \mathbf{x}_i - \mathbf{x}_i^k \rangle + \frac{1}{2\alpha_i^k} \|\mathbf{x}_i - \mathbf{x}_i^k\|^2 + \hat{r}_i(\mathbf{x}_i). \quad (2.23)$$

Let

$$\hat{\Phi}(\mathbf{x}) = F(\mathbf{x}) + \sum_{i=1}^s \hat{r}_i(\mathbf{x}_i). \quad (2.24)$$

Then problem (1.1) becomes $\min_{\mathbf{x}} \hat{\Phi}(\mathbf{x})$ under the assumption that either $\mathcal{X}_i = \mathbb{R}^{n_i}$ or $r_i = 0$ for all i . When Φ and thus $\hat{\Phi}$ is strongly convex with modulus $\mu > 0$, then from the discussion in section 2.2 of [58], it follows that

$$\hat{\Phi}(\mathbf{x}) - \hat{\Phi}(\mathbf{x}^*) \leq \frac{1}{\mu} \|\mathbf{g}\|^2, \forall \mathbf{g} \in \partial \hat{\Phi}(\mathbf{x}). \quad (2.25)$$

Using this result, we establish the following theorem.

THEOREM 2.6. *Let $\{\mathbf{x}^k\}$ be generated from Algorithm 1 with $\alpha_i^k = \alpha_k < \frac{2}{L}$, $\forall i, k$. Under Assumptions 1 and 2, if either $\mathcal{X}_i = \mathbb{R}^{n_i}$ or $r_i = 0$ for all i , Φ is strongly convex with constant μ , and all \hat{r}_i 's in (2.22) are convex, then*

$$\mathbb{E}[\Phi(\mathbf{x}^{k+1}) - \Phi(\mathbf{x}^*)] \leq \frac{\gamma(\alpha_k)}{1 + \gamma(\alpha_k)} \mathbb{E}[\Phi(\mathbf{x}^k) - \Phi(\mathbf{x}^*)] + \frac{s\nu(\alpha_k)}{1 + \gamma(\alpha_k)} \sigma_k^2, \quad (2.26)$$

where

$$\gamma(\alpha_k) = \frac{3L}{\mu^2} \left(\frac{1}{\alpha_k^2} + sL^2 \right) \frac{1}{\frac{1}{2\alpha_k} - \frac{L}{4}}, \quad \nu(\alpha_k) = \frac{\gamma(\alpha_k)\alpha_k}{2 - L\alpha_k} + \frac{3L}{\mu^2}. \quad (2.27)$$

REMARK 5. *This theorem does not require Assumptions 3 or 4. Although it requires the strong convexity of Φ , it does not require the convexity of F .*

Proof. Note

$$\frac{1}{\alpha_k} (\mathbf{x}_i^k - \mathbf{x}_i^{k+1}) - \tilde{\mathbf{g}}_i^k \in \partial \hat{r}_i(\mathbf{x}_i^{k+1}),$$

or equivalently

$$\frac{1}{\alpha_k} (\mathbf{x}_i^k - \mathbf{x}_i^{k+1}) + \nabla_{\mathbf{x}_i} F(\mathbf{x}^{k+1}) - \tilde{\mathbf{g}}_i^k \in \partial \hat{r}_i(\mathbf{x}_i^{k+1}) + \nabla_{\mathbf{x}_i} F(\mathbf{x}^{k+1}).$$

Hence, from (2.25) it holds that

$$\begin{aligned} \hat{\Phi}(\mathbf{x}^{k+1}) - \hat{\Phi}(\mathbf{x}^*) &\leq \frac{1}{\mu} \sum_{i=1}^s \left\| \frac{1}{\alpha_k} (\mathbf{x}_i^k - \mathbf{x}_i^{k+1}) + \nabla_{\mathbf{x}_i} F(\mathbf{x}^{k+1}) - \tilde{\mathbf{g}}_i^k \right\|^2 \\ &\leq \frac{3}{\mu} \sum_{i=1}^s \left(\left\| \frac{1}{\alpha_k} (\mathbf{x}_i^k - \mathbf{x}_i^{k+1}) \right\|^2 + \left\| \nabla_{\mathbf{x}_i} F(\mathbf{x}^{k+1}) - \tilde{\mathbf{g}}_i^k \right\|^2 + \|\delta_i^k\|^2 \right). \end{aligned}$$

Noting $\left\| \nabla_{\mathbf{x}_i} F(\mathbf{x}^{k+1}) - \tilde{\mathbf{g}}_i^k \right\|^2 \leq L^2 \|\mathbf{x}^{k+1} - \mathbf{x}^k\|^2$, we have from the above inequality that

$$\hat{\Phi}(\mathbf{x}^{k+1}) - \hat{\Phi}(\mathbf{x}^*) \leq \frac{3}{\mu} \left(\frac{1}{\alpha_k^2} + sL^2 \right) \|\mathbf{x}^{k+1} - \mathbf{x}^k\|^2 + \frac{3}{\mu} \sum_{i=1}^s \|\delta_i^k\|^2. \quad (2.28)$$

Letting $\mathbf{x} = \mathbf{x}^k$ in (2.12) gives

$$\begin{aligned}\hat{\Phi}(\mathbf{x}^{k+1}) - \hat{\Phi}(\mathbf{x}^k) &\leq - \sum_{i=1}^s \langle \boldsymbol{\delta}_i^k, \mathbf{x}_i^{k+1} - \mathbf{x}_i^k \rangle - \left(\frac{1}{\alpha_k} - \frac{L}{2}\right) \|\mathbf{x}^{k+1} - \mathbf{x}^k\|^2 \\ &\leq \frac{\alpha_k}{2 - L\alpha_k} \sum_{i=1}^s \|\boldsymbol{\delta}_i^k\|^2 - \left(\frac{1}{2\alpha_k} - \frac{L}{4}\right) \|\mathbf{x}^{k+1} - \mathbf{x}^k\|^2,\end{aligned}\quad (2.29)$$

which together with (2.28) implies

$$\begin{aligned}&\hat{\Phi}(\mathbf{x}^{k+1}) - \hat{\Phi}(\mathbf{x}^*) \\ &\leq \frac{L}{2} \|\mathbf{x}^{k+1} - \mathbf{x}^*\|^2 \\ &\leq \frac{L}{\mu} (\hat{\Phi}(\mathbf{x}^{k+1}) - \hat{\Phi}(\mathbf{x}^*)) \\ &\leq \frac{3L}{\mu^2} \left(\frac{1}{\alpha_k^2} + sL^2\right) \|\mathbf{x}^{k+1} - \mathbf{x}^k\|^2 + \frac{3L}{\mu^2} \sum_{i=1}^s \|\boldsymbol{\delta}_i^k\|^2 \\ &\leq \frac{3L}{\mu^2} \left(\frac{1}{\alpha_k^2} + sL^2\right) \frac{1}{\frac{1}{2\alpha_k} - \frac{L}{4}} \left(\Phi(\mathbf{x}^k) - \Phi(\mathbf{x}^{k+1}) + \frac{\alpha_k}{2 - L\alpha_k} \sum_{i=1}^s \|\boldsymbol{\delta}_i^k\|^2\right) + \frac{3L}{\mu^2} \sum_{i=1}^s \|\boldsymbol{\delta}_i^k\|^2,\end{aligned}\quad (2.30)$$

where the first inequality follows from the gradient Lipschitz continuity of F , the second one from the strong convexity of Φ , the third one from (2.28), and the fourth one from (2.29). Taking expectation on both sides of (2.30), using (2.1b), and arranging terms give

$$\mathbb{E}[\hat{\Phi}(\mathbf{x}^{k+1}) - \hat{\Phi}(\mathbf{x}^*)] \leq \gamma(\alpha_k) \mathbb{E}[\hat{\Phi}(\mathbf{x}^k) - \hat{\Phi}(\mathbf{x}^{k+1})] + s\nu(\alpha_k) \sigma_k^2,$$

which is equivalent to the desired result (2.26). Noting $\Phi(\mathbf{x}^k) = \hat{\Phi}(\mathbf{x}^k)$ and $\Phi(\mathbf{x}^*) = \hat{\Phi}(\mathbf{x}^*)$ completes the proof. \square

From (2.26), one can see that the convergence rate of Algorithm 1 depends on how fast σ_k decreases. If $\sigma_k \equiv 0$, i.e., the deterministic case, (2.26) implies linear convergence. Generally, σ_k does not vanish. However, one can decrease it by increasing the batch size m_k . This point will be discussed in Remark 8 below. Directly from (2.26) and using the following lemma, we can get a convergence result in Theorem 2.8 below, whose proof follows [13].

LEMMA 2.7. *Let $e_k = \max\{a_k, \eta^k\}$. If a_{k+1}/a_k has a finite limit, then e_{k+1}/e_k also has a finite limit, which equals η or the limit of a_{k+1}/a_k , and the limit is no less than η .*

THEOREM 2.8. *Assume the assumptions in Theorem 2.6. Suppose $0 < \underline{\alpha} = \inf_k \alpha_k \leq \sup_k \alpha_k = \bar{\alpha} < \frac{2}{L}$. If $\sigma_{k+1}^2/\sigma_k^2$ has a finite limit, then*

$$\mathbb{E}[\Phi(\mathbf{x}^{k+1}) - \Phi(\mathbf{x}^*)] \leq \eta^k (\Phi(\mathbf{x}^1) - \Phi(\mathbf{x}^*)) + \mathcal{O}(e_k), \quad (2.31)$$

where $\eta = \frac{\gamma(\bar{\alpha})}{1 + \gamma(\bar{\alpha})}$ and $e_k = \max\{\sigma_k^2, (\eta + \epsilon)^k\}$ for any $\epsilon > 0$.

Proof. Note that $\gamma(\alpha)$ and $\nu(\alpha)$ defined in (2.27) are both increasing with respect to $\alpha \in (0, \frac{2}{L})$. Hence it follows from (2.26) that

$$\mathbb{E}[\Phi(\mathbf{x}^{k+1}) - \Phi(\mathbf{x}^*)] \leq \frac{\gamma(\bar{\alpha})}{1 + \gamma(\bar{\alpha})} \mathbb{E}[\Phi(\mathbf{x}^k) - \Phi(\mathbf{x}^*)] + \frac{s\nu(\bar{\alpha})}{1 + \gamma(\underline{\alpha})} \sigma_k^2.$$

Let $\tau = \frac{s\nu(\bar{\alpha})}{1 + \gamma(\underline{\alpha})}$. Then using the above inequality recursively yields

$$\mathbb{E}[\Phi(\mathbf{x}^{k+1}) - \Phi(\mathbf{x}^*)] \leq \eta^k (\Phi(\mathbf{x}^1) - \Phi(\mathbf{x}^*)) + \tau \sum_{j=1}^k \eta^{k-j} \sigma_j^2.$$

From Lemma 2.7, we have $\lim_{k \rightarrow \infty} e_{k+1}/e_k \geq \eta + \epsilon$, and thus there is a sufficiently large integer K such that $e_{k+1}/e_k \geq \eta + \epsilon/2, \forall k \geq K$. Let $\mu_k = \sum_{j=1}^k \eta^{k-j} \sigma_j^2$ and choose a sufficiently large number B such that $\mu_k \leq B e_k, \forall k \leq K$ and $(B-1)\epsilon \geq 2\eta$. Note that such B must exist since K is finite and $\epsilon > 0$. Suppose that for some $k \geq K$, it holds $\mu_k \leq B e_k$. Then

$$\mu_{k+1} = \eta \mu_k + \sigma_{k+1}^2 \leq \eta B e_k + e_{k+1} \leq \frac{\eta B}{\eta + \epsilon/2} e_{k+1} + e_{k+1} \leq B e_{k+1},$$

where the last inequality uses the conditions $(B-1)\epsilon \geq 2\eta$ and $e_{k+1} \geq 0$. Hence, by induction, we conclude $\mu_k \leq B e_k, \forall k$. This completes the proof. \square

REMARK 6. If $m_k = \mathcal{O}(k)$ in Algorithm 1, then according to Remark 8, we have the convergence rate $\mathcal{O}(\max\{((\eta+1)/2)^k, 1/k\})$ by letting $\epsilon = \frac{1-\eta}{2}$ in Theorem 2.8. If σ_k decreases linearly, then we have a linear convergence rate.

2.2. Nonconvex case. In this section, we do not assume any convexity of F or r_i 's. First, we analyze Algorithm 1 for the smooth case with $r_i = 0$ and $\mathcal{X}_i = \mathbb{R}^{n_i}$ for all i . Then, we impose a more restrictive condition on σ_k and analyze the algorithm for the unconstrained nonsmooth and constrained smooth cases. We start our analysis with the following lemma, which can be found in Lemma A.5 of [30] and Proposition 1.2.4 of [4].

LEMMA 2.9. For two nonnegative scalar sequence $\{a_k\}$ and $\{b_k\}$, if $\sum_{k=1}^{\infty} a_k = +\infty$ and $\sum_{k=1}^{\infty} a_k b_k < +\infty$, then

$$\liminf_{k \rightarrow \infty} b_k = 0.$$

Furthermore, if $|b_{k+1} - b_k| \leq B \cdot a_k$ for some constant $B > 0$, then

$$\lim_{k \rightarrow \infty} b_k = 0.$$

THEOREM 2.10. Let $\{\mathbf{x}^k\}$ be generated from Algorithm 1 with $r_i = 0$ and $\mathcal{X}_i = \mathbb{R}^{n_i}$ for all i and with $\alpha_i^k = c_i^k \beta_k$, where c_i^k, β_k are positive scalars such that $\beta_k \geq \beta_{k+1}, \forall k$,

$$0 < \inf_k c_i^k \leq \sup_k c_i^k < \infty, \forall i,$$

and

$$\sum_{k=1}^{\infty} \beta_k = +\infty, \quad \sum_{k=1}^{\infty} \beta_k^2 < +\infty. \quad (2.32)$$

Under Assumptions 1 through 3, if $\sigma = \sup_k \sigma_k < \infty$, then

$$\lim_{k \rightarrow \infty} \mathbb{E} \|\nabla \Phi(\mathbf{x}^k)\| = \lim_{k \rightarrow \infty} \mathbb{E} \|\nabla F(\mathbf{x}^k)\| = 0. \quad (2.33)$$

Proof. From the Lipschitz continuity of $\nabla_{\mathbf{x}_i} F$, it holds that

$$\begin{aligned}
& F(\mathbf{x}_{\leq i}^{k+1}, \mathbf{x}_{> i}^k) - F(\mathbf{x}_{\leq i}^{k+1}, \mathbf{x}_{\geq i}^k) \\
& \leq \langle \mathbf{g}_i^k, \mathbf{x}_i^{k+1} - \mathbf{x}_i^k \rangle + \frac{L}{2} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\|^2 \\
& = -\alpha_i^k \langle \mathbf{g}_i^k, \tilde{\mathbf{g}}_i^k \rangle + \frac{L}{2} (\alpha_i^k)^2 \|\tilde{\mathbf{g}}_i^k\|^2 \\
& = -\left(\alpha_i^k - \frac{L}{2} (\alpha_i^k)^2\right) \|\mathbf{g}_i^k\|^2 + \frac{L}{2} (\alpha_i^k)^2 \|\boldsymbol{\delta}_i^k\|^2 - (\alpha_i^k - L(\alpha_i^k)^2) \langle \mathbf{g}_i^k, \boldsymbol{\delta}_i^k \rangle \\
& = -\left(\alpha_i^k - \frac{L}{2} (\alpha_i^k)^2\right) \|\mathbf{g}_i^k\|^2 + \frac{L}{2} (\alpha_i^k)^2 \|\boldsymbol{\delta}_i^k\|^2 - (\alpha_i^k - L(\alpha_i^k)^2) (\langle \mathbf{g}_i^k - \nabla_{\mathbf{x}_i} F(\mathbf{x}^k), \boldsymbol{\delta}_i^k \rangle + \langle \nabla_{\mathbf{x}_i} F(\mathbf{x}^k), \boldsymbol{\delta}_i^k \rangle) \\
& \leq -\left(\alpha_i^k - \frac{L}{2} (\alpha_i^k)^2\right) \|\mathbf{g}_i^k\|^2 + \frac{L}{2} (\alpha_i^k)^2 \|\boldsymbol{\delta}_i^k\|^2 - (\alpha_i^k - L(\alpha_i^k)^2) \langle \nabla_{\mathbf{x}_i} F(\mathbf{x}^k), \boldsymbol{\delta}_i^k \rangle \\
& \quad + L(\alpha_i^k + L(\alpha_i^k)^2) \max_j \alpha_j^k (\|\boldsymbol{\delta}_i^k\|^2 + \sum_{j=1}^s (\|\mathbf{g}_j^k\|^2 + \|\boldsymbol{\delta}_j^k\|^2)),
\end{aligned} \tag{2.34}$$

where the last inequality follows from the following argument:

$$\begin{aligned}
& -(\alpha_i^k - L(\alpha_i^k)^2) \langle \mathbf{g}_i^k - \nabla_{\mathbf{x}_i} F(\mathbf{x}^k), \boldsymbol{\delta}_i^k \rangle \\
& \leq |\alpha_i^k - L(\alpha_i^k)^2| \|\boldsymbol{\delta}_i^k\| \|\mathbf{g}_i^k - \nabla_{\mathbf{x}_i} F(\mathbf{x}^k)\| \\
& \leq L |\alpha_i^k - L(\alpha_i^k)^2| \|\boldsymbol{\delta}_i^k\| \|\mathbf{x}_{\leq i}^{k+1} - \mathbf{x}_{\leq i}^k\| \\
& \leq L |\alpha_i^k - L(\alpha_i^k)^2| \|\boldsymbol{\delta}_i^k\| \sqrt{\sum_{j=1}^s \|\alpha_j^k \tilde{\mathbf{g}}_j^k\|^2} \\
& \leq L(\alpha_i^k + L(\alpha_i^k)^2) \max_j \alpha_j^k (\|\boldsymbol{\delta}_i^k\|^2 + \sum_{j=1}^s (\|\mathbf{g}_j^k\|^2 + \|\boldsymbol{\delta}_j^k\|^2)).
\end{aligned}$$

Summing (2.34) over i and using (2.1b) give

$$\begin{aligned}
F(\mathbf{x}^{k+1}) - F(\mathbf{x}^k) & \leq -\sum_{i=1}^s \left(\alpha_i^k - \frac{L}{2} (\alpha_i^k)^2\right) \|\mathbf{g}_i^k\|^2 - \sum_{i=1}^s (\alpha_i^k - L(\alpha_i^k)^2) \langle \nabla_{\mathbf{x}_i} F(\mathbf{x}^k), \boldsymbol{\delta}_i^k \rangle \\
& \quad + \sum_{i=1}^s \left(\frac{L}{2} (\alpha_i^k)^2 \|\boldsymbol{\delta}_i^k\|^2 + L(\alpha_i^k + L(\alpha_i^k)^2) \max_j \alpha_j^k (\|\boldsymbol{\delta}_i^k\|^2 + \sum_{j=1}^s (\|\mathbf{g}_j^k\|^2 + \|\boldsymbol{\delta}_j^k\|^2)) \right).
\end{aligned} \tag{2.35}$$

Note that since \mathbf{x}^k is independent of Ξ_k . Hence, from Lemma 2.1, we have

$$\mathbb{E} \langle \nabla_{\mathbf{x}_i} F(\mathbf{x}^k), \boldsymbol{\delta}_i^k \rangle \leq A \cdot \max_j \alpha_j^k \cdot \mathbb{E} \|\nabla_{\mathbf{x}_i} F(\mathbf{x}^k)\| \leq M_\rho A \cdot \max_j \alpha_j^k,$$

where M_ρ is defined in (2.2). Taking expectation over (2.35), we have

$$\begin{aligned}
& \mathbb{E} F(\mathbf{x}^{k+1}) - \mathbb{E} F(\mathbf{x}^k) \\
& \leq -\sum_{i=1}^s \left(\alpha_i^k - \frac{L}{2} (\alpha_i^k)^2\right) \mathbb{E} \|\mathbf{g}_i^k\|^2 + \sum_{i=1}^s M_\rho A (\alpha_i^k + L(\alpha_i^k)^2) \max_j \alpha_j^k \\
& \quad + \sum_{i=1}^s \left(\frac{L}{2} (\alpha_i^k)^2 \sigma^2 + L(\alpha_i^k + L(\alpha_i^k)^2) \max_j \alpha_j^k ((s+1)\sigma^2 + sM_\rho^2) \right) \\
& \leq -\sum_{i=1}^s (c\beta_k - \frac{LC^2}{2} \beta_k^2) \mathbb{E} \|\mathbf{g}_i^k\|^2 + \sum_{i=1}^s M_\rho A C \beta_k (C\beta_k + LC^2 \beta_k^2) \\
& \quad + \sum_{i=1}^s \left(\frac{LC^2}{2} \beta_k^2 \sigma^2 + L\beta_k (C\beta_k + LC^2 \beta_k^2) ((s+1)\sigma^2 + sM_\rho^2) \right),
\end{aligned} \tag{2.36}$$

where

$$c = \min_i \inf_k c_i^k, \quad C = \max_i \sup_k c_i^k. \quad (2.37)$$

Note that F is lower bounded, $\mathbb{E}\|\mathbf{g}_i^k\|^2 \leq M_\rho^2$, and $0 < c \leq C < \infty$. Summing (2.36) over k and using (2.32), we have

$$\sum_{k=1}^{\infty} \beta_k \mathbb{E}\|\mathbf{g}_i^k\|^2 < \infty, \forall i.$$

Furthermore,

$$\begin{aligned} |\mathbb{E}\|\mathbf{g}_i^{k+1}\|^2 - \mathbb{E}\|\mathbf{g}_i^k\|^2| &\leq \mathbb{E}[\|\mathbf{g}_i^{k+1} + \mathbf{g}_i^k\| \cdot \|\mathbf{g}_i^{k+1} - \mathbf{g}_i^k\|] \\ &\leq 2LM_\rho \mathbb{E}\|(\mathbf{x}_{j<i}^{k+2}, \mathbf{x}_{j\geq i}^{k+1}) - (\mathbf{x}_{j<i}^{k+1}, \mathbf{x}_{j\geq i}^k)\| \\ &= 2LM_\rho \mathbb{E} \sqrt{\sum_{j<i} \|\alpha_j^{k+1} \tilde{\mathbf{g}}_j^{k+1}\|^2 + \sum_{j\geq i} \|\alpha_j^k \tilde{\mathbf{g}}_j^k\|^2} \\ &\leq 2LM_\rho C \beta_k \mathbb{E} \sqrt{\sum_{j<i} \|\tilde{\mathbf{g}}_j^{k+1}\|^2 + \sum_{j\geq i} \|\tilde{\mathbf{g}}_j^k\|^2} \\ &\leq 2LM_\rho C \beta_k \sqrt{\mathbb{E}[\sum_{j<i} \|\tilde{\mathbf{g}}_j^{k+1}\|^2 + \sum_{j\geq i} \|\tilde{\mathbf{g}}_j^k\|^2]} \\ &\leq 2LM_\rho C \beta_k \sqrt{2s(M_\rho^2 + \sigma^2)}, \end{aligned} \quad (2.38)$$

where we have used the monotonicity of β_k in the third inequality and the Jensen's inequality in the first and fourth inequalities. According to Lemma 2.9, we have $\mathbb{E}\|\mathbf{g}_i^k\|^2 \rightarrow 0$, $\forall i$, as $k \rightarrow \infty$ and thus $\mathbb{E}\|\mathbf{g}_i^k\| \rightarrow 0$, $\forall i$, as $k \rightarrow \infty$ by Jensen's inequality. Hence,

$$\begin{aligned} \mathbb{E}\|\nabla_{\mathbf{x}_i} F(\mathbf{x}^k)\| &\leq \mathbb{E}\|\nabla_{\mathbf{x}_i} F(\mathbf{x}^k) - \mathbf{g}_i^k\| + \mathbb{E}\|\mathbf{g}_i^k\| \\ &\leq L \cdot \mathbb{E}\|\mathbf{x}_{<i}^{k+1} - \mathbf{x}_{<i}^k\| + \mathbb{E}\|\mathbf{g}_i^k\| \\ &\leq LC \sqrt{2s(M_\rho^2 + \sigma^2)} \beta_k + \mathbb{E}\|\mathbf{g}_i^k\| \rightarrow 0, \forall i, \text{ as } k \rightarrow \infty, \end{aligned}$$

where the last inequality is obtained following the same argument for (2.38). Therefore, the desired result is obtained. \square

REMARK 7. *The above proof only needs the boundedness of $\mathbb{E}\|\mathbf{g}_i^k\|^2$, $\forall i, k$, instead of the stronger Assumption 3.*

We next analyze Algorithm 1 for the non-smooth case with the notation in (2.22) through (2.24).

THEOREM 2.11. *Let $\{\mathbf{x}^k\}$ be generated from Algorithm 1 with α_i^k 's being the same as those in Theorem 2.10. Under Assumptions 1 through 4, if either $\mathcal{X}_i = \mathbb{R}^{n_i}$ or $r_i = 0$ for all i , and*

$$\sum_{k=1}^{\infty} \beta_k \sigma_k^2 < \infty, \quad (2.39)$$

then there exists an index subsequence \mathcal{K} such that

$$\lim_{\substack{k \rightarrow \infty \\ k \in \mathcal{K}}} \mathbb{E}[\text{dist}(\mathbf{0}, \partial \hat{\Phi}(\mathbf{x}^k))] = 0, \quad (2.40)$$

where $\text{dist}(\mathbf{y}, \mathcal{X}) = \min_{\mathbf{x} \in \mathcal{X}} \|\mathbf{x} - \mathbf{y}\|$.

Proof. From the optimality of \mathbf{x}_i^{k+1} for (2.23), it holds that

$$\hat{r}_i(\mathbf{x}_i^{k+1}) - \hat{r}_i(\mathbf{x}_i^k) \leq -\langle \tilde{\mathbf{g}}_i^k, \mathbf{x}_i^{k+1} - \mathbf{x}_i^k \rangle - \frac{1}{2\alpha_i^k} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\|^2.$$

In addition, from the Lipschitz continuity condition (2), it follows that

$$F(\mathbf{x}_{\leq i}^{k+1}, \mathbf{x}_{> i}^k) - F(\mathbf{x}_{< i}^{k+1}, \mathbf{x}_{\geq i}^k) \leq \langle \tilde{\mathbf{g}}_i^k, \mathbf{x}_i^{k+1} - \mathbf{x}_i^k \rangle + \frac{L}{2} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\|^2.$$

Since $\alpha_i^k \rightarrow 0, \forall i$ as $k \rightarrow \infty$, one can take sufficiently large \tilde{k} such that $\alpha_i^k < \frac{1}{2L}, \forall i, k \geq \tilde{k}$. Summing up the above two inequalities and assuming $k \geq \tilde{k}$, we have

$$\begin{aligned} & \hat{\Phi}(\mathbf{x}_{\leq i}^{k+1}, \mathbf{x}_{> i}^k) - \hat{\Phi}(\mathbf{x}_{< i}^{k+1}, \mathbf{x}_{\geq i}^k) \\ & \leq -\langle \tilde{\boldsymbol{\delta}}_i^k, \mathbf{x}_i^{k+1} - \mathbf{x}_i^k \rangle - \left(\frac{1}{2\alpha_i^k} - \frac{L}{2}\right) \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\|^2 \end{aligned} \quad (2.41)$$

$$\leq \frac{\alpha_i^k}{1 - L\alpha_i^k} \|\tilde{\boldsymbol{\delta}}_i^k\|^2 - \frac{1}{2} \left(\frac{1}{2\alpha_i^k} - \frac{L}{2}\right) \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\|^2 \quad (2.42)$$

$$= \frac{\alpha_i^k}{1 - L\alpha_i^k} \|\tilde{\boldsymbol{\delta}}_i^k\|^2 - \left(\frac{\alpha_i^k}{4} - \frac{L}{4}(\alpha_i^k)^2\right) \|\tilde{\mathbf{h}}_i^k\|^2$$

where we have used the Young's inequality in the second inequality. Summing the above inequality over i yields

$$\hat{\Phi}(\mathbf{x}^{k+1}) - \hat{\Phi}(\mathbf{x}^k) \leq \sum_{i=1}^s \frac{\alpha_i^k}{1 - L\alpha_i^k} \|\tilde{\boldsymbol{\delta}}_i^k\|^2 - \sum_{i=1}^s \left(\frac{\alpha_i^k}{4} - \frac{L}{4}(\alpha_i^k)^2\right) \|\tilde{\mathbf{h}}_i^k\|^2. \quad (2.43)$$

Taking expectation on both sides of the above inequality gives

$$\begin{aligned} \mathbb{E}\hat{\Phi}(\mathbf{x}^{k+1}) - \mathbb{E}\hat{\Phi}(\mathbf{x}^k) & \leq \sum_{i=1}^s \frac{\alpha_i^k}{1 - L\alpha_i^k} \sigma_k^2 - \sum_{i=1}^s \left(\frac{\alpha_i^k}{4} - \frac{L}{4}(\alpha_i^k)^2\right) \mathbb{E}\|\tilde{\mathbf{h}}_i^k\|^2 \\ & \leq \sum_{i=1}^s 2C\beta_k \sigma_k^2 - \sum_{i=1}^s \left(\frac{c}{4}\beta_k - \frac{L}{4}C^2\beta_k^2\right) \mathbb{E}\|\tilde{\mathbf{h}}_i^k\|^2, \end{aligned} \quad (2.44)$$

where in the second inequality, we have assumed $\alpha_i^k < \frac{1}{2L}$ by taking $k \geq \tilde{k}$, and c and C are defined in (2.37). Note that from Lemma 2.2, we have that $\mathbb{E}\|\tilde{\mathbf{h}}_i^k\|^2$ is bounded for all i and k . In addition, by (2.32) and (2.39) and recalling that Φ , and thus $\hat{\Phi}$, is lower bounded, we have from (2.44) that

$$\sum_{k=0}^{\infty} \beta_k \mathbb{E}\|\tilde{\mathbf{h}}_i^k\|^2 < \infty, \forall i.$$

Hence, from Lemma 2.9, there must be a subsequence $\{\mathbb{E}\|\tilde{\mathbf{h}}_i^k\|^2\}_{k \in \mathcal{K}}$ converging to zero for all i .

Furthermore, note that

$$\tilde{\mathbf{h}}_i^k = \frac{1}{\alpha_i^k} (\mathbf{x}_i^k - \mathbf{x}_i^{k+1}) \in \tilde{\mathbf{g}}_i^k + \partial \hat{r}_i(\mathbf{x}_i^{k+1}), \forall i,$$

which is equivalent to

$$\tilde{\mathbf{h}}_i^k + \nabla_{\mathbf{x}_i} F(\mathbf{x}^{k+1}) - \tilde{\mathbf{g}}_i^k \in \nabla_{\mathbf{x}_i} F(\mathbf{x}^{k+1}) + \partial \hat{r}_i(\mathbf{x}_i^{k+1}), \forall i.$$

Hence,

$$\begin{aligned}
\text{dist}(\mathbf{0}, \partial\hat{\Phi}(\mathbf{x}^{k+1}))^2 &\leq \sum_{i=1}^s \|\tilde{\mathbf{h}}_i^k + \nabla_{\mathbf{x}_i} F(\mathbf{x}^{k+1}) - \tilde{\mathbf{g}}_i^k\|^2 \\
&\leq 2 \sum_{i=1}^s (\|\tilde{\mathbf{h}}_i^k\|^2 + \|\nabla_{\mathbf{x}_i} F(\mathbf{x}^{k+1}) - \tilde{\mathbf{g}}_i^k\|^2) \\
&\leq 4 \sum_{i=1}^s \left(\|\tilde{\mathbf{h}}_i^k\|^2 + \|\nabla_{\mathbf{x}_i} F(\mathbf{x}^{k+1}) - \mathbf{g}_i^k\|^2 + \|\boldsymbol{\delta}_i^k\|^2 \right),
\end{aligned}$$

and thus taking expectation over the last inequality gives

$$\mathbb{E}\text{dist}(\mathbf{0}, \partial\hat{\Phi}(\mathbf{x}^{k+1}))^2 \leq 4 \sum_{i=1}^s \left(\mathbb{E}\|\tilde{\mathbf{h}}_i^k\|^2 + \mathbb{E}\|\nabla_{\mathbf{x}_i} F(\mathbf{x}^{k+1}) - \mathbf{g}_i^k\|^2 + \mathbb{E}\|\boldsymbol{\delta}_i^k\|^2 \right).$$

Following the argument same as that at the end of the proof of Theorem 2.10, one can easily show that $\mathbb{E}\|\nabla_{\mathbf{x}_i} F(\mathbf{x}^{k+1}) - \mathbf{g}_i^k\|^2 \rightarrow 0$, $\forall i$, as $k \rightarrow \infty$. Taking another subsequence of \mathcal{K} if necessary, we have $\mathbb{E}\|\boldsymbol{\delta}_i^k\|^2 \leq \sigma_k^2 \rightarrow 0$, $\forall i$, as $\mathcal{K} \ni k \rightarrow \infty$ from (2.39). Hence, the right-hand side of the last inequality converges to zero as $\mathcal{K} \ni k \rightarrow \infty$. Applying Jensen's inequality completes the proof. \square

From (2.43) in the above proof, if σ_k decreases in a faster way, we can show that (2.40) holds for \mathcal{K} being the whole index sequence. This result is summarized below.

COROLLARY 2.12. *Let $\{\mathbf{x}^k\}$ be generated from Algorithm 1 with α_i^k 's satisfying $0 < \inf_k \alpha_i^k \leq \sup_k \alpha_i^k < \frac{1}{L}$, $\forall i$. Under Assumptions 1 through 4, if either $\mathcal{X}_i = \mathbb{R}^{n_i}$ or $r_i = 0$ for all i and $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$, then*

$$\lim_{k \rightarrow \infty} \mathbb{E}[\text{dist}(\mathbf{0}, \partial\hat{\Phi}(\mathbf{x}^k))] = 0,$$

which says that the optimality condition for (1.1) asymptotically holds in expectation.

Proof. Under the conditions that $0 < \inf_k \alpha_i^k \leq \sup_k \alpha_i^k < \frac{1}{L}$, $\forall i$ and $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$, it follows from (2.43) that the subsequence \mathcal{K} in the proof of Theorem 2.11 can be taken as the whole sequence. Noting $\sigma_k \rightarrow 0$ as $k \rightarrow \infty$, we immediately get the desired result from Theorem 2.11. \square

REMARK 8. *One way to make $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$ is to asymptotically increase m_k at a sufficiently fast rate. Let*

$$\boldsymbol{\delta}_{i,\ell}^k = \nabla_{\mathbf{x}_i} f(\mathbf{x}_{<i}^{k+1}, \mathbf{x}_{\geq i}^k; \xi_{k,\ell}) - \mathbf{g}_i^k.$$

Then, $\boldsymbol{\delta}_i^k = \frac{1}{m_k} \sum_{\ell=1}^{m_k} \boldsymbol{\delta}_{i,\ell}^k$. As in Assumption 1, assume $\|\mathbb{E}[\boldsymbol{\delta}_{i,\ell}^k | \boldsymbol{\Xi}_{[k-1]}]\| \leq A \cdot \max_j \alpha_j^k$ and also $\mathbb{E}\|\boldsymbol{\delta}_{i,\ell}^k\|^2 \leq \sigma^2$ for some constants A and σ . Then following the proof of [18] on page 11, one can show that $\mathbb{E}\|\boldsymbol{\delta}_i^k\|^2 \leq \mathcal{O}(1/m_k)$. Hence, taking $m_k = \lceil k^{1+\epsilon} \rceil$ for any $\epsilon > 0$ guarantees $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$, where $\lceil a \rceil$ denotes the smallest integer that is no less than a .

3. Numerical experiments. In this section, we report the simulation results of Algorithm 1, dubbed as BSG, on both convex and nonconvex problems to demonstrate its advantages. The tested convex problems include stochastic least squares (3.2) and logistic regression (3.3). The tested nonconvex problems include low-rank tensor recovery (3.4) and bilinear logistic regression (3.5).

3.1. Parameter settings. For convex problems (3.2) and (3.3), we let $(\pi_1^k, \dots, \pi_s^k)$ be a random shuffling of $(1, \dots, s)$ and set the stepsize of BSG to $\alpha_i^k = \min(\frac{\theta}{\sqrt{k}}, \frac{1}{L_i^k})$, $\forall i, k$, where the value of θ was specified in each test, and L_i^k was the Lipschitz constant of

$$\frac{1}{m_k} \sum_{\ell=1}^{m_k} \nabla_{\mathbf{x}_{\pi_i^k}} f(\mathbf{x}_{\pi_{<i}^k}^{k+1}, \mathbf{x}_i, \mathbf{x}_{\pi_{>i}^k}^k; \xi_{k,\ell}) \quad (3.1)$$

TABLE 3.1

Objective values of BSG, SG, and SBMD on solving stochastic least squares (3.2) with data following the Gaussian distribution. Bold numbers are best.

N Total Samples	BSG	SG	SBMD-10	SBMD-50	SBMD-100
4000	6.45e-3	6.03e-3	67.49	4.79	1.03e-1
6000	5.69e-3	5.79e-3	53.84	1.43	1.43e-2
8000	5.57e-3	5.65e-3	42.98	4.92e-1	6.70e-3
10000	5.53e-3	5.58e-3	35.71	2.09e-1	5.74e-3

with respect to \mathbf{x}_i . In addition, m_k was set the same for all k .

For nonconvex problems (3.4) and (3.5), we used the fixed updated order by letting $\pi_i^k = i, \forall i, k$, and set $\alpha_i^k = \min(\frac{\theta}{\sqrt{k} \log k}, \frac{1}{L_i^k})$ for smooth nonconvex problems, i.e., problems (3.4) with $\lambda = 0$ and (3.5), and set $\alpha_i^k = \frac{1}{L_i^k}$ for non-smooth nonconvex ones, i.e., problem (3.4) with $\lambda > 0$, where L_i^k was the Lipschitz constant of (3.1). In addition, for smooth nonconvex cases, m_k was set the same for all k , and for non-smooth nonconvex case, we asymptotically increased it by $m_k = m_1 + \lceil \frac{k-1}{10} \rceil, \forall k$.

We compared BSG with the SG (stochastic gradient) method and the SBMD (stochastic block mirror descent) method [9] on problems (3.2) and (3.3). Step size α_k for SG and SBMD was set in the same way as α_i^k in the above. Specifically, $\alpha_k = \min(\frac{\theta}{\sqrt{k}}, \frac{1}{L_k})$, where L_k was the Lipschitz constant of $\frac{1}{m_k} \sum_{\ell=1}^{m_k} \nabla_{\mathbf{x}} f(\mathbf{x}; \xi_{k,\ell})$ for SG and $\frac{1}{m_k} \sum_{\ell=1}^{m_k} \nabla_{\mathbf{x}_{i_k}} f(\mathbf{x}_{\neq i_k}^k, \mathbf{x}_{i_k}; \xi_{k,\ell})$ with respect to the selected block \mathbf{x}_{i_k} for SBMD. On solving (3.4) and (3.5), we compared BSG with the BCGD (block coordinate gradient descent) method [56], whose step size λ_i^k was taken as the reciprocal of the Lipschitz constant of $\nabla_{\mathbf{x}_i} F(\mathbf{x}_{<i}^{k+1}, \mathbf{x}_i, \mathbf{x}_{>i}^k)$ with respect to \mathbf{x}_i for all i and k . Throughout our tests, all compared algorithms were supplied with the same randomly generated starting point.

3.2. Stochastic least squares. We tested BSG, SG, and SBMD on the stochastic least squares problem:

$$\min_{\mathbf{x}} \mathbb{E}_{\mathbf{a}, b} \frac{1}{2} (\mathbf{a}^\top \mathbf{x} - b)^2, \quad (3.2)$$

where \mathbf{a} and b were random variables. In this test, entries of \mathbf{a} independently followed the standard Gaussian distribution, and $b = \mathbf{a}^\top \hat{\mathbf{x}} + \eta$ where η was independent of \mathbf{a} and followed the Gaussian distribution $\mathcal{N}(0, 0.01)$, and $\hat{\mathbf{x}} \in \mathbb{R}^{200}$ was a deterministic vector. It is easy to show that $\hat{\mathbf{x}}$ was the solution to (3.2), and the optimal objective value was 0.005. We first generated a Gaussian random vector $\hat{\mathbf{x}}$. Then we generated N samples of \mathbf{a} and b according to their distributions, one at a time, and for each sample, we performed one update of the three algorithms, i.e., $m_k = 1$ in (3.1). All three algorithms started from the same Gaussian randomly generated point and used $\theta = 0.1$. To compare their solutions, we generated another 100,000 samples (\mathbf{a}, b) following the same distribution and calculated the empirical loss. The entire process was repeated 100 times independently, and average empirical losses were shown in Table 3.1 for different N 's. ‘‘SBMD- t ’’ denotes SBMD algorithm that independently selected t coordinates at each iteration. Since ‘‘SBMD-200’’ becomes SG method, its results were identical to those of SG and thus not reported. From the results, we see that SBMD performed consistently better by updating more coordinates at each iteration. BSG was better than SG except for $N = 4000$.

3.3. Logistic regression. We tested BSG, SG, and SBMD on the logistic regression problem:

$$\min_{\mathbf{w}, b} \frac{1}{N} \sum_{\ell=1}^N \log(1 + \exp(-y_\ell (\mathbf{x}_\ell^\top \mathbf{w} + b))). \quad (3.3)$$

First, we compared the three algorithms on synthetic data. We randomly generated $N = 2000$ samples of dimension 200 with half of them belonging to the ‘‘+1’’ class and the other half to the ‘‘-1’’ class. Each

FIG. 3.1. Objective values of BSG, SG, and SBMD for solving logistic regression (3.3) on Gaussian randomly generated samples.

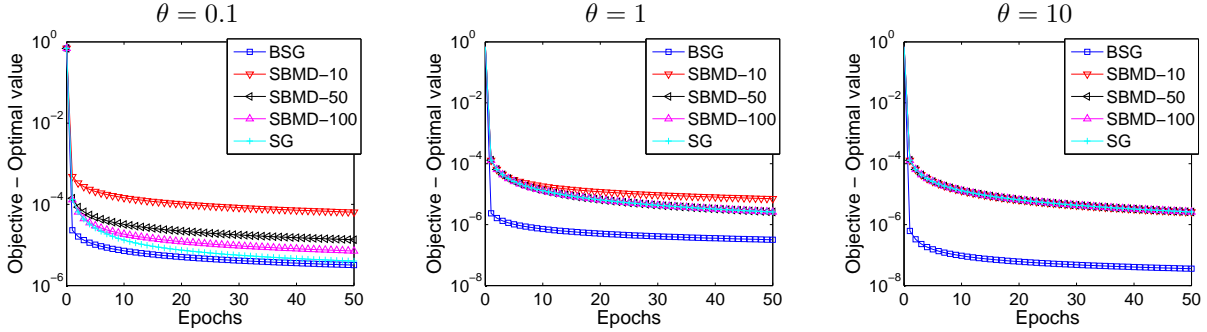
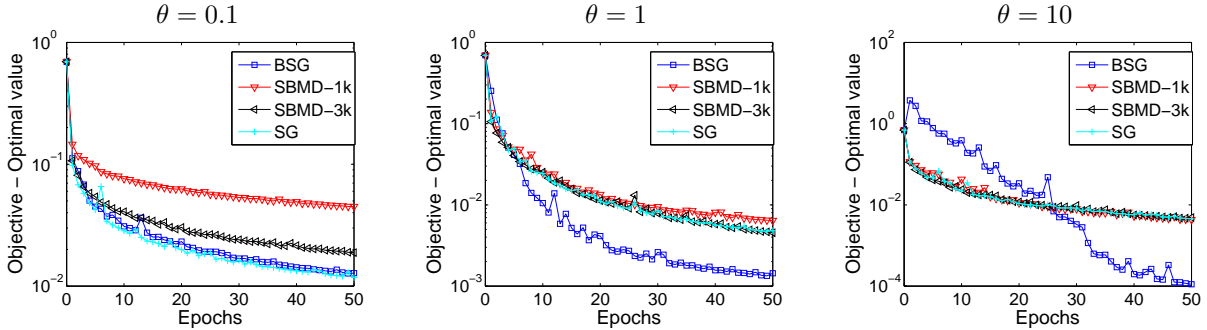


FIG. 3.2. Objective values of BSG, SG, and SBMD for solving logistic regression (3.3) on the *gisette* dataset.



sample in the positive class has components independently sampled from the Gaussian distribution $\mathcal{N}(5, 1)$ and those in the negative class from $\mathcal{N}(-5, 1)$. We ran BSG, SG, and SBMD each to 50 epochs, where one epoch was equivalent to going through all samples once. At each iteration of the algorithms, we uniformly randomly selected one sample, i.e., $m_k = 1$ in (3.1). Three different values of θ were tested. Figure 3.1 plots the gap between the optimal objective value and those given by different algorithms for solving (3.3), where the optimal objective value was accurately obtained by running FISTA [2] to 5,000 iteration. From the figure, we see that when θ was small, SBMD performed consistently better if more coordinates were updated at each iteration, and when θ was large, its performance was almost irrelevant to the numbers of updated coordinates. In addition, the proposed BSG method performed the best, and it reached a much lower objective within the same number of epochs, especial when a large θ was used.

Secondly, we compared the three algorithms on the *gisette* dataset², which has 6,000 training samples of dimension 5,000. At each iteration, SBMD updated 1,000 or 3,000 coordinates. Figure 3.2 plots the results of the three algorithms on problem (3.3) with three different values of θ . From the figures, we see that BSG performed almost the same as SG when $\theta = 0.1$, and they both outperformed SBMD within the same number of epochs. When θ becomes larger, BSG reaches much lower objective values than SG and SBMD, and the latter two performed almost the same.

²Available from <http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/>

FIG. 3.3. 3D shape of the original $60 \times 60 \times 60$ tensor \mathcal{M} (on the left) and the corresponding recovered one (on the right) by BSG.

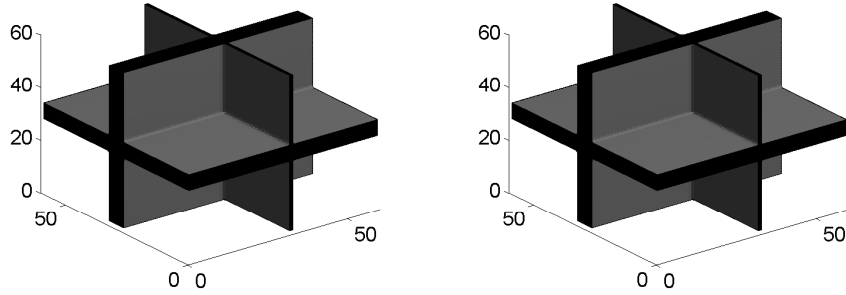
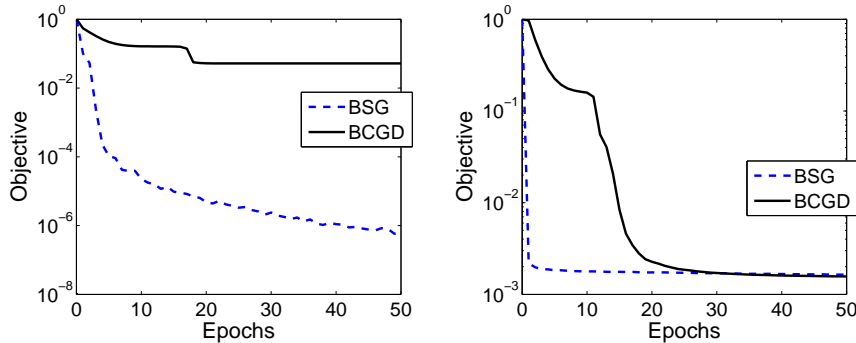


FIG. 3.4. Objective values of BSG and BCGD for solving (3.4) on a $32 \times 32 \times 32$ low-rank tensor. Left: ℓ_1 penalty parameter $\lambda = 0$; Right: $\lambda = \frac{1}{N}$



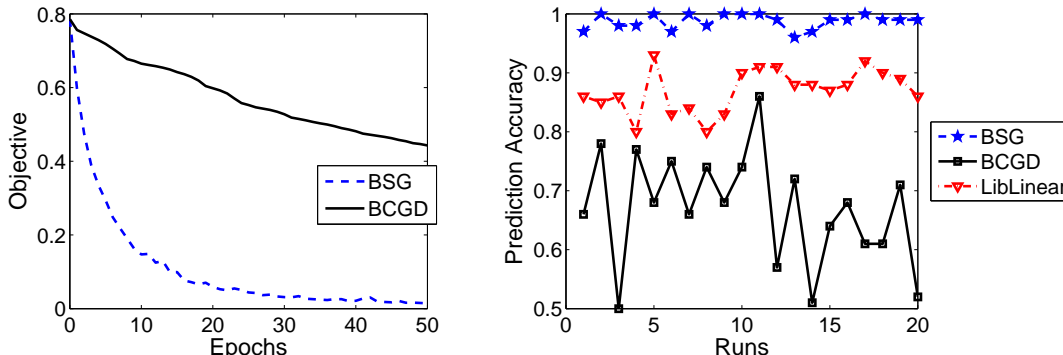
3.4. Low-rank tensor recovery. We compared BSG and BCGD on the low-rank tensor recovery problem:

$$\min_{\mathbf{X}} \frac{1}{2N} \sum_{\ell=1}^N (\mathcal{A}_{\ell}(\mathbf{X}_1 \circ \mathbf{X}_2 \circ \mathbf{X}_3) - b_{\ell})^2 + \sum_{i=1}^3 \lambda \|\mathbf{X}_i\|_1, \quad (3.4)$$

where $b_{\ell} = \mathcal{A}_{\ell}(\mathcal{M}) = \langle \mathcal{G}_{\ell}, \mathcal{M} \rangle$, $\ell = 1, \dots, N$ with \mathcal{M} being the underlying low-rank tensor. We generated each element of \mathcal{G}_{ℓ} according to the standard Gaussian distribution. In Figure 3.3, we tested BSG on recovering a low-rank tensor of size $60 \times 60 \times 60$ from $N = 40,000$ Gaussian random measurements by solving (3.4) with $\lambda = 0$. The original tensor had 10 middle slices of all *one*'s along each mode and all the other elements to be *zero*. For this test, the data size exceeded the memory of our workstation, and thus BCGD could not be tested. Figure 3.3 plots the original tensor and the recovered one by BSG with 50 epochs and sample size $m_k = 256, \forall k$. Its relative error is about 1.93%.

In Figure 3.4, we compared BSG and BCGD for solving (3.4) on a smaller tensor of size $32 \times 32 \times 32$. It has the same shape as those in Figure 3.3, and it has 6 middle slices of all *one*'s along each mode and all other elements to be *zero*. We generated $N = 15,000$ Gaussian random measurements. The left plot in Figure 3.4 shows the objectives by BSG with $m_k = 64, \forall k$ and BCGD for solving (3.4) with $\lambda = 0$, and the right plot corresponds to $\lambda = \frac{1}{N}$ and $m_k = 64 + \lceil \frac{k-1}{10} \rceil, \forall k$ for BSG. From the figure, we see that BSG significantly outperformed BCGD in the beginning. In the smooth case, BCGD was trapped at some local solution, and in the nonsmooth case, BCGD eventually reached slightly lower objective than that of BSG.

FIG. 3.5. Left: objective values of (3.5) (lower is better) given by BSG and BCGD on the BCI EEG data within 50 epochs; Right: prediction accuracies (higher is better) of 20 independent runs by BSG and BCGD within 30 epochs and also by LIBLINEAR.



3.5. Bilinear logistic regression. We compared BSG and BCGD on the bilinear logistic regression problem:

$$\min_{\mathbf{U}, \mathbf{V}, b} \frac{1}{N} \sum_{\ell=1}^N \log(1 + \exp[-y_{\ell}(\text{tr}(\mathbf{U}^{\top} \mathbf{X}_{\ell} \mathbf{V}) + b)]), \quad (3.5)$$

where $\{(\mathbf{X}_{\ell}, y_{\ell})\}, \ell = 1, \dots, N$, were given training samples with class labels $y_{\ell} \in \{+1, -1\}$. The bilinear logistic regression appears to be first used in [10] for EEG data classification. It applies the matrix format of the original data, in contrast to the standard linear logistic regression which collapses each feature matrix into a vector. It has been shown that the bilinear logistic regression outperforms the standard linear logistic regression in many applications such as brain-computer interface [10] and visual recognition [53].

In this test, we used the EEG dataset IVb from BCI competition III³ and the dataset concerns motor imagery with uncued classification task. The 118 channel EEG was recorded from a healthy subject sitting in a comfortable chair with arms resting on armrests. Visual cues (letter presentation) were shown for 3.5 seconds, during which the subject performed: left hand, right foot, or tongue. The data was sampled at 100 Hz, and the cues of “left hand” and “right foot” were marked in the training data. We chose all the 210 marked data points, and for each data point We randomly subsampled 100 temporal slices independently for 10 times to get, in total, 2,100 samples of size 118×100 .

We set $m_k = 64, \forall k$ in (3.1) for BSG and used the same random starting point for BSG and BCGD. The left plot of Figure 3.5 depicts their convergence behaviors. From the figure, we see that BSG significantly outperformed BCGD within 50 epochs. Running BCGD to more epochs, we observed that BCGD could later reach a similar objective as that of BSG. The right plot of Figure 3.5 shows the prediction accuracy of the solutions of BSG and BCGD, both of which ran to 30 epochs, and that of the result returned by LIBLINEAR [12], which solved linear logistic regression to its default tolerance. We ran the three methods 20 times. For each run, we randomly chose 2,000 samples for training and the remaining ones for testing. From the figure, we see that the *bilinear* logistic regression problem solved by BSG gave consistently higher prediction accuracies than the *linear* logistic regression problem. The low accuracies given by BCGD were results of its non-convergence in 30 epochs, which can be observed from the left plot of Figure 3.5. Running to more epochs, BCGD will eventually give similar predictions as BSG. However, that will take much more time.

³<http://www.bbc.de/competition/iii/>

4. Conclusions. We have proposed a BSG (block stochastic gradient) method and analyzed its convergence for both convex and nonconvex problems. The method has the same order of convergence rate as that of the SG (stochastic gradient) method for convex programming, and its convergence has been established in terms of the expected violation of first-order optimality conditions for the nonconvex case. Numerical results demonstrate its advantages over SG and a BSMD (block stochastic mirror descent) method for solving certain convex problems and also over the BCGD (block coordinate gradient descent) method for nonconvex problems.

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Appendix A. Proof of some lemmas. We give the proofs of some lemmas in the paper.

A.1. Proof of Lemma 2.1. The result in (2.3) can be shown by

$$\begin{aligned}
\mathbb{E}\langle \mathbf{u}^k, \boldsymbol{\delta}_i^k \rangle &= \mathbb{E}_{\Xi_{[k-1]}} \left[\mathbb{E}[\langle \mathbf{u}^k, \boldsymbol{\delta}_i^k \rangle | \Xi_{[k-1]}] \right] \\
&= \mathbb{E}_{\Xi_{[k-1]}} \left[\langle \mathbb{E}[\mathbf{u}^k | \Xi_{[k-1]}], \mathbb{E}[\boldsymbol{\delta}_i^k | \Xi_{[k-1]}] \rangle \right] \\
&\leq \mathbb{E}_{\Xi_{[k-1]}} \left[\left\| \mathbb{E}[\mathbf{u}^k | \Xi_{[k-1]}] \right\| \cdot \left\| \mathbb{E}[\boldsymbol{\delta}_i^k | \Xi_{[k-1]}] \right\| \right] \\
&\leq A(\max_j \alpha_j^k) \mathbb{E}_{\Xi_{[k-1]}} \left[\left\| \mathbb{E}[\mathbf{u}^k | \Xi_{[k-1]}] \right\| \right] \\
&\leq A(\max_j \alpha_j^k) \mathbb{E} \|\mathbf{u}^k\|,
\end{aligned}$$

where the second equality follows from the conditional independence between \mathbf{u}^k and $\boldsymbol{\delta}_i^k$, and the last inequality follows from the Jensen's inequality.

A.2. Proof of Lemma 2.2. We first show the following lemma.

LEMMA A.1. *For any function ψ and positive scalar α , it holds that*

$$\left\| \frac{1}{\alpha} (\mathbf{x} - \mathbf{prox}_{\alpha\psi}(\mathbf{x} - \alpha\mathbf{y})) \right\|^2 \leq 2\|\mathbf{y}\|^2 + 2\|\partial\psi(\mathbf{prox}_{\alpha\psi}(\mathbf{x} - \alpha\mathbf{y}))\|^2, \quad (\text{A.1})$$

where $\|\partial\psi(\mathbf{z})\| = \sup_{\mathbf{u}} \{\|\mathbf{u}\| : \mathbf{u} \in \partial\psi(\mathbf{z})\}$, and if $\partial\psi(\mathbf{z}) = \emptyset$, we let $\|\partial\psi(\mathbf{z})\| = +\infty$ by convention.

Proof. Let $\mathbf{z} = \mathbf{prox}_{\alpha\psi}(\mathbf{x} - \alpha\mathbf{y})$. Then $\mathbf{0} \in \mathbf{y} + \frac{1}{\alpha}(\mathbf{z} - \mathbf{x}) + \partial\psi(\mathbf{z})$, namely, for some $\mathbf{u} \in \partial\psi(\mathbf{z})$, it holds $\frac{1}{\alpha}(\mathbf{x} - \mathbf{z}) = \mathbf{y} + \mathbf{u}$. Hence,

$$\left\| \frac{1}{\alpha} (\mathbf{x} - \mathbf{z}) \right\|^2 \leq 2\|\mathbf{y}\|^2 + 2\|\mathbf{u}\|^2,$$

which completes the proof. \square

Now we are ready to prove Lemma 2.2. For $i \in \mathcal{I}_1$, we have

$$\tilde{\mathbf{h}}_i^k = \frac{1}{\alpha_i^k} \left(\mathbf{x}_i^k - \mathbf{prox}_{\alpha_i^k r_i}(\mathbf{x}_i^k - \alpha_i^k \tilde{\mathbf{g}}_i^k) \right),$$

and thus from Lemma A.1 and Remark 3, it follows that

$$\mathbb{E} \|\tilde{\mathbf{h}}_i^k\|^2 \leq 2\mathbb{E} \|\tilde{\mathbf{g}}_i^k\|^2 + 2L_{r_i}^2 \leq 4\mathbb{E} \|\mathbf{g}_i^k\|^2 + 4\mathbb{E} \|\boldsymbol{\delta}_i^k\|^2 + 2L_{r_i}^2 \leq 4M_\rho^2 + 4\sigma_k^2 + 2L_{r_i}^2,$$

where we have used the Cauchy-Schwarz inequality in the second inequality. For $i \in \mathcal{I}_2$, we have

$$\begin{aligned}
\mathbb{E} \|\tilde{\mathbf{h}}_i^k\|^2 &= \frac{1}{(\alpha_i^k)^2} \left\| \mathbf{x}_i^k - \mathcal{P}_{\mathcal{X}_i}(\mathbf{x}_i^k - \alpha_i^k(\tilde{\mathbf{g}}_i^k + \tilde{\nabla} r_i(\mathbf{x}_i^k))) \right\|^2 \\
&\leq \|\tilde{\mathbf{g}}_i^k + \tilde{\nabla} r_i(\mathbf{x}_i^k)\|^2 \leq 4M_\rho^2 + 4\sigma_k^2 + 2L_{r_i}^2,
\end{aligned}$$

where we have used the nonexpansiveness of the projection operator in the first inequality. This completes the proof of Lemma 2.2.

A.3. Proof of Lemma 2.3. The result can be proved by induction. First, consider the case of $a \leq 1$. When $k = 1$, (2.5) obviously holds. Suppose it holds for some $k \geq 1$. Then

$$\begin{aligned} A_{k+1} - \frac{c}{k+1} &\leq \left(1 - \frac{a}{k}\right) \frac{c}{k} + \frac{b}{k^2} - \frac{c}{k+1} \\ &= \frac{1}{k^2} \left(b - \left(a + \frac{k}{k+1}\right) c \right) \\ &\leq \frac{1}{k^2} \left(b - \left(a + \frac{k}{k+1}\right) \frac{b}{a} \right) < 0, \end{aligned}$$

which shows $A_{k+1} \leq c/(k+1)$.

Secondly, consider $a > 1$. When $k = \lfloor a \rfloor + 1$, it holds from

$$A_{\lfloor a \rfloor + 1} \leq \left(1 - \frac{a}{\lfloor a \rfloor}\right) A_{\lfloor a \rfloor} + \frac{b}{(\lfloor a \rfloor)^2} \leq \frac{b}{(\lfloor a \rfloor)^2} \leq \frac{2b}{(\lfloor a \rfloor + 1)(a-1)}.$$

Suppose (2.5) holds for some $k \geq \lfloor a \rfloor + 1$. Then

$$\begin{aligned} A_{k+1} - \frac{c}{k+1} &\leq \left(1 - \frac{a}{k}\right) \frac{c}{k} + \frac{b}{k^2} - \frac{c}{k+1} \\ &= \frac{1}{k^2} \left(b - \left(a + \frac{k}{k+1}\right) c \right) \\ &\leq \frac{1}{k^2} \left(b - \left(a + \frac{k}{k+1}\right) \frac{2b}{a-1} \right) < 0, \end{aligned}$$

which indicates $A_{k+1} \leq c/(k+1)$ and completes the proof.

A.4. Proof of Lemma 2.7. Suppose $\lim_{k \rightarrow \infty} a_{k+1}/a_k = \nu$. We prove the result for the cases of $\nu < \eta$ and $\nu = \eta$, and the case of $\nu > \eta$ can be shown in a similar way as that of $\nu < \eta$.

Case 1: $\nu < \eta$. Since $\lim_{k \rightarrow \infty} a_{k+1}/a_k = \nu$, for $\epsilon = \frac{\eta - \nu}{2} > 0$, there exists a sufficiently large integer $K_1 > 0$ such that $a_{k+1}/a_k \leq \nu + \epsilon = \frac{\nu + \eta}{2}$, $\forall k \geq K_1$. Therefore, $a_k \leq \left(\frac{\nu + \eta}{2}\right)^{k-K_1} a_{K_1}$, $\forall k \geq K_1$. Since $\nu < \eta$, we can choose another sufficiently large integer $K \geq K_1$ to have $a_k \leq \eta^k$, $\forall k \geq K$. Hence, $e_k = \eta^k$, $\forall k \geq K$, and the limit of e_{k+1}/e_k is η .

Case 2: $\nu = \eta$. If $a_k \geq \eta^k$, $\forall k$, or $a_k \leq \eta^k$, $\forall k$, then the result is obvious. Otherwise, there must exist a sequence $\{n_\ell\}_{\ell=1}^\infty$ such that

$$\begin{cases} a_k \geq \eta^k, & \text{if } n_{2m-1} \leq k < n_{2m} \\ a_k < \eta^k, & \text{if } n_{2m} \leq k < n_{2m+1} \end{cases}$$

Note that if $n_{2m-1} \leq k < k+1 < n_{2m}$ or $n_{2m} \leq k < k+1 < n_{2m+1}$, then it is easy to have $e_{k+1}/e_k \rightarrow \eta$ as $k \rightarrow \infty$. In addition,

$$\frac{e_{n_{2m}}}{e_{n_{2m-1}}} = \frac{\eta^{n_{2m}}}{a_{n_{2m-1}}} \leq \frac{\eta^{n_{2m}}}{\eta^{n_{2m-1}}} = \eta,$$

and

$$\frac{e_{n_{2m}}}{e_{n_{2m-1}}} = \frac{\eta^{n_{2m}}}{a_{n_{2m-1}}} \geq \frac{a_{n_{2m}}}{a_{n_{2m-1}}} \rightarrow \eta, \text{ as } m \rightarrow \infty.$$

Hence, $\lim_{m \rightarrow \infty} e_{n_{2m}}/e_{n_{2m-1}} = \eta$, and in the same way, one can show $\lim_{m \rightarrow \infty} e_{n_{2m+1}}/e_{n_{2m+1}-1} = \eta$. Therefore, the limit of e_{k+1}/e_k is η . This completes the proof.

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