

On the exact separation of rank inequalities for the maximum stable set problem

Stefano Coniglio⁺, Stefano Gualandi⁺

⁺Lehrstuhl II für Mathematik, RWTH Aachen University, Aachen, Germany

^{*}AntOptima SA, Lugano, Switzerland

{stefano.coniglio,stefano.gualandi}@gmail.com

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Abstract

When addressing the maximum stable set problem on a graph $G = (V, E)$, rank inequalities prescribe that, for any subgraph $G[U]$ induced by $U \subseteq V$, at most as many vertices as the stability number of $G[U]$ can be part of a stable set of G . These inequalities are very general, as many of the valid inequalities known for the problem can be seen as rank inequalities in which $G[U]$ is restricted to a specific topology. In spite of their generality, their exact separation (without introducing topological restrictions) has not yet been investigated, at least to our knowledge.

In this work, we first formulate the separation problem of rank inequalities as a bilevel integer program. Starting from it, we derive what we call rounded fractional rank inequalities, a weaker version of rank inequalities which we show to contain clique, odd hole, odd antihole, odd wheel (with unit weights), antiweb, and a (well characterized) subset of web inequalities. We show that these inequalities can be separated exactly by solving a single level mixed-integer linear program. We then propose a restriction of rank inequalities where the right-hand side is fixed to a given constant, which we again show to be separated exactly by solving a single level integer linear program. By iteratively increasing the right-hand side, the exact separation of the whole family is achieved. Computational results adopting the separation algorithms that we develop show that rounded fractional rank inequalities and rank inequalities with a small, fixed right-hand side yield a substantial bound improvement over that provided by the fractional clique polytope.

Keywords: Maximum stable set problem, Rank inequalities, Cutting planes, Integer programming, Combinatorial optimization, Bilevel programming

1 Introduction

Consider an undirected graph $G = (V, E)$ with vertex set V and edge set E , and let $n = |V|$. The *maximum stable set problem* calls for a stable set of G of maximum cardinality, that is for the largest subset of V where no two vertices share an edge. The problem is one of Karp’s 21 \mathcal{NP} -hard problems [Kar72] and, to date, is among the most challenging “fundamental” problems in combinatorial optimization when tackled via integer programming techniques. From a polyhedral point of view, the focus is on studying:

$$STAB(G) = \text{conv} \left\{ x \in \{0, 1\}^n : x \text{ denotes a stable set of } G \right\}. \quad (1)$$

As customary, we refer to the *stability number* of G , i.e., to the size of one of its largest stable sets, as $\alpha(G)$, namely, $\alpha(G) = \max \{ \sum_{i \in V} x_i : x \in STAB(G) \}$.

In this paper, we address the class of so-called *Rank Inequalities* (RIs), introduced by Chvátal in [Chv75]. In their general form, RIs prescribe that, for any subgraph $G[U]$ induced by $U \subseteq V$, at most as many vertices

as the stability number of $G[U]$ can be part of a stable set of G . Formally:

$$\sum_{i \in U} x_i \leq \alpha(G[U]) \quad \forall U \subseteq V. \quad (2)$$

By construction, RIs are all the inequalities with binary coefficients in the left-hand side that are valid for $STAB(G)$. Indeed, an inequality $\pi x \leq \pi_0$ is valid for some $Q \subseteq \mathbb{R}^n$ if and only if $\pi_0 \geq \max\{\pi x : x \in Q\}$. The definition of RIs follows when letting $Q = STAB(G)$ and restricting ourselves to $\pi \in \{0, 1\}^n$. In the rest of the paper, we will refer to the closure of RIs as:

$$RSTAB(G) = \left\{ x \in \mathbb{R}_+^n : \sum_{i \in U} x_i \leq \alpha(G[U]) \quad \forall U \subseteq V \right\}. \quad (3)$$

The study of RIs is of interest since, when restricting $G[U]$ to some specific topologies, we obtain many of the valid inequalities for $STAB(G)$ that are known in the literature. We will cover the most relevant ones in Section 2. For an illustration of a RI, see Figure 1. The figure adopts the so-called Chvátal graph, which we will use as running example throughout the paper.

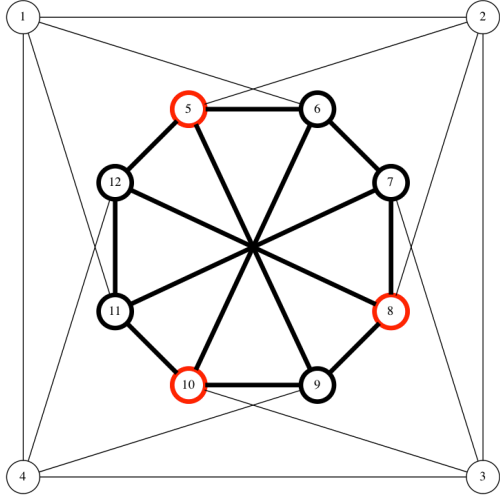


Figure 1: The Chvátal graph (the smallest triangle-free 4-colorable 4-regular graph, see [Chv70]) with, in bold, the subgraph $G[U]$ induced by $U = \{5, 6, 7, 8, 9, 10, 11, 12\}$ and, in red, one of its maximum stable sets, yielding $\alpha(G[U]) = 3$. Note that, due to G being triangle-free, $FSTAB(G) = QSTAB(G)$ (see Section 2). The corresponding (unique) LP solution is $x_i^* = \frac{1}{2}$ for all $i \in V$, yielding an upper bound of 6. x^* violates the RI $\sum_{i \in U} x_i \leq \alpha(G[U]) = 3$, due to $\sum_{i \in U} x_i^* = 4$. Note that $G[U]$ is isomorphic to the web $W(8, 3)$ (see Section 2).

To the best of our knowledge, there are no computational attempts in the literature at separating *exactly* the general family of rank inequalities for the maximum stable set problem without topological restrictions. This paper aims at bridging such gap. After providing some background material and outlining related work in Section 2, in Section 3 we point out how, by their very definition, RIs entail a bilevel separation problem where, at the second level, a maximum stable set problem is solved. Since the latter is, in principle, an instance of the very problem for the solution of which RIs are of interest, it is clear that a different strategy not relying on this bilevel problem is needed. We consider two approaches. In the first one, reported in Section 4, we propose what we call *Rounded Fractional Rank Inequalities* (RFRIs), obtained by relaxing the right-hand side of RIs so to arrive, via a Linear Programming (LP) duality argument, at a separation problem that can be cast as a single level Mixed-Integer Linear Program (MILP). We show that many of the most relevant inequalities for the maximum stable set problem are RFRIs. In the second approach, as reported in Section 5, we propose an exact way to optimize over $RSTAB(G)$ by separating *Rank Inequalities with a fixed right-hand side* $\bar{\alpha}$ (RIs- $\bar{\alpha}$). With those inequalities, by progressively increasing the value of $\bar{\alpha}$, the exact optimization over $RSTAB(G)$ is achieved. We establish some complexity results and show again how to cast the corresponding separation problem as a single level MILP. In Section 6, we discuss on the implementation of our separation algorithms and illustrate the results of computational experiments aimed at assessing the tightness of the two families of inequalities in practice. Section 7 draws some concluding remarks and outlines ideas for future developments.

2 Related work

In this section, we introduce some of the most relevant relaxations and valid inequalities for the maximum stable set problem. For more references, primarily from an integer programming (and branch-and-cut oriented) point of view, we refer the reader to [Reb09].

Given any relaxation $P(G) \supseteq STAB(G)$, we will adopt the notation $\alpha_P(G) = \max \{ \sum_{i \in V} x_i : x \in P(G) \}$. Among the different relaxations that can be considered for the maximum stable set problem, the arguably “minimal” LP relaxation which is exact when integrality restrictions are introduced is:

$$FSTAB(G) = \{ x \in \mathbb{R}_+^n : x_i + x_j \leq 1, \forall \{i, j\} \in E \}. \quad (4)$$

The inequalities $x_i + x_j \leq 1$ for all $\{i, j\} \in E$ are typically called *edge inequalities*. As it can be easily verified, $x_i^* = \frac{1}{2}$ for all $i \in V$ is always a feasible solution. It follows that $FSTAB(G)$ provides a typically weak bound, with $\alpha_{FSTAB}(G) \geq \frac{n}{2}$ even for graphs with $\alpha(G) = 1$. Better formulations and tighter bounds can be obtained by including a number of valid inequalities. We describe some of them in the following. For an example of the corresponding topologies, see Figure 2.

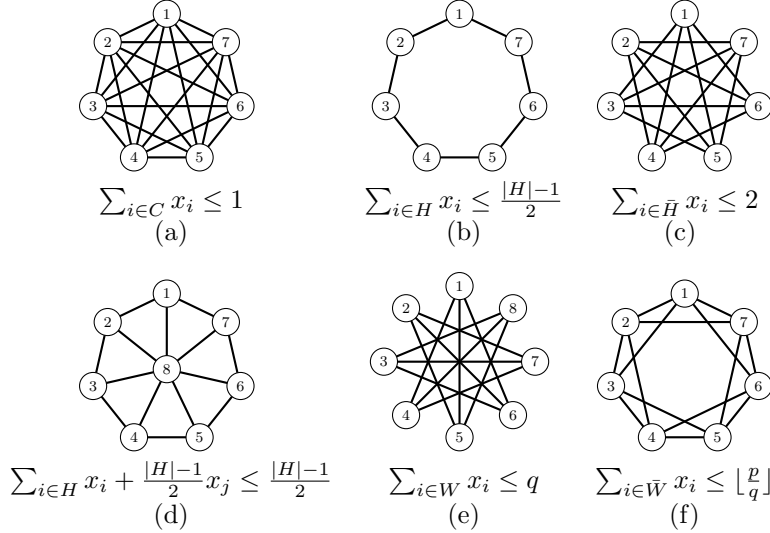


Figure 2: Six subgraphs with the corresponding inequality: (a) a clique with vertex set C , (b) an odd hole induced by H , (c) an odd antihole induced by \bar{H} , (d) an odd wheel with vertex set $H \cup \{j\}$, (e) a web $W(p, q)$ with vertex set W , $p = 8, q = 3$, (f) an antiweb $\bar{W}(p, q)$ with vertex set \bar{W} , $p = 7, q = 3$.

In [Pad73], Padberg introduces *clique inequalities* $\sum_{i \in C} x_i \leq 1$, where $C \subseteq V$ induces a clique of G , i.e., a complete subgraph. They are shown to be facet defining if and only if the corresponding clique is maximal. Due to $\alpha(G[C]) = 1$, clique inequalities are RIs. The same paper also introduces *odd hole inequalities* $\sum_{i \in H} x_i \leq \frac{|H|-1}{2}$, where $H \subseteq V$ induces an odd hole of G , i.e., a chordless cycle with an odd number of components, and *odd antihole inequalities* $\sum_{i \in \bar{H}} x_i \leq 2$, where $\bar{H} \subseteq V$ induces an odd antihole, i.e., an odd hole in the complement graph \bar{G} of G . In general, none of the two is facet defining, unless G is itself, respectively, an odd hole or an odd antihole. It can be easily verified that $\alpha(G[H]) = \frac{|H|-1}{2}$ and $\alpha(G[\bar{H}]) = 2$. Hence, both inequalities are RIs. In the same work, via a lifting argument, Padberg introduces *odd wheel inequalities* $\sum_{i \in H} x_i + \frac{|H|-1}{2}x_j \leq \frac{|H|-1}{2}$, where $H \cup \{j\}$ induces an odd wheel of G , i.e., an odd hole induced by $H \subseteq V$ with an extra vertex $j \in V \setminus H$ connected to all the vertices $i \in H$. Similarly to the previous cases, such inequalities induce a facet when G is itself an odd wheel, but not in general. Since the coefficient of x_j is larger than 1, odd wheel inequalities are not RIs. Nevertheless, due to $\alpha(G[H \cup \{j\}]) = \frac{|H|-1}{2}$, the weaker inequality $\sum_{i \in H} x_i + x_j \leq \frac{|H|-1}{2}$ is a RI.

In [TJ75], Trotter introduces *web inequalities* $\sum_{i \in W} x_i \leq q$ where $W \subseteq V$ induces a web $W(p, q)$ of G , i.e., a subgraph with p vertices $\{1, \dots, p\}$ with, adopting modulo p arithmetic, an edge between each two vertices i and $j \in \{i+q, \dots, i-q\}$. Web inequalities generalize clique, odd hole, and odd antihole inequalities due to $W(p, 1)$ being a clique and, for any integer $s \geq 2$, $W(2s+1, s)$ and $W(2s+1, 2)$ being, respectively, an odd hole and an odd antihole. If G is a web, the inequalities are facet defining if and only if p and q are relatively prime, but not in the general case. Trotter also introduces *antiweb inequalities* $\sum_{i \in \bar{W}} x_i \leq \lfloor \frac{p}{q} \rfloor$, where \bar{W} induces an antiweb $\bar{W}(p, q)$ of G , i.e., a web $W(p, q)$ of \bar{G} . Again, if G is itself an antiweb, these inequalities are facet defining if and only if $q \geq 2$ and p and q are relatively prime, but not in the general case. Due to $\alpha(W(p, q)) = q$ and $\alpha(\bar{W}(p, q)) = \lfloor \frac{p}{q} \rfloor$, see [TJ75], web and antiweb inequalities are RIs.

Since RIs generalize all the inequalities that we discussed (except for odd wheel inequalities with a coefficient greater than one) and most of them are not facet defining, it is clear that RIs cannot be facet defining as a family, even if $G[U] = G$ (consider, for instance, the case of $G = W(p, q)$ with p multiple of q). For sufficient conditions for which the rank inequality $\sum_{i \in V} x_i \leq \alpha(G)$ is facet defining for G , see [Chv75], [BZ77], and [Bal97]. For necessary and sufficient conditions for which, given a RI corresponding to a subgraph $G[U]$ and facet defining for it, the RI is also facet defining for the original graph G , see [BZ77].

As usual, we refer to the formulation that is obtained when all the (maximal) clique inequalities are considered as:

$$QSTAB(G) = \left\{ x \in \mathbb{R}_+^n : \sum_{i \in C} x_i \leq 1 \quad \forall C \in \mathcal{K} \right\}, \quad (5)$$

where \mathcal{K} , adopting a notation that will be used throughout the paper, is the collection of all the subsets of V inducing inclusion-wise maximal cliques of G . Note that, since any two vertices sharing an edge form a clique, the edge inequalities of $FSTAB(G)$ are always satisfied in $QSTAB(G)$. Hence, $QSTAB(G) \subseteq FSTAB(G)$.

A nonpolyhedral relaxation of $STAB(G)$, which is at least as tight as $QSTAB(G)$, is the so-called ϑ -body of G :

$$THETA(G) = \left\{ x \in \mathbb{R}^n : \exists Y \in \mathbb{R}^{n \times n} \text{ with } \begin{cases} Y - xx^T \succeq 0 \\ \text{diag}(Y) = x \\ Y_{ij} = 0 \quad \forall \{i, j\} \in E \end{cases} \right\}. \quad (6)$$

Lovász's number $\vartheta(G) = \max\{\sum_{i \in V} x_i : x \in THETA(G)\}$ is an upper bound on the stability number of G [GLS88]. The corresponding SemiDefinite Program (SDP) can be solved in polynomial time to an arbitrary (fixed) precision by employing an interior point method, as it is done, for instance, in [GR03]. For more recent results on the derivation of tight bounds for the maximum stable set problem via $THETA(G)$ and closely related sets, we refer the reader to [GLRS13, GLRS11]. Recent work on branch-and-cut methods for the problem can be found in [ROT⁺11].

As we mentioned in Section 1, to the best of our knowledge there are no known approaches to optimize exactly over $RSTAB(G)$ without introducing topological restrictions over the induced subgraph $G[U]$. The only work where rank inequalities are separated (although heuristically) can be found in the paper by Rossi and Smriglio [RS01] (their technique is then readopted in [ROT⁺11]). It relies on the *edge projection* operator, introduced by Mannino and Sassano in [MS96] (although, therein, it is only used to produce a combinatorial upper bound for the maximum stable set problem). The operator produces a smaller graph $G' = (V', E')$ by first removing an edge $e = \{v, w\}$ and the common neighborhood of its endpoints $\delta(v) \cap \delta(w)$ and, then, adding an edge $\{v', w'\}$ between every pair of remaining vertices $v' \in \delta(v) \setminus \delta(w)$ and $w' \in \delta(w) \setminus \delta(v)$. If e meets certain (typically only sufficient, for practical reasons) conditions, it can be shown that $\alpha(G) = \alpha(G') + 1$. Given a rank inequality $\sum_{i \in U} x_i \leq \alpha(G'[U])$ for some $U \subseteq V'$, valid for $STAB(G')$ but, in general, not for $STAB(G)$, we can lift it to $\sum_{i \in U} x_i + \sum_{i \in \delta(v) \cap \delta(w)} x_i + x_v + x_w \leq \alpha(G'[U]) + 1$, valid for $STAB(G)$. The (practical) method that the authors adopt performs a sequence of edge projections, thus constructing a sequence $G', G'', \dots, G^{(k)}$ of graphs, and then (heuristically) finds a violated clique inequality $\sum_{i \in V^{(k)}} x_i \leq 1$ in $G^{(k)}$, thus only considering a subset of the RIs valid for $STAB(G^{(k)})$, which is then lifted to be valid for $STAB(G)$. Although the method allows for the generation of a subset of RIs without a specific topological

restriction, due to its heuristic nature it can halt before all the violated RIs have been found. Hence, it does not allow to optimize exactly over $RSTAB(G)$.

3 Preliminaries: on the separation of rank inequalities

Let x^* denote an optimal solution to some relaxation of the maximum stable set problem, such as $FSTAB(G)$ or $QSTAB(G)$. Given x^* , the separation problem for rank inequalities calls for a subset U of vertices such that $\sum_{i \in U} x_i^* > \alpha(G[U])$, or for a proof that none such subset exists.

As noted in [LS90], due to how RIs are defined, even checking whether a given inequality is a RI is difficult. Formally, we can establish the following:

Proposition 3.1. *Given a graph $G = (V, E)$ and a vector $(\pi, \pi_0) \in \mathbb{R}^{n+1}$, the membership problem of verifying whether $\pi x \leq \pi_0$ is a rank inequality is \mathcal{NP} -hard.*

Proof. The recognition version of the maximum stable set problem, which asks whether $\alpha(G) \geq k$ for some input graph G and integer k , can be solved by solving the membership problem for RIs a linear number of times. It suffices to call a routine that solves the latter at most $n - k + 1$ times, with input G and the inequality $\pi x \leq \pi_0$ with $\pi_i = 1$ for all $i \in V$ and π_0 taking increasing values from k to n . By definition, $\pi x \leq \pi_0$ is a RI for our choice of π and π_0 if and only if $\pi_0 = \alpha(G)$. Hence, as soon as the routine for the membership problem returns the answer “yes” for some π_0 , we have $\pi_0 = \alpha(G)$ and $\pi_0 \geq k$, thus providing answer “yes” to the recognition version of the maximum stable set problem. \square \square

This result is not surprising. Indeed, given $x^* \in \mathbb{R}^n$, one can formally cast the decision problem of finding a RI of violation greater than a given $k > 0$ as $\exists U \subseteq V (\forall \text{ stable sets } S \subseteq U (\sum_{i \in U} x_i^* - |S| \geq k))$. In this form, the problem is easily recognizable as belonging to Σ_2^P , the second level of the polynomial hierarchy. For the connection between Σ_2^P and the separation of cutting planes with a special structure, we refer the reader to [LRW13].

What is more, when it comes to finding a maximally violated RI, the corresponding separation problem becomes almost naturally a bilevel problem. It calls for, at the upper level, the selection of a subset of vertices $U \subseteq V$ and, at the lower level, for the identification of a stable set of maximum cardinality $\alpha(G[U])$ in the subgraph $G[U]$ induced by U . The upper level choice has to be made so as to maximize the cut violation, i.e., the quantity $\sum_{i \in U} x_i^* - \alpha(G[U])$. Let $u \in \{0, 1\}^n$ be the characteristic vector of U . Formally, we are thus solving:

$$\max \quad \sum_{i \in V} x_i^* u_i - \lambda \tag{7}$$

$$\text{s.t.} \quad \lambda = \max \left\{ \sum_{i \in U} z_i : z \in STAB(G[U]) \right\} \tag{8}$$

$$u \in \{0, 1\}^n \tag{9}$$

$$\lambda \geq 0 \tag{10}$$

$$z \in [0, 1]^{|U|}. \tag{11}$$

The second level problem imposes $\lambda = \alpha(G[U])$ by guaranteeing that z be the characteristic vector of a stable set of $G[U]$ of maximum cardinality. Note that, due to employing $STAB(G)$, the integrality on the second level variable z does not need to be explicitly imposed.

Ideally, we would like to cast the separation problem as a single level MILP. In general, this can be done for bilevel programs when necessary and sufficient optimality conditions for the second level problem are known and when such conditions can be written by employing linear constraints and mixed-integer variables. Whenever a full polyhedral description of the second level problem is available, as, for instance, in [CT13] or [LMS98], LP duality directly allows for the construction of one such set of optimality conditions. Unfortunately, this is typically not the case for \mathcal{NP} -hard problems, for which a closed-form description of their convex hull is usually not known. The same holds for RIs for which, in the separation problem, the

second level problem is itself an instance of the maximum stable set problem which we aim at solving in the first place. In the next two sections, we will propose two alternative approaches that do not rely on solving Problem (7)–(11) directly.

4 Exact separation of rank inequalities with a relaxed right-hand side

The idea that we pursue in this section is of relaxing the right-hand side $\alpha(G[U])$ of rank inequalities (at the cost of weakening them) so to obtain a bilevel separation problem for which the second level problem has known optimality conditions.

4.1 A general relaxation of rank inequalities

Consider a graph G and some relaxation $P(G) \supseteq STAB(G)$. Let $\alpha_P(G) = \max \{ \sum_{i \in V} x_i : x \in P(G) \}$. Since $\alpha(G) \leq \alpha_P(G)$, the inequality

$$\sum_{i \in U} x_i \leq \alpha_P(G[U]) \quad \forall U \subseteq V, \quad (12)$$

which corresponds to a weaker version of the rank inequality $\sum_{i \in U} x_i \leq \alpha(G[U])$ with an at least as large right-hand side, is valid. Constraints (12) are easily characterized:

Proposition 4.1. *Constraints (12) correspond to all the (undominated) inequalities with binary left-hand side coefficients that are valid for $P(G)$.*

Proof. This is due to an inequality $\pi x \leq \pi_0$ being valid for a set $Q \subseteq \mathbb{R}^n$ if and only if $\pi_0 \geq \max \{ \pi x : x \in Q \}$. The claim follows when adding the restriction $\pi \in \{0, 1\}^n$ and letting $Q = P(G)$. \square \square

The result implies that the closure of Constraints (12) is, at most, as tight as $P(G)$. From a cutting plane point of view, if $P(G)$ is adopted as the starting relaxation of $STAB(G)$, Constraints (12) are completely unnecessary. They might still be of interest if $P(G)$ is not a polyhedron though, such as, e.g., when choosing $P(G) = THETA(G)$. In that case, their closure (which is always a polyhedron, for their number is finite, being upperbounded by $2^{|V|}$) corresponds to the tightest (by construction) polyhedral relaxation of $P(G)$ which only employs inequalities with binary left-hand side coefficients. Although the topic of constructing tight approximations of the ϑ -body is of clear interest, it is outside of the scope of this paper. We refer the interested reader to [GRS13, GLRS09, GLRS13].

To obtain new inequalities that are not implied by $P(G)$, we round the relaxed right-hand side to the nearest integer. We obtain:

$$\sum_{i \in U} x_i \leq \lfloor \alpha_P(G[U]) \rfloor \quad \forall U \subseteq V. \quad (13)$$

Due to their derivation, we can precisely characterize Constraints (13):

Proposition 4.2. *Constraints (13) correspond to all the (undominated) rank-1 Chvátal-Gomory cuts valid for $P(G)$ with binary left-hand side coefficients.*

Proof. A cut $\pi x \leq \pi_0$ is an (undominated) rank-1 Chvátal-Gomory cut valid for the integer hull of a set $Q \subseteq \mathbb{R}_+^n$ if and only if $\pi \in \mathbb{Z}^n$ and $\pi_0 = \lfloor \max \{ \pi x : x \in Q \} \rfloor$. Constraints (13) are obtained when the restriction $\pi \in \{0, 1\}^n$ is added, letting $Q = P(G)$. \square \square

When $P(G)$ is a polyhedron, the separation problem for Constraints (13), also including the rounding aspect, can be cast as a single level MILP:

Proposition 4.3. Let $P(G) = \{x \in \mathbb{R}_+^n : Ax \leq b\}$, where $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m$, with $P(G) \supseteq STAB(G)$. The separation problem for a maximally violated Constraint (13) asks for a subset $U \subseteq V$ which solves:

$$\max_{\substack{U \subseteq V \\ \lambda \in \mathbb{Z}_+}} \left\{ \sum_{i \in U} x_i^* - \lambda : \lambda = \left\lfloor \max_{z \in [0,1]^{|U|}} \left\{ \sum_{i \in U} z_i : z \in P(G[U]) \right\} \right\rfloor \right\}. \quad (14)$$

Let $u \in \{0,1\}^n$ be the incidence vector of $U \subseteq V$. Problem (14) can be cast as the single level MILP:

$$\max \sum_{i \in V} x_i^* u_i - \lambda \quad (15)$$

$$\sum_{j=1}^m a_{ji} y_j \geq u_i \quad \forall i \in V \quad (16)$$

$$\lambda > \sum_{j=1}^m y_j - 1 \quad (17)$$

$$y_j \geq 0 \quad \forall j = 1, \dots, m \quad (18)$$

$$u_i \in \{0,1\} \quad \forall i \in V \quad (19)$$

$$\lambda \in \mathbb{Z}_+. \quad (20)$$

Proof. Restate the second level problem in (14) as $\max\{\sum_{i \in V} u_i z_i : Az \leq b, z \in \mathbb{R}_+^n\}$. The corresponding LP dual reads $\min\{\sum_{j=1}^m y_j : yA \geq u, y \in \mathbb{R}_+^m\}$. By LP duality and due to Constraints (16) and (18), for any $u \in \{0,1\}^n$ we have $\sum_{j=1}^m y_j \geq \max\{\sum_{i \in V} u_i z_i : Az \leq b, z \in \mathbb{R}_+^n\}$. Due to Constraint (17), $\lambda \geq \lfloor \sum_{j=1}^m y_j \rfloor$. The constraint is strict so to prevent λ from taking value $\sum_{j=1}^m y_j - 1$ when $\sum_{j=1}^m y_j$ is integer. Since the objective function in (15) is a decreasing function of λ , we will have $\lambda = \lfloor \sum_{j=1}^m y_j \rfloor$ in any optimal solution. For the same reason, in any optimal solution we will also have $\lfloor \sum_{j=1}^m y_j \rfloor = \lfloor \max\{\sum_{i \in V} u_i z_i : Az \leq b, z \in \mathbb{R}_+^n\} \rfloor = \lfloor \alpha_P(G[U]) \rfloor$. \square

We remark that this result can be generalized to more complicated relaxations of $STAB(G)$ which cannot be solved by linear programming. For instance, when choosing $P(G) = THETA(G)$, semidefinite programming duality allows to obtain a similar reformulation amounting to a single level (mixed-integer semidefinite) program. Note that, in that case, as well as for any $P(G) \supseteq STAB(G)$, the closure of Constraints (13) is, due to their finiteness, always a polyhedral set. Due to the greater solvability of the arising separation problem, in the following we will solely focus on polyhedral relaxations.

4.2 Rounded fractional rank inequalities

Among the different choices for $P(G)$, we will now focus on $P(G) = QSTAB(G)$, the so-called *fractional stable set polytope* of G . We refer to the corresponding specialized version of Constraints (13) as *Rounded Fractional Rank Inequalities* (RFRIs):

$$\sum_{i \in U} x_i \leq \lfloor \alpha_{QSTAB}(G[U]) \rfloor \quad \forall U \subseteq V. \quad (21)$$

We are interested in comparing the tightness of RFRIs to that of RIs, with special focus on the RIs with specific topology that we discussed in Section 2. A first, general results is:

Lemma 4.4. Given $U \subseteq V$, the RFRI corresponding to $G[U]$ is a RI if and only if $\alpha(G[U]) = \lfloor \alpha_{QSTAB}(G[U]) \rfloor$, i.e., $\alpha_{QSTAB}(G[U]) - \alpha(G[U]) < 1$.

This is the case, for instance, when $G[U]$ is a perfect graph (since, as already proven in 1975 by Chvátal [Chv75], $QSTAB(G[U]) = STAB(G[U])$ if $G[U]$ is perfect). Due to cliques being perfect graphs, we directly have:

Proposition 4.5. *RFRI subsume clique inequalities.*

To analyze the cases of odd hole, odd antihole, web, and antiweb inequalities, we leverage a result from [SU11]. Recall that an automorphism of a graph $G = (V, E)$ is a bijection $\mu : V \rightarrow V$ such that, if $\{v, w\} \in E$, then $\{\mu(v), \mu(w)\} \in E$. We say that a graph $G = (V, E)$ is *vertex-transitive* if, for any pair of vertices $v, w \in V$, there is an automorphism μ of G for which $\mu(v) = w$ (roughly speaking, this means all the vertices in the graph are “topologically equivalent” or that the graph is “symmetric”). Let $\chi_f(G)$ be the fractional chromatic number of G . We have:

Proposition 4.6. [SU11] *For any vertex-transitive graph $G = (V, E)$, $\chi_f(G) = \frac{|V|}{\alpha(G)}$.*

Let $\omega(G)$ be the size of the largest clique in G . By taking the complement of G and noting that $\chi_f(\bar{G}) = \alpha_{QSTAB}(G)$ and $\alpha(\bar{G}) = \omega(G)$, we have:

Corollary 4.7. *For any vertex-transitive graph $G = (V, E)$, $\alpha_{QSTAB}(G) = \frac{|V|}{\omega(G)}$.*

All the inequalities with a given topology that we considered, with the sole exception of odd wheel inequalities, correspond to vertex-transitive subgraphs. We analyze them in the following.

Proposition 4.8. *RFRI subsume odd hole inequalities.*

Proof. Let $H \subseteq V$ induce an odd hole $G[H]$. Since $\omega(G[H]) = \alpha(\bar{G}[H]) = 2$, where $\bar{G}[H]$ is the odd antihole induced by H , due to Corollary 4.7 we have $\alpha_{QSTAB}(G[H]) = \frac{|H|}{2}$. The claim follows due to Lemma 4.4 since, for an odd $|H|$, $\alpha(G[H]) = \frac{|H|-1}{2} = \left\lfloor \frac{|H|}{2} \right\rfloor = \lfloor \alpha_{QSTAB}(G[H]) \rfloor$. \square

Proposition 4.9. *RFRI subsume odd antihole inequalities.*

Proof. Let $\bar{H} \subseteq V$ induce an odd antihole $G[\bar{H}]$. Note that $\omega(G[\bar{H}]) = \alpha(\bar{G}[\bar{H}]) = \frac{|\bar{H}|-1}{2}$, where $\bar{G}[\bar{H}]$ is the odd hole induced by \bar{H} . By virtue of Corollary 4.7, we have $\alpha_{QSTAB}(G[\bar{H}]) = \frac{|\bar{H}|}{\frac{|\bar{H}|-1}{2}} = 2 \frac{|\bar{H}|}{|\bar{H}|-1}$. The claim follows due to Lemma 4.4 since, for any $\bar{H} : |\bar{H}| > 3$, $\left\lfloor 2 \frac{|\bar{H}|}{|\bar{H}|-1} \right\rfloor = 2 = \alpha(G[\bar{H}])$. \square

Proposition 4.10. *RFRI subsume web inequalities corresponding to webs $W(p, q)$ if and only if the remainder $p - \lfloor \frac{p}{q} \rfloor q$ of the integer division between p and q is strictly smaller than the quotient $\lfloor \frac{p}{q} \rfloor$.*

Proof. Let $W \subseteq V$ induce a web $W(p, q)$. Note that $\omega(W(p, q)) = \alpha(\bar{W}(p, q)) = \lfloor \frac{p}{q} \rfloor$. As a consequence of Corollary 4.7, we have $\alpha_{QSTAB}(W(p, q)) = \frac{p}{\lfloor \frac{p}{q} \rfloor}$. Since $\alpha(W(p, q)) = q$, by applying Lemma 4.4 we have a RFRI if and only if $\frac{p}{\lfloor \frac{p}{q} \rfloor} - q < 1$. The claim follows upon multiplication of the expression by $\lfloor \frac{p}{q} \rfloor$, which yields $p - \lfloor \frac{p}{q} \rfloor q < \lfloor \frac{p}{q} \rfloor$. \square

Proposition 4.11. *RFRI subsume antiweb inequalities.*

Proof. Let $\bar{W} \subseteq V$ induce an antiweb $\bar{W}(p, q)$. After noting $\omega(\bar{W}(p, q)) = \alpha(W(p, q)) = q$, the application of Corollary 4.7 yields $\alpha_{QSTAB}(\bar{W}(p, q)) = \frac{p}{q}$. The claim follows from Lemma 4.4, since $\alpha(\bar{W}(p, q)) = \left\lfloor \frac{p}{q} \right\rfloor$. \square

Since odd wheels are not vertex-transitive, for odd wheel inequalities (with unit left-hand side coefficients) we use a combinatorial argument.

Proposition 4.12. *RFRI subsume odd wheel inequalities with unit left-hand side coefficients.*

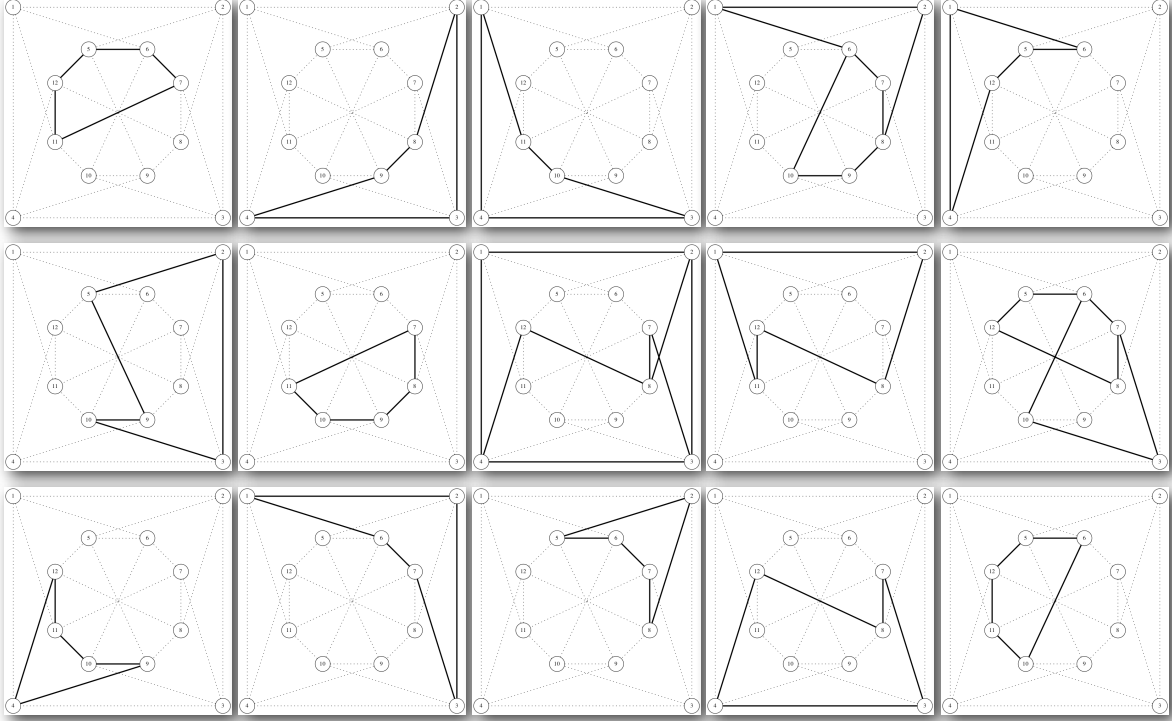


Figure 3: The set of the 15 rounded fractional rank inequalities which are generated when optimizing over their closure for the Chvátal graph, adopting a cutting plane method. They yield a bound of 4.8, as opposed to $\alpha_{QSTAB}(G) = 6$ and $\vartheta(G) = 4.895$.

Proof. Given $H \subseteq V$ inducing an odd hole and an extra vertex $j \in V \setminus H$ sharing an edge with each vertex in H , the odd wheel inequality $\sum_{i \in H} x_i + \frac{|H|-1}{2} x_j \leq \frac{|H|-1}{2}$ is not a RI (due to having a nonbinary coefficient) and, hence, not a RFRI. Consider the weaker inequality $\sum_{i \in H} x_i + x_j \leq \frac{|H|-1}{2}$. Note that $G[H \cup \{j\}]$ contains $|H|$ cliques, all of size three, each containing the vertex j and two vertices in H , so that j is covered $|H|$ times, while each vertex in H is covered twice. We show $\alpha_{QSTAB}(G[H \cup \{j\}]) = \frac{|H|}{2}$, achieved for $x_i^* = \frac{1}{2}$ for all $i \in H$, $x_j = 0$. The solution is feasible since the sum of x_i^* over each clique is one. In the dual, the solution $y_j^* = \frac{1}{2}$ is feasible since the sum of y_j^* over each vertex $i \in H$ is 1, while the sum over j is $\frac{|H|}{2} \geq 1$. Since both solutions have value $\frac{|H|}{2}$, they are optimal. For an odd $|H|$, we then have $\lfloor \alpha_{QSTAB}(G[H \cup \{j\}]) \rfloor = \frac{|H|-1}{2}$. Since $\alpha(G[H \cup \{j\}]) = \frac{|H|-1}{2}$, the claim follows due to Lemma 4.4. \square \square

On top of those results, when generating RFRI with a cutting plane method, we observe that their family contains a number of *template-free* inequalities, not adhering to any specific topological structure. We also observe many instances where $G[U]$ is not even a regular graph –as opposed to the case of all the “classical” inequalities that we mentioned, where $G[U]$ is regular, with the only exception of odd wheel inequalities. A graphical example is reported in Figure 3.

For $j = 1, \dots, |\mathcal{K}|$, let $C_j \in \mathcal{K}$ be a subset of vertices inducing a maximal clique of G . When separating

RFRIs, the single level MILP formulation of Proposition 4.3 translates into:

$$\max \sum_{i \in V} x_i^* u_i - \lambda \quad (22)$$

$$\lambda > \sum_{C_j \in \mathcal{C}} q_j - 1 \quad (23)$$

$$\sum_{C_j \in \mathcal{K}: i \in C_j} q_j \geq u_i \quad \forall i \in V \quad (24)$$

$$q_j \geq 0 \quad \forall C_j \in \mathcal{K} \quad (25)$$

$$u_i \in \{0, 1\} \quad \forall i \in V \quad (26)$$

$$\lambda \in \mathbb{Z}_+, \quad (27)$$

where the variables q_j are the variables of the fractional clique cover problem, dual to the fractional stability number problem of computing $\alpha_{QSTAB}(G[U])$, the right-hand side of Constraint (23) (minus the constant) is the corresponding objective function value, and Constraints (24) are the corresponding clique cover constraints.

Although Problem (22)–(27) contains exponentially many variables (or columns) q_j , one per maximal clique of G , it can be suitably solved via a branch-and-price algorithm based on column generation. Note that, for sparse graphs, for which the number of maximal cliques is typically small, the columns of the problem can be quite efficiently enumerated *a priori*, thus allowing for the direct solution of Problem (22)–(27) via state-of-the-art branch-and-cut solvers.

5 Exact separation of rank inequalities with a fixed right-hand side

In this section, as a second approach to overcome the difficulty of separating rank inequalities via the bilevel problem that we illustrated in Section 3, we focus on rank inequalities with a fixed right-hand side. Formally, we address *Rank Inequalities with a fixed right-hand side* $\alpha(G[U]) = \bar{\alpha}$ (RIs- $\bar{\alpha}$), namely:

$$\sum_{i \in U} x_i \leq \bar{\alpha} \quad \forall U \subseteq V : \alpha(G[U]) = \bar{\alpha}. \quad (28)$$

We will show how to cast the corresponding separation problem as a single level Integer Linear Program (ILP). Due to $\alpha(G[U])$ being a bounded integer, the technique that we will describe allows to optimize exactly over $RSTAB(G)$ by progressively increasing the right-hand side $\bar{\alpha}$ from 1 to, at most, n .

Note that, when fixing the right-hand side to an (increasing) constant, rank inequalities can be seen as an (increasingly complex) generalization of clique inequalities. As much as clique inequalities are induced by cliques, which are subgraphs with a K_2 -free complement (i.e., without missing edges), rank inequalities for $\alpha(G[U]) = \bar{\alpha}$ correspond to subgraphs with a $K_{\bar{\alpha}+1}$ -free complement. For instance, we have subgraphs with a K_3 -free complement (i.e., without missing triangles) for $\bar{\alpha} = 2$, subgraphs with a K_4 -free complement for $\bar{\alpha} = 3$, and so on. We will leverage this observation in the following. For an illustration, see Figure 4.

5.1 On the separation of RIs- $\bar{\alpha}$

For the ease of tractability, but without loss of generality, rather than considering Constraints (28), we will address the following set of inequalities:

$$\sum_{i \in U} x_i \leq \bar{\alpha} \quad \forall U \subseteq V : \alpha(G[U]) \leq \bar{\alpha}, \quad (29)$$

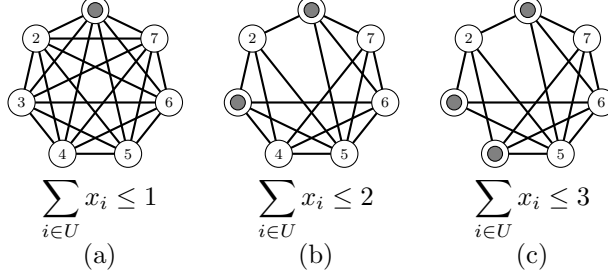


Figure 4: Three induced subgraphs $G[U]$ with a maximum stable set, highlighted in gray, and the corresponding rank inequality: (a) a clique with a stable set yielding $\alpha(G[U]) = 1$, (b) a graph with a triangle-free complement, with a stable set yielding $\alpha(G[U]) = 2$, and (c) a graph with a K_4 -free complement, with a stable set yielding $\alpha(G[U]) = 3$.

corresponding to all the induced subgraphs with a stability number of *at most* $\bar{\alpha}$, rather than equal to $\bar{\alpha}$. Note that, when considering all the values of $\bar{\alpha}$, the closure of Constraints (29) is the same of that of Constraints (28). Indeed, it is clear that, for any $\bar{\alpha}$, any of Constraints (28) also appears among Constraints (29). It is also clear that, if an inequality $\sum_{i \in U} x_i \leq \bar{\alpha}$ is found among Constraints (29) but not among Constraints (28), then $\alpha(G[U]) < \bar{\alpha}$, thus showing that this inequality is dominated by a tighter one among those in (28).

The separation problem for Constraints (29) corresponds to the following combinatorial optimization problem:

Definition. [MWS-BSN]: *Given a graph $G = (V, E)$ and a weight vector $x^* \in \mathbb{R}^n$, the Maximum Weighted Subgraph with Bounded Stability Number problem (MWS-BSN) calls for a subset of vertices $U \subseteq V$ of maximum weight inducing a subgraph $G[U]$ with stability number smaller than or equal to $\bar{\alpha}$.*

In order to arrive at a characterization of MWS-BSN, we rely on the following simple observation:

Lemma 5.1. *For any $U \subseteq V$, $\alpha(G[U]) \leq \bar{\alpha}$ if and only if, for all stable sets S of G where $|S| \geq \bar{\alpha} + 1$, $|S \cap U| \leq \bar{\alpha}$.*

Let $u \in \{0, 1\}^n$ be the incidence vector of $U \subset V$ and let $\mathcal{S}_{\bar{\alpha}+1}$ be the collection of stable sets of G of cardinality $\bar{\alpha} + 1$. From Lemma 5.1, we derive the following inequalities, valid for the MWS-BSN problem:

$$\sum_{i \in S} u_i \leq |S| - 1 = \bar{\alpha} \quad \forall S \in \mathcal{S}_{\bar{\alpha}+1}. \quad (30)$$

Due to the similarity with cover inequalities for the 0-1 knapsack problem, with a little abuse of language we refer to Constraints (30) as *Stable Set Cover Inequalities* (SSCIs).

Note that Constraints (30) are not facet defining for MWS-BSN. This is because any SSCI corresponding to a stable set S that is not maximal can be lifted by introducing, in its left-hand side, a new variable with a unit coefficient for each vertex $j \in S' \setminus S$, where S' is a maximal stable set containing S . To profit from this observation, let $\mathcal{S}_{\bar{\alpha}+1}^{\geq}$ be the collection of maximal stable sets of G of cardinality greater than or equal to $\bar{\alpha} + 1$. A stronger version of Constraints (30) is thus:

$$\sum_{i \in S} u_i \leq \bar{\alpha} \quad \forall S \in \mathcal{S}_{\bar{\alpha}+1}^{\geq}. \quad (31)$$

We refer to Constraints (31) as *Tightened Stable Set Cover Inequalities* (T-SSCIs).

Differently from SSCIs, we can show that T-SSCIs are facet defining for the MWS-BSN problem. We do it in two steps.

Proposition 5.2. *Constraints (31) are facet defining for the convex hull of feasible solutions to MWS-BSN when restricted to $G[S]$, i.e., to the subspace where $u_i = 0$ for all $i \in V \setminus S$.*

Proof. When restricting ourselves to $G[S]$, $G[S]$ itself is a stable set and, hence, any subset $S' \subseteq S$ of at most $\bar{\alpha}$ vertices yields a feasible solution. It follows that three group of constraints suffice to characterize the convex hull, constituting a totally unimodular constraint matrix: $\sum_{i \in S} u_i \leq \bar{\alpha}$, $u_i \geq 0, \forall i \in S$, and $u_i \leq 1, \forall i \in S$. Since, by definition of Constraints (31), $|S| \geq \bar{\alpha} + 1$, the inequality $\sum_{i \in S} u_i \leq \bar{\alpha}$ is not implied or dominated by any of the other constraints and, hence, it is facet defining. \square \square

Proposition 5.3. *Constraints (31) are facet defining for the MWS-BSN problem.*

Proof. Let $j_1, \dots, j_{|V \setminus S|}$ be an ordering of $V \setminus S$. Let M be the set of integer solutions to MWS-BSN and let M^ℓ be the subset of M restricted to $u_{j_k} = 0$ for all $k \in \{\ell + 1, \dots, |V \setminus S|\}$, where $\{\ell + 1, \dots, |V \setminus S|\}$ is considered equal to \emptyset for $\ell + 1 > |V \setminus S|$. We adopt a sequential lifting argument as found, e.g., in [NW88]. Starting from the inequality $\sum_{i \in S} u_i \leq \bar{\alpha}$ which, as of Proposition 5.3, is a facet of $\text{conv}(M^0)$, at each lifting iteration ℓ we obtain a facet of $\text{conv}(M^\ell)$ and, for $\ell = |V \setminus S|$, a facet of $\text{conv}(M)$.

At iteration ℓ , given the lifted inequality $\sum_{i \in S} u_i + \sum_{k \in \{1, \dots, \ell-1\}} \lambda_{j_k} u_{j_k} \leq \bar{\alpha}$, valid for $\text{conv}(M^{\ell-1})$, we compute the (largest) coefficient λ_{j_ℓ} for which the new inequality $\sum_{i \in S} u_i + \sum_{k \in \{1, \dots, \ell-1\}} \lambda_{j_k} u_{j_k} + \lambda_{j_\ell} u_{j_\ell} \leq \bar{\alpha}$ is valid for $\text{conv}(M^\ell \cap \{u_{j_\ell} = 1\})$ (and thus for $\text{conv}(M^\ell)$). The lifting problem reads:

$$\Lambda_\ell = \max_{u \in \{0,1\}^n} \sum_{i \in S} u_i + \sum_{k \in \{1, \dots, \ell-1\}} \lambda_{j_k} u_{j_k} \quad (32)$$

$$\text{s.t. } u_{j_\ell} = 1 \quad (33)$$

$$u_{j_k} = 0 \quad \forall k \in \{\ell + 1, \dots, |V \setminus S|\} \quad (34)$$

$$\alpha(G[\{i \in V : u_i = 1\}]) \leq \bar{\alpha}. \quad (35)$$

Since S is maximal (by definition of Constraints (31)) and $j_\ell \notin S$, $\exists i \in S : \{i, j_\ell\} \in E$. Let then S' be a subset of S containing vertex i , of cardinality $|S'| = \bar{\alpha}$. Since S' is a stable set and $\{i, j_\ell\} \in E$, $\alpha(G[S' \cup \{j_\ell\}]) = \alpha(G[S']) = |S'| = \bar{\alpha}$. Letting $u_{j_\ell} = 1$ and $u_i = 1$ for all $i \in S'$ we thus have a feasible solution to the lifting problem of value $\bar{\alpha}$, showing that $\Lambda_\ell \geq \bar{\alpha}$. Since the lifted inequality is valid if and only if $\Lambda_\ell + \lambda_{j_\ell} \leq \bar{\alpha}$, this implies $\lambda_{j_\ell} \leq 0$.

To show that $\lambda_{j_\ell} = 0$ for all $\ell \in \{1, \dots, |V \setminus S|\}$, first note that, if $\lambda_{j_k} = 0$ for all $k \in \{1, \dots, \ell-1\}$, then $\Delta_\ell \leq \bar{\alpha}$. Due to the previous argument, this implies $\Delta_\ell = \bar{\alpha}$ and, hence, $\lambda_{j_\ell} = 0$. Also note that, for $\ell = 1$, no terms $\lambda_{j_k} u_{j_k}$ appear in the objective function and, hence, we have $\lambda_{j_1} = 0$. The claim then follows by induction (if $\lambda_{j_1}, \dots, \lambda_{j_{\ell-1}} = 0$, then $\lambda_{j_\ell} = 0$), showing that, at the end of the lifting procedure, any of Constraints (31) is lifted into itself, thus being facet defining. \square \square

When separating RIs- $\bar{\alpha}$, we can cast MWS-BSN as the following single level ILP:

$$\max \sum_{i \in V} x_i^* u_i - \underbrace{\bar{\alpha}}_{\text{const}} \quad (36)$$

$$\text{s.t. } \sum_{i \in S} u_i \leq \bar{\alpha} \quad \forall S \in \mathcal{S}_{\bar{\alpha}+1}^{\geq} \quad (37)$$

$$u \in \{0, 1\}^n, \quad (38)$$

which can be solved via branch-and-cut, relying on a separation procedure for T-SSCIs, i.e., for Constraints (37). For more details and computational observations, we refer the reader to Section 6.

5.2 On the separation of T-SSCIs within the separation of RIs- $\bar{\alpha}$

Given a vector $u^* \in [0, 1]^n$ as a tentative, possibly infeasible solution to MWS-BSN, the separation problem for a maximally violated T-SSCI is:

Definition. [T-SSC-SEP]: Given a graph $G = (V, E)$ and a vector of vertex weights $u^* \in \mathbb{R}^n$, the SEPa-ration problem for T-SSCs (*T-SSC-SEP*) calls for a (maximal) stable set S of G of maximum weight with a cardinality greater than or equal to $\bar{\alpha} + 1$.

Not surprisingly, the separation of T-SSCIs is a hard problem. Indeed:

Lemma 5.4. *T-SSC-SEP is \mathcal{NP} -hard.*

Proof. The decision version of the maximum stable set problem asking whether a graph contains a stable set of size at least $\bar{\alpha} + 1$ has answer “yes” if and only if T-SSC-SEP admits a feasible solution. \square \square

Note that, by relying on the so-called equivalence between optimization and separation [GLS81], Proposition 5.3 and Lemma 5.4 imply the following:

Proposition 5.5. *MWS-BSN is \mathcal{NP} -hard.*

Let $s \in \{0, 1\}^n$ be the incidence vector of S . If we lift the maximality requirement on S , the following ILP solves T-SSC-SEP:

$$\max \sum_{i \in V} u_i^* s_i - \underbrace{\bar{\alpha}}_{const} \quad (39)$$

$$\sum_{i \in V} s_i \geq \bar{\alpha} + 1 \quad (40)$$

$$s_i + s_j \leq 1 \quad \forall \{i, j\} \in E \quad (41)$$

$$s \in \{0, 1\}^n. \quad (42)$$

Constraints (41) are edge inequalities, as in $FSTAB(G)$, guaranteeing that s be the incidence vector of a stable set with, due to Constraints (40), a cardinality of, at least, $\bar{\alpha} + 1$. The formulation is arguably the smallest correct formulation for the problem, although a tighter one can be obtained by introducing any of the inequalities discussed in this paper.

Any optimal solution S to Problem (39)–(42) gives a violated inequality if its total weight $\sum_{i \in S} u_i^*$ exceeds $\bar{\alpha}$ or, if it does not, proves that no violated T-SSCI exists. Note that the maximality of S , which is required to establish the facet definingness of T-SSCIs, is not guaranteed when solving Problem (39)–(42). This issue can be circumvented in at least two ways. Along the lines of [ACG14], we can separate $\hat{u} := u^* + \epsilon e$, rather than u^* , where e is the all-one vector and $\epsilon > 0$ is a sufficiently small, but finite, rational number (see Section 6 for more details). Alternatively, maximality can be easily achieved *a posteriori* by a greedy algorithm which, iteratively, adds vertices to S as long as they do not induce edges.

Similarly to what we noted for RFRIs, when separating RIs- $\bar{\alpha}$ we observe the generation of a number of inequalities corresponding to subgraphs without a specific topology, as it is illustrated in Figure 5 for the case of $\bar{\alpha} = 3$.

6 Computational study

In this section, we provide some details on the implementation of the separation algorithms that we considered and report on the results obtained when computationally investigating the tightness of RFRIs and RIs- $\bar{\alpha}$ with an increasingly larger right-hand side $\bar{\alpha}$.

Our algorithms are coded in C, using Gurobi 5.6 as MILP solver. We adopt the `parallel` setting, with 8 threads and default parameters. In all the separation problems, we set `solutionlimit=1`, imposing a violation cutoff of 0.01. For the separation of T-SSCIs, we use Cliquer 1.21. The value $\vartheta(G)$, which is computed for the sake of comparisons, is obtained via DSDP 5.8. All the results are produced within a time limit of 7200 seconds (two hours) on an Intel i7-3770 CPU @ 3.40GHz desktop computer with 8 cores, with 16GB RAM.

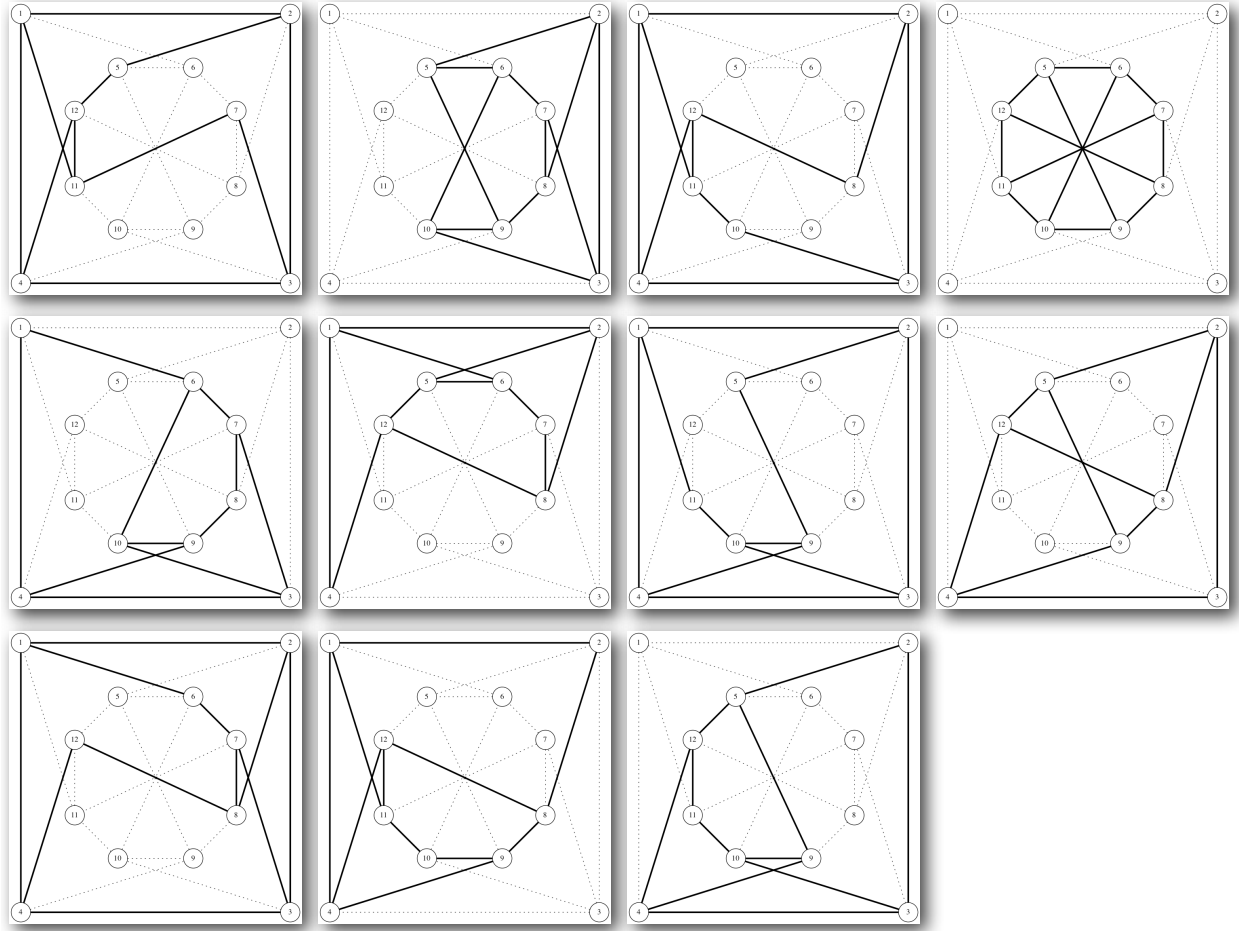


Figure 5: The set of the 11 rank inequalities which are generated when optimizing over the closure of rank inequalities with $\bar{\alpha} = 3$ over the Chvátal graph, adopting a cutting plane method. They yield a bound of 4.5, as opposed to $\alpha_{QSTAB}(G) = 6$ and $\vartheta(G) = 4.895$.

6.1 Algorithmic setup

In principle, we would like to generate rank inequalities that are *maximal*, i.e., such that, for any $i \in V \setminus U$, $\alpha(G[U \cup \{i\}]) > \alpha(G[U])$. This is because any rank inequality that is not maximal is clearly dominated by one of its maximal counterparts. Along the lines of [ACG14], it can be shown that a maximal RFRI or RI- $\bar{\alpha}$ can be obtained by separating $\hat{x} := x^* + \epsilon e$ rather than x^* , where e is the all-one vector and $\epsilon > 0$ is a sufficiently small, but finite, rational number. This is because, when separating \hat{x} , the objective function of the separation problem of either RFRI or RI- $\bar{\alpha}$ becomes $\max f(u) + \epsilon g(u)$, with $g(u) = \sum_{i \in V} u_i$ and, respectively for RFRI and RI- $\bar{\alpha}$, $f(u) = \sum_{i \in V} x_i^* u_i - \lfloor \alpha_{QSTAB}(G[U]) \rfloor$ and $f(u) = \sum_{i \in V} x_i^* u_i - \bar{\alpha}$. Since $f(u)$ is, in both cases, a discrete function and $g(u)$ is bounded, it suffices to select an ϵ smaller than the ratio between $\max g(u) - \min g(u)$ and the smallest difference between two different values in the image of $f(u)$. Experimentally, we observe that Gurobi (and its primal heuristic) finds the separation problem for RFRI much harder when \hat{x} is adopted, rather than x^* . As a consequence, we will forsake maximality when generating RFRI, resorting to just separating x^* . On the contrary, the adoption of \hat{x} seems very beneficial when generating RI- $\bar{\alpha}$, leading to overall stronger inequalities with a much denser left-hand side.

Another necessary condition for a rank inequality to be undominated is that of $G[U]$ be connected. This is because, if $G[U]$ contains k connected components $G[U_1], \dots, G[U_k]$, then the maximum stable set problem decomposes into $\alpha(G[U]) = \sum_{j=1}^k \alpha(G[U_j])$. Hence, the inequality $\sum_{i \in U} x_i \leq \alpha(G[U])$ can be obtained as a linear combination with unit weights of the k inequalities $\sum_{i \in U_k} x_i \leq \alpha(G[U_k])$. The same holds for RFRI. Since the LP for $\alpha_{QSTAB}(G[U])$ decomposes over the k connected components of G , we have $\alpha_{QSTAB}(G[U]) = \sum_{j=1}^k \alpha_{QSTAB}(G[U_j])$, from which we deduce $\lfloor \alpha_{QSTAB}(G[U]) \rfloor \geq \sum_{j=1}^k \lfloor \alpha_{QSTAB}(G[U_j]) \rfloor$. Aiming at solving the separations problems in a reasonable amount of computing time, connectivity constraints are likely too cumbersome to handle, either when enforced via lazy constraints or via an extended formulation. We circumvent this issue via postprocessing. Whenever an inequality has been generated, we identify in linear time the connected components $G[U_1], \dots, G[U_k]$ of $G[U]$ and introduce a RFRI or RI- $\bar{\alpha}$ for each of them, recomputing the corresponding right-hand side accordingly, i.e., either as $\lfloor \alpha_{QSTAB}(G[U_j]) \rfloor$ or $\alpha(G[U_j])$.

When separating RFRI, we assume that the collection \mathcal{K} of subsets of vertices inducing maximal cliques of G be *precomputed*. As we mentioned, this is a reasonable assumption for sparse graphs (like those that we will adopt), which have few cliques that can be enumerated within a reasonable computational effort. This way, we can solve the separation problem for RFRI via branch-and-bound, without resorting to column generation and branch-and-price.

Observe that, due to solving the separation problem for RFRI within a solution limit via `solutionlim=1`, the requirement on the optimality of the solution that is found is clearly lifted. It is thus possible, in Problem (22)–(27), that the fractional clique cover subproblem (see Constraints (16), (18)) does not achieve a value that is optimal for the chosen u . As a consequence, the right-hand side $\lfloor \alpha_{QSTAB}(G[U]) \rfloor$ could be overestimated, leading to a weaker inequality. To prevent the introduction of dominated inequalities, once an inequality has been found we always recompute $\alpha_{QSTAB}(G[U])$ by linear programming and then round it down to the nearest integer, even if $G[U]$ has a single connected component.

We always restrict the separation problems for both RFRI and RI- $\bar{\alpha}$ to the support of x^* , i.e., to the subgraph induced by the only vertices in V for which $x_i^* > 0$. We do the same when separating T-SSCI, restricting ourselves to the support of u^* . For RFRI and RI- $\bar{\alpha}$, a simple argument also allows to fix $u_i = 0$ for all $i \in V$ where $x_i^* = 1$. This is because, if $x_i^* = 1$, assuming that the LP relaxation of the maximum stable set problem that we are solving contains, at least, all the edge inequalities (which is always the case starting from $QSTAB(G)$), we have $x_j^* = 0$ for all $\{i, j\} \in E$. As a consequence, when restricting ourselves to the support of x^* , vertex i becomes isolated. Due to looking for inequalities where $G[U]$ is connected, it is thus correct to fix $u_i = 0$.

When solving the separation problem MWS-BSN for RI- $\bar{\alpha}$, we carry out the separation of T-SSCI inequalities with Cliquer 1.21. What is more, we restrict ourselves to generating T-SSCI only from an incumbent solution to MWS-BSN. This way, u^* is always a binary vector. This is a desirable property since, in case of a binary u^* and when restricting ourselves to its support, the separation problem for T-SSCI just amounts to looking for a stable set S of G of maximum cardinality, rather than for a stable

set of maximum weight and a cardinality of at least $\bar{\alpha} + 1$. This is because, if a violated inequality exists (in the subspace of the support of u^*), the cardinality constraint on the size of the stable set S , imposing $|S| \geq \bar{\alpha} + 1$, is automatically satisfied in any optimal solution. For our instances, the arising problem is solved quite efficiently with Cliquer. To better exploit this observation, rather than resorting to the separation of $\hat{u} = u^* + \epsilon e$ (as mentioned in Section 5), which would destroy the binarity of the vector that is being separated, we carry out a greedy maximalization *a posteriori* (after randomly shuffling, at each separation problem, the vertex labels of the graph).

To speedup the cutting plane algorithm for RI- $\bar{\alpha}$, we also introduce a simple greedy heuristic for their separation, loosely inspired by Cliquer. We sort the vertices in nonincreasing order of x^* and add them to U one at a time, until a maximal clique is formed (thus introducing only stable sets of cardinality 1). We then add, in the previously found order, the next $\bar{\alpha} - 1$ nodes (after which we have a stability number of, at most, $\bar{\alpha}$). Then, for each vertex currently not in U , we add it to U only if it does not form a stable set of cardinality $\bar{\alpha} + 1$. If it does, we skip it and continue to the next vertex. The algorithm runs in $O(n \log n + n^{\bar{\alpha}+1})$, where $O(n \log n)$ accounts for sorting and $O(n^{\bar{\alpha}})$ is the number of operations needed to check whether a new vertex increases the stability number of the current subgraph over $\bar{\alpha}$, which is executed $O(n)$ times. By construction, if this heuristic finds a solution, the corresponding inequality is maximal. Whenever the heuristic terminates without finding a violated inequality, Gurobi is called to solve the ILP (36)–(38).

6.2 Computational results

We compare the results obtained when optimizing over the RFRI and RI closures, the latter with $\bar{\alpha} = 2$, $\bar{\alpha} = \{2, 3\}$, $\bar{\alpha} = \{2, 3, 4\}$, and $\bar{\alpha} = \{2, 3, 4, 5\}$. This way, we can assess how tighter the RI closure becomes when simultaneously allowing for inequalities with an increasingly larger right-hand side.

As we will show, except for RIs with $\bar{\alpha} = 2$, the running times are typically quite large, hitting the time limit of two hours in many cases. Nevertheless, we remark that computational efficiency is not our primary interest here. Rather, we focus on assessing the *quality of the bounds* produced by the various inequalities. As a reference, we will compare such bounds with those obtained when optimizing over $QSTAB(G)$ via linear programming and over $THETA(G)$ via semidefinite programming.

Differently from other contributions [RS01, ROT⁺11], the separation of RIs (and also of RFRI, which is quite natural due to the way they are defined) is carried out on top of *all* clique inequalities, i.e., adopting $QSTAB(G)$ as the initial LP relaxation. Due to clique inequalities typically providing a large bound improvement by themselves, by adopting $QSTAB(G)$ as our starting relaxation we can assess the contribution of RIs in a clearer way, preventing ourselves from registering very large bound improvements which, after a closer analysis, could be produced almost only by clique inequalities.

We consider three groups of instances. The first one contains uniform random graphs with 60 to 80 nodes and an edge density between 5% and 50%. The second one is a subset of the largest instances among the random graphs that are used in [GR03] to solve the maximum stable set problem via SDP techniques. They are very sparse, with a density between 1% and 5%. The third group is a subset of the structured graphs that are typically adopted in the maximum stable set literature, see, for instance, [RS01, ROT⁺11].

The results are reported at the end of the paper, in Table 1. For each set of inequalities that is separated, we report the Upper Bound (UB) that has been found, the running time in seconds (Time), and the number of cuts that were generated (Cuts). For each of the three groups of instances, we also report the average % closed gap (Avg ClGap) that each set of inequalities closes w.r.t. $QSTAB(G)$, namely: $(1 - \frac{UB - \alpha(G)}{\alpha_{QSTAB(G)} - \alpha(G)})100$.

Overall, we observe that the bounds obtained via RFRI and RI- $\bar{\alpha}$ improve upon $\vartheta(G)$ in many occurrences. This is the case of both RFRI, where we register better bounds for 17 instances out of 30, and of RI- $\bar{\alpha}$ with $\bar{\alpha} \geq 3$, with better bounds for 16, 19, and 19 instances with, respectively, $\bar{\alpha} = \{2, 3\}$, $\{2, 3, 4\}$, and $\{2, 3, 4, 5\}$. On average, for the first two groups of instances, RFRI close more than 10% extra gap than RIs with $\bar{\alpha} = 2$, although at the cost of a much larger computing time. RIs with $\bar{\alpha} = \{2, 3\}$ improve over RFRI for the first and last groups of instances and achieve almost the same closed gap for the second one. Allowing for a larger $\bar{\alpha}$ up to 4 and 5 yields even tighter relaxations, closing with $\bar{\alpha} = \{2, 3, 4, 5\}$ almost

80% of the gap for the first group and 75% for the second one. As to the third group of instances, although the bounds that we obtain are still significantly better than those obtained with $QSTAB(G)$, they are not tighter than those achieved with $THETA(G)$. This is most likely due to very large computing times which are registered for those graphs and that, in almost all the cases, prevent a fast enough convergence within the time limit to a good bound. As an example, for RIs with $\bar{\alpha} = 2$ the time limit is hit in three cases out of six on this group, as opposed to being hit only for one instance out of 24 for the union of the other two groups. Interestingly though, we observe that the value of an optimal solution (to the maximum stable set problem) is found for one instance of this group, the instance `hamming6-4`, in less than 2 seconds by generating only 11 RIs with $\bar{\alpha} = 2$.

7 Concluding remarks

In this work, we have addressed the exact separation of the general family of rank inequalities. After highlighting the bilevel nature of their separation problem, we have proposed a general scheme to produce weaker inequalities with a right-hand side coefficient that can be computed by rounding the solution of a linear program. We then focused on the special case where the right-hand side $\alpha(G[U])$ is relaxed into $\lfloor \alpha_{QSTAB}(G[U]) \rfloor$, the fractional stability number of $G[U]$ rounded down to the nearest integer, thus introducing the class of *Rounded Fractional Rank Inequalities* (RFRIs). We have shown how to optimize over the corresponding closure with a single level MILP and addressed the relationship between such inequalities and well known ones. Then, we addressed the closure of *Rank Inequalities with a given right-hand side* $\alpha(G[U]) = \bar{\alpha}$ (RIs- $\bar{\alpha}$). We have investigated the complexity of the corresponding separation problem and shown how to solve it as an ILP with exponentially many facet defining inequalities.

Overall, RFRIs and RIs- $\bar{\alpha}$ with a small right-hand side yield a substantial bound improvement over that provided by the fractional clique polytope $QSTAB(G)$. In a number of cases, such bound is also tighter than $\vartheta(G)$, the bound given by $THETA(G)$.

Future work should focus on developing fast heuristics for the separation of RIs- $\bar{\alpha}$ with a small right-hand side. Due to the bound improvement that rank inequalities have shown to yield as of our experiments, the effectiveness of such algorithms could allow to add RIs- $\bar{\alpha}$ with $\bar{\alpha} = 2$ (or even $\bar{\alpha} = 3$) to the set of cutting planes that are routinely generated to solve the maximum stable set problem to optimality.

	$\alpha(G)$ $\alpha_Q(G)$		$\vartheta(G)$	RFRI			RI $_{\bar{\alpha}=2}$			RI $_{\bar{\alpha}=\{2,3\}}$			RI $_{\bar{\alpha}=\{2,3,4\}}$			RI $_{\bar{\alpha}=\{2,3,4,5\}}$		
	$\alpha(G)$	$\alpha_Q(G)$	$\vartheta(G)$	Cuts	UB	Time	Cuts	UB	Time	Cuts	UB	Time	Cuts	UB	Time	Cuts	UB	Time
r-60-5-13	31	31.50	31.07	6	31.00	0.1	4	31.00	0.0	4	31.00	0.0	4	31.00	0.0	4	31.00	0.0
r-60-10-13	23	25.63	23.67	440	23.04	tlim	50	23.47	1.0	98	23.14	39.5	131	23.00	108.7	136	23.00	120.3
r-60-15-13	18	21.20	19.54	892	19.06	tlim	68	19.70	3.3	176	19.20	111.0	287	18.97	686.4	418	18.81	2550.2
r-60-20-13	7	9.25	7.50	961	8.50	tlim	437	8.15	1192.6	955	7.78	tlim	1031	7.74	tlim	1039	7.74	tlim
r-60-25-13	14	16.50	14.67	875	14.72	tlim	119	15.24	13.4	342	14.70	476.3	604	14.38	3454.9	744	14.23	tlim
r-70-5-13	35	36.00	35.53	7	35.33	1.5	3	35.50	0.1	4	35.50	0.3	4	35.50	0.5	6	35.00	0.9
r-70-10-13	26	28.66	26.86	944	26.18	tlim	60	26.78	4.4	138	26.29	106.7	232	26.01	641.5	237	26.00	711.6
r-70-15-13	21	23.82	21.91	1083	21.62	tlim	96	22.16	9.2	246	21.63	412.6	404	21.37	3255.6	508	21.24	tlim
r-70-20-13	17	20.52	18.23	1041	18.55	tlim	109	19.18	25.8	332	18.46	1248.7	511	18.16	tlim	510	18.16	tlim
r-70-25-13	14	18.13	15.72	962	16.17	tlim	134	16.66	62.2	412	16.04	2059.6	601	15.69	tlim	604	15.68	tlim
r-80-5-13	39	39.50	39.02	10	39.00	0.3	3	39.00	0.0	3	39.00	0.0	3	39.00	0.0	3	39.00	0.0
r-80-10-13	27	30.50	28.55	1092	28.28	tlim	80	29.02	13.8	215	28.38	733.2	354	27.95	5674.0	376	27.89	tlim
r-80-15-13	22	26.74	23.65	1263	24.14	tlim	140	24.76	38.8	355	23.97	2101.3	481	23.66	tlim	468	23.69	tlim
r-80-20-13	18	22.78	20.05	1732	20.61	tlim	160	21.06	64.8	428	20.40	3188.7	533	20.21	tlim	534	20.20	tlim
r-80-25-13	16	19.85	17.07	1483	17.83	tlim	183	18.19	231.0	496	17.61	tlim	497	17.61	tlim	497	17.61	tlim
Avg ClGap			67.60		66.38			54.03			68.82			74.67			79.22	
g150.4	59	67.00	61.80	1544	61.01	tlim	88	62.09	45.8	238	60.80	tlim	239	60.80	tlim	239	60.80	tlim
g150.5	55	64.00	58.73	1945	58.25	tlim	147	58.56	107.5	311	57.73	tlim	310	57.73	tlim	311	57.73	tlim
g170.3	71	78.50	73.34	1036	72.22	tlim	85	73.53	66.4	181	72.16	tlim	181	72.16	tlim	184	72.16	tlim
g200.2	96	100.00	97.17	241	96.00	234.9	23	97.00	12.1	46	96.00	321.7	47	96.00	352.8	48	96.00	393.5
g200.3	83	94.50	86.52	1422	85.26	tlim	126	86.60	307.5	208	85.28	tlim	208	85.29	tlim	209	85.26	tlim
g300.2	122	141.00	129.47	1623	128.65	tlim	138	130.43	1960.6	160	130.06	tlim	160	130.06	tlim	160	130.06	tlim
g350.2	133	161.00	143.43	3204	145.35	tlim	278	146.11	tlim	276	146.11	tlim	280	146.11	tlim	277	146.11	tlim
g400.1	191	201.50	194.79	297	193.55	tlim	28	195.50	109.0	58	193.71	tlim	57	193.75	tlim	59	193.66	tlim
Avg ClGap			65.85		74.88			62.86			74.30			74.23			74.37	
brock200_1	21	38.02	27.46	556	37.25	tlim	267	35.59	tlim	267	35.59	tlim	267	35.59	tlim	267	35.59	tlim
C125.9	34	43.06	37.81	2422	39.45	tlim	188	39.75	409.2	322	39.20	tlim	320	39.21	tlim	322	39.21	tlim
C250.9	44	71.37	56.24	6915	69.99	tlim	375	66.05	tlim	375	66.05	tlim	375	66.05	tlim	375	66.05	tlim
hamming6-4	4	5.33	5.33	13076	4.92	5592.6	11	4.00	1.5	11	4.00	1.5	11	4.00	1.5	11	4.00	1.5
keller4	11	14.83	14.01	1623	14.53	tlim	314	13.80	tlim	318	13.80	tlim	314	13.80	tlim	314	13.80	tlim
MANN_a9	16	18.00	17.48	252	17.14	tlim	1	18.00	0.1	1	18.00	0.7	1	18.00	2.7	1	18.00	13.1
sanr200_0.9	45	59.82	49.27	9624	57.33	tlim	366	55.14	tlim	366	55.13	tlim	366	55.14	tlim	366	55.14	tlim
Avg ClGap			26.48		16.18			19.51			19.94			19.92			19.92	

Table 1: Bounds (UB) produced by Rounded Fractional Rank Inequalities (RFRI) and Rank Inequalities with a given right-hand side of $\bar{\alpha}$ (RI $_{\bar{\alpha}}$), with an increasing $\bar{\alpha}$ from 2 to 5. They are compared to the stability number $\alpha(G)$, the fractional stability number $\alpha_{QSTAB}(G)$, and the ϑ -number $\vartheta(G)$. Computing times in seconds (Time) and total number of generated cutting planes (Cuts) are also reported. Bounds that are tighter than those provided by $\vartheta(G)$ are highlighted in bold. Closed Gap, averaged in geometric mean (Avg ClGap), is reported for the three classes of instances: small uniform random graphs that we generated, larger uniform random taken from [GR03], and structured instances from the literature [RS01].

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