DIFFERENTIAL PROPERTIES OF EUCLIDEAN PROJECTION ONTO POWER CONE

LE THI KHANH HIEN

Abstract. In this paper, we study differential properties of Euclidean projection onto the power cone $K_n^{(p,q)} = \{(x, y, z) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n, \|z\| \le x^p y^q\}$, where $0 < p, q < 1, p + q = 1$. Projections onto certain power cones are examples of semismooth but non-strongly-semismooth projection onto a convex cone.

1. Introduction

Up until 2000, conic optimization likely means optimization over symmetric cones, which are self-dual and homogeneous. Many authors recently have attention to nonsymmetric cones, such as homogeneous cone [7], matrix norm cone [11], doubly nonnegative cone [17]. In this paper, we consider the following power cone

$$K_n^{(p,q)} = \{(x, y, z) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n, \|z\| \le x^p y^q\}$$

where $0 < p, q < 1, p + q = 1$. The cone $K_n^{p,q}$ is self-dual but not homogeneous in general. When $p = q = \frac{1}{2}$ the power cone is the rotated second order cone, which have a broad range of applications, see [1, 16]. The 3-dimensional power cones $K^{(p,q)} = \{(x, y, z), x, y \in \mathbb{R}_+, z \in \mathbb{R}, |z| \le x^p y^q\}$ recently have been employed in reformulation many convex optimization problems to conic form, see [4, 5].

While interior point methods [18] solve a convex optimization problem directly without using information about the Euclidean projection, many other methods approach the problem indirectly and need differential properties of the projection for updating iterates or studying convergence rate. We refer the readers to [6, 8, 13, 15, 21] for a look at these algorithms and the importance of the projection. To that end, we study differential properties of projection onto the power cone $K_n^{p,q}$. We hope that they would be useful in using algorithms other than interior point methods for conic optimization problems involving power cones. The projection onto power cone is semismooth but not always strongly semismooth. To the best knowledge of the author, this is the first example of semismooth but non-strongly-semismooth projection onto a convex cone so far. Semismoothness but non-strongly semismoothness of the projection leads to the fact that some smoothing Newton methods may achieve locally superlinear convergence but may not achieve quadratic convergence.

The paper is organized as follows. We give some preliminaries in Section 2. Directional derivative formula, first order Fréchet derivative formula and strongly semismoothness are proved in Section 3.

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1.1. **Notations.** For $x \in \mathbb{R}$, we denote $[x]_+ = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$. We use $\mathrm{Diag}(y)$ to denote a $n \times n$ diagonal matrix with its diagonal being the vector $y \in \mathbb{R}^n$. For a Fréchet differentiable map $F : \mathbb{R}^n \to \mathbb{R}^m$ we use $\mathbf{J}F$ to denote its derivative. For mappings $g_1(x), g_2(x)$, we write $g_1(x) = O(g_2(x))$ if $\limsup_{g_2(x) \to 0} \frac{g_1(x)}{g_2(x)} < \infty$. We write $g(x) = O(1)$ if $g(x)$ is bounded. For $t \in \mathbb{R}$, we denote $s(t) = \begin{cases} 1 & \text{if } \ t > 0 \\ 0 & \text{if } \ t \leq 0 \end{cases}$.

2. Preliminaries

In this section, we give some basic definitions and propositions necessary for the next section.

2.1. **Strong semismoothness.**

Definition 2.1. (see [21]) A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is said to be semismooth at x if it is locally Lipschitz at x and the limit

$$\lim_{\substack{V \in \partial F(x+th') \\ h' \to h, \ t \downarrow 0}} Vh'$$

exists for any $h \in \mathbb{R}^n$, where $\partial F(x + th')$ is the Clarke generalized Jacobian of F at $x + th'$ [10, §2.6].

It was proved in [21, Theorem 2.3] that a locally Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^m$ is semismooth at x if and only if

$$\lim_{x+h \in D_F, h \to 0} \frac{F'(x+h;h) - F'(x;h)}{\|h\|} = 0,$$

where $F'(x;h)$ denotes the directional derivative

$$F'(x;h) = \lim_{t \downarrow 0} \frac{F(x+th) - F(x)}{t}$$

and D_F denotes the set of Fréchet-differentiable points of F. Together with [12, Proposition 3.1.3], it follows that a locally Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^m$ is semismooth at x if and only if

$$\lim_{x+h \in D_F, h \to 0} \frac{\|F(x+h) - F(x) - F'(x+h;h)\|}{\|h\|} = 0.$$

From this property, a strongly semimooth function is defined in a similar manner.

Definition 2.2. A locally Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^m$ is said to be strongly semismooth at x if

$$\limsup_{x+h \in D_F, h \to 0} \frac{\|F(x+h) - F(x) - F'(x+h;h)\|}{\|h\|^2} < \infty.$$

Proposition 2.1. *If a locally Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^m$ is directional differentiable and satisfies*

$$\limsup_{x+h\in D_F, h\to 0} \frac{\|F'(x+h;h) - F'(x;h)\|}{\|h\|^2} < \infty \tag{1}$$

then F is strongly semismooth at x.

See [13, Theorem 7.4.3] for the proof.

The expression (1) is used to define strong semismoothness in some references, see e.g., [3], [13, Definition 7.4.2], [19, Definition 4]. For the purpose of studying locally quadratically convergent rate of many smoothing Newton algorithms, see e.g.,[6, 8, 14, 20], definition 2.2 is used in proving strongly semismooth property of Euclidean projection. Recently, a locally Lipschitz definable function is shown to be semismooth by Bolte et al [3]. Projection onto the power cone $K_n^{(p,q)}$ is locally Lipschitz definable (see [8, Part 2.3, example 3.5]), hence is semismooth. Nevertheless, it is not strongly semismooth in some cases.

2.2. **Power cone and its Euclidean projection.**

Definition 2.3. For a given closed convex cone K, its dual cone is defined as

$$K^\sharp = \{s \in \mathbb{R}^n : \langle s, x\rangle \geq 0 \;\forall x \in K\},$$

and its polar cone is $K^o = -K^\sharp$.

Proposition 2.2. *The dual cone of $K_n^{(p,q)}$ is also a power cone*

$$K^\sharp = \left\{(u,v,w) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n : \left(\frac{u}{p}\right)^p \left(\frac{v}{q}\right)^q \geq \|w\|\right\}.$$

The proof is totally similar to [4, Theorem 4.3.1]. We omit the details here.

We use Π_K to denote the Euclidean projection onto a closed convex cone K, i.e.,

$$\Pi_K(z) = \arg\min_{x\in K} \frac{1}{2}\|x - z\|^2. \tag{2}$$

A function $f : \mathrm{int}(K) \to \mathbb{R}$ is said to be a barrier of K if $f(x_k) \to \infty$ for any sequence $\{x_k\} \subset \mathrm{int}(K)$ converging to a point of the boundary of K. For a given barrier f of K and $\mu > 0$, we consider the corresponding barrier problem of the convex minimization problem (2)

$$\min\left\{\frac{1}{2}\|x - z\|^2 + \mu^2 f(x) : x \in \mathrm{int}(K)\right\}.$$

We denote $p_\mu(z)$ to be the solution of this barrier problem. By [9, Proposition 3.1], $p_\mu(z)$ is the unique solution of the following equation

$$p_\mu(z) + \mu^2 \nabla f(p_\mu(z)) = z. \tag{3}$$

When we take μ to 0, every limit point of $p_\mu(z)$ is the unique minimizer $\Pi_K(z)$ of (2), see [2, Proposition 4.1.1], [8, Theorem 3.1]. We now use the barrier

$$f(x,y,z) = -\log\left(x^{2p}y^{2q} - \|z\|^2\right) \tag{4}$$

of $K_n^{(p,q)}$ to find its projection. For simplicity, we write K for $K_n^{(p,q)}$.

Proposition 2.3. *Let $(x^o, y^o, z^o) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$ be a given point and $(\overline{x}, \overline{y}, \overline{z})$ be its projection onto K. Denote*

$$\Phi(x^o, y^o, z^o, r) = \tfrac{1}{2}\left(x^o + \sqrt{(x^o)^2 + 4pr(\|z^o\| - r)}\right)^p \left(y^o + \sqrt{(y^o)^2 + 4qr(\|z^o\| - r)}\right)^q - r.$$

(i) *If $(x^o, y^o, z^o) \notin K, K^o$ and $z^o \neq 0$ then its projection onto the power cone K is*

$$\begin{cases} \overline{x} = \dfrac{1}{2}\left(x^o + \sqrt{(x^o)^2 + 4pr(\|z^o\| - r)}\right), \\ \overline{y} = \dfrac{1}{2}\left(y^o + \sqrt{(y^o)^2 + 4qr(\|z^o\| - r)}\right), \\ \overline{z}_l = z_l^o \dfrac{r}{\|z^o\|}, \quad l = 1, \dots, n, \end{cases} \tag{5}$$

where $r = r(x^o, y^o, z^o)$ is the unique solution of the following system:

$$E(x^o, y^o, z^o) : \begin{cases} \Phi(x^o, y^o, z^o, r) = 0 \\ 0 < r < \|z^o\| \end{cases} \tag{6}$$

(ii) *If $(x^o, y^o, z^o) \notin K, K^o$ and $z^o = 0$ then its projection onto K is*

$$\begin{cases} \overline{x} = [x^o]_+ \\ \overline{y} = [y^o]_+ \\ \overline{z} = 0. \end{cases}$$

(iii) *If $(x^o, y^o, z^o) \in K$ then $(\overline{x}, \overline{y}, \overline{z}) = (x^o, y^o, z^o)$ and if $(x^o, y^o, z^o) \in K^o$ then $(\overline{x}, \overline{y}, \overline{z}) = 0$.*

Proof. Denote $M = x^{2p}y^{2q} - \|z\|^2, A = x^{2p}y^{2q}$. Gradient of the barrier (4) is

$$\nabla f(x, y, z) = \left(-2px^{-1}AM^{-1}, -2qy^{-1}AM^{-1}\ ,\ 2M^{-1}z_l\big|_{l=1,\dots,n}\right).$$

The equation (3) for the power cone with $p_\mu(x^o, y^o, z^o) = (x, y, z)$ now becomes

$$\begin{cases} x - \mu^2(2px^{-1}AM^{-1}) = x^o \\ y - \mu^2(2qy^{-1}AM^{-1}) = y^o \\ z_l + \mu^2 2M^{-1}z_l = z_l^o \quad (l = 1, \dots, n) \\ x > 0, y > 0, M > 0 \end{cases} \tag{7}$$

The third equation of (7) yields $z_l = 0$ if $z_l^o = 0$, and $\dfrac{z_l^o}{z_l} = 1 + 2\mu^2 M^{-1}$ if $z_l^o \neq 0$.

(i) If $z^o \neq 0$ then there exists $z_l^o \neq 0$; thus $\|z\| > 0$ and

$$1 + 2\mu^2 M^{-1} = \frac{z_l^o}{z_l} = \frac{\|z^o\|}{\|z\|} \text{ when } z_l^o \neq 0. \tag{8}$$

Hence $AM^{-1} = (M + \|z\|^2)M^{-1} = 1 + \|z\|^2 M^{-1} = 1 + \dfrac{1}{2\mu^2}\|z\|(\|z^o\| - \|z\|)$. The first equation of (7) now is equivalent to $x^2 - x^o x - 2\mu^2 p - p\|z\|(\|z^o\| - \|z\|) = 0$, which has the following positive solution

$$x = \frac{1}{2}\left(x^o + \sqrt{(x^o)^2 + 4p\|z\|(\|z^o\| - \|z\|) + 8p\mu^2}\right). \tag{9}$$

Similarly,

$$y=\frac{1}{2}\left(y^o+\sqrt{(y^o)^2+4q\,\|z\|\,(\|z^o\|-\|z\|)+8q\mu^2}\right). \tag{10}$$

Taking μ to 0, $p_\mu(x^o,y^o,z^o)=(x,y,z)$ converges to the projection $(\overline{x},\overline{y},\overline{z})$. From (8), (9) and (10) we get

$$\begin{cases}\overline{x}=\frac{1}{2}\left(x^o+\sqrt{(x^o)^2+4p\,\|\overline{z}\|\,(\|z^o\|-\|\overline{z}\|)}\right)\\ \overline{y}=\frac{1}{2}\left(y^o+\sqrt{(y^o)^2+4q\,\|\overline{z}\|\,(\|z^o\|-\|\overline{z}\|)}\right)\\ \overline{z}_l=z_l^o\,\frac{\|\overline{z}\|}{\|z^o\|}, l=1,\dots,n\end{cases}$$

and $\|\overline{z}\|$ is a solution of the first equation of the system (6) since $(\overline{x},\overline{y},\overline{z})$ belongs to the boundary of K. We can prove that $0<\|\overline{z}\|<\|z^o\|$; for simplicity we leave the proof of this fact in an appendix, see Proposition A.1.

Now we prove the system (6) has unique solution. Denote
$g_p(r)=\sqrt{(x^o)^2+4pr(\|z^o\|-r)}, g_q(r)=\sqrt{(y^o)^2+4qr(\|z^o\|-r)}$,
$f_p(r)=\frac{1}{2}\left(x^o+g_p(r)\right), f_q(r)=\frac{1}{2}\left(y^o+g_q(r)\right)$.
We have

$$\Phi(x^o,y^o,z^o,r)=f_p^p(r)f_q^q(r)-r=0,$$

and

$$\begin{aligned}\Phi'_r &= f_p^pf_q^q\left(p\frac{f'_p}{f_p}+q\frac{f'_q}{f_q}\right)-1\\ \Phi''_{rr} &= \prod_{i\in\{p,q\}}f_i^i\left(\sum_{i\in\{p,q\}}i\frac{f'_i}{f_i}\right)^2+\prod_{i\in\{p,q\}}f_i^i\sum_{i\in\{p,q\}}i\,\frac{f''_if_i-(f'_i)^2}{(f_i)^2}.\end{aligned}$$

By Cauchy-Schwarz inequality,

$$\left(\sum_{i\in\{p,q\}}i\frac{f'_i}{f_i}\right)^2\le\sum_{i\in\{p,q\}}(\sqrt{i})^2\sum_{i\in\{p,q\}}\left(\sqrt{i}\frac{f'_i}{f_i}\right)^2=\sum_{i\in\{p,q\}}i\frac{(f'_i)^2}{(f_i)^2}.$$

Furthermore, for $i\in\{p,q\}$, $f''_i=i\dfrac{-2g_i-2i(\|z\|-2r)^2/g_i}{g_i^2}<0,\ f_i>0$. Hence $\Phi''_{rr}<0$. This implies that Φ is a strictly concave function with respected to r over $0<r<\|z^o\|$. Moreover, $\Phi(x^o,y^o,z^o,0)\ge0$ and $\Phi(x^o,y^o,z^o,\|z^o\|)=[x^o]_+^p[y^o]_+^q-\|z^o\|<0$ (otherwise, $x^o>0,y^o>0,(x^o)^p(y^o)^q\ge\|z^o\|$ then $(x^o,y^o,z^o)\in K$ is a contradiction). Therefore, (6) has unique solution.

(ii) If $z^o=0$ then the third equation of (7) implies $z_l=0,l=1,\dots,n$, which yields that $\overline{z}_l=0, AM^{-1}=1$. Then from the first equation of (7) we get $x=\frac{1}{2}\left(x^o+\sqrt{(x^o)^2+8p\mu^2}\right)$. Totally similarly $y=\frac{1}{2}\left(y^o+\sqrt{(y^o)^2+8q\mu^2}\right)$. We derive the formula for the projection in this case by taking μ to 0.

(iii) This case is trivial. □

3. Differential properties of Euclidean projection onto $K_n^{(p,q)}$

In this section, we reuse the notations
$g_p = \sqrt{x^2 + 4pr(\|z\| - r)}, g_q = \sqrt{y^2 + 4qr(\|z\| - r)},\ f_p = \frac{1}{2}(x + g_p), f_q = \frac{1}{2}(y + g_q)$.

Theorem 3.1. *If $(x, y, z) \notin \mathrm{bd}(K), \mathrm{bd}(K^o)$ then the projection $\Pi_K(x, y, z)$ is continuously differentiable. Furthermore,*

(i) *If $(x, y, z) \notin K, K^o$ and $z \neq 0$, the projection is twice continuously differentiable and its Jacobian is determined by the following rows :*

$$\begin{aligned}
\Pi'_x &= \left(\frac{1}{2} + \frac{x}{2g_p} + \frac{p^2(\|z\| - 2r)rL}{g_p^2}, \frac{pq(\|z\| - 2r)rL}{g_pg_q}, \frac{z_l}{\|z\|}\frac{prL}{g_p}\bigg|_{l=1,\dots,n}\right)\\
\Pi'_y &= \left(\frac{pq(\|z\| - 2r)rL}{g_pg_q}, \frac{1}{2} + \frac{y}{2g_q} + \frac{q^2(\|z\| - 2r)rL}{g_q^2}, \frac{z_l}{\|z\|}\frac{qrL}{g_q}\bigg|_{l=1,\dots,n}\right)\\
\Pi'_{z_l} &= \left(\frac{z_l}{\|z\|}\frac{prL}{g_p}, \frac{z_l}{\|z\|}\frac{qrL}{g_q}, \frac{r}{\|z\|} + \frac{rz_l^2}{\|z\|^3}TL, \frac{rz_lz_k}{\|z\|^3}TL\bigg|_{k\in\{1,\dots,n\}\setminus\{l\}}\right),
\end{aligned} \tag{11}$$

where r is the unique solution of the system $E(x, y, z)$, see (6), and

$$\begin{aligned}
L &= \frac{1}{1 - r(\|z\| - 2r)\left(\frac{p^2}{g_pf_p} + \frac{q^2}{g_qf_q}\right)} = \frac{1}{\frac{1}{2} + \frac{px}{2g_p} + \frac{qy}{2g_q} + \frac{r^2p^2}{g_pf_p} + \frac{r^2q^2}{g_qf_q}}\\
&= \frac{2(\|z\| - r)}{\|z\| + (\|z\| - 2r)\left(\frac{px}{g_p} + \frac{qy}{g_q}\right)}\\
T &= 2r(\|z\| - r)\left(\frac{p^2}{g_pf_p} + \frac{q^2}{g_qf_q}\right) - 1 = -\frac{px}{g_p} - \frac{qy}{g_q}
\end{aligned} \tag{12}$$

(ii) *If $(x, y, z) \notin K, K^o$ and $z = 0$, the Jacobian is the diagonal matrix*

$$\mathbf{D} = \mathrm{Diag}\left(s(x), s(y), \quad d\,|_{l=1,\dots,n}\right),$$

$$d = \begin{cases} 1 & \textit{if } (x - y)(p - q) > 0\\ 0 & \textit{if } (x - y)(p - q) < 0\\ \left(\dfrac{-2x}{y} + 1\right)^{-1} & \textit{if } x < 0, y > 0, p = q = 1/2\\ \left(\dfrac{-2y}{x} + 1\right)^{-1} & \textit{if } x > 0, y < 0, p = q = 1/2. \end{cases} \tag{13}$$

(iii) *$\boldsymbol{J}\Pi_K(x, y, z) = I$ if $(x, y, z) \in \mathrm{int}(K)$ and $\boldsymbol{J}\Pi_K(x, y, z) = 0$ if $(x, y, z) \in \mathrm{int}(K^o)$.*

Before proving this Theorem, we prove the following lemma.

Lemma 3.1. *Let $\{(x_t, y_t, z_t)\}$ be a sequence lying outside K, K^o such that $x_t \neq 0, y_t \neq 0, z_t \neq 0$ and (x_t, y_t, z_t) converges to $(x, y, 0) \notin K, K^o$. Let $0 < r_t < \|z_t\|$ be the unique solution of the boundary equation $\Phi(x_t, y_t, z_t, r_t) = 0$(see (6)). Then*

$$\lim_{(x_t,y_t,z_t)\to(x,y,0)} \frac{r_t}{\|z_t\|} = d, \quad \lim_{(x_t,y_t,z_t)\to(x,y,0)} T = d_1, \ \textit{and} \quad \lim_{(x_t,y_t,z_t)\to(x,y,0)} L = d_2,$$

where d is defined in (13), T, L *are defined in* 12 *(with (x,y,z) being replaced by (x_t,y_t,z_t), and*

$$d_1 = \begin{cases} 2q-1 & \text{if } x>0, y<0 \\ 2p-1 & \text{if } x<0, y>0 \end{cases}, \quad d_2 = \begin{cases} 0 & \text{if } (x-y)(p-q)>0 \\ \text{some nonzero constant} & \text{otherwise} \end{cases}$$

Proof. We assume that $x>0, y<0$, the case $x<0, y>0$ is proved totally similar. We rewrite the equation $\Phi(x_t,y_t,z_t,r)=0$ as

$$\begin{aligned} r_t &= \frac{1}{2}\left(x_t + \sqrt{x_t^2 + 4pr_t(\|z_t\|-r_t)}\right)^p \left(y_t + \sqrt{y_t^2 + 4qr_t(\|z_t\|-r_t)}\right)^q \\ &= \frac{1}{2}\left(x_t + \sqrt{x_t^2 + 4pr_t(\|z_t\|-r_t)}\right)^p \frac{(4qr_t(\|z_t\|-r_t))^q}{\left(\sqrt{y_t^2+4qr_t(\|z_t\|-r_t)}-y_t\right)^q} \end{aligned}$$

This implies

$$r_t^{p-q} = \frac{1}{2}\frac{\left(x_t + \sqrt{x_t^2 + 4pr_t(\|z_t\|-r_t)}\right)^p}{\left(\sqrt{y_t^2+4qr_t(\|z_t\|-r_t)}-y_t\right)^q}(4q)^q\left(\frac{\|z_t\|}{r_t}-1\right)^q.$$

By noting that $r_t \to 0$ since $0<r_t<\|z_t\|$, the left hand side of this equation converges to 0 if $p>q$, to ∞ if $p<q$ and is constant 1 if $p=q$. The limit of $\frac{r_t}{\|z_t\|}$ follows then. Finally, from $g_p \to x>0, g_q \to -y>0, \dfrac{r_t}{\|z_t\|} \to d$ we get the limit of T and L easily. □

Now we prove Theorem 3.1.

Proof. (i) It is not difficult to see that the function $\Phi(x,y,z,r)$ is twice continuously differentiable whenever $0<r<\|z\|$. For a given point (x,y,z,r) such that $(x,y,z)\notin K, K^o, \|z\|>0, 0<r<\|z\|$ and r is the unique solution of the boundary equation $\Phi(x,y,z,r)=0$ (see (6), Proposition 2.3 (i)), the function $\Phi(u,v,w,\rho)$ is a twice continuously differentiable in the following neighbourhood of (x,y,z,r):

$$\begin{aligned} \mathcal{N} = \{(u,v,w,\rho) | (u,v,w) \notin K, K^o, \|(u,v)-(x,y)\|<\varepsilon, \|w-z\|<\delta<\|z\|-r, \\ 0<\rho<\|z\|-\delta\}, \end{aligned}$$

where δ is a constant such that $0<r<\|z\|-\delta$, since $(u,v,w,\rho)\in\mathcal{N}$ implies $0<\rho<\|w\|$. Applying implicit function theorem, we deduce that the function $r(x,y,z)$ such that $\Phi(x,y,z,r(x,y,z))=0$ is twice continuously differentiable in an open neighbourhood of (x,y,z). From formula (5) of Proposition 2.3 (i), the projection $(\overline{x},\overline{y},\overline{z})$ of (x,y,z) on to K, which is determined by

$$\begin{cases} \overline{x} = \frac{1}{2}\left(x + \sqrt{x^2+4pr(x,y,z)(\|z\|-r(x,y,z))}\right), \\ \overline{y} = \frac{1}{2}\left(y + \sqrt{y^2+4qr(x,y,z)(\|z\|-r(x,y,z))}\right), \\ \overline{z}_l = z_l\dfrac{r(x,y,z)}{\|z\|}, \quad l=1,\dots,n, \end{cases}$$

is also twice continuously differentiable. Moreover, by noting that

$$r'_x = -(\Phi'_r)^{-1}\Phi'_x = \frac{r\left(p\frac{(f_p)'_x}{f_p} + q\frac{(f_q)'_x}{f_q}\right)}{1 - r\left(p\frac{(f_p)'_r}{f_p} + q\frac{(f_q)'_r}{f_q}\right)} = \frac{prL}{g_p}, \text{ similarly } r'_y = \frac{qrL}{g_q}$$

and

$$r'_{z_l} = -(\Phi'_r)^{-1}\Phi'_{z_l} = \frac{r\left(p\frac{(f_p)'_{z_l}}{f_p} + q\frac{(f_q)'_{z_l}}{f_q}\right)}{1 - r\left(p\frac{(f_p)'_r}{f_p} + q\frac{(f_q)'_r}{f_q}\right)} = r^2\frac{z_l}{\|z\|}\left(\frac{p^2}{g_p f_p} + \frac{q^2}{g_q f_q}\right)L, \quad l = 1, \ldots, n,$$

where $L = \dfrac{1}{1 - r(\|z\| - 2r)\left(\frac{p^2}{g_p f_p} + \frac{q^2}{g_q f_q}\right)}$, we can get the formula (11) easily.

Now we prove equalities in (12). From $f_p = \frac{1}{2}(x + g_p), g_p^2 - x^2 = 4pr(\|z\| - r)$, we get

$$2r(\|z\| - r)\frac{p^2}{g_p f_p} = 2r(\|z\| - r)\frac{2p^2(g_p - x)}{g_p 4pr(\|z\| - r)} = p - \frac{px}{g_p}.$$

Similarly, $2r(\|z\| - r)\frac{q^2}{g_q f_q} = q - \frac{qy}{g_q}$. Then we get the equality of T. Moreover,

$$\begin{aligned} L^{-1} &= 1 - r(\|z\| - r)\left(\tfrac{p^2}{g_p f_p} + \tfrac{q^2}{g_q f_q}\right) + r^2\left(\tfrac{p^2}{g_p f_p} + \tfrac{q^2}{g_q f_q}\right) \\ &= 1 - \tfrac{T+1}{2} + r^2\left(\tfrac{p^2}{g_p f_p} + \tfrac{q^2}{g_q f_q}\right) \\ &= \tfrac{1}{2} + \tfrac{px}{2g_p} + \tfrac{qy}{2g_q} + \tfrac{r^2p^2}{g_p f_p} + \tfrac{r^2q^2}{g_q f_q} \\ &= \tfrac{1}{2}\left(1 + \tfrac{px}{g_p} + \tfrac{qy}{g_q} + \tfrac{r}{\|z\|-r}\left(1 - \tfrac{px}{g_p} - \tfrac{qy}{g_q}\right)\right), \end{aligned}$$

where the last equality is from

$$\frac{p^2r^2}{f_p} = \frac{2p^2r^2(g_p - x)}{(g_p + x)(g_p - x)} = \frac{pr(g_p - x)}{2g_p(\|z\| - r)}, \text{similarly } \frac{q^2r^2}{f_q} = \frac{qr(g_q - x)}{2g_q(\|z\| - r)}.$$

Then we get the equalities of L.

(ii) We consider case $x > 0, y < 0, z = 0$, the case of $x < 0, y > 0, z = 0$ is proved totally similar. Now we prove $\mathbf{D}$ is the derivative of $\Pi_K(x, y, z)$ at $(x, y, 0)$ by verifying the definition

$$\lim_{\|h\|\to 0} \frac{\Pi_K(x + h_1, y + h_2, h_3) - \Pi_K(x, y, 0) - \mathbf{D}[h]}{\|h\|} = 0, \tag{14}$$

where $h = (h_1, h_2, h_3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$. Denote the nominator of the fraction in (14) by $B = (B_1, B_2, B_3^1, \ldots, B_3^n)$. We consider the following two cases:

Case 1 If $\|h_3\| = 0$ then from Proposition 2.3 (ii)

$$B = \left(\frac{h_1 + \sqrt{(x + h_1)^2} - \sqrt{x^2}}{2} - h_1, \frac{h_2 + \sqrt{(y + h_2)^2} - \sqrt{y^2}}{2}, 0\right) = 0.$$

Case 2 If $\|h_3\| \neq 0$ then using projection's formulas in Proposition 2.3 (i) we get

$$B_1 = \frac{1}{2}\left(h_1 + \sqrt{(x + h_1)^2 + 4pr_h(\|h_3\| - r_h)} - \sqrt{x^2}\right) - h_1,$$

$$B_2 = \frac{1}{2}\left(h_2 + \sqrt{(y + h_2)^2 + 4qr_h(\|h_3\| - r_h)} - \sqrt{y^2}\right),$$

and $B_3^l = h_3^l \frac{r_h}{\|h_3\|} - dh_3^l$ for $l = 1, \ldots, n$, where $0 < r_h < \|h_3\|$ is the unique solution of $\Phi(x + h_1, y + h_2, h_3, r_h) = 0$. Applying Lemma 3.1 we get $\frac{r_h}{\|h_3\|} \to d$. Therefore,

$$\lim_{\|h\| \to 0} \frac{B_3^l}{\|h\|} = \lim_{\|h\| \to 0} \frac{h_3^l}{\|h\|} \left(\frac{r_h}{\|h_3\|} - d \right) = 0.$$

Furthermore,

$$\lim_{\|h\| \to 0} \frac{B_1}{\|h\|} = \lim_{\|h\| \to 0} 2p \frac{r_h}{\|h\|} \frac{\|h_3\| - r_h}{\sqrt{(x + h_1)^2 + 4pr_h(\|h_3\| - r_h)} + (x + h_3)} = 0,$$
$$\lim_{\|h\| \to 0} \frac{B_2}{\|h\|} = \lim_{\|h\| \to 0} 2q \frac{r_h}{\|h\|} \frac{\|h_3\| - r_h}{\sqrt{(y + h_2)^2 + 4qr(\|h_3\| - r_h)} - (y + h_2)} = 0.$$

Hence in Case 2 we also get $\lim_{\|h\| \to 0} \frac{B}{\|h\|} = 0$.

We now prove the projection is continuously differentiable in this situation by proving that $\lim_{(x_t, y_t, z_t) \to (x, y, 0)} \mathbf{J}\Pi(x_t, y_t, z_t) = \mathbf{J}\Pi(x, y, 0)$.

The case $z_t = 0$ is trivial. We consider $z_t \neq 0$. The Jacobian $\mathbf{J}\Pi(x_t, y_t, z_t)$ is given in formulas (11), (12) with x, y, z, r being replaced by x_t, y_t, z_t, r_t.

For simplicity in notation, we write z_l, z_k for the l-th and k-th coordinate of z_t. Now we find limit of $\frac{r_t z_l z_k}{\|z_t\|^3} TL$. We consider 3 cases.

- $p > q$. By Lemma 3.1, $L \to 0, T \to 2q - 1$. Therefore, $\frac{r_t z_l z_k}{\|z_t\|^3} TL \to 0$.
- $p = q = \frac{1}{2}$. In this case, $T \to 2q - 1 = 0$. Hence $\frac{r_t z_l z_k}{\|z_t\|^3} TL \to 0$.
- $p < q$. By Lemma 3.1, $\frac{r_t}{\|z_t\|} \to 0$, which implies $\frac{r_t z_l z_k}{\|z_t\|^3} TL \to 0$.

In all cases, we have $\lim_{(x_t, y_t, z_t) \to (x, y, 0)} \frac{r_t z_l z_k}{\|z_t\|^3} TL = 0$ for $l, k = 1, \ldots, n$. Together with $r_t \to 0, g_p \to x, q_q \to -y$, and Lemma 3.1, it is easily to see that $\lim_{(x_t, y_t, z_t) \to (x, y, 0)} \mathbf{J}\Pi(x_t, y_t, z_t) = \mathbf{J}\Pi(x, y, 0)$. □

Theorem 3.2. *Projection onto the power cone K is directional differentiable everywhere; and*

(i) If $(x, y, z) \in bd(K)$ and $(x, y, z) + t(\mathfrak{h}_1, \mathfrak{h}_2, \kappa) \in K$ for some t then

$$\Pi'((x, y, z); (\mathfrak{h}_1, \mathfrak{h}_2, \kappa)) = (\mathfrak{h}_1, \mathfrak{h}_2, \kappa),$$

else if $(x, y, z) \in bd(K)$ and $z \neq 0$ then

$$\Pi'((x, y, z); (\mathfrak{h}_1, \mathfrak{h}_2, \kappa)) = \left(\mathfrak{h}_1 - \frac{p}{x} c, \mathfrak{h}_2 - \frac{q}{y} c, \kappa_l + \frac{z_l}{\|z\|^2} c \bigg|_{l=1,\ldots,n} \right),$$

where $c = \frac{\|z\|^2}{1 + \|z\|^2 P} \left(\frac{p}{x} \mathfrak{h}_1 + \frac{q}{y} \mathfrak{h}_2 \right) - \frac{1}{1 + \|z\|^2 P} \sum_{l=1}^{n} \kappa_l z_l$ *and* $P = \frac{p^2}{x^2} + \frac{q^2}{y^2}$, *else if* $(x,y,z) \in bd(K)$ *and* $z = 0$ *then*

$$\Pi'((x,y,0);(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)) = (s(x)\mathfrak{h}_1, s(y)\mathfrak{h}_2, \kappa)\,.$$

(ii) If $(x,y,z) \in bd(K^o)$ *and* $(x,y,z) + t(\mathfrak{h}_1,\mathfrak{h}_2,\kappa) \in K^o$ *for some* t *then*

$$\Pi'((x,y,z);(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)) = 0,$$

else if $(x,y,z) \in bd(K^o)$ *and* $z \neq 0$ *then*

$$\Pi'((x,y,z);(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)) = \left(\frac{p}{x}c, \frac{q}{y}c\,,\, -\frac{z_l}{\|z\|^2}c\bigg|_{l=1,\dots,n}\right),$$

else if $(x,y,z) \in bd(K^o)$ *and* $z = 0$ *then*

$$\Pi'((x,y,0);(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)) = ((1-s(-x))\mathfrak{h}_1, (1-s(-y))\mathfrak{h}_2, 0)\,.$$

Proof. (i) By definition

$$\Pi'((x,y,z);(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)) = \lim_{t\downarrow 0}\frac{\Pi_K(x+t\mathfrak{h}_1, y+t\mathfrak{h}_2, z+t\kappa) - \Pi_K(x,y,z)}{t}.$$

The case $(x+t\mathfrak{h}_1, y+t\mathfrak{h}_2, z+t\kappa) \in K$ for some t is trivial, since then $(x+t_1\mathfrak{h}_1, y+t_1\mathfrak{h}_2, z+t_1\kappa) = \frac{t-t_1}{t}(x,y,z) + \frac{t_1}{t}(x+t\mathfrak{h}_1, y+t\mathfrak{h}_2, z+t\kappa) \in K$ for all $0 < t_1 < t$. Now we consider $(x+t\mathfrak{h}_1, y+t\mathfrak{h}_2, z+t\kappa) \notin K$.

<u>Case $z \neq 0$</u> We use the notation

$$g_p(t) = \sqrt{(x+t\mathfrak{h}_1)^2 + 4pr(\|z+t\kappa\| - r)}, g_q(t) = \sqrt{(y+t\mathfrak{h}_2)^2 + 4qr(normz + t\kappa - r)}$$

Using formula of projection in Proposition 2.3 (i), we get

$\Pi'((x,y,z);(\mathfrak{h}_1,\mathfrak{h}_2,\kappa))$

$$= \lim_{t\downarrow 0}\frac{\left(\frac{1}{2}(-x+t\mathfrak{h}_1+g_p(t)), \frac{1}{2}(-y+t\mathfrak{h}_2+g_q(t)), (z_l+t\kappa_l)\frac{r(t)}{\|z+t\kappa\|} - z_l\Big|_{l=1,\dots,n}\right)}{t},$$

where $0 < r = r(t) < \|z+t\kappa\|$ is the unique solution of

$$\Theta = \frac{1}{2}(x+t\mathfrak{h}_1+g_p(t))^p(y+t\mathfrak{h}_2+g_q(t))^q - r(t) = 0. \tag{15}$$

We note that

$$\frac{-x+t\mathfrak{h}_1+g_p(t)}{2t} = \mathfrak{h}_1 + \frac{g_p(t)-(x+t\mathfrak{h}_1)}{2t} = \mathfrak{h}_1 - \frac{2pr(t)}{g_p(t)+x+t\mathfrak{h}_1}\,\frac{r(t)-\|z+t\kappa\|}{t},$$

similarly,

$$\frac{-y+t\mathfrak{h}_2+g_q(t)}{2t} = \mathfrak{h}_2 - \frac{2qr(t)}{g_q(t)+y+t\mathfrak{h}_2}\,\frac{r(t)-\|z+t\kappa\|}{t},$$

and

$$\frac{(z_l+t\kappa_l)\frac{r(t)}{\|z+t\kappa\|} - z_l}{t} = \kappa_l\frac{r(t)}{\|z+t\kappa\|} + \frac{z_l}{\|z+t\kappa\|}\,\frac{r-\|z+t\kappa\|}{t}.$$

Hence, we now find $c_1 = \lim_{t\downarrow 0}\frac{r(t)-\|z+t\kappa\|}{t}$.

Let $f_p(t)=\frac{1}{2}(x+t\mathfrak{h}_1+g_p(t)), f_q(t)=\frac{1}{2}(y+t\mathfrak{h}_2+g_q(t))$. From the equation (15),

$$r'(t)=\frac{\Theta'_t}{\Theta'_r}=\frac{\frac{1}{2}r(t)\left(\Omega_p(t)+\Omega_q(t)\right)}{1-r(t)\sum\limits_{i\in\{p,q\}}\frac{i^2(\|z+t\kappa\|-2r)}{g_i(t)f_i(t)}},$$

where $\Omega_p(t)=\frac{p}{f_p(t)}\left(\mathfrak{h}_1+\frac{1}{2}\frac{2(x+t\mathfrak{h}_1)\mathfrak{h}_1+4p\frac{r(t)}{\|z+t\kappa\|}\sum\limits_{l=1}^{n}\kappa_l(z_l+t\kappa_l)}{g_p(t)}\right)$, and

$$\Omega_q(t)=\frac{q}{f_q(t)}\left(\mathfrak{h}_2+\frac{1}{2}\frac{2(y+t\mathfrak{h}_2)\mathfrak{h}_2+4q\frac{r(t)}{\|z+t\kappa\|}\sum\limits_{l=1}^{n}\kappa_l(z_l+t\kappa_l)}{g_p(t)}\right).$$

We note that $r(t)$ is nothing else but the norm of the z-component of the projection $\Pi_K(x+t\mathfrak{h}_1,y+t\mathfrak{h}_2,z+t\kappa)$. Furthermore, $(x+t\mathfrak{h}_1,y+t\mathfrak{h}_2,z+t\kappa)\to(x,y,z)\in\mathrm{bd}(K)$. Hence $r(t)\to\|z\|$. Moreover, $g_p(t)\to x, g_q(t)\to y, f_p(t)\to x, f_q(t)\to y$, we deduce

$$\begin{aligned}\lim_{t\downarrow 0}r'(t)&=\frac{1}{2}\|z\|\frac{\frac{p}{x}\left(\mathfrak{h}_1+\frac{\mathfrak{h}_1x+2p\sum\limits_{l=1}^{n}\kappa_lz_l}{x}\right)+\frac{q}{y}\left(\mathfrak{h}_2+\frac{\mathfrak{h}_2y+2q\sum\limits_{l=1}^{n}\kappa_lz_l}{y}\right)}{1+\|z\|^2P}\\&=\frac{\|z\|}{1+\|z\|^2P}\left(\frac{p}{x}\mathfrak{h}_1+\frac{q}{y}\mathfrak{h}_2+P\sum_{l=1}^{n}\kappa_lz_l\right).\end{aligned}$$

Applying L'Hopital rule,

$$\begin{aligned}c_1&=\lim_{t\downarrow 0}\left(r'(t)-\frac{1}{\|z+t\kappa\|}\sum_{l=1}^{n}\kappa_l(z_l+t\kappa_l)\right)\\&=\frac{\|z\|}{1+\|z\|^2P}\left(\frac{p}{x}\mathfrak{h}_1+\frac{q}{y}\mathfrak{h}_2\right)-\frac{1}{\|z\|(1+\|z\|^2P)}\sum_{l=1}^{n}\kappa_lz_l.\end{aligned}$$

The result follows then easily.

<u>Case $z=0$</u>. The case $\kappa=0$ is trivial, then we consider $\kappa\neq 0$. Let us assume $x>0, y=0$ as the result can be proved totally similarly for $x=0, y>0$. Using Proposition 2.3 (i),

$$\begin{aligned}\Pi'((x,0,0);(\mathfrak{h}_1,\mathfrak{h}_2,\kappa))&=\lim_{t\downarrow 0}\frac{\Pi_K(x+t\mathfrak{h}_1,t\mathfrak{h}_2,t\kappa)-\Pi_K(x,0,0)}{t}\\&=\lim_{t\downarrow 0}\frac{\left(\frac{1}{2}(-x+t\mathfrak{h}_1+g_p(t)),\frac{1}{2}(t\mathfrak{h}_2+g_q(t)),t\kappa_l\frac{r(t)}{\|t\kappa\|}\Big|_{l=1,\dots,n}\right)}{t},\end{aligned}\tag{16}$$

where we reuse the notations

$$g_p(t)=\sqrt{(x+t\mathfrak{h}_1)^2+4pr(t)(\|t\kappa\|-r(t))}, g_q(t)=\sqrt{(t\mathfrak{h}_2)^2+4qr(t)(\|t\kappa\|-r(t))},$$

and $0<r(t)<\|t\kappa\|$ is the unique solution of the equation

$$\frac{1}{2}(x+t\mathfrak{h}_1+g_p(t))^p(t\mathfrak{h}_2+g_q(t))^q-r(t)=0.$$

This equation yields

$$r(t)^p = \frac{1}{2}(x + t\mathfrak{h}_1 + g_p(t))^p \left(\mathfrak{h}_2 \frac{t}{r(t)} + \sqrt{\mathfrak{h}_2^2 \frac{t^2}{r(t)^2} + 4q\left(\|\kappa\| \frac{t}{r(t)} - 1\right)}\right)^q. \tag{17}$$

Suppose that $\mathfrak{h}_2 > 0$, then $\frac{(x+t\mathfrak{h}_1)^p (t\mathfrak{h}_2)^q - \|t\kappa\|}{t} = \left(\frac{x}{t} + \mathfrak{h}_1\right)^p \mathfrak{h}_2^q - \|\kappa\| \to +\infty$ when $t \downarrow 0$. This implies $(x + t\mathfrak{h}_1)^p (t\mathfrak{h}_2)^q > \|t\kappa\|$ for sufficiently small t and as a consequence $(x + t\mathfrak{h}_1, t\mathfrak{h}_2, t\kappa)$ belongs to K. Therefore, $\mathfrak{h}_2 \le 0$.

Since $0 < r \le \|t\kappa\|$ then $\left\{\frac{r}{t}\right\}$ have finite non-negative limit points when $t \downarrow 0$. Assuming $\frac{t}{r(t)} \to +\infty$, if $\mathfrak{h}_2 = 0$ then $\mathfrak{h}_2 \frac{t}{r(t)} + \sqrt{\mathfrak{h}_2^2 \frac{t^2}{r(t)^2} + 4q\left(\|\kappa\| \frac{t}{r(t)} - 1\right)} \to +\infty$, and if $\mathfrak{h}_2 < 0$ this value equals $\frac{4q\left(\|\kappa\| - \frac{r(t)}{t}\right)}{\sqrt{\mathfrak{h}_2^2 + 4q\left(\|\kappa\| - \frac{r(t)}{t}\right)\frac{r(t)}{t}} - \mathfrak{h}_2} \to \frac{2q\|\kappa\|}{-\mathfrak{h}_2} > 0$. Furthermore, the left hand side of (17) converges to 0 and $g_p(t) \to x$. We imply $\dfrac{r(t)}{t} \to \|\kappa\|$.

Rewriting

$$\frac{-x + t\mathfrak{h}_1 + g_p(t)}{2t} = \mathfrak{h}_1 - \frac{2pr(t)}{g_p(t) + x + t\mathfrak{h}_1} \frac{r(t) - \|t\kappa\|}{t}; \frac{g_q(t)}{t} = \sqrt{\mathfrak{h}_2^2 + 4q\frac{r(t)}{t}\left(\|\kappa\| - \frac{r(t)}{t}\right)}$$

we then can easily get the result from the expression(16).

(ii) By Moreau decomposition,

$$(x, y, z) = \Pi_K(x, y, z) - \Pi_{K^\sharp}(-x, -y, -z). \tag{18}$$

we deduce that

$$\Pi'_K((x, y, z); (\mathfrak{h}_1, \mathfrak{h}_2, \kappa)) = (\mathfrak{h}_1, \mathfrak{h}_2, \kappa) + \Pi'_{K^\sharp}((-x, -y, -z); (-\mathfrak{h}_1, -\mathfrak{h}_2, -\kappa)).$$

By Proposition 2.2 the dual cone $K^\sharp$ is also a power cone. Furthermore, $(x, y, z) \in K^o$ implies $(-x, -y, -z) \in K^\sharp$. Therefore, we can use formula established in part (i) for directional derivative $\Pi'_{K^\sharp}((-x, -y, -z); (-\mathfrak{h}_1, -\mathfrak{h}_2, -\kappa))$ of $\Pi_{K^\sharp}$ at $(-x, -y, -z)$ along the direction $(-\mathfrak{h}_1, -\mathfrak{h}_2, -\kappa)$ and deduce the result. □

Theorem 3.3. *The Euclidean projection $\Pi(x, y, z)$ onto the power cone K is not strongly semismooth at (x, y, z) in the following cases*

(A) $(x, y, z) \notin K, K^o, z = 0$ *and* $\frac{1}{3} < p, q < \frac{2}{3}, p, q \neq \frac{1}{2}$,
(B) $(x, y, z) \neq 0, xy = 0, z = 0$ *and* $(|x| - |y|)(p - q) < 0$,

and it is strongly semismooth in the remaining cases.

Proof. To prove non-strongly semismoothness, we provide a sequence of $(\mathfrak{h}_1, \mathfrak{h}_2, \kappa)$ such that the definition 2.2 is not satisfied, i.e., $(x + \mathfrak{h}_1, y + \mathfrak{h}_2, z + \kappa) \in D_{\Pi_K}$, $(\mathfrak{h}_1, \mathfrak{h}_2, \kappa) \to 0$ and the following value

$$\frac{\Pi_K(x + \mathfrak{h}_1, y + \mathfrak{h}_2, z + \kappa) - \mathbf{J}\Pi_K(x + \mathfrak{h}_1, y + \mathfrak{h}_2, z + \kappa)(\mathfrak{h}_1, \mathfrak{h}_2, \kappa) - \Pi_K(x, y, z)}{\|(\mathfrak{h}_1, \mathfrak{h}_2, \kappa)\|^2},$$

whose nominator is denoted by $(C_1, C_2, C_1^z, \dots, C_n^z)$, converges to ∞. Let $z = 0$ and $\kappa \neq 0$. Using Proposition 2.3 and Theorem 3.1, we get

$$\begin{aligned} C_l^z &= -\frac{\kappa_l}{\|\kappa\|}\frac{prL}{g_p}\mathfrak{h}_1 - \frac{\kappa_l}{\|\kappa\|}\frac{qrL}{g_q}\mathfrak{h}_2 - \sum_{k=1}^{n}\frac{r\kappa_l\kappa_k}{\|\kappa\|^3}TL\kappa_k, \\ &= -\frac{\kappa_l}{\|\kappa\|}\frac{prL}{g_p}\mathfrak{h}_1 - \frac{\kappa_l}{\|\kappa\|}\frac{qrL}{g_q}\mathfrak{h}_2 - \frac{\kappa_l}{\|\kappa\|}rTL, \quad l = 1, \dots, n, \end{aligned} \tag{19}$$

where $g_p = \sqrt{(x+\mathfrak{h}_1)^2 + 4pr(\|\kappa\| - r)}$, $g_q = \sqrt{(y+\mathfrak{h}_2)^2 + 4qr(\|\kappa\| - r)}$, T, L are defined in (12) with (x, y, z) being replaced by $(x + \mathfrak{h}_1, y + \mathfrak{h}_2, \kappa)$, and r is the unique solution of $E(x + \mathfrak{h}_1, y + \mathfrak{h}_2, \kappa)$ (see (6))

$$\frac{1}{2}\left(x + \mathfrak{h}_1 + g_p\right)^p \left(y + \mathfrak{h}_2 + \sqrt{(y+\mathfrak{h}_2)^2 + 4qr(\|\kappa\| - r)}\right)^q = r. \tag{20}$$

The case $(x, y, z) \notin K, K^o, z = 0$ includes 2 sub-cases $(x > 0, y < 0, z = 0)$ and $(x < 0, y > 0, z = 0)$. To prove the first situation of non-strongly semismooth (A), we can assume $(x > 0, y < 0, z = 0)$ as it is totally similar for the second sub-case.

Case $x > 0, y < 0, z = 0$ and $\frac{1}{3} < q < \frac{1}{2}$. Using $g_q^2 - (y + \mathfrak{h}_2)^2 = 4qr(\|\kappa\| - r)$ and dividing both sides of (20) by $r^q \|\kappa\|^{2q}$, we get

$$\frac{1}{2}\frac{(x + \mathfrak{h}_1 + g_p)^p (4q)^q}{(-y - \mathfrak{h}_2 + g_q)^q}\left(\frac{1 - \frac{r}{\|\kappa\|}}{\|\kappa\|}\right)^q = \frac{r^{1-q}}{\|\kappa\|^{2q}} = \left(\frac{r}{\|\kappa\|}\right)^{1-q}\frac{1}{\|\kappa\|^{3q-1}}. \tag{21}$$

It follows from Lemma 3.1 that $\frac{r}{\|\kappa\|} \to 1$. The equation (21) yields that $\frac{1 - \frac{r}{\|\kappa\|}}{\|\kappa\|} \to \infty$. On the other hand, from (12)

$$\frac{L}{\|\kappa\|} = \frac{2\left(1 - \frac{r}{\|\kappa\|}\right)}{\|\kappa\|\left(1 + \left(1 - 2\frac{r}{\|\kappa\|}\right)\left(\frac{p(x+\mathfrak{h}_1)}{g_p} + \frac{q(y+\mathfrak{h}_2)}{g_q}\right)\right)}. \tag{22}$$

Thus $\frac{L}{\|\kappa\|} \to \infty$. Hence if we choose $\mathfrak{h}_1 = \mathfrak{h}_2 = 0, \kappa_1 = k \neq 0, \kappa_2 = \dots = \kappa_n = 0$, together with $T \to 2q - 1 \neq 0$ from Lemma 3.1, we then imply $\frac{C_1^z}{\|\kappa\|^2} \to \infty$. Therefore, the projection is not strongly semismooth in this case.

Case $x > 0, y < 0, z = 0$ and $\frac{1}{2} < q < \frac{2}{3}$. Lemma 3.1 gives $\frac{r}{\|\kappa\|} \to 0$. Dividing both sides of (20) by $r^q \|\kappa\|^{2p} = r^q \|\kappa\|^{2-2q}$ we get

$$\frac{1}{2}\frac{(x + \mathfrak{h}_1 + g_p)^p (4q)^q}{(-y - \mathfrak{h}_2 + g_q)^q}\left(1 - \frac{r}{\|\kappa\|}\right)^q \|\kappa\|^{3q-2} = \frac{r^p}{\|\kappa\|^{2p}}. \tag{23}$$

This implies $\frac{r}{\|\kappa\|^2} \to \infty$. Furthermore, from Lemma 3.1 $T \to 2q - 1 \neq 0, L \to d_2 \neq 0$. Thus if we choose $\mathfrak{h}_1 = \mathfrak{h}_2 = 0, \kappa_1 = k \neq 0, \kappa_2 = \dots = \kappa_n = 0$, then $\frac{C_1^z}{\|\kappa\|^2} \to \infty$. Hence the projection is not strongly semismooth.

To prove (B), we can assume that $x \neq 0, y = 0, z = 0$ as the case $x = 0, y \neq 0, z = 0$ can be proved totally similarly.

Case $(x<0, y=0, z=0)$ and $p<q$. We choose $\mathfrak{h}_1=0, \mathfrak{h}_2=h, \kappa_1=h, \kappa_2=\dots\kappa_n=0$, where h is a positive number converging to 0. From (20) we have

$$\frac{1}{2}\frac{(4pr(h-r))^p}{(-x+g_p)^p}\left(h+\sqrt{h^2+4qr(h-r)}\right)^q=r.$$

Dividing both sides of this equation by $r^p h^p h^q$, we get

$$\frac{(4p)^p}{2(-x+g_p)^p}\left(1-\frac{r}{h}\right)^p\left(1+\sqrt{1+4q\frac{r}{h}\left(1-\frac{r}{h}\right)}\right)^q=\frac{r^q}{h^{p+q}}=\left(\frac{r}{h}\right)^q h^{-p}.$$

The left hand side of the received equation is bounded, thus $\frac{r}{h}\to 0$, otherwise the right hand side goes to ∞. Then the left hand side converges to a non-zero number. Furthermore, since $\left(\frac{r}{h}\right)^q h^{-p}=\left(\frac{r}{h^2}\right)^q h^{q-p}$, we imply that $\frac{r}{h^2}\to+\infty$, otherwise the right hand side converges to 0. Now we find limit of $L=\dfrac{2(\|\kappa\|-r)}{\|\kappa\|+(\|\kappa\|-2r)\left(\frac{px}{g_p}+\frac{qh}{g_q}\right)}$ and $T=-\frac{px}{g_p}-\frac{qh}{g_q}$ (see (12)).

We note that $g_p\to -x$, $\frac{g_q}{h}=\frac{\sqrt{h^2+4qr(h-r)}}{h}\to 1$ and $\frac{f_q}{h}=\frac{h+g_q}{2h}\to 1$. Hence $T\to p-q$ and $L\to q^{-1}>0$. Therefore, in this case

$$\frac{C_1^z}{\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|^2}=\frac{C_1^z}{2h^2}=-\frac{rL}{2h^2}\left(\frac{q}{g_q}h+T\right)\to\infty$$

We conclude that the projection is not strongly semismooth in this case.

Case $(x>0, y=0, z=0)$ and $p<q$. We have proved that the projection on to $K^\sharp$ is not strongly semismooth at $(-x,-y,-z)$ as $-x<0, -y=0, -z=0, p<q$ and $K^\sharp$ is also a power cone. Using Moreau decomposition (18) we deduce non-strongly semismoothness of the projection on to K.

Now we prove projection onto power cone is strongly semismooth in remaining cases.

The cases $(x,y,z)\in\operatorname{int}(K)$ and $(x,y,z)\in\operatorname{int}(K^o)$ are trivial. When $(x,y,z)\notin K, K^o$ and $z\neq 0$ the projection is twice continuously differentiable by Theorem 3.1, it thus is strongly semismooth. We now consider the other cases.

Case $(x,y,z)\in\operatorname{bd}(K), z\neq 0$. We now prove that

$$\Pi'((x+\mathfrak{h}_1,y+\mathfrak{h}_2,z+\kappa);(\mathfrak{h}_1,\mathfrak{h}_2,\kappa))-\Pi'((x,y,z);(\mathfrak{h}_1,\mathfrak{h}_2,\kappa))=O(\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|^2). \tag{24}$$

If $(x+\mathfrak{h}_1,y+\mathfrak{h}_2,z+\kappa)\in\operatorname{int}(K)$ then the left hand side of(24) equals 0, we thus consider $(x+\mathfrak{h}_1,y+\mathfrak{h}_2,z+\kappa)\notin K$. Denote $D_1=\lim_{(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\to 0}\mathbf{J}\Pi(x+\mathfrak{h}_1,y+\mathfrak{h}_2,z+\kappa)$. Using formulas of $\mathbf{J}\Pi_K$ in Theorem 3.1 with (x,y,z) being replaced by $(x+\mathfrak{h}_1,y+\mathfrak{h}_2,z+\kappa)$, and noting that $\overline{x}>0, \overline{y}>0, r\to z$ (r equals $\|\overline{z+\kappa}\|$, which is the z-coordinate of $\Pi_K(x+\mathfrak{h}_1,y+\mathfrak{h}_2,z+\kappa)=(\overline{x+\mathfrak{h}_1},\overline{y+\mathfrak{h}_2},\overline{z+\kappa})$), we deduce the rows of D_1

$$\begin{aligned}
\lim_{(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\to 0}\Pi'_{x+\mathfrak{h}_1}&=\left(1-\tfrac{p^2\|z\|^2}{x^2(1+\|z\|^2P)},-\tfrac{qp\|z\|^2}{xy(1+\|z\|^2P)},\tfrac{pz_l}{x(1+\|z\|^2P)}\Big|_{l=1,\dots,n}\right)\\
\lim_{(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\to 0}\Pi'_{y+\mathfrak{h}_2}&=\left(-\tfrac{qp\|z\|^2}{xy(1+\|z\|^2P)},1-\tfrac{q^2\|z\|^2}{y^2(1+\|z\|^2P)},\tfrac{qz_l}{y(1+\|z\|^2P)}\Big|_{l=1,\dots,n}\right)\\
\lim_{(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\to 0}\Pi'_{z_l+\kappa_l}&=\left(\tfrac{pz_l}{x(1+\|z\|^2P)},\tfrac{qz_l}{y(1+\|z\|^2P)},1-\tfrac{(z_l)^2}{\|z\|^2(1+\|z\|^2P)},\tfrac{-z_kz_l}{\|z\|^2(1+\|z\|^2P)}\Big|_{k\neq l}\right),
\end{aligned}$$

where P is defined in Theorem 3.2. Furthermore, while finding limit of $\mathbf{J}\Pi(x+\mathfrak{h}_1, y+\mathfrak{h}_2, z+\kappa)$ when $(\mathfrak{h}_1, \mathfrak{h}_2, \kappa) \to 0$ we see that all of involving convergence are linear since all of the values $r - \|z\| = \|\overline{z+\kappa}\| - \|z\|$, $\|z+\kappa\| - \|z\|$, $|z_l + \kappa_l| - |\kappa_l|$, $|x+\mathfrak{h}_1| - x$, and $|y+\mathfrak{h}_2| - y$ are $O(\|(\mathfrak{h}_1, \mathfrak{h}_2, \kappa\|)$. Hence, $\mathbf{J}\Pi(x+\mathfrak{h}_1, y+\mathfrak{h}_2, z+\kappa)$ converges linearly to D_1, i. e.,

$$\|\mathbf{J}\Pi(x+\mathfrak{h}_1, y+\mathfrak{h}_2, z+\kappa) - D_1\| = O(\|(\mathfrak{h}_1, \mathfrak{h}_2, \kappa)\|).$$

On the other hand, by directly calculation we can verify that

$$\Pi'((x,y,z);(\mathfrak{h}_1, \mathfrak{h}_2, \kappa))) = D_1[(\mathfrak{h}_1, \mathfrak{h}_2, \kappa)].$$

Thus

$$\begin{aligned}
&\Pi'((x+\mathfrak{h}_1, y+\mathfrak{h}_2, z+\kappa);(\mathfrak{h}_1, \mathfrak{h}_2, \kappa)) - \Pi'((x,y,z);(\mathfrak{h}_1, \mathfrak{h}_2, \kappa)) \\
&\quad = \mathbf{J}\Pi_K((x+\mathfrak{h}_1, y+\mathfrak{h}_2, z+\kappa))[(\mathfrak{h}_1, \mathfrak{h}_2, \kappa)] - D_1[(\mathfrak{h}_1, \mathfrak{h}_2, \kappa)] \\
&\quad = O(\|(\mathfrak{h}_1, \mathfrak{h}_2, \kappa)\|^2).
\end{aligned}$$

By Proposition 2.1, the projection is strongly semismooth in this case.

Case $(x,y,z) \in \mathrm{bd}(K^o), z \neq 0$.

This case is dual part of the previous case. Strong semismoothness of the projection follows from $(-x,-y,-z) \in \mathrm{bd}(K^\sharp)$, $z \neq 0$, Moreau decomposition (18), Proposition 2.2 and the result of the previous case.

Now we consider $(x,y,z) \notin K, K^o$ and $z = 0$. Let's assume $x > 0, y < 0, z = 0$ as $x < 0, y > 0, z = 0$ is proved totally similarly.

We need to prove that

$$(C_1, C_2, C_1^z, \ldots, C_n^z) = O(\|(\mathfrak{h}_1, \mathfrak{h}_2, \kappa)\|^2). \tag{25}$$

If $\kappa = 0$ then it is not difficult to verify that the left hand side of (25) is 0 by using formulas of Π_K, $\mathbf{J}\Pi_K$ in Proposition 2.3 and Theorem 3.1 with (x,y,z) being replaced by $(x+\mathfrak{h}_1, y+\mathfrak{h}_2, \kappa)$, we thus consider $\kappa \neq 0$. Now we have formula (19) as well as

$$\begin{aligned}
C_1 &= \tfrac{1}{2}(-x+g_p) - \left(\tfrac{x+\mathfrak{h}_1}{2g_p} + \tfrac{p^2(\|\kappa\|-2r)rL}{g_p^2}\right)\mathfrak{h}_1 - \tfrac{pq(\|\kappa\|-2r)rL}{g_p g_q}\mathfrak{h}_2 - \sum_{l=1}^n \tfrac{\kappa_l}{\|\kappa\|}\tfrac{prL}{g_p}\kappa_l \\
&= \tfrac{1}{2}(-x+g_p) - \tfrac{x\mathfrak{h}_1}{2g_p} - \tfrac{\mathfrak{h}_1^2}{2g_p} - \tfrac{prL}{g_p}\left(\tfrac{p(\|\kappa\|-2r)\mathfrak{h}_1}{g_p} + \tfrac{q(\|\kappa\|-2r)}{g_q}\mathfrak{h}_2 + \|\kappa\|\right) \\
C_2 &= \tfrac{1}{2}(y+g_q) - \tfrac{pq(\|\kappa\|-2r)rL}{g_p g_q}\mathfrak{h}_1 - \left(\tfrac{y+\mathfrak{h}_2}{2g_q} + \tfrac{q^2(\|\kappa\|-2r)rL}{g_q^2}\right)\mathfrak{h}_2 - \sum_{l=1}^n \tfrac{\kappa_l}{\|\kappa\|}\tfrac{qrL}{g_q}\kappa_l \\
&= \tfrac{1}{2}(y+g_q) - \tfrac{y\mathfrak{h}_2}{2g_q} - \tfrac{\mathfrak{h}_2^2}{2g_q} - \tfrac{qrL}{g_q}\left(\tfrac{p(\|\kappa\|-2r)\mathfrak{h}_1}{g_p} + \tfrac{q(\|\kappa\|-2r)}{g_q}\mathfrak{h}_2 + \|\kappa\|\right)
\end{aligned} \tag{26}$$

Noting that $0 < r < \|\kappa\|$, $g_p \to x > 0$, $g_q \to -y > 0$, we have

$$\frac{1}{2}(-x+g_p) = \frac{g_p^2 - x^2}{2(g_p+x)} = \frac{x\mathfrak{h}_1}{g_p+x} + \frac{\mathfrak{h}_1^2 + 4pr(\|\kappa\|-r)}{2(g_p+x)} = \frac{x\mathfrak{h}_1}{g_p+x} + O(\|(\mathfrak{h}_1, \mathfrak{h}_2, \kappa)\|^2).$$

This implies

$$\begin{aligned}
\tfrac{1}{2}(-x+g_p) - \tfrac{x\mathfrak{h}_1}{2g_p} &= x\mathfrak{h}_1\tfrac{-x+g_p}{2(g_p+x)g_p} + O(\|(\mathfrak{h}_1, \mathfrak{h}_2, \kappa)\|^2) \\
&= \tfrac{x\mathfrak{h}_1}{(g_p+x)g_p}\left(\tfrac{x\mathfrak{h}_1}{g_p+x} + O(\|(\mathfrak{h}_1, \mathfrak{h}_2, \kappa)\|^2)\right) + O(\|(\mathfrak{h}_1, \mathfrak{h}_2, \kappa)\|^2) \\
&= O(\|(\mathfrak{h}_1, \mathfrak{h}_2, \kappa)\|^2).
\end{aligned} \tag{27}$$

Totally similarly, from

$$\frac{1}{2}(y+g_q)=\frac{g_q^2-y^2}{2(g_q-y)}=\frac{y\mathfrak{h}_2}{g_q-y}+\frac{\mathfrak{h}_2^2+4qr(\|\kappa\|-r)}{2(g_q-y)}=\frac{y\mathfrak{h}_2}{g_q-y}+O(\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|^2),$$

we can prove $\frac{1}{2}(y+g_q)-\frac{y\mathfrak{h}_2}{2g_q}=O(\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|^2)$. We also note that L is bounded by Lemma 3.1. Hence $C_1=O(\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|^2)$ and $C_2=O(\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|^2)$ follows then.

We now consider C_l^z (see (19)), $l=1,\dots,n$.

Case $x>0, y<0, z=0, q\le\frac{1}{3}, p\ge\frac{2}{3}$. From $q\le\frac{1}{3}$ and the equation (21), we deduce $1-\frac{r}{\|\kappa\|}=O(\|\kappa\|)$. Together with the equation (22) we imply $L=O(\|\kappa\|)$. On the other hand, T is bounded by Lemma 3.1. It then yields that $C_l^z=O(\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|^2)$. Therefore, the projection is strongly semismooth.

Case $x>0, y<0, q\ge\frac{2}{3}, p\le\frac{1}{3}$. From $q\ge\frac{2}{3}$ and the equation (23) we deduce $r=O(\|\kappa\|^2)$. Together with boundedness of T,L (see Lemma 3.1), we deduce $C_l^z=O(\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|^2)$. The projection is also strongly semismooth in this case.

Now we prove strongly semismoothness when $xy=0, z=0, (x,y,z)\ne 0$ and $(|x|-|y|)(p-q)\ge 0$. We can assume that $x\ne 0, y=0, z=0$ as the case $x=0, y\ne 0, z=0$ can be proved totally similarly.

Case $x>0, y=0, z=0, p\ge q$. Similarly to(19), (26), we rewrite $C_1, C_2, C_l^z (l=1,\dots,n)$ as

$$C_1=\tfrac{1}{2}(-x+g_p)-\tfrac{x\mathfrak{h}_1}{2g_p}-\tfrac{\mathfrak{h}_1^2}{2g_p}-\tfrac{prL}{g_p}\left(\tfrac{p(\|\kappa\|-2r)\mathfrak{h}_1}{g_p}+\tfrac{q(\|\kappa\|-2r)}{g_q}\mathfrak{h}_2+\|\kappa\|\right)$$
$$C_2=\tfrac{1}{2}(g_q)-\tfrac{\mathfrak{h}_2^2}{2g_q}-\tfrac{qrL}{g_q}\left(\tfrac{p(\|\kappa\|-2r)\mathfrak{h}_1}{g_p}+\tfrac{q(\|\kappa\|-2r)}{g_q}\mathfrak{h}_2+\|\kappa\|\right)$$
$$C_l^z=-\tfrac{\kappa_l}{\|\kappa\|}\tfrac{prL}{g_p}\mathfrak{h}_1-\tfrac{\kappa_l}{\|\kappa\|}\tfrac{qrL}{g_q}\mathfrak{h}_2-\tfrac{\kappa_l}{\|\kappa\|}rTL, l=1,\dots,n,$$

Let's reuse the notations $f_p=\frac{1}{2}(x+\mathfrak{h}_1+g_p), f_q=\frac{1}{2}(\mathfrak{h}_2+g_q)$ and from (20) we get

$$f_p^p\left(\frac{f_q}{r^2}\right)^q=r^{1-2q}.$$

The right hand side of this equation is bounded as $q\le\frac{1}{2}$. Together with $f_p\to x$, we imply $f_q=O(r^2)$. Furthermore, from (12)

$$\frac{L}{g_q}=\frac{1}{\frac{1}{2}g_q+\frac{p(x+\mathfrak{h}_1)g_q}{2g_p}+\frac{q\mathfrak{h}_2}{2}+\frac{r^2p^2g_q}{g_pf_p}+\frac{r^2q^2}{f_q}}$$

It follows that $\frac{L}{g_q}=O(1)$. On the other hand, $\frac{\mathfrak{h}_2}{g_q}=\frac{\mathfrak{h}_2}{\sqrt{\mathfrak{h}_2^2+4qr(\|\kappa\|-r)}}=O(1)$, which implies $T=-\frac{p(x+\mathfrak{h}_1)}{g_p}-\frac{q\mathfrak{h}_2}{g_q}$ is bounded. Therefore, we have:

(1) $\frac{(\|\kappa\|-2r)}{g_q}\mathfrak{h}_2=O(\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|)$. Moreover, totally similarly to (27) we can prove $\frac{1}{2}(-x+g_p)-\frac{x\mathfrak{h}_1}{2g_p}=O(\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|^2$. And $C_1=O(\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|^2)$ follows easily.

(2) $\frac{1}{2}g_q-\frac{1}{2}\frac{\mathfrak{h}_2^2}{g_q}=\frac{g_q-\mathfrak{h}_2}{g_q}f_q=O(\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|^2)$. Then $C_2=O(\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|^2)$ follows easily.

(3) $g_q=\sqrt{\mathfrak{h}_2^2+4qr(\|\kappa\|-r)}=O(\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|)$ implies $rTL=O(\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|)O(g_q)=O(\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|^2)$. Then $C_l^z=O(\|(\mathfrak{h}_1,\mathfrak{h}_2,\kappa)\|^2)$ for $l=1,\dots,n$ follows easily.

We have proved that the projection is strongly semismooth in this case.

$x < 0, y = 0, z = 0, p \le q$: This case is the dual part of previous case. Strongly semismoothness follows from previous case, Moreau decomposition and Proposition 2.2.

We finish the proof by the final case when (x, y, z) is the origin.

Case $(x, y, z) = 0$. The values of T, L of (12) now become :

$$T = -\frac{p\mathfrak{h}_1}{g_p} - \frac{q\mathfrak{h}_2}{g_q}, L = \frac{2(\|\kappa\| - r)}{\|\kappa\| + (\|\kappa\| - 2r)\left(\frac{p\mathfrak{h}_1}{g_p} + \frac{q\mathfrak{h}_2}{g_q}\right)}.$$

Therefore, from (19)

$$C_l^z = -\frac{\kappa_l}{\|\kappa\|} rL \left(\frac{p\mathfrak{h}_1}{g_p} + \frac{q\mathfrak{h}_2}{g_q} + T\right) = 0.$$

By noting $g_p^2 - \mathfrak{h}_1^2 = 4pr(\|\kappa\| - r)$, similarly to (26) we get

$$\begin{aligned} C_1 &= \tfrac{1}{2} g_p - \tfrac{\mathfrak{h}_1^2}{2g_p} - \tfrac{prL}{g_p}\left(\tfrac{p(\|\kappa\|-2r)\mathfrak{h}_1}{g_p} + \tfrac{q(\|\kappa\|-2r)}{g_q}\mathfrak{h}_2 + \|\kappa\|\right) \\ &= \tfrac{pr}{g_p}\left(2(\|\kappa\| - r) - L\left(\tfrac{p(\|\kappa\|-2r)\mathfrak{h}_1}{g_p} + \tfrac{q(\|\kappa\|-2r)\mathfrak{h}_2}{g_q} + \|\kappa\|\right)\right) \\ &= 0. \end{aligned}$$

And totally similarly $C_2 = 0$. We conclude that the projection is strongly semismooth at the origin. □

4. Conclusion

We have formulated projection onto power cone, its directional derivatives, its first order Fréchet derivative and characterized its strongly semismoothness. By a similar way to what we have done, we can find formula of Euclidean projection onto more general power cone - the high dimensional power cone

$$K_\alpha^{(m,n)} = \left\{(x_1, \ldots, x_m, z_1, \ldots, z_n) \in \mathbb{R}_+^m \times \mathbb{R}^n : \prod_{i=1}^m x_i^{\alpha_i} \ge ||z||_2\right\},$$

where $0 < \alpha_i < 1, \sum_{i=1}^m \alpha_i = 1$, and its first order Fréchet derivative. We give these formulas in Appendix part.

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Appendix A.

Proposition A.1. *Let $(x^o, y^o, z^o) \notin K, K^o$ and $z^o \neq 0$. Then its projection $\Pi_K(x^o, y^o, z^o)$, which is denoted by $(\overline{x}, \overline{y}, \overline{z})$, satisfies $0 < \|\overline{z}\| < \|z^o\|$.*

Proof. From (8) we have $0 < \|z\| < \|z^o\|$, this implies $0 \leq \|\overline{z}\| \leq \|z^o\|$.

Suppose $\|\overline{z}\| = \|z^o\|$, from (9), (10) $\overline{x} = [x^o]_+, \overline{y} = [y^o]_+$. If $x^o \leq 0$ or $y^o \leq 0$ then $\overline{x} = 0$ or $\overline{y} = 0$ and $\|\overline{z}\|^2 = (\overline{x})(\overline{y}) = 0$. This contradicts to $z^o \neq 0$. Therefore, $\overline{x} = x^o > 0, \overline{y} = y^o > 0$. But then $\|z^o\|^2 = \|\overline{z}\|^2 = (\overline{x})(\overline{y}) = x^o y^o$, which implies that $(x^o, z^o) \in K$: contradiction. Therefore $\|\overline{z}\| < \|z^o\|$.

Suppose $\overline{z} = 0$, (9) and (10)imply $\overline{x} = [x^o]_+, \overline{y} = [y^o]_+$. Denote $\Pi_{K^o}(x^o, y^o, z^o) = (\overline{\overline{x}}, \overline{\overline{y}}, \overline{\overline{z}})$. If $x^o > 0$ then $\overline{x} = x^o$. By Moreau decomposition, $\overline{\overline{x}} = x^o - \overline{x} = 0$. Moreover, $-(\overline{\overline{x}}, \overline{\overline{z}}) \in K^\sharp$ implies $\| - \overline{\overline{z}}\| \leq \left(\dfrac{-\overline{\overline{x}}}{p}\right)^p \left(\dfrac{-\overline{\overline{y}}}{q}\right)^q = 0$. Hence $\overline{\overline{z}} = 0$ and $z^o = \overline{z} + \overline{\overline{z}} = 0$: contradiction. Therefore, $x^o \leq 0$. Similarly $y^o \leq 0$. One implies that $\overline{x} = 0, \overline{y} = 0$ and $x^o = x^o - \overline{x} = \overline{\overline{x}}, y^o = y^o - \overline{y} = \overline{\overline{y}}$. On the other hand, we have $z^o = z^o - \overline{z} = \overline{\overline{z}}$. Hence $(x^o, y^o, z^o) \in K^o$: contradiction. Therefore $\|\overline{z}\| > 0$. □

Proposition A.2. *Let $(x^o, z^o) \in \mathbb{R}^n \times \mathbb{R}^m$ be a given point and $(\overline{x}, \overline{z})$ be its projection onto $K_\alpha^{(m,n)}$.*

(i) *If* $(x^o, z^o) \notin K, K^o$ *and* $z^o \neq 0$ *then its projection onto* $K_\alpha^{(m,n)}$ *is*

$$\begin{cases} \overline{x}_i = \dfrac{1}{2}\Big(x_i^o + \sqrt{(x_i^o)^2 + 4\alpha_i r(\|z^o\| - r)}\Big), \quad i = 1, \dots, m, \\ \overline{z}_l = z_l^o \dfrac{r}{\|z^o\|}, \quad l = 1, \dots, n. \end{cases}$$

where $r = r(x, y)$ *is the unique solution of the following system:*

$$E(x,z) : \begin{cases} \dfrac{1}{2}\prod_{i=1}^{m} \Big(x_i^o + \sqrt{(x_i^o)^2 + 4\alpha_i r(\|z^o\| - r)}\Big)^{\alpha_i} - r = 0 \\ 0 < r < \|z^o\| \end{cases} \tag{28}$$

(ii) *If* $(x^o, z^o) \notin K, K^o$ *and* $z^o = 0$ *then its projection onto* K *is*

$$\begin{cases} \overline{x}_i &= [x_i^o]_+, \quad i = 1, \dots, m \\ \overline{z}_l &= 0, \quad l = 1, \dots, n. \end{cases}$$

Theorem A.1. *The Euclidean projection on to* $K_\alpha^{(m,n)}$ *is continuously differentiable at* (x, z) *if and only if*

(1) $(x, z) \in \mathrm{int}(K)$ *or* $(x, z) \in \mathrm{int}(K^o)$,
(2) $(x, z) \notin K, K^o$ *and* $z \neq 0$,
(3) $(x, z) \notin K, K^o, z = 0$ *and* $x_i \neq 0, i = 1, \dots, m$.

In the second case, the projection is furthermore twice continuously differentiable and its Jacobian is defined by the following rows :

$$\begin{aligned} \Pi'_{x_i} &= \left(\frac{\alpha_j \alpha_i (\|z\| - 2r) r L^{(m,n)}}{g_j g_i}\Big|_{j \neq i}, \frac{1}{2} + \frac{x_i}{2g_i} + \frac{\alpha_i^2 (\|z\| - 2r) r L^{(m,n)}}{g_i^2}, \frac{z_l}{\|z\|} \frac{\alpha_i r L^{(m,n)}}{g_i}\Big|_{l=1,\dots,n} \right) \\ \Pi'_{z_l} &= \left(\frac{z_l}{\|z\|} \frac{\alpha_j r L^{(m,n)}}{g_j}\Big|_{j=1,\dots,m}, \frac{r}{\|z\|} + \frac{r z_l^2}{\|z\|^3} T^{(m,n)} L^{(m,n)}, \frac{r z_l z_k}{\|z\|^3} T^{(m,n)} L^{(m,n)}\Big|_{k \neq l} \right) \end{aligned}$$

where r *is the unique solution of the system* $E(x, z)$*, see* (28)*, and*

$$\begin{aligned} g_i &= \sqrt{x_i^2 + 4\alpha_i r(\|z\| - r)}, f_i = \tfrac{1}{2}(x_i + g_i), i = 1, \dots, m \\ L^{(m,n)} &= \frac{1}{1 - r(\|z\| - 2r)\sum_{j=1}^{m} \frac{\alpha_j^2}{g_j f_j}}, \quad T^{(m,n)} = 2r(\|z\| - r)\sum_{j=1}^{m} \frac{\alpha_j^2}{g_j f_j} - 1. \end{aligned}$$

In the third case, the Jacobian of the projection is the diagonal matrix

$$\mathbf{D}^{(m,n)} = \mathrm{Diag}\left(s(x_i)\ |_{i=1,\dots,m}, \quad , \quad d^{(m,n)}\ |_{l=1,\dots,n} \right),$$

where

$$d^{(m,n)} = \begin{cases} 1 & \textit{if } \sum\limits_{x_i>0} \alpha_i > \sum\limits_{x_i<0} \alpha_i \\ 0 & \textit{if } \sum\limits_{x_i>0} \alpha_i < \sum\limits_{x_i<0} \alpha_i \\ \left(\left(\dfrac{\prod\limits_{x_i<0} (-x_i)^{\alpha_i}}{\prod\limits_{x_i<0} \alpha_i^{\alpha_i} \prod\limits_{x_i>0} (x_i)^{\alpha_i}} \right)^{\frac{1}{\sum_{x_i<0} \alpha_i}} + 1 \right)^{-1} & \textit{if } \sum\limits_{x_i>0} \alpha_i = \sum\limits_{x_i<0} \alpha_i \ . \end{cases}$$

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