A primal-simplex based Tardos' algorithm

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Abstract

In the mid-eighties Tardos proposed a strongly polynomial algorithm for solving linear programming problems for which the size of the coefficient matrix is polynomially bounded by the dimension. Combining Orlin's primal-based modification and Mizuno's use of the simplex method, we introduce a modification of Tardos' algorithm considering only the primal problem and using simplex method to solve the auxiliary problems. The proposed algorithm is strongly polynomial if the coefficient matrix is totally unimodular and the auxiliary problems are non-degenerate.

Keyword: Tardos' algorithm, simplex method, strongly polynomial algorithm, total unimodularity

1   Introduction

In the mid-eighties Tardos [1, 2] proposed a strongly polynomial algorithm for solving linear programming problems $\min\{c^\top x \mid Ax = b,\ x \geq 0\}$ for which the size of the coefficients of A are polynomially bounded by the dimension. Such instances include minimum cost flow, bipartite matching, multicommodity flow, and vertex packing in chordal graphs. The basic strategy of Tardos' algorithm is to identify the coordinates equal to zero at optimality. The algorithm involves solving several auxiliary dual problems by the ellipsoid or interior-point methods. By successively identifying such vanishing coordinates, the problem is made smaller and an optimal solution is obtained inductively. Orlin [6] proposed a modification of Tardos' algorithm considering only the primal problem; that is, identifying the coordinates strictly positive at optimality. He observed that the right-hand side coefficients of the auxiliary problems might be impractically large.

In 2014, Mizuno [5] modified Tardos' algorithm by using a dual simplex method to solve the auxiliary problems. He observed that this approach is strongly polynomial if A is totally unimodular and the auxiliary problems are non-degenerate; that is, the basic variables are strictly positive for every basic feasible solution. The strong polynomiality is a consequence of Kitahara and Mizuno [3, 4] results which extend in part Ye's result [8] for Markov decision problems and bounds the number of distinct basic feasible solutions generated by the simplex method.

Combining Orlin's and Mizuno's approaches, we introduce a modification of the algorithm proposed by Mizuno considering only the primal problem. The proposed algorithm is strongly polynomial if A is totally unimodular and the auxiliary problems are non-degenerate. As it involves only the primal and does not suffer from impractically large right-hand side coefficients, the proposed algorithm improves the implementability of the approach. While the proposed algorithm and the complexity analysis is focusing on the case where A is totally unimodular, the algorithm could be enhanced to handle general matrices. The enhanced algorithm would be strongly polynomiality if the absolute value of any subdeterminant of A is polynomially bounded by the dimension.

2 A primal-simplex based Tardos' algorithm

2.1 Formulation and main result

Consider the following formulation:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\top}\boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b},\, \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\boldsymbol{c} \in \mathbb{R}^n$ are given. The optimal solution of (1), if any, is assumed without loss of generality to be unique. Otherwise $\boldsymbol{c}$ could be perturbed by $(\epsilon, \epsilon^2, \ldots, \epsilon^n)$ for a sufficiently small $\epsilon > 0$. Alternatively, the simplex method can be performed using a lexicographical order if a tie occurs when choosing an entering variable by Dantzig's rule. Let $K^* \subseteq N = \{1, 2, \ldots, n\}$ be the optimal basis of (1). The proposed algorithm inductively builds a subset $\bar{K} \subseteq K^*$ through solving an auxiliary problem. If $\bar{K} = K^*$ we obtained the optimal solution. Otherwise, we obtain a smaller yet equivalent problem by deleting the variables corresponding to $\bar{K}$. Thus, the optimal solution is obtained inductively. For clarity of the exposition of the algorithm and of the proof of Theorem 1, we assume in the remainder of the paper that A is totally unimodular; that is, all its subdeterminants are equal to either $-1, 0$ or 1.

Theorem 1. *The primal-simplex based Tardos' algorithm is strongly polynomial if A is totally unimodular and all the auxiliary problems are non-degenerate; that is, all the basic variables are strictly positive for every basic feasible solution.*

Proof. See Section 3. $\qquad\square$

2.2 A primal-simplex based Tardos' algorithm

Step 0 (initialization):

Let $\bar{K} := \emptyset$ and its complement $K := N$.

Step 1 (reduction):

If $\bar{K} \neq \emptyset$, remove the variables corresponding to $\bar{K}$ in the following way. Let $G \in \mathbb{R}^{m \times m}$ be a nonsingular submatrix of A such that its first $|\bar{K}|$ columns form $A_{\bar{K}}$ and $H = G^{-1}$. Let H_1 consists of the first $|\bar{K}|$ rows of H, H_2 denote the remainder, and consider the following *reduced* problem:

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c'}^{\top} \boldsymbol{x'} \\ \text{subject to} & A' \boldsymbol{x'} = \boldsymbol{b'}, \, \boldsymbol{x'} \geq \boldsymbol{0}, \end{array} \tag{2}$$

where $A' = H_2 A_K$, $\boldsymbol{b'} = H_2 \boldsymbol{b}$, $\boldsymbol{c'} = \boldsymbol{c}_K - (H_1 A_K)^{\top} \boldsymbol{c}_{\bar{K}}$, and $\boldsymbol{x'} = \boldsymbol{x}_K$. If $\bar{K} = \emptyset$, set $A' := A$, $\boldsymbol{b'} := \boldsymbol{b}$, and $\boldsymbol{c'} := \boldsymbol{c}$. GO TO STEP 2.

Step 2 (scaling and rounding):

Let $m' = m - |\bar{K}|$ and $n' = n - |\bar{K}|$. For a basis $L \subseteq K$ of A' and $\bar{L} = K \setminus L$, rewrite (2) as:

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c'}^{\top} \boldsymbol{x'} \\ \text{subject to} & \boldsymbol{x'}_L + (A'_L)^{-1} A'_{\bar{L}} \boldsymbol{x'}_{\bar{L}} = (A'_L)^{-1} \boldsymbol{b'}, \, \boldsymbol{x'} \geq \boldsymbol{0}. \end{array} \tag{3}$$

If $(A'_L)^{-1} \boldsymbol{b'} = \boldsymbol{0}$, stop. Otherwise, consider the following *scaled* problem:

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c'}^{\top} \boldsymbol{x'} \\ \text{subject to} & \boldsymbol{x'}_L + (A'_L)^{-1} A'_{\bar{L}} \boldsymbol{x'}_{\bar{L}} = (A'_L)^{-1} \boldsymbol{b'}/k, \, \boldsymbol{x'} \geq \boldsymbol{0}, \end{array} \tag{4}$$

where $k = \|{A'}^{\top}(A'{A'}^{\top})^{-1} \boldsymbol{b'}\|_2/(m' + (n')^2)$. Then, consider the following *rounded* problem:

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c'}^{\top} \boldsymbol{x'} \\ \text{subject to} & \boldsymbol{x'}_L + (A'_L)^{-1} A'_{\bar{L}} \boldsymbol{x'}_{\bar{L}} = \lceil (A'_L)^{-1} \boldsymbol{b'}/k \rceil, \, \boldsymbol{x'} \geq \boldsymbol{0}. \end{array} \tag{5}$$

If (5) is infeasible, stop. Otherwise, solve (5) using the simplex method with Dantzig's rule. If (5) is unbounded, stop. Otherwise, let $\boldsymbol{x''}$ denote the optimal solution and L'' the optimal basis. If $\bar{K} \cup L''$ is an optimal basis of the original problem (1), stop. Otherwise, GO TO STEP 3.

Step 3 (iteration):

Set $\bar{K} := \bar{K} \cup J$ and $K := K \setminus J$ where $J = \{\, i \,|\, x''_i \geq n', \, i \in K \,\}$. If $|K| = n - m$, stop. Otherwise, GO TO STEP 1.

2.3 Annotations of the proposed algorithm

If $\bar{K} \subseteq K^*$ and the optimal solution of (1) is unique, we can remove the non-negativity constraints for x_i for $i \in \bar{K}$. In Step 1, the reduced problem (2) is obtained by expressing $\boldsymbol{x}_{\bar{K}}$ as $H_1 \boldsymbol{b} - H_1 A_K \boldsymbol{x}_K$ and substituting $H_1 \boldsymbol{b} - H_1 A_K \boldsymbol{x}_K$ for $\boldsymbol{x}_{\bar{K}}$ in the objective function. Therefore, the optimal solution for (2) yields the optimal solution for (1) via $\boldsymbol{x}_{\bar{K}} = H_1 \boldsymbol{b} - H_1 A_K \boldsymbol{x}_K$. The constant term in the objective function is removed for simplicity. Note that the matrices A' and $[I, (A'_L)^{-1} A'_{\bar{L}}]$ involved in (2), (3), (4), and (5) are totally unimodular if A is totally unimodular, see Theorem 19.5 in Schrijver [7].

In step 2, the scaling factor k is strictly positive if $(A'_L)^{-1} \boldsymbol{b}' \neq \boldsymbol{0}$ and, see Lemma 2, $\|\lceil (A'_L)^{-1} \boldsymbol{b}'/k \rceil \|_\infty$ is polynomially bounded above in m' and n', which is a key fact for showing the strong polynomiality. Although the proposed algorithm builds the simplex tableau associated to (3) and the reduced problem (2) from scratch at each iteration, it is essentially for clarity of the exposition and can be ignored. In particular, one can observe that $L'' \setminus J$ can be used as the basis L for (3) at the next iteration, thus enabling a warm start. By performing Phase one of the two-phase simplex method for the rounded problem (5), we can check the feasibility of (5) and compute an initial basic feasible solution, unless it is infeasible.

In Step 3, $J \neq \emptyset$ by Lemma 1; that is, the size of K is strictly decreasing. Thus, the proposed algorithm terminates after at most m iterations. If (1) has an optimal solution, $\bar{K} \subseteq K^*$ by Corollary 1.

The stopping conditions of the proposed algorithm are:

○ if $(A'_L)^{-1} \boldsymbol{b}' = \boldsymbol{0}$, the simplex tableau associated to (3) yields either the optimality of $\boldsymbol{x}' = \boldsymbol{0}$ or the unboundedness of the reduced problem (2).

○ since the rounded problem (5) is a relaxation of the scaled problem (4),

– the scaled problem (4) and the original problem (1) are both infeasible if (5) is infeasible

– the scaled problem (4) is unbounded or infeasible if (5) is unbounded. In both cases, the original problem (1) has no optimal solution.

○ if $|K| = n - m$ in Step 3, the problem (1) is infeasible as otherwise the algorithm finds an optimal basis in Step 2.

3 Proof of Theorem 1

Lemma 1 states that the set $J = \{i \mid x''_i \geq n', i \in K\}$ used in Step 3 is never empty and thus, the proposed algorithm solves the rounded problem (5) at most m times.

Lemma 1. $J \neq \emptyset$ as any solution $\boldsymbol{x}''$ of the rounded problem (5) satisfies $\|\boldsymbol{x}''\|_\infty \geq n'$.

Proof. Let $\boldsymbol{x}''$ be any solution of the rounded problem (5). Then

$$A'\boldsymbol{x}'' = A'_L\lceil (A'_L)^{-1}\boldsymbol{b}'/k\rceil.$$

Since, for any $\boldsymbol{g}$, the minimal l_2-norm point satisfying $A'\boldsymbol{x}' = \boldsymbol{g}$ is $A'^T(A'A'^T)^{-1}\boldsymbol{g}$, we have

$$
\begin{aligned}
\|\boldsymbol{x}''\|_2 &\geq \|A'^T(A'A'^T)^{-1}A'_L\lceil (A'_L)^{-1}\boldsymbol{b}'/k\rceil\|_2 \\
&\geq \|A'^T(A'A'^T)^{-1}\boldsymbol{b}'/k\|_2 - \|A'^T(A'A'^T)^{-1}A'_L\boldsymbol{d}\|_2 \\
&= (m'+(n')^2) - \|A'^T(A'A'^T)^{-1}A'\begin{pmatrix} \boldsymbol{d} \\ \boldsymbol{0}_{\bar{L}} \end{pmatrix}\|_2 \\
&\geq (n')^2 + m' - \|\boldsymbol{d}\|_2,
\end{aligned}
$$

where $k = \|A'^\top(A'A'^\top)^{-1}\boldsymbol{b}'\|_2/(m'+(n')^2)$ and $\boldsymbol{d} = (A'_L)^{-1}\boldsymbol{b}'/k - \lceil (A'_L)^{-1}\boldsymbol{b}'/k\rceil$. Since $\|\boldsymbol{d}\|_\infty < 1$, we obtain that

$$\|\boldsymbol{x}''\|_\infty \geq \|\boldsymbol{x}''\|_2/n' > ((n')^2 + m' - m')/n' = n'.$$

$\square$

Corollary 1 is a direct consequence of Theorem 2 and shows that $\bar{K} \subseteq K^*$.

Theorem 2 (Theorem 10.5 in Schrijver [7]). *Let A be an $m \times n$-matrix, and let Δ^* be such that for each nonsingular submatrix B of A all entries of B^{-1} are at most Δ^* in absolute value. Let $\boldsymbol{c}$ be a column n-vector, and let $\boldsymbol{b}''$ and $\boldsymbol{b}^*$ be column m-vectors such that $P'' : \max\{\boldsymbol{c}^\top\boldsymbol{x} \mid A\boldsymbol{x} \leq \boldsymbol{b}''\}$ and $P^* : \max\{\boldsymbol{c}^\top\boldsymbol{x} \mid A\boldsymbol{x} \leq \boldsymbol{b}^*\}$ are finite. Then, for each optimal solution $\boldsymbol{x}''$ of P'', there exists an optimal solution $\boldsymbol{x}^*$ of P^* with $\|\boldsymbol{x}'' - \boldsymbol{x}^*\|_\infty \leq n\Delta^*\|\boldsymbol{b}'' - \boldsymbol{b}^*\|_\infty$.*

Corollary 1. *Let $\boldsymbol{x}''$ be an optimal solution of the rounded problem (5), and $J = \{i \mid x_i'' \geq n', i \in K\}$ as defined in Step 3 of the proposed algorithm. If the scaled problem (4) is feasible, the i-th coordinate of the optimal solution of the scaled problem (4) is strictly positive for $i \in J$. Furthermore, the same holds for the reduced problem (2) and the original problem (1) as the scaling factor k is strictly positive.*

Proof. Define $\tilde{A} \in \mathbb{R}^{(2m'+n')\times n'}$, $\tilde{\boldsymbol{b}}''$, and $\tilde{\boldsymbol{b}}^* \in \mathbb{R}^{2m'+n'}$ as:

$$\tilde{A} = \begin{bmatrix} E \\ -E \\ -I \end{bmatrix}, \ \tilde{\boldsymbol{b}}^* = \begin{bmatrix} (A_L^{-1}\boldsymbol{b}')/k \\ -(A_L^{-1}\boldsymbol{b}')/k \\ \boldsymbol{0} \end{bmatrix}, \ \text{ and } \tilde{\boldsymbol{b}}'' = \lceil \tilde{\boldsymbol{b}}^*\rceil.$$

where $E = [I, (A'_L)^{-1}A'_{\bar{L}}]$. With this notation, the rounded problem (5), respectively the scaled problem (4), can be restated as $P'' : \max\{-\boldsymbol{c}'^\top\boldsymbol{x} \mid \tilde{A}\boldsymbol{x} \leq \tilde{\boldsymbol{b}}''\}$, respectively $P^* : \max\{-\boldsymbol{c}'^\top\boldsymbol{x} \mid \tilde{A}\boldsymbol{x} \leq \tilde{\boldsymbol{b}}^*\}$. Since E is totally unimodular, $\tilde{A}$ is totally unimodular, and thus $\Delta^* = 1$ in Theorem 2. In addition, note that

$\|\tilde{\boldsymbol{b}}'' - \tilde{\boldsymbol{b}}^*\|_\infty < 1$. Recall that the scaled problem (4) and P^* share the same unique optimal solution $\boldsymbol{x}^*$ as the optimal solution of the original problem (1) is assumed to be unique. Therefore, since $\boldsymbol{x}''$ is an optimal solution of P'', we observe that $\|\boldsymbol{x}'' - \boldsymbol{x}^*\|_\infty < n'$ by Theorem 2 and thus, $x_i^* > 0$ for $i \in J$. $\qquad\square$

Finally, we show the strong polynomiality of the proposed algorithm using Kitahara and Mizuno [3, 4] results showing that the number of different basic feasible solutions generated by the primal simplex method with the most negative pivoting rule – Dantzig's rule – or the best improvement pivoting rule is bounded by:
$$n\lceil m\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta})\rceil$$
where m is the number of constraints, n is the number of variables, and γ and δ are, respectively, the minimum and the maximum values of all the positive elements of the primal basic feasible solutions. Thus, we need to estimate the values γ and δ for the introduced auxiliary problems.

Since the coefficient matrices used in the proposed algorithm are totally unimodular and the right hand side vector of the rounded problem (5) is integer, we have $\delta = 1$. For γ, we use Lemma 2.

Lemma 2. *For the auxiliary problem* (5), *we have* $\gamma \le \gamma^* = m(mn(m+n^2)+1)$.

Proof. Note that the right-hand side vector for (5) is $\lceil (A_L')^{-1}\boldsymbol{b}'/k\rceil$. By the total unimodularity, we observe that

$$\|\lceil (A_L')^{-1}\boldsymbol{b}'/k\rceil\|_\infty \le \|(A_L')^{-1}\boldsymbol{b}'/k\|_\infty + 1 \le m'\|\boldsymbol{b}'\|_\infty/k + 1.$$

The numerator $\|A'^\top(A'A'^\top)^{-1}\boldsymbol{b}'\|_2$ of k is bounded below by $\|\boldsymbol{b}'\|_\infty/n'$ implying $\|\lceil (A_L')^{-1}\boldsymbol{b}'/k\rceil\|_\infty \le m'n'(m'+(n')^2)+1$. Thus, by Cramer's rule and the total unimodularity of the coefficient matrix of (5), the l_∞-norm of a basic solution of (5) is bounded above by $m'(m'n'(m'+(n')^2)+1)$. $\qquad\square$

The two-phase simplex algorithm is called at most m times. Thus, the number of auxiliary problems solved by the proposed algorithm is bounded above by $2m$ as each call corresponds to 2 auxiliary problems : one for each phase. Therefore, if all the auxiliary problems are non-degenerate, the total number of basic solutions generated by the algorithm is bounded above by $2m[n\lceil m\gamma^*\log(m\gamma^*)\rceil]$; that is by
$$2mn\lceil(m^4n + m^3n^3 + m^2)\log(m^4n + m^3n^3 + m^2)\rceil$$
which completes the proof of Theorem 1. Alternatively, since $m \le n$, this bound can be restated as $O(m^4n^4\log n)$. While assuming the non-degeneracy of the auxiliary problems is needed to use Kitahara-Mizuno's bound, the number of degenerate updates of bases at a single basic solution is typically not too large in practice.

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