A tight iteration-complexity upper bound for the MTY predictor-corrector algorithm via redundant Klee-Minty cubes

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Abstract

It is an open question whether there is an interior-point algorithm for linear optimization problems with a lower iteration-complexity than the classical bound $\mathcal{O}(\sqrt{n}\log(\frac{\mu_1}{\mu_0}))$. This paper provides a negative answer to that question for a variant of the Mizuno-Todd-Ye predictor-corrector algorithm. In fact, we prove that for any $\epsilon > 0$, there is a redundant Klee-Minty cube for which the aforementioned algorithm requires $n^{(\frac{1}{2}-\epsilon)}$ iterations to reduce the barrier parameter by at least a constant. This is provably the first case of an adaptive step interior-point algorithm, where the classical iteration-complexity upper bound is shown to be tight.

1 Introduction

The paper of Karmarkar [5] in 1984 launched the field of interior-point methods (IPMs). Since then, IPMs have changed the landscape of optimization theory and been extended successfully for linear, nonlinear, and conic linear optimization [10].

For linear optimization problems (LO), to reduce the barrier parameter from μ_1 to μ_0, the best known iteration-complexity upper bound is $\mathcal{O}(\sqrt{n}\log(\frac{\mu_1}{\mu_0}))$. In practice however, IPMs require much less iterations than predicted by the theory. It has been conjectured that the required number of iterations grows logarithmically in the number of variables [4]. Sonnevend et al. [13] showed that for two distinct special classes of LO problems, we have the complexity upper bounds $\mathcal{O}(n^{\frac{1}{4}}\log(\frac{\mu_1}{\mu_0}))$ and $\mathcal{O}(n^{\frac{3}{8}}\log(\frac{\mu_1}{\mu_0}))$. Using an "anticipated" iteration-complexity analysis, [6] gives an $\mathcal{O}(n^{\frac{1}{4}}\log(\frac{\mu_1}{\mu_0}))$ iteration-complexity bound for the Mizuno-Todd-Ye predictor-corrector (MTY P-C) algorithm. Huhn and Borgwardt [3] presents a thorough probabilistic analysis of the iteration-complexity of IPMs and establish that under the rotation-symmetry-model with certain probabilistic assumptions, the average iteration-complexity is strongly polynomial.

Another direction of research regarding the iteration-complexity of IPMs is to construct worst-case examples. Sonnevend et al. [13] showed that a variant of MTY predictor-corrector algorithm requires $\Omega(n^{\frac{1}{3}})$ iterations to reduce the duality gap by $\log n$ for certain LO problems. A similar result has been obtained by Todd et al. [15] for the primal-dual affine scaling algorithm and has been later extended by Todd and Ye [16] for long step primal-dual IPMs; they showed that these algorithms take $\Omega(n^{\frac{1}{3}})$ iterations to reduce the duality gap by a constant.

In a series of papers [1, 2, 8, 9], LO problems have been constructed with central paths making a large number of sharp turns with the intuitive idea that for a path-following algorithm each turn should should lead to an extra Newton step. These constructions share the common feature; that is, the (dual) feasible set is a perturbed Klee-Minty (KM) cube and the central path visits all the vertices of the KM cube. In [9], for instance, the authors show that the central path makes $\Omega(\frac{\sqrt{n}}{\sqrt{\log n}})$ sharp turns.

A curvature integral developed by [13, 14] accurately estimates the number of iterations of a variant of MTY predictor-corrector algorithm, see Section 2. This curvature integral is one of the main tools in our paper and we will refer to this curvature as *Sonnevend's curvature*.

In this paper, we build our work upon the KM construction in [9]. The main argument of the paper can be summarized as follows: We first prove that a KM construction [9] with a carefully chosen neighborhood of the central path which depends on the dimension of the cube, visits every vertices of the cube in such a way that following the central path within that neighborhood requires an exponential number of steps. From Theorem 2.1, this yields a large lower bound for the Sonnevend curvature. Then by using a modified hybrid version of that construction as well as Theorem 2.1 once again, we are able to conclude that for any $\epsilon > 0$, there is a redundant hybrid version of the KM cube for which the MTY predictor-corrector algorithm requires $\Omega\left(n^{\left(\frac{1}{2}-\epsilon\right)}\log\left(\frac{\mu_1}{\mu_0}\right)\right)$ where $\log\frac{\mu_1}{\mu_0} = \mathcal{O}(\log n)$. Hence by a rigorous analysis, our modified KM construction provides the first case of an IPM, the MTY predictor-corrector algorithm, for which the classical iteration-complexity upper bound is essentially tight.

In the rest of this section, the basic terminology used in this paper is presented. Let A be an $m \times n$ matrix of full rank. For $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, we consider the standard form primal and dual linear optimization problems,

$$\begin{array}{llll} \min & c^T x & & \\ \text{s.t.} & Ax & = b & \\ & x & \geq 0, & \end{array} \qquad \begin{array}{lll} \max & b^T y & \\ \text{s.t.} & A^T y + s & = c \\ & s & \geq 0, \end{array} \tag{1}$$

where $x, s \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ are vectors of variables. Denote the sets of primal and dual feasible solutions by $\mathcal{P} = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$ and $\mathcal{D} = \{(y,s) \in \mathbb{R}^m \times \mathbb{R}^n : A^T y + s = c,\ s \geq 0\}$; the sets of strictly feasible primal and dual solutions by $\mathcal{P}^+$ and $\mathcal{D}^+$, respectively. Without loss of generality, see e.g., [12], we may assume that $\mathcal{P}^+ \neq \emptyset$ and $\mathcal{D}^+ \neq \emptyset$. For a parameter $\mu > 0$ and a vector $w > 0$, the w-weighted path equations are given by

$$\begin{aligned} Ax &= b, \quad x \geq 0 \\ A^T y + s &= c, \quad s \geq 0 \\ xs &= \mu w, \end{aligned} \tag{2}$$

where uv denotes $[u_1 v_1, \ldots, u_n v_n]^T$ for $u, v \in \mathbb{R}^n$. For $w = e$, with e being the all-one vector, equation (2) gives the central path equations.

2 IPMs and Sonnevend's curvature of the central path

First we briefly review the relevant algorithms to this paper. Roughly speaking, path-following IPMs differ by the way the barrier parameter $\mu^+ := (1-\theta)\mu$ is chosen and for what values of μ, the Newton steps are calculated. While for short-step IPMs, we have $\theta = \Omega(\frac{1}{\sqrt{n}})$, predictor-corrector type algorithms allow a larger θ, hence a larger reduction in μ. Given $\mu > 0$, and $\beta > 0$, we define the β-neighborhood of the point on the central path corresponding to μ as

$$\mathcal{N}(\beta, \mu) := \{(x,s) \in \mathcal{P}^+ \times \mathcal{D}^+ : \left\| \frac{xs}{\mu} - e \right\| \leq \beta\}. \tag{3}$$

The β-neighborhood of the central path is defined as $\mathcal{N}(\beta) := \bigcup_{\mu > 0} \mathcal{N}(\beta, \mu)$.

Both the algorithm of [14] and the MTY predictor-corrector algorithm use two nested neighborhoods $\mathcal{N}(\beta_0)$ and $\mathcal{N}(\beta_1)$ for $0 < \beta_0 < \beta_1 < 1$. The MTY predictor-corrector algorithm alternates between two search directions: The predictor search direction is used within the smaller neighborhood $\mathcal{N}(\beta_0)$ and it aims to reduce μ to zero. Let (x,s) be the current iterate, $(\Delta x, \Delta s)$ the predictor search direction and $(x^+, s^+) := (x + \theta\Delta x, s + \theta\Delta s)$. The MTY predictor-corrector algorithm and the algorithm in [14] differ in the way the value of θ is determined. In the MTY predictor-corrector algorithm, θ is determined as being the largest step for which (x^+, s^+) stays within the larger neighborhood $\mathcal{N}(\beta_1)$. In the algorithm of [14], the value of θ is determined as the largest number for which $\left\| \frac{x^+ s^+}{\mu^+} - \xi \right\| \leq \beta_1$, where $\xi = \frac{xs}{\mu}$. Then a pure centering step is taken which will take the iterate back to the smaller neighborhood $\mathcal{N}(\beta_0)$ in such a way that the normalized duality gap $\mu = \frac{x^T s}{n}$ does not change. Both algorithms can take long steps, in fact, it is known that [11, 14] as $k \to \infty$, $\theta_k \to 1$, where θ_k is the step length of the predictor direction at iteration k.

For the rest of the paper, we will refer to the both algorithms as MTY predictor-corrector algorithm.

Sonnevend's curvature, introduced in [13], is closely related to the iteration-complexity of a variant of the MTY predictor-corrector algorithm. Let $\kappa(\mu) = \|\mu \dot{x} \dot{s}\|^{1/2}$. Stoer et al. [14] proved that their predictor-corrector algorithm has a complexity bound, which can be expressed in terms of $\kappa(\mu)$.

Theorem 2.1. [14] *Let the nested neighborhood parameters β_0, β_1 of the MTY predictor-corrector algorithm satisfy $\beta_0 + \beta_1 < \frac{1}{2}$. Let N be the number of iterations of the MTY predictor-corrector algorithm to reduce the barrier parameter from μ_1 to μ_0. Suppose $\kappa(\mu) \geq \nu$ for some constant $\nu > 0$ on $\mu \in [\mu_0, \mu_1]$. Then for some "universal" constants C_1 and C_2 that depend only on the neighborhood of the central path, we have*

$$C_3 \int_{\mu_0}^{\mu_1} \frac{\kappa(\mu)}{\mu} d\mu - 1 \leq N \leq C_1 \int_{\mu_0}^{\mu_1} \frac{\kappa(\mu)}{\mu} d\mu + C_2 \log\left(\frac{\mu_1}{\mu_0}\right) + 2. \quad (4)$$

Constant C_3 depends on ν as well as the neighborhood of the central path.

The following proposition states the basic properties of Sonnevend's curvature.

Proposition 2.2. [13] *The following holds.*

1. *We have* $\kappa(\mu) = \left\| \frac{\mu \dot{s}(\mu)}{s(\mu)} - \left(\frac{\mu \dot{s}(\mu)}{s(\mu)}\right)^2 \right\|^{\frac{1}{2}}$.

2. *We have* $\left\| \frac{\mu \dot{s}(\mu)}{s(\mu)} \right\| \leq \sqrt{n}$ *and* $\kappa(\mu) \leq \sqrt{n}$ *implying that*

$$\int_{\mu_0}^{\mu_1} \frac{\kappa(\mu)}{\mu} d\mu = \mathcal{O}\left(\sqrt{n} \log\left(\frac{\mu_1}{\mu_0}\right)\right).$$

3 KM cube construction

First we recall the KM construction in [9] and review its fundamental properties.

$$
\begin{array}{lrll}
\max & -y_m & & \\
\text{s.t.} & 0 \leq \; y_1 & \leq 1 & \\
& \rho y_{k-1} \leq \; y_k & \leq 1 - \rho y_{k-1} & \text{for } k = 2 \ldots m. \\
& 0 & \leq d_1 + y_1 & \text{repeated } h_1 \text{ times} \\
& 0 & \leq d_2 + y_2 & \text{repeated } h_2 \text{ times} \\
& & \vdots & \\
& 0 & \leq d_m + y_m & \text{repeated } h_m \text{ times} .
\end{array} \tag{5}
$$

Certain variants of the simplex method take $2^m - 1$ to solve this problem. The simplex path for these variants starts from $(0, ..., 0, 1)T$, it visit all the vertices ordered by the decreasing value of the last coordinate y_m until reaching the optimal point, which is the origin.
As in [9], we fix $\rho(m) := \frac{m}{2(m+1)}$ and $d := \left(\frac{1}{\sqrt{\rho^{m-1}}}, \frac{1}{\sqrt{\rho^{m-2}}}, \ldots, \frac{1}{\sqrt{\rho}}, 0\right)$. We denote the m-dimensional KM cube by $\mathcal{KM}(m, \rho(m))$. See Figure 1 for $\mathcal{KM}(m, \rho(m))$ with $m = 2$.

Let the slack variables $\bar{s}_k = 1 - \rho y_{k-1} - y_k$ and $s_k = y_k - \rho y_{k-1}$ for $k = 2, \ldots, n$ with the convention $\bar{s}_1 = 1 - y_1$ and $s_1 = y_1$. There is a one-to-one correspondence between the vertices of $\mathcal{KM}(m, \rho(m))$ with the m-tuples $v^i \in \{0, 1\}^m$, $i = 1, \ldots, 2^m$ as follows. Each vertex of $\mathcal{KM}(m, \rho(m))$ is determined by whether exactly one of $s_i = 0$ or $\bar{s}_i = 0$ for each $i = 1, \ldots, m$ in (5). If $s_i = 0$, the i-th coordinate of the corresponding m-tuple in $\{0, 1\}^m$ is 0; if $\bar{s}_i = 1$, it is 1. For our purpose, we describe the relevant terms of $\mathcal{KM}(m, \rho(m))$ inductively as follows:

First we describe the order of the set of the vertices $\mathcal{V}(m)$ of $\mathcal{KM}(m, \rho(m))$ which the simplex path visits. Note that $\mathcal{V}(m)$ is an encoding of the vertices of $\mathcal{KM}(m, \rho(m))$, they are not the actual vertex points in $\mathbb{R}^m$. For $m = 2$, let

$$\mathcal{V}(2) = \{v^1, v^2, v^3, v^4\} = \{(0, 1), (1, 1), (1, 0), (0, 0)\}. \tag{6}$$

Figure 1 shows the vertices of the $\mathcal{KM}(m, \rho(m))$. Then let

$$\mathcal{V}(m + 1) = \{(v^{2^m}, 1), (v^{2^m - 1}, 1), \ldots, (v^1, 1), (v^1, 0), (v^2, 0), \ldots, (v^{2^m}, 0)\}. \tag{7}$$

It can be shown [9] that there exists a redundant $\mathcal{KM}(m, \rho(m))$ whose central path, denoted by $\mathcal{CP}(m)$, visits the vertices in the order given in the set $\mathcal{V}(m)$. Figure 2 and 3 show the central path for $m = 2$ and $m = 3$.

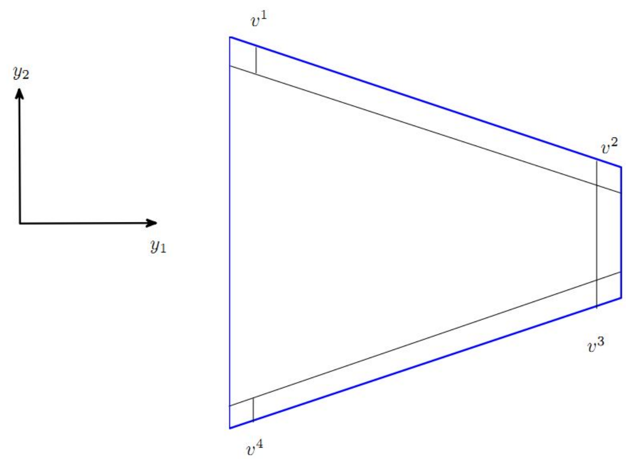

Figure 1: $\mathcal{V}(2) = \{v^1, v^2, v^3, v^4\} = \{(0,1),(1,1),(1,0),(0,0)\}$ shows the vertices of the $\mathcal{KM}(m, \rho(m))$ cube for $m = 2$.

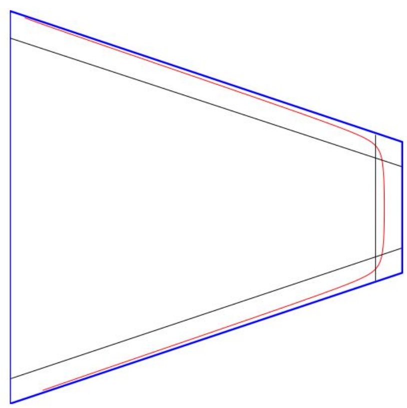

Figure 2: The central path visits the vertices $\mathcal{V}(2) = \{v^1, v^2, v^3, v^4\}$ of the $\mathcal{KM}(m, \rho(m))$ cube for $m = 2$ in the given order as μ decreases.

Next we define inductively a tube along the edges of the simplex path in $\mathcal{KM}(m, \rho(m))$ as follows. Let $\delta \leq \frac{1}{4(m+1)}$. Let $\mathcal{T}_\delta^U(2) = \{y : \mathbb{R}^2 : \bar{s}_2 \leq \delta\}$, $\mathcal{T}_\delta^L(2) = \{y : \mathbb{R}^2 : s_2 \leq \delta\}$ and $\mathcal{C}_\delta(m) = \{y : \mathbb{R}^2 : \bar{s}_m \geq \delta,\ s_m \geq \delta\}$ for $m \geq 2$. Note that $\mathcal{T}_\delta^U(2)$ and $\mathcal{T}_\delta^L(2)$ corresponds to a tube for the upper and lower facets of $\mathcal{KM}(2, \rho(2))$, respectively, while $\mathcal{C}_\delta(2)$ corresponds to the central part of $\mathcal{KM}(2, \rho(2))$, see Figure 1. By $\mathcal{T}_\delta(m)$, denote the union $\mathcal{T}_\delta^L(m) \cup \mathcal{T}_\delta^U(m) \cup \mathcal{C}_\delta(m)$. Then for $m \geq 2$, define $\mathcal{T}_\delta^U(m+1) = \{y : \mathbb{R}^{m+1} : \bar{s}_{m+1} \leq \delta,\ (y_1, \ldots, y_m) \in \mathcal{T}_\delta(m)\}$ and $\mathcal{T}_\delta^L(m+1) = \{y : \mathbb{R}^{m+1} : s_{m+1} \leq \delta,\ (y_1, \ldots, y_m) \in \mathcal{T}_\delta(m)\}$. Notice that $\mathcal{T}_\delta^U(3)$ is a tube that corresponds to the upper facet of $\mathcal{KM}(3, \rho(3))$ where $y_3 = 1 - \rho y_2$. Similarly $\mathcal{T}_\delta^L(3)$ is a tube that corresponds to the lower facet of $\mathcal{KM}(3, \rho(3))$ where $y_3 = \rho y_2$. Also these upper and lower facets are $\mathcal{KM}(2, \rho(3))$ cubes themselves, see Figure 3. Hence by

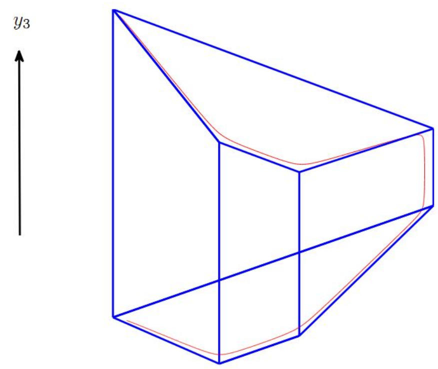

Figure 3: Central path in the redundant cube $\mathcal{KM}(m, \rho(m))$ cube for $m = 2$

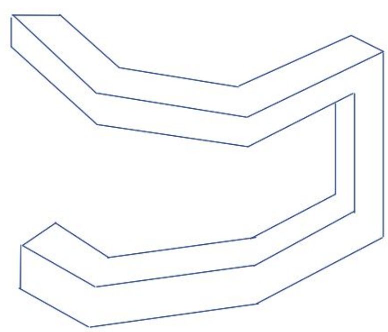

Figure 4: Illustration of the tube $\mathcal{T}_\delta(m)$ for $m = 3$.

identifying the first m coordinates of $(y_1, \ldots, y_m, y_{m+1})$ inside $\mathcal{KM}((m+1), \rho(m+1))$ with $(y_1, \ldots, y_m) \in \mathcal{KM}(m, \rho(m+1))$, and considering the assumption that δ is decreasing in m, we can write $\mathcal{T}_\delta^U(m+1) \subset \mathcal{T}_\delta(m)$ and $\mathcal{T}_\delta^L(m+1) \subset \mathcal{T}_\delta(m)$, see Figure 4.

We also define a δ-neighborhood of a vertex of $\mathcal{KM}(m, \rho(m))$ by whether exactly one of $s_i \leq \delta$ or $\bar{s}_i \leq \delta$ for each $i = 1, \ldots, m$ in (5). Figure 1 displays the δ-neighborhoods of the vertices $\mathcal{V}(2) = \{v^1, v^2, v^3, v^4\}$ of the $\mathcal{KM}(m, \rho(m))$ cube for $m = 2$.

The following proposition is essentially Proposition 2.2 in [9].

Proposition 3.1. *In (5), one can choose the parameters in such a way that the central path $\mathcal{CP}(m)$ in $\mathcal{KM}(m, \rho(m))$ stay inside the tube $\mathcal{T}_\delta(m)$. In particular, one can choose $\rho = \frac{m}{2(m+1)}$, $\delta \leq \frac{1}{4(m+1)}$ so that $n = \mathcal{O}(m2^{2m})$. As μ decreases, the central path visits the δ-neighborhoods of the vertices given in the order by (7). Moreover, the number of inequalities n is linear in $\frac{1}{\delta}$.*

Proof. See Proposition 2.2 in [9]. □

Now for $\mathcal{KM}(m, \rho(m))$, we identify two regions R_δ^U and R_δ^L within tube $\mathcal{T}_\delta(m)$ in such a way that going from R_δ^U to R_δ^L (an vice versa) with line segments staying inside tube $\mathcal{T}_\delta(m)$ requires $\Omega(2^{m-1})$ number of iterations. Let

$$R_\delta^U := \{y \in \mathcal{KM}(m, \rho(m)) : s_1 \leq \delta,\ s_2 \leq \delta, \ldots, s_{m-1} \leq \delta,\ \bar{s}_m \leq \delta\} \tag{8}$$

and

$$R_\delta^L := \{y \in \mathcal{KM}(m, \rho(m)) : s_1 \leq \delta,\ s_2 \leq \delta, \ldots, s_{m-1} \leq \delta,\ s_m \leq \delta\}. \tag{9}$$

We have the following.

Proposition 3.2. *For $\mathcal{KM}(m, \rho(m))$, let $y^U \in R_\delta^U$ and $y^L \in R_\delta^L$. Then staying inside the tube $\mathcal{T}_\delta(m)$, one requires at least 2^{m-1} line segments to reach y^U from y^L and vice versa.*

Proof. With the parameters chosen as in Proposition 3.1, we first show $\mathcal{T}_\delta^U(m)$ and $\mathcal{T}_\delta^L(m)$ do not intersect for any m. Suppose by contradiction that there is a $y \in \mathcal{T}_\delta^U(m) \cap \mathcal{T}_\delta^L(m)$. From the definition of $\mathcal{T}_\delta^U(m)$ and $\mathcal{T}_\delta^L(m)$, we have $\bar{s}_m = 1 - \rho y_{m-1} - y_m \leq \delta$ and $s_m = y_m - \rho y_{m-1} \leq \delta$. Adding these two inequalities, we get $1 - 2\rho y_{m-1} \leq 2\delta$. By the choice of ρ and δ, it is easy to see that, this will lead to the contradiction $y_{m-1} > 1$. Hence $\mathcal{T}_\delta^U(m) \cap \mathcal{T}_\delta^L(m) = \emptyset$.

The rest of the proof is by induction on m. For $m = 2$, let $y^U \in R_\delta^U$ and $y^L \in R_\delta^L$ with $\delta \leq \frac{1}{4(m+1)}$. Then, for y^U we have $s_1 = y_1 \leq \delta$ and $\bar{s}_2 \leq \delta$ which implies that $y_2 \geq 1 - \delta - \rho\delta \geq 1 - 2\delta = \frac{5}{6}$. Analogously, for y^L we have $s_1 = y_1 \leq \delta$ and $s_2 \leq \delta$ which implies $y_2 \leq \delta + \rho y_1 \leq 2\delta = \frac{1}{6}$. Clearly, staying inside the tube $\mathcal{T}_\delta(2)$, it takes at least 2 iterations to reach a point with $y_2 \leq \frac{1}{6}$ from a point with $y_2 \geq \frac{5}{6}$, see Figure 1.

As inductive step, suppose that to reach any point in R_δ^L from a point in R_δ^U with $R_\delta^L \subset \mathcal{KM}(m-1, \rho(m-1))$ and $R_\delta^U \subset \mathcal{KM}(m-1, \rho(m-1))$ one requires at least 2^{m-2} steps with line segments staying inside $\mathcal{T}_\delta(m-1)$. Let $y^U \in R_\delta^U$ and $y^L \in R_\delta^L$ inside $\mathcal{T}_\delta(m) \subset \mathcal{KM}(m, \rho(m))$. We distinguish two points p^1 and p^2 such that

$$p^1 \in \{y \in \mathcal{KM}(m, \rho(m)) : s_1 \leq \delta,\ s_2 \leq \delta, \ldots, \bar{s}_{m-1} \leq \delta,\ \bar{s}_m \leq \delta\}$$

and

$$p^2 \in \{y \in \mathcal{KM}(m, \rho(m)) : s_1 \leq \delta,\ s_2 \leq \delta, \ldots, \bar{s}_{m-1} \leq \delta,\ s_m \leq \delta\}.$$

Note that the point p^1 belongs to the δ-neighborhood of the vertex $v^{2^{m-1}} = (0, 0, \ldots, 0, 1, 1)$ and the point p^2 belongs to the δ-neighborhood of the vertex point $v^{2^{m-1}+1} = (0, 0, \ldots, 0, 1, 0)$.

Then, using the inductive definition of $\mathcal{T}_\delta^U(m)$ and $\mathcal{T}_\delta^L(m)$, it is easy to see that $y^U, p^1 \in \mathcal{T}_\delta^U(m)$ and $p^2, y^L \in \mathcal{T}_\delta^L(m)$. By inductive hypothesis, one needs at least 2^{m-2} line segments to reach p^1 from y^U staying inside the tube $\mathcal{T}_\delta^U(m) \subset \mathcal{T}_\delta(m-1)$. Similarly one needs at least 2^{m-2} line segments to reach y^L from p^2 staying inside the tube $\mathcal{T}_\delta^L(m) \subset \mathcal{T}_\delta(m-1)$. Moreover since by the first part of the proof, we have $\mathcal{T}_\delta^U(m) \cap \mathcal{T}_\delta^L(m) = \emptyset$, it follows that to reach y^L from y^U, one needs to traverse within $\mathcal{T}_\delta(m-1)$ twice, each time requiring at least 2^{m-2} steps. This proves that one requires at least 2^{m-1} line segments to reach y^U from y^L, hence the proof is complete. $\square$

4 Neighborhood of the KM cube central path

In Section 3, we showed that with $n = \mathcal{O}(m2^{2m})$ redundant constraints, the central path $\mathcal{CP}(m)$ stays inside a tube $\mathcal{T}_\delta(m)$. Moreover, we proved that it will take at least 2^{m-1} line segments to reach a point in R_δ^L close to the optimal solution of (5) from a point in R_δ^U close to the analytic center of $\mathcal{KM}(m, \rho(m))$. However, path-following IPMs algorithms including the MTY predictor-corrector algorithm, use the neighborhood $\mathcal{N}(\beta)$ as opposed to the tube neighborhood $\mathcal{T}_\delta(m)$ we used in Section 3. In this section we analyze the $\mathcal{N}(\beta)$ neighborhood for the cube $\mathcal{KM}(m, \rho(m))$ and prove that for $\beta = \Omega(\frac{1}{m+1})$, we have $\mathcal{N}(\beta) \subset \mathcal{T}_\delta(m)$. In other words, with appropriately chosen neighborhood parameters of $\mathcal{KM}(m, \rho(m))$, all the iterates of the MTY predictor-corrector algorithm stay inside the tube $\mathcal{T}_\delta(m)$. Hence, we can draw the conclusion that for $\mathcal{KM}(m, \rho(m))$, the MTY predictor-corrector algorithm will require $\Omega(2^{m-1})$ iterations with the neighborhood $\mathcal{N}(\beta)$, where $\beta = \Omega(\frac{1}{m+1})$.

In order to find the largest β for which $\mathcal{N}(\beta) \subset \mathcal{T}_\delta(m)$, we will use weighted paths. The following lemma is essentially Lemma 4.1 in [14].

Lemma 4.1. *Fix μ and let $w > 0$ such that $\|w - e\| \leq \epsilon$. Let $(x(w), y(w), s(w))$ denote the w-weighted path which is the solution set of (2). Let $\Delta s_i = s_i(w) - s_i$, where the s_i values are the coordinates of the central path point for $i = 1, \ldots, n$. Then we have $\left|\frac{\Delta s_i}{s_i}\right| \leq 2\epsilon$ for $i = 1, \ldots, n$.*

When we apply the information in Lemma 4.1 to $\mathcal{KM}(m, \rho(m))$, we obtain the following result.

Lemma 4.2. *There exists a $\mathcal{KM}(m, \rho(m))$ with $n = \mathcal{O}(m2^{2m})$ such that all the w-weighted paths with $\|w - e\| \leq \beta := \frac{\delta}{4}$ stay inside the tube $\mathcal{T}_\delta(m)$ with $\delta \leq \frac{1}{4(m+1)}$.*

Proof. Let $\delta \leq \frac{1}{4(m+1)}$. Then, from Proposition 3.1, we know that there exists $\mathcal{KM}(m, \rho(m))$ with $n = \mathcal{O}(m2^{2m})$ so that the central path stays inside the tube $\mathcal{T}_{\frac{\delta}{2}}(m)$. Choose $\beta = \frac{\delta}{4}$ for

$\mathcal{KM}(m, \rho(m))$ so that $\|w - e\| \le \beta$. Since for all the slacks, we have $s_i \le 1$ or $\bar{s}_i \le 1$, Lemma 4.1 implies that $s_i(w) \le s_i + \frac{\delta}{2}$ and $\bar{s}_i(w) \le \bar{s}_i + \frac{\delta}{2}$. Then whenever $s_i \le \frac{\delta}{2}$ or $\bar{s}_i \le \frac{\delta}{2}$, we have $\bar{s}_i(w) \le \delta$ and $s_i(w) \le \delta$. Since a tube $\mathcal{T}_\delta(m)$ with a general δ inside $\mathcal{KM}(m, \rho(m))$ is determined by these slacks, it follows that all w-weighted paths stay inside the tube $\mathcal{T}_\delta(m)$ with $\delta \le \frac{1}{4(m+1)}$. This concludes the proof. □

Next lemma proves a result analogous to Lemma 4.2 tailored for R^U_δ and R^L_δ.

Lemma 4.3. *Let $\delta \le \frac{1}{4(m+1)}$ and fix $\beta := \frac{\delta}{4}$. Suppose that $y(\mu_1) \in R^U_{\delta/2}$ for some μ_1. Then $\mathcal{N}(\beta, \mu_1) \subset R^U_\delta$. Similarly if for some μ_0, $y(\mu_0) \in R^L_{\delta/2}$, then $\mathcal{N}(\beta, \mu_0) \subset R^L_\delta$.*

Proof. Suppose that for some μ_1, $y(\mu_1) \in R^U_{\delta/2}$, i.e., $s_1 \le \frac{\delta}{2}$, $s_2 \le \frac{\delta}{2}, \dots, s_{m-1} \le \frac{\delta}{2}, \bar{s}_m \le \frac{\delta}{2}$. Let $y \in \mathcal{N}(\beta, \mu_1)$. Then, for $w := \frac{xs}{\mu_1}$, we have $\|w - e\| \le \beta$. Since for all the slacks in $\mathcal{KM}(m, \rho(m))$, we have $s_i \le 1$ or $\bar{s}_i \le 1$, Lemma 4.1 implies that $s_i(w) \le s_i + \frac{\delta}{2}$ and $\bar{s}_i(w) \le \bar{s}_i + \frac{\delta}{2}$. Then whenever $s_i \le \frac{\delta}{2}$ or $\bar{s}_i \le \frac{\delta}{2}$, we have $\bar{s}_i(w) \le \delta$ and $s_i(w) \le \delta$. This proves $y \in R^U_\delta$, which implies $\mathcal{N}(\beta, \mu_1) \subset R^U_\delta$. The proof of the rest of the claim is similar. □

In the rest of this section, we aim to find an interval $[\mu_0, \mu_1]$ and an upper bound for $\log(\frac{\mu_1}{\mu_0})$ such that the neighborhoods $\mathcal{N}(\beta, \mu_1) \subset R^U_\delta$ and $\mathcal{N}(\beta, \mu_0) \subset R^L_\delta$ for some δ and β.

Let $\delta \le \frac{1}{4(m+1)}$ and $(y_1(\mu_1), \dots, y_m(\mu_1))$ be a central path $\mathcal{CP}(m)$ point such that $s_1 = \frac{\delta}{2}$, $s_2 \le \frac{\delta}{2}, \dots, \bar{s}_m \le \frac{\delta}{2}$. Note that any point satisfying $s_1 = \frac{\delta}{2}$, $s_2 \le \frac{\delta}{2}, \dots, \bar{s}_m \le \frac{\delta}{2}$ is inside the $\frac{\delta}{2}$-neighborhood of the vertex point $(0, 0, \dots, 0, 1)$, hence Proposition 3.1 guarantees the existence of a central path point $(y_1(\mu_1), \dots, y_m(\mu_1))$. Then, by using Theorem 3.7 in [9], one can show that $\mu_1 \le \frac{\rho^{m-1}\delta}{2}$. Let us fix $\mu_1 = \frac{\rho^{m-1}\delta}{2}$ and let $\beta := \frac{\delta}{4}$. Then Lemma 4.3 implies that the neighborhood $\mathcal{N}(\beta, \mu_1)$ stays inside the region R^U_δ. Hence any point inside the neighborhood $\mathcal{N}(\beta, \mu_1)$ also stays inside the region R^U_δ.

Next, we will find a μ_0 such that the neighborhood $\mathcal{N}(\beta, \mu_0)$ is within the region R^L_δ. Let $(y_1(\mu_0), \dots, y_m(\mu_0))$ be the central path point such that $y_m(\mu_0) = \frac{\rho^{m-1}\delta}{2}$. Note that since the objective function in (5) is $-y_m$, a central point satisfying $y_m(\mu) = \frac{\rho^{m-1}\delta}{2}$ exists and is unique. Since from (5), we have $\rho y_i \le y_{i+1}$ for $i = 1, \dots, (m-1)$, we obtain $y_1(\mu) \le \frac{\delta}{2}$, $y_2(\mu) \le \frac{\delta}{2}, \dots, y_m(\mu) \le \frac{\delta}{2}$, which in turn implies that $s_1(\mu) \le \frac{\delta}{2}$, $s_2(\mu) \le \frac{\delta}{2}, \dots, s_m(\mu) \le \frac{\delta}{2}$. Then, using Lemma 4.3 once again, we conclude that the neighborhood $\mathcal{N}(\beta, \mu_0)$ stays inside the region R^L_δ for $\beta = \frac{\delta}{4}$. For the central path (2), the duality gap $c^T x(\mu) - b^T y(\mu) = n\mu$. It is well-known (see e.g.,[12]) that $b^T y(\mu)$ is monotonically increasing and $c^T x(\mu)$ is monotonically decreasing along the central path. In our case, $b^T y(\mu) = -y_m(\mu)$ is increasing to 0 and $c^T x(\mu)$ is monotonically decreasing to 0, i.e., $c^T x(\mu) > 0$ for all $\mu > 0$. Then $n\mu = c^T x(\mu) - b^T y(\mu) >$

y_m implies that $\mu > \frac{y_m}{n}$ for any point on the central path. Hence for the central path point for which $y_m(\mu) = \frac{\rho^{m-1}\delta}{2}$, it follows that $\mu_0 > \frac{\rho^{m-1}\delta}{2n}$. Then using the fact that $n = \mathcal{O}(m2^{2m})$, we have $\log(\frac{\mu_1}{\mu_0}) = \mathcal{O}(m)$. The following corollary summarizes our findings.

Corollary 4.4. *Let the neighborhood parameters be given as $\beta_0 < \beta_1 = \frac{1}{16(m+1)}$ for the MTY predictor-corrector algorithm. Then there exists a $\mathcal{KM}(m, \rho(m))$ with $n = \mathcal{O}(m2^{2m})$ for which MTY predictor-corrector algorithm requires at least $\Omega(2^{m-1})$ predictor steps to reduce the barrier parameter from μ_1 to μ_0 where $\log(\frac{\mu_1}{\mu_0}) = \mathcal{O}(m)$.*

Proof. Let $\delta := \frac{1}{4(m+1)}$ and $\beta_1 = \frac{\delta}{4} = \frac{1}{16(m+1)}$. We know from Lemma 4.2 that, there exists a $\mathcal{KM}(m, \rho(m))$ with $n = \mathcal{O}(m2^{2m})$ such that $\mathcal{N}(\beta) \subset \mathcal{T}_\delta(m)$. Lemma 4.3 shows that there is an interval $[\mu_0, \mu_1]$ such that the neighborhoods $\mathcal{N}(\beta, \mu_1) \subset R_\delta^U$ and $\mathcal{N}(\beta, \mu_0) \subset R_\delta^L$. Hence starting from an iterate (x^1, y^1, s^1) and μ_1 such that $(x^1, y^1, s^1) \in \mathcal{N}(\beta, \mu_1) \subset R_\delta^U$, in order to reach an iterate (x^0, y^0, s^0) and μ_0 such that $(x^0, y^0, s^0) \in \mathcal{N}(\beta, \mu_0) \subset R_\delta^L$; Proposition 3.2 and Proposition 4.2 imply that one needs $\Omega(2^{m-1})$ steps. Since the number of corrector steps is constant, it follows that the number of predictor steps is $\Omega(2^{m-1})$. Moreover the discussion after Lemma 4.3 proves that, we can choose the interval $[\mu_0, \mu_1]$ so that $\log(\frac{\mu_1}{\mu_0}) = \mathcal{O}(m)$. This completes the proof. □

5 A worst-case iteration-complexity lower bound for the Sonnevend curvature

In Section 4, we proved that the MTY predictor-corrector algorithm requires $\Omega(2^{m-1})$ iterations using the larger neighborhood $\mathcal{N}(\beta_1)$ with $\beta_1 = \Omega(\frac{1}{m+1})$. Our goal, in this section, is to derive a lower bound for the Sonnevend curvature using the tools from the previous section. To this end, we need to examine the constants in Theorem 2.1 more closely.

Lemma 5.1. *Let β_1 be the large neighborhood constant so that $\beta_1 \leq \dfrac{1}{400}$ and N be the number of iterations of the MTY predictor-corrector algorithm to reduce the barrier parameter from μ_1 to μ_0. Then*

$$N \leq \frac{4\sqrt{2}}{\sqrt{\beta_1}} \int_{\mu_0}^{\mu_1} \frac{\kappa(\mu)}{\mu} d\mu + \frac{1}{2\log(1 + \frac{\sqrt{\beta_1}}{4})} \log\left(\frac{\mu_1}{\mu_0}\right). \tag{10}$$

Proof. See Theorem 2.4 and its proof in [14]. □

The next theorem shows that on the interval $[\mu_0, \mu_1]$, the total Sonnevend curvature is in comparable order to the number of sharp turns of the central path.

Theorem 5.2. *There is an integer $m_0 > 0$ such that for any $m \geq m_0$, there exists a $\mathcal{KM}(m, \rho(m))$ and interval $[\mu_0, \mu_1]$ such that the Sonnevend curvature satisfies*

$$\int_{\mu_0}^{\mu_1} \frac{\kappa(\mu)}{\mu} d\mu \geq \left(\frac{\sqrt{n}}{\sqrt{\log n}\sqrt{\log(n+1)}} - \frac{8\sqrt{\log n + 1}}{\log(2)} \right) \log\left(\frac{\mu_1}{\mu_0}\right).$$

Proof. Let $\beta_1 = \frac{1}{16(m+1)}$ and choose the parameters of $\mathcal{KM}(m, \rho(m))$ as $\rho = \frac{m}{2(m+1)}$ and $\delta = \frac{1}{8(m+1)}$ so that $n = \mathcal{O}(m2^{2m})$. Write $n = \tau m 2^{2m}$ for some constant $\tau > 0$ and we calculate $\log\left(\frac{\mu_1}{\mu_0}\right) = \log n = \log \tau + \log m + 2m$. This shows that for large enough m, $\log\left(\frac{\mu_1}{\mu_0}\right) = \mathcal{O}(m)$. Since we can extend the interval $[\mu_0, \mu_1]$ so that it still includes all the sharp turns, we will assume that $\log\left(\frac{\mu_1}{\mu_0}\right) = \Theta(m)$. Then Corollary 4.4 applies and we have $N \geq 2^{m-1}$. Now using the bound $\log(1+\omega) \geq (\log 2)\omega$ for $0 \leq \omega \leq 1$, we get from (10)

$$\frac{1}{2\log(1+\frac{\sqrt{\beta_1}}{4})} \leq \frac{8\sqrt{m+1}}{\log 2}.$$

Using the fact that $m \leq \log n$, a straightforward calculation shows that

$$\frac{\int_{\mu_0}^{\mu_1} \frac{\kappa(\mu)}{\mu} d\mu}{\log\left(\frac{\mu_1}{\mu_0}\right)} = \Omega\left(\frac{\sqrt{n}}{\sqrt{\log n}\sqrt{\log(n+1)}} - \frac{8\sqrt{\log n + 1}}{\log(2)} \right). \tag{11}$$

The proof is complete. ☐

Corollary 5.3. *For any $\epsilon > 0$, there is an integer $m_0 > 0$ such that for any $m \geq m_0$, there exists a $\mathcal{KM}(m, \rho(m))$ and interval $[\mu_0, \mu_1]$ such that $\int_{\mu_0}^{\mu_1} \frac{\kappa(\mu)}{\mu} d\mu \geq n^{(\frac{1}{2}-\epsilon)} \log\left(\frac{\mu_1}{\mu_0}\right)$, where $\log\left(\frac{\mu_1}{\mu_0}\right) = \mathcal{O}(m)$.*

Proof. The claim follows from Theorem 5.2 for large m. ☐

Remark 5.4. *Corollary 5.3 yields a negative answer to the question raised by [17], i.e., whether there exists an $\alpha_1 < \frac{1}{2}$ with $\log\left(\frac{\mu_1}{\mu_0}\right) = \Omega(1)$ such that $\int_{\mu_0}^{\mu_1} \frac{\kappa(\mu)}{\mu} d\mu \leq n^{\alpha_1} \log\left(\frac{\mu_1}{\mu_0}\right)$ for the class of LO problems.*

6 An iteration-complexity lower bound for MTY predictor-corrector algorithm with constant neighborhood opening

In practice, the MTY predictor-corrector algorithm operates in a larger neighborhood where β_1 is a constant. In order to conclude an iteration-complexity lower bound for MTY predictor-corrector algorithm with constant neighborhood opening β_1 by using Theorem 2.1, we need to show that there is a constant $\nu > 0$ with $\kappa(\mu) \geq \nu$ for $\mu \in [\mu_0, \mu_1]$ for $\mathcal{KM}(m, \rho(m))$. While this appears to hold numerically, proving it is much more difficult. To go around this difficulty, we exploit a trick introduced by [13]. The idea is to use one dimensional LO problems, where it is easier to calculate the central path and its corresponding $\kappa(\mu)$; and to use LO problems with scaled objectives with block diagonal constraints. For the details, we refer the reader to Appendix section 8.

Recall that by Corollary 5.3, we know there exists a $\mathcal{KM}(m, \rho(m))$ and an interval $[\mu_0, \mu_1]$ such that $\displaystyle\int_{\mu_0}^{\mu_1} \frac{\kappa(\mu)}{\mu} d\mu \geq n^{\left(\frac{1}{2}-\epsilon\right)} \log\left(\frac{\mu_1}{\mu_0}\right)$. Here $n = \mathcal{O}(m2^{2m})$ and $\frac{\mu_1}{\mu_0} = \mathcal{O}(\log n)$. Now by using Lemma 8.4 and Proposition 8.2, we can embed $\mathcal{KM}(m, \rho(m))$ in a block diagonal LO problem at the expense of increasing the size of the problem by at most $\overline{n} := n + \mathcal{O}(m + \log m)$. Denote by $\overline{\mathcal{KM}}(\overline{m})$ this hybrid construction with $\mathcal{KM}(m, \rho(m))$ embedded in. Since $\overline{n} = \mathcal{O}(n)$, we have the following:

Theorem 6.1. *For any $\epsilon > 0$, there exists a positive integer m_0 such that for any $\overline{m} \geq m_0$, there exists an LO problem $\overline{\mathcal{KM}}(\overline{m})$ and an interval $[\mu_0, \mu_1]$ with the following properties:*

- $\displaystyle\frac{\mu_1}{\mu_0} = \mathcal{O}\left(m2^{2m}\right)$.
- *Let $\beta_0 < \beta_1 \leq \frac{1}{400}$ be the constant neighborhood $\mathcal{N}(\beta)$ parameters. Then the MTY predictor-corrector algorithm on this neighborhood requires $\Omega\left(n^{\left(\frac{1}{2}-\epsilon\right)} \log\left(\frac{\mu_1}{\mu_0}\right)\right)$ predictor steps.*

Proof. Consider the $\mathcal{KM}(m, \rho(m))$ cube from Corollary 5.3. Then by using Lemma 8.4 and Proposition 8.2, we can embed $\mathcal{KM}(m, \rho(m))$ in a block diagonal LO problem with size $\overline{n} := n + \mathcal{O}(m + \log m)$ and $\overline{m} = \mathcal{O}(m)$. Note that since the interval $[\mu_0, \mu_1]$ comes from $\mathcal{KM}(m, \rho(m))$, the first claim in the theorem follows from Corollary 5.3. Also, since for $\overline{\mathcal{KM}}(\overline{m})$, there exists a constant $\nu > 0$ for all $\mu \in [\mu_0, \mu_1]$ with the corresponding $\overline{\kappa}(\mu) \geq \nu$, Theorem 2.1 implies the first claim. This completes the proof. □

7 Conclusion and future work

It is an open question whether there is an interior-point algorithm for LO problems with $\mathcal{O}(n^{\alpha_1} \log(\frac{\mu_1}{\mu_0}))$ iteration-complexity upper bound for $\alpha_1 < \frac{1}{2}$ to reduce the barrier parameter from μ_1 to μ_0. In this regard, a related open question raised by Stoer et al. [14], was whether there is an $\alpha_1 < \frac{1}{2}$ with $\displaystyle\int_{\mu_0}^{\mu_1} \frac{\kappa(\mu)}{\mu} d\mu \leq n^{\alpha_1} \log\left(\frac{\mu_1}{\mu_0}\right)$ for all LO problems. This paper provides a negative answer to the latter question. We also show that for the MTY predictor-corrector algorithm, the classical iteration-complexity upper bound is tight. Future work would be to investigate whether an analogous result could be derived to the case of long step IPMs.

In this paper we establish that for the central path of the carefully constructed redundant Klee-Minty cubes, both the geometric curvature and the Sonnevend curvature of the central path are essentially in the order of $\Omega(\sqrt{n})$. In a recent work, Mut and Terlaky [7] show the existence of another class of LO problems where a large geometric curvature of the central path implies a large Sonnevend curvature. These two important cases suggest that it might be possible to prove this implication in a more general setting.

8 Appendix

Lemma 8.1. *For large enough r, there is 1-dimensional LO problem with $(r+1)$ constraints for which $\tau_1\sqrt{r} \leq \kappa(\mu) \leq \tau_2\sqrt{r}$ for any $\mu \in [\alpha_1, \alpha_2]$, where $\alpha_1 = \frac{1}{r-\frac{\sqrt{r}}{4}}$ and $\alpha_2 = \frac{1}{r-\sqrt{r}}$ for some constants $\tau_1, \tau_2 \geq 0$.*

Proof. Consider the problem $\min\{\ y:\ y \leq 1$ and, $y \geq 0$ counted r times$\}$. The construction is given in [13], p:551. Consider the interval $[\alpha_1, \alpha_2]$, where $\alpha_1 = \frac{1}{r-\frac{\sqrt{r}}{4}}$ and $\alpha_2 = \frac{1}{r-\sqrt{r}}$. Let $s_0(\mu) = 1 - y(\mu)$. Then it is shown in [13], p:551 that, $\frac{\dot{s}_0(\mu)}{s_0(\mu)} \geq \frac{r^2}{3\sqrt{r}}$ on $[\alpha_1, \alpha_2]$. This implies $\frac{\mu\dot{s}_0(\mu)}{s_0(\mu)} = \Omega(\sqrt{r})$ on $[\alpha_1, \alpha_2]$. Then, from Proposition 2.2 part *1.*, we have $\kappa(\mu) = \Omega(\sqrt{r})$ for all $\mu \in [\alpha_1, \alpha_2]$. The proof is complete. □

Proposition 8.2. *Consider the LO problems*

$$\begin{array}{lll} \min & (c^1)^T x & \\ \text{s.t.} & A^1 x^1 & = b^1 \\ & x^1 & \geq 0, \end{array} \qquad \textit{and} \qquad \begin{array}{lll} \min & (c^2)^T x & \\ \text{s.t.} & A^2 x^2 & = b^2 \\ & x^2 & \geq 0, \end{array} \tag{12}$$

with the corresponding $\kappa^1(\mu)$ and $\kappa^2(\mu)$ on the interval $[\mu_0, \mu_1]$. Then for the problem

$$\begin{array}{lll} \min & c^T x & \\ \text{s.t.} & Ax & = b \\ & x & \geq 0, \end{array} \tag{13}$$

with the corresponding $\overline{\kappa}(\mu)$ where $c = \begin{bmatrix} c^1 \\ c^2 \end{bmatrix}$, $b = \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$ and $A = \begin{bmatrix} A^1 & 0 \\ 0 & A^2 \end{bmatrix}$, on $[\mu_0, \mu_1]$, we have $\overline{\kappa}(\mu) \geq \kappa^i(\mu)$ for $i = 1, 2$.

Proof. Let $\big(x^1(\mu), y^1(\mu), s^1(\mu)\big)$ and $\big(x^2(\mu), y^2(\mu), s^2(\mu)\big)$ be the central paths in (12). Then the term $\overline{\kappa}(\mu)$ for the combined problem (13) becomes $\overline{\kappa}(\mu) = \big|\big|[\mu\dot{x}^1\dot{s}^1, \mu\dot{x}^2\dot{s}^2]\big|\big|^{\frac{1}{2}} \geq \kappa^i(\mu)$ for $i = 1, 2$. □

Proposition 8.3. *Let $\eta > 0$ and consider the central path (2) and its $\kappa(\mu)$. Let $(\hat{A}, \hat{b}, \hat{c})$ be another problem instance, where $(\hat{A}, \hat{b}, \hat{c}) = (A, \frac{b}{\eta}, c)$ with its corresponding $\hat{\kappa}(\mu)$. Then, we have*

$$\hat{\kappa}(\mu) = \kappa(\eta\mu), \;\; \mu \in \left[\frac{\mu_0}{\eta}, \frac{\mu_1}{\eta}\right]. \tag{14}$$

Proof. Using (2), it is straightforward to verify that the central path $(\hat{x}(\mu), \hat{y}(\mu), \hat{s}(\mu))$ of the new problem satisfies $\hat{x}(\mu) = \dfrac{x(\eta\mu)}{\eta}$, $\hat{y}(\mu) = y(\eta\mu)$ and $\hat{s}(\mu) = s(\eta\mu)$. Using the definition of $\kappa(\mu)$, we get $\hat{\kappa}(\mu) = \kappa(\eta\mu)$. Hence the claim follows. □

Lemma 8.4. *Given an interval $[\mu_0, \mu_1]$ and a constant $\nu > 0$, there exists an LO problem of size $n = \Theta\left(\log(\frac{\mu_1}{\mu_0})\right)$ such that $\overline{\kappa}(\mu) \geq \nu$ for all $\mu \in [\mu_0, \mu_1]$. The hidden constant in $n = \Theta\left(\log(\frac{\mu_1}{\mu_0})\right)$ depends on ν.*

Proof. Let a constant $\nu > 0$ and an interval $[\mu_0, \mu_1]$ be given. For the given $\nu > 0$, by Lemma 8.1, there exists an LO problem with its $\kappa(\mu) \geq \nu$ on an interval $\mu \in [\alpha_1, \alpha_2]$. By applying Proposition 8.3 for $\eta := \frac{\alpha_1}{\left(\frac{\alpha_2}{\alpha_1}\right)^i \mu_0}$ for $i = 0, 1, \ldots, k$, we find $(k-1)$ scaled LO problems with their corresponding $\kappa^i(\mu)$, $i = 0, 1, \ldots, k-1$ such that $\kappa^i(\mu) = \kappa(\eta\mu)$ on $\mu \in \left[(\frac{\alpha_2}{\alpha_1})^i\mu_0, (\frac{\alpha_2}{\alpha_1})^{i+1}\mu_0\right]$, for $i = 0, 1, \ldots, k-1$. Then by using Proposition 8.2, we can obtain a block diagonal LO problem with its $\overline{\kappa}(\mu) \geq \kappa^i(\mu) \geq \nu$ for $i = 0, 1, \ldots, k-1$ for any $\mu \in \left[\mu_0, \left(\frac{\alpha_2}{\alpha_1}\right)^k \mu_0\right]$. In order to have $\overline{\kappa}(\mu) \geq \nu$ for any $\mu \in [\mu_0, \mu_1]$, it is then enough to have $\left(\frac{\alpha_2}{\alpha_1}\right)^k \mu_0 \geq \mu_1$. This is true if and only if $k \log\left(\frac{\alpha_2}{\alpha_1}\right) \geq \log\left(\frac{\mu_1}{\mu_0}\right)$. Since by Lemma 8.1, the ratio $\frac{\alpha_2}{\alpha_1}$ is a constant depending only on the given ν, the number of blocks k needed is $\Theta\left(\log(\frac{\alpha_2}{\alpha_1})\right)$. Also since the size of the LO problem with its $\kappa(\mu)$ is a constant only determined by ν, the size of the problem is $n = \Theta\left(\log(\frac{\mu_1}{\mu_0})\right)$ to achieve $\overline{\kappa}(\mu) \geq \nu$ for all $\mu \in [\mu_0, \mu_1]$. This completes the proof. □

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