THE ROBUST STABILIZATION PROBLEM FOR DISCRETE TIME DESCRIPTOR SYSTEMS

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Abstract. We investigate here the robust stabilization problem for the descriptor discrete time systems and build an optimal solution in the case when both the nominal system and the perturbations are given in terms of left coprime factorizations. Moreover our formulas are given straight from the original datas, using solely the stabilizing solutions of two Riccati equations, thus making our approach appropriate for computatonal use.

Key words. Descriptor systems, Riccati equations, $\mathbb{H}^\infty$ control problem.

1. Introduction. With the rise and ubiquity of digital and data technology, the processes that need to be controlled have become more and more complex, and consequently the associated mathematical models more complicated. As, in practice, any system could be subjected to various kinds of perturbations, not to say that the mathematical and the physical models do not ever match completely, the problem of internally stabilizing the system could become more intricate, and harder to solve especially for the complex ones. Some systems might also be allowed not to be proper and thus the need for finding a single controller that stabilizes a more general class is a current issue that is required to be solved. The robust stabilization problem for standard proper systems was solved long time ago: in [1], a solution to the suboptimal problem for continuous-time systems is given, while in [2] an optimal solution is built for both continuous and discrete-time systems. The main drawback is that they avoid the improper and polynomial cases which, sometimes, can be of great interest.

1.1. Personal contributions. Our contributions can be summarized as follows:
- i We give analytical formulas for computing the stability radius, as well as the associated controller for descriptor systems. This means that we actually remove the assumption of proper systems and we also allow for improper and even polynomial rational matrix functions.
- ii Our relations are given straight from the original datas, using solely the stabilizing solutions of two Riccati equations.
- iii Finally, our approach avoids preliminary decompositions and also matrix inversions, thus gaining a strong numerical reliability.

2. Preliminaries. By $\mathbb{D}$ and $\mathbb{D}_c$ we denote the open unit disk and the exterior of the closed-unit disk respectively, and by $C_{o;1}$ the unit circle, centered in origin. $\bar{\mathbb{C}}$ represents the complex-plane adjunctioned with the point at infinity. A constant matrix $A \in \mathbb{C}^{n \times n}$ is said to be hermitic, if $A = A^*$, where

$$A^* := \bar{A}^T. \tag{2.1}$$

$\mathbb{R}\mathbb{H}^\infty$ stands for the *Banach* space of all stable discrete time rational matrix functions, i.e. those who have all their poles insight the open unit disk, and $\mathbb{R}\mathbb{L}^\infty$ stands for the *Banach* space of all rational matrix functions with no poles on $C_{o;1}$. We also denote by G^* the adjoint of G, i.e.

$$G^*(z) = \bar{G}^T(\frac{1}{\bar{z}}), \tag{2.2}$$

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where the bar above takes the complex conjugate. $rank_n(\cdot)$ represents the normal rank of G, i.e. the rank of $G(z)$ for all $z \in \bar{\mathbb{C}}$ except for a finite number of points. $\Lambda(A - zE)$ stands for the spectrum of the matrix-pencil $A - zE$ (finite and infinite, multiplicities counted). By I we denote the unit and by 0 the null matrix respectively. For a given linear operator $\mathcal{L}$, $\rho(\mathcal{L})$ represents its spectral radius, and for any square matrix $A \in \mathbb{C}^{n \times n}$, $sgn(\cdot)$ is its signature. Also by $l^{2,n}$ we denote the *Hylbert* space of $\mathbb{Z}$-defined functions $u : \mathbb{Z} \to \mathbb{C}^n$ for which $\sum_{k=-\infty}^{\infty} \bar{u}^T[k]u[k] < \infty$, and by $l^{2,n}_{+[-]}$ the *Hylbert* space of $\mathbb{Z}_+[\mathbb{Z}_-]$-defined functions $u : \mathbb{Z}_+[\mathbb{Z}_-] \to \mathbb{C}^n$ for which $\sum_{k=0[-\infty]}^{\infty[-1]} \bar{u}^T[k]u[k] < \infty$.

2.1. Centered realizations. Any general $p \times m$ rational matrix function (rmf), which might also be improper or polynomial, can be represented as

$$G(z) =: C(zE - A)^{-1}B(\alpha - \beta z) + D =: \left[\begin{array}{c|c} A - zE & B \\ \hline C & D \end{array} \right]_{z_0}, \qquad (2.3)$$

where $z_0 := \frac{\alpha}{\beta} \in \mathbb{C}$ is a fixed point, A, E, B, C, D are matrices of dimension $n \times n$, $n \times n$, $n \times m$, $p \times n$, $p \times m$ respectively, the matrix pencil $A - zE$ is regular (i.e. is square and $\det(A - zE) \not\equiv 0$), and n is *the order* of the realization. Such a representation is called a *centered realization* at z_0 and has been introduced in [6] for solving certain control problems involving improper systems. One may notice that, setting $\alpha = 1$ and $\beta = 0$, z_0 turns into the point at infinity, and moreover the realization (2.3) becomes a generalized state-space realization. In order to provide symmetry, we may assume that $\beta = \bar{\alpha}$, which means that $z_0 \in C_{o;1}$ -assumption in force for the rest of the paper-.

However, when coping with improper (or polynomial) systems, centered realizations have some advantages over the generalized realizations which make them suitable for our approach. For instance, centered realizations allow a flexibility in choosing $z_0 \in \mathbb{C}$ always different from the poles of G. As long as z_0 is not a pole of G -this assumption will also be in force for the rest of the paper- the minimal order of a centered realization always equals the McMillan degree of G, and further we can see from (2.3) that $D = G(z_0)$. All these facilities make centered realizations closer to state space realizations for proper systems rather than for descriptor ones.

Finally, one may switch from centered realizations to generalized state space ones solely by some simple formulas (Section 5 in [3]), or directly from the transfer function matrix in a much similar way to the standard case ([10]).

In conclusion, all the advantages presented above, as well as the possibilities of transfering from generalized to centered realization or viceversa, and even to write a centered realization directly from the rmf description make *centered realizations* a viable choice in dealing with state space representations of improper or polynomial systems.

In what follows we recall the notions of stabilizability, $C^c_{o;1}$-controlability, detectability and $C^c_{o;1}$-observability associated with this kind of realizations, and consequently we present a very important property regarding them which enables us working with improper (or even polynomial) state spaces systems in a very similar fashion as with the proper state space ones (for further details see [3]).

DEFINITION 2.1. *Let G be a general rmf, having a realization* (2.3).
We say that $(A - zE; B)$ is stabilizable if

$$rank \begin{bmatrix} A - zE & B \end{bmatrix} = n, \qquad (2.4)$$

for all $z \in \mathbb{C} - \mathbb{D}$ and

$$rank \begin{bmatrix} E & B \end{bmatrix} = n. \qquad (2.5)$$

Next we say that $(A - zE; B)$ is $C^c_{o;1}$-controllable provided

$$rank \begin{bmatrix} A - zE & B \end{bmatrix} = n, \forall z \in C_{o;1}. \tag{2.6}$$

Finally $(C; A - zE)$ is said to be detectable ($C^c_{o;1}$-observable) if $(A^* - zE^*; C^*)$ is stabilizable ($C^c_{o;1}$-controllable).

LEMMA 2.2. Let $(A - zE; B)$ be stabilizable. Than there is always a constant matrix F such that we have

$$\Lambda(A - zE + BF(\alpha - \bar{\alpha}z)) \subset \mathbb{D}. \tag{2.7}$$

2.2. The Riccati equation. We recall here the Riccati equation associated with a centered realization (2.3) (see [8] for more details).

DEFINITION 2.3. Let

$$G(z) =: C(zE - A)^{-1}B(\alpha - \beta z) + D =: \left[\begin{array}{c|c} A - zE & B \\ \hline C & D \end{array} \right]_{z_0}, \tag{2.8}$$

and suppose $(A - zE; B)$ is stabilizable.

The Riccati equation

$$A^*XA - E^*XE - ((\bar{\alpha}A - \alpha E)^*XB + L)R^{-1}((\bar{\alpha}A - \alpha E)^*XB + L)^* + Q = 0, \tag{2.9}$$

where $L \in \mathbb{C}^{n \times m}$, $Q = Q^* \in \mathbb{C}^{n \times n}$ and $R = R^* \in \mathbb{C}^{m \times m}$ (R invertible) is called the descriptor discrete time algebraic Riccati equation (DDTARE).

We say that $X_s = X_s^*$ is a stabilizing solution of (2.9) provided

$$\Lambda(A - zE + BF_s(\alpha - \bar{\alpha}z)) \subset \mathbb{D}, \tag{2.10}$$

where the matrix $F_s := -R^{-1}((\bar{\alpha}A - \alpha E)^*X_sB + L)^*$ is called the stabilizing feedback.

The next result is nothing but a slightly modified version of Theorem 12 in [8].

LEMMA 2.4. Suppose $(A - zE; B)$ is stabilizable and $(C; A - zE)$ is $C^c_{o;1}$-observable. Than the DDTARE

$$A^*XA - E^*XE - (\bar{\alpha}A - \alpha E)^*XBB^*X(\bar{\alpha}A - \alpha E) + C^*C = 0 \tag{2.11}$$

has an invertible Hermitical stabilizing solution.

3. Problem formulation and main results. In this section we present our problem and state the main results of this paper.

3.1. Problem formulation. Let G be a general $p \times m$ rmf (possibly improper or polynomial) given by the following stabilizable and detectable realization

$$G(z) =: C(zE - A)^{-1}B(\alpha - \beta z) + D =: \left[\begin{array}{c|c} A - zE & B \\ \hline C & D \end{array} \right]_{z_0}, \tag{3.1}$$

and let $M(z)$ and $N(z)$ be the associated *normalized left coprime factorization* (see Definition 4.1 in *Appendix*). We define a stabilizing controller for $G(z)$ as a $m \times p$ rmf, $K(z)$, for which the closed-loop feedback of G and K is *internally stable*. For any $\sigma > 0$ we introduce the following class of systems

$$D_\sigma^{cf} := \{G_\Delta | G_\Delta(z) = (M(z) + \Delta_M(z))^{-1}(N(z) + \Delta_N(z))\}, \qquad (3.2)$$

where

$$\Delta(z) := \begin{bmatrix} \Delta_N(z) & -\Delta_M(z) \end{bmatrix} \in \mathbb{RH}^\infty,$$
$$\|\Delta\|_\infty < \sigma.$$

DEFINITION 3.1. *The largest $\sigma = \sigma_{max}$ for which there exists a single stabilizing controller for all $G_\Delta \in D_{\sigma_{max}}^{cf}$ is called the maximum stability margin for G. We shall also denote by K_{opt} the associated controller.*

To this end, given any rational matrix function $G(z)$ (possibly improper or polynomial) our problem actually consists in computing the maximum stability margin σ_{max} corresponding to G as well as the associated controller $K_{opt}(z)$.

It is useful to notice that making $\sigma \to 0$, we get $G_\Delta \to G$. To this end, from a geometrical viewpoint D_σ^{cf} might be seen as a ball having G in its center and whose elements are exactly the G_Δ systems given in (3.2) for $\|\Delta\|_\infty = \alpha$ and $\alpha \in (0, \sigma)$. This is why σ_{max} is often called the *stability radius* associated with G.

We are now in position to state the main results of this paper. For a completely general rmf (possibly improper or polynomial), we provide ellegant analtical formulas for computing both the *maximum stability margin σ_{max}* and the *optimal controller K_{opt}* associated with it. We want to emphasize again that the proposed relations also satisfy the computational demands of numerical reliability.

THEOREM 3.2 (Maximum Stability Margin). *Let $G(z)$ be a general rmf given by the following stabilizable and detectable realization*

$$G(z) =: C(zE - A)^{-1}B(\alpha - \bar\alpha z) + D =: \left[\begin{array}{c|c} A - zE & B \\ \hline C & D \end{array} \right]_{z_0}. \qquad (3.3)$$

We have:

  i *The following Riccati equations*

$$A^*XA - E^*XE - (\bar\alpha A - \alpha E)^*XBB^*X(\bar\alpha A - \alpha E) + C^*C = 0, \qquad (3.4)$$
$$AYA^* - EYE^* - (\bar\alpha A - \alpha E)YC^*CY(\bar\alpha A - \alpha E)^* + BB^* = 0 \qquad (3.5)$$

  have (unique) stabilizing solutions X_s and Y_s respectively.

  ii *The maximum stability margin for G is*

$$\sigma_{max} = \{1 + \rho[(\bar\alpha A - \alpha E)^*X_s(\bar\alpha A - \alpha E)Y_s]\}^{-1/2}. \qquad (3.6)$$

Proof. The proof is deferred to *Appendix*. $\square$

THEOREM 3.3 (Optimal Controller). *Under the same hypothesis as in Theorem 3.2, the optimal controller for G is given by*

$$K_{opt} = \left[\begin{array}{c|c} A_{opt} - zE_{opt} & B_{opt} \\ \hline C_{opt} & D_{opt} \end{array} \right]_{z_0}, \qquad (3.7)$$

where

$$A_{opt} := A + \alpha[K_sC - (K_sD + B)B^*X_0(\bar\alpha A - \alpha E)], \qquad (3.8)$$
$$E_{opt} := E + \bar\alpha[K_sC - (K_sD + B)B^*X_0(\bar\alpha A - \alpha E)], \qquad (3.9)$$
$$B_{opt} := -K_s, \qquad (3.10)$$
$$C_{opt} := -B^*X_0(\bar\alpha A - \alpha E), \qquad (3.11)$$
$$D_{opt} := 0, \qquad (3.12)$$

and where $X_0 = X_0^*$ is given by

$$X_0 = \sigma_{max}^{-2} X_s [(\sigma_{max}^{-2} - 1)I + (\bar{\alpha}A - \alpha E)^* X_s (\bar{\alpha}A - \alpha E) Y_s]^{-1}. \qquad (3.13)$$

Proof. The proof is deferred to *Appendix.* □

4. Appendix. This section is devoted to the proof of Theorems 3.2 and 3.3. To do this, we shall need some preliminary notions and results at first.

4.1. The normalized coprime factorization problem. We present here the notion of normalized stable coprime factorization and state an useful result in computing such kind of factorization for a general rmf. As we shall see further, this is going to be a key element in solving our problem.

DEFINITION 4.1. *Let $G(z)$ be a general rational matrix function. Two elements $M(z)$ and $N(z)$ belonging to $\mathbb{R}\mathbb{H}^\infty$ are said to be a stable left coprime factorization (slcf) of G provided*

$$G(z) = M^{-1}(z)N(z) \qquad (4.1)$$

and moreover

$$N(z)X(z) + M(z)Y(z) \equiv I_p \qquad (4.2)$$

holds for two stable rmfs $X(z)$ and $Y(z)$. The slcf in called, in addition, normalized (and we denote it by nslcf) if

$$M(z)M^*(z) + N(z)N^*(z) \equiv I_p. \qquad (4.3)$$

LEMMA 4.2. *Let $G(z)$ be a general $p \times m$ rmf (possibly improper or polynomial) given by a detectable and $C_{o;1}^c$-controllable realization (2.3), having $D = 0$. The following statements hold*

1. *The Riccati equation*

$$AXA^* - EXE^* - (\bar{\alpha}A - \alpha E)XC^*CX(\bar{\alpha}A - \alpha E)^* + BB^* = 0. \qquad (4.4)$$

   has a (unique) stabilizing solution $X_s = X_s^*$.
2. *A nslcf of G is given by*

$$\begin{bmatrix} N(z) & M(z) \end{bmatrix} := \left[\begin{array}{c|cc} A_K - zE_K & B & -K_s \\ \hline C & 0 & I_p \end{array} \right]_{z_0}, \qquad (4.5)$$

   where

$$K_s := -(\bar{\alpha}A - \alpha E)X_s C^*, \qquad (4.6)$$
$$A_K := A + \alpha K_s C, \qquad (4.7)$$
$$E_K := E + \bar{\alpha} K_s C. \qquad (4.8)$$

Proof.

1. Taking $A_l \leftarrow A^T$, $E_l \leftarrow E^T$, $B_l \leftarrow \begin{bmatrix} C^T & 0 \end{bmatrix}$, $C_l \leftarrow B^*$ and $D_l \leftarrow \begin{bmatrix} 0 & I_m \end{bmatrix}$ in Theorem 12 in [8] the conclusion follows immediately.
2. It is a mere adaptation of *Theorem 3* in [5].

□

4.2. Switching form robust stabilization to disturbance feedforward.
We present here an important result that links our problem to the new solved *2-block H^∞* one (the disturbance feedforward). It is useful to notice that each class of uncertainties (3.2) can be represented as an upper linear fractional transformation (ULFT) of a nominal system T^{cf} and a perturbation Δ

$$ULFT(T^{cf}, \Delta)(z) := T_{22}^{cf}(z) + T_{21}^{cf}(z)K(z)(I - T_{11}^{cf}(z)K(z))^{-1}T_{12}^{cf}(z) \qquad (4.9)$$

where

$$T^{cf}(z) := \left[\begin{array}{c|c} T_{11}^{cf}(z) & T_{12}^{cf}(z) \\ \hline T_{21}^{cf}(z) & T_{22}^{cf}(z) \end{array} \right] = \left[\begin{array}{c|c} \left[\begin{array}{c} 0 \\ M^{-1}(z) \end{array} \right] & \left[\begin{array}{c} I_m \\ G(z) \end{array} \right] \\ \hline M^{-1}(z) & G(z) \end{array} \right]. \qquad (4.10)$$

To this end, the feedback connection of G and K is equivalent to $ULFT(T^{cf}; \Delta)$, where both Δ and $ULFT(T^{cf}; \Delta)$ are stable.

The next result is a mere adaptation of *Theorem 7.4.1* in [2], also using *Proposition 12* in [9].

LEMMA 4.3 (Small Gain Theorem). *Let*

$$G_i(z) = \left[\begin{array}{c|c} A_i - zE_i & B_i \\ \hline C_i & D_i \end{array} \right]_{z_0}, \qquad (4.11)$$

*be two rmf of dimensions $p \times m$ amd $m \times p$ respectively, having $\Lambda(A_i - zE_i) \subset \mathbb{D}$, $i = 1 : 2$. Also assume $S := I_m - D_2D_1$ is nonsingular. If $\|G_1\|_\infty < \frac{1}{\gamma}$ and $\|G_2\|_\infty \leq \gamma$ (for some $\gamma > 0$), than the closed loop feedback in **figure below** —– is internally stable.*

THEOREM 4.4. *A controller $K(z)$ is a solution to the robust stabilization problem with respect to the class of systems D_σ^{cf} for $\sigma < \sigma_{max}$ iff $K(z)$ is a solution to the corresponding $\frac{1}{\sigma}$-nonstrict H^∞ problem formulated for $T^{cf}(z)$.*

Proof.

$\Longleftarrow$ Since $K(z)$ is a solution to the $\frac{1}{\sigma}$-nonstrict H^∞ control problem formulated for $T^{cf}(z)$, we have that $T^{cl}(z)$ is stable and, moreover, $\|T^{cl}\|_\infty \leq \frac{1}{\sigma}$. As $\Delta(z)$ is stable and $\|\Delta\|_\infty < \sigma$ it follows from Lemma 4.3 that $K(z)$ is also a solution for the robust stabilization problem formulated for G.

$\Longrightarrow$ This implication follows immediately by applying Theorem 9.1 in [1].

$\square$

DEFINITION 4.5. *An union of five algebraic elements*

$$\Sigma := (A - zE, B; Q, L, R), \qquad (4.12)$$

where $A \in \mathbb{C}^{n \times n}$, $E \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times m}$, $Q \in \mathbb{C}^{n \times n}$, $L \in \mathbb{C}^{n \times m}$, and $R \in \mathbb{C}^{m \times m}$ is called an algebraic quintet.

LEMMA 4.6. *Consider the $(p_1 + p_2) \times (m_1 + m_2)$ system*

$$T(z) =: \left[\begin{array}{c|cc} A - zE & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right]_{z_0}, \qquad (4.13)$$

and suppose it satisfies the assumptions:

A.1 D_{21} *is square and invertible,*

A.2 $\Lambda(A - zE - B_1 D_{21}^{-1} C_2(\alpha - \bar{\alpha}z)) \subset \mathbb{D}$,

A.3 $(A - zE, B_2)$ *is stabilizable,*

A.4 $rank \begin{bmatrix} A - zE & B_2(\alpha - \bar{\alpha}z) \\ C_1 & D_{12} \end{bmatrix} = n + m_2 \; \forall z \in C_{o;1}$.

Also let $J := diag(-I_{m_1}; I_{m_2})$ *and* Σ *as in (4.12), where*

$$
\begin{aligned}
Q : &= C_1^* C_1, \\
L : &= \begin{bmatrix} L_1 & L_2 \end{bmatrix} := \begin{bmatrix} C_1^* D_{11} & C_1^* D_{12} \end{bmatrix}, \\
R : &= \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^* & R_{22} \end{bmatrix} := \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}
\tag{4.14}
$$

The 2-block γ-suboptimal H^∞ problem for T has a solution iff the Riccati equation

$$
A^* X A - E^* X E - ((\bar{\alpha}A - \alpha E)^* X B + L) R^{-1}((\bar{\alpha}A - \alpha E) X B + L)^* + Q = 0 \tag{4.15}
$$

has a stabilizing solution $X = X^* \geq 0$ and $sgn(R) = J$.

Proof. From [9] (Theorem 4 and Proposition 20) we deduce that the above conditions are both necessary and sufficient. □

4.3. An auxiliary operator-based result. We state here necessary and sufficient conditions for the existence of a robustly stabilizing controller in terms of linear operators. As a result, a formula for the maximum stability margin raises in operatorial form. We make first some preparations.

DEFINITION 4.7. *Let X be a Hilbert space. A linear operator $\mathcal{R} : X \to X$ is coercive ($\mathcal{R} >> 0$) if there exists a $\delta > 0$ such that $\langle \mathcal{R}u, u \rangle_X \geq \delta \|u\|_X^2, \; \forall u \in X$.*

Suppose G is a general stable $p \times m$ rmf given by the stabilizable and detectable realization

$$
G(z) =: \left[\begin{array}{c|c} A - zE & B \\ \hline C & D \end{array} \right]_{z_0} = C(zE - A)^{-1} B(\alpha - \bar{\alpha}z) + D. \tag{4.16}
$$

DEFINITION 4.8. *The linear bounded operator $\Psi : l_-^{2;m} \to \mathbb{C}^n$*

$$
\Psi u := \sum_{i=-\infty}^{-1} (AE^{-1})^{-i-1} B(\alpha u[i] - \bar{\alpha}u[i+1]) \tag{4.17}
$$

is called the causal direct time controlability operator.

Similarly, the linear bounded operator $\Phi : \mathbb{C}^n \to l_+^{2;p}$

$$
(\Phi\xi)[k] := CE^{-1}(AE^{-1})^k \xi \tag{4.18}
$$

is called the causal direct time observability operator.

Next we recall the notions of *Hankel* and *Toeplitz* operators (for further details, see [7]).

DEFINITION 4.9. *Let $\mathcal{G}$ be a linear bounded operator from $l^{2,m}$ to $l^{2,p}$. We define the causal Hankel operator associated with $\mathcal{G}$ as $\mathbb{H}_{\mathcal{G}} := P_+^p \mathcal{G}|_{l_-^{2;m}}$. Similarly, we define the causal Toeplitz operator associated with $\mathcal{G}$ as $\mathbb{T}_{\mathcal{G}} := P_+^p \mathcal{G}|_{l_+^{2;m}}$. Here P_+^p denotes the projection on l_+^p.*

Consider now the algebraic quintet $\Sigma := (A - zE, B; Q, L, R)$ having $\Lambda(A - zE) \subset \mathbb{D}$ and associate with it the Popov function

$$\Pi_\Sigma(z) = \left[\begin{array}{cc|c} A - zE & 0 & B \\ Q(\alpha - \bar\alpha z) & E^* - zA^* & L \\ \hline L^* & B^* & R \end{array} \right]_{z_0}, \tag{4.19}$$

partitioned in accordance with (4.14). Denote by $\mathcal{R}_{\Sigma_e}$ its input output operator and let $\mathcal{R}_\Sigma := \mathbb{T}_{\mathcal{R}_{\Sigma_e}}$. $\mathcal{R}_\Sigma$ will called the Σ Topelitz operator.

LEMMA 4.10. *Let Σ and Π_Σ be as above. Suppose R is nonsingular and $\Pi_{\Sigma_{22}}(z) > 0$ for all $z \in C_{o;1}$. There exists a rmf $H \in \mathbb{R}\mathbb{H}^\infty$ such that*

$$\begin{bmatrix} I & H^*(z) \end{bmatrix} \Pi_\Sigma(z) \begin{bmatrix} I \\ H(z) \end{bmatrix} < 0, \tag{4.20}$$

for all $z \in C_{o;1}$ if and only if $-\mathcal{R}_{\Sigma_{11}}^X >> 0$.

   Proof.

$\implies$ Since $\Pi_{\Sigma_{22}}(z) > 0$ for all $z \in C_{o;1}$, and since $C_{o;1}$ is a closed set, we infer form Parseval's identity that $\mathcal{R}_{\Sigma_{22}}$ is coercive. Now suppose there is a stable rmf H such that we have (4.20). Denote by $\mathcal{H}_e$ and $\mathcal{H}$ the input output operator of the system H and its Toeplitz operator respectively.
   Further let

$$\mathcal{T}_{\Sigma_e} := \begin{bmatrix} I & \mathcal{H}_e^* \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{R}_{\Sigma_{e_{11}}} & \mathcal{R}_{\Sigma_{e_{12}}} \\ \mathcal{R}_{\Sigma_{e_{12}}}^* & \mathcal{R}_{\Sigma_{e_{22}}} \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathcal{H}_e & I \end{bmatrix} =: \begin{bmatrix} \mathcal{T}_{\Sigma_{e_{11}}} & \mathcal{T}_{\Sigma_{e_{12}}} \\ \mathcal{T}_{\Sigma_{e_{12}}}^* & \mathcal{T}_{\Sigma_{e_{22}}} \end{bmatrix}, \tag{4.21}$$

and we notice that $\mathcal{T}_{\Sigma_{e_{11}}}$ is the input output operator associated with the left hand side of (4.20). Therefore $-\mathcal{T}_{\Sigma_{e_{11}}} >> 0$, $\mathcal{T}_{\Sigma_{e_{22}}} = \mathcal{R}_{\Sigma_{e_{22}}} >> 0$, and taking into account that $H \in \mathbb{R}\mathbb{H}^\infty$ we get from Corollary 2.4.13 in [2]

$$\mathcal{T}_\blacksquare := \mathbb{T}_{\mathcal{T}_{\Sigma_e}} = \begin{bmatrix} I & \mathcal{H}^* \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{R}_{\Sigma_{11}} & \mathcal{R}_{\Sigma_{12}} \\ \mathcal{R}_{\Sigma_{12}}^* & \mathcal{R}_{\Sigma_{22}} \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathcal{H} & I \end{bmatrix} =: \begin{bmatrix} \mathcal{T}_{\Sigma_{11}} & \mathcal{T}_{\Sigma_{12}} \\ \mathcal{T}_{\sigma_{12}}^* & \mathcal{T}_{\Sigma_{22}} \end{bmatrix}. \tag{4.22}$$

From the above equation we can express $\mathcal{R}_\Sigma$ function of $\mathcal{T}_\Sigma$ and deduce that $\mathcal{R}_{\Sigma_{11}}^X = \mathcal{T}_{\Sigma_{11}} - \mathcal{T}_{\Sigma_{12}} \mathcal{T}_{\Sigma_{22}}^{-1} \mathcal{T}_{\Sigma_{12}}^*$. Hence the conclusion follows.

$\impliedby$ Notice that $\mathcal{R}_{\Sigma_{e_{22}}}$ is a constant invertible matrix, and $\mathcal{R}_{\Sigma_{e_{22}}}^{-1} \mathcal{R}_{\Sigma_{e_{12}}}^* = 0$ for all $k \in \mathbb{Z} - \mathbb{N}$. Denote $h := -\mathcal{R}_{\Sigma_{22}}^{-1} \mathcal{R}_{\Sigma_{12}}^*$ and notice that its $\mathcal{Z}$ transform $H(z) := \mathcal{Z}(h)_{|(z)}$ is a stable rmf. For any $w \in l_+^{2;m}$ we have

$$\begin{bmatrix} I & 0 \\ \mathcal{R}_{\Sigma_{22}}^{-1} \mathcal{R}_{\Sigma_{12}}^* & I \end{bmatrix} \begin{bmatrix} I \\ h \end{bmatrix} w = \begin{bmatrix} I \\ 0 \end{bmatrix} w. \tag{4.23}$$

Compute

$$w^* \mathcal{R}_{\Sigma_{11}}^X w = w^* \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \mathcal{R}_{\Sigma_{11}}^X & 0 \\ 0 & \mathcal{R}_{\Sigma_{22}} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} w \tag{4.24}$$

$$= w^* \begin{bmatrix} I & h^* \end{bmatrix} \begin{bmatrix} I & \mathcal{R}_{\Sigma_{12}} \mathcal{R}_{\Sigma_{22}}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{R}_{\Sigma_{11}}^X & 0 \\ 0 & \mathcal{R}_{\Sigma_{22}} \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathcal{R}_{\Sigma_{22}}^{-1} \mathcal{R}_{\Sigma_{12}}^* & I \end{bmatrix} \begin{bmatrix} I \\ h \end{bmatrix} w.$$

On the other hand, since h and w are causal, we have

$$w^* \begin{bmatrix} I & h^* \end{bmatrix} \mathcal{R}_\Sigma \begin{bmatrix} I \\ h \end{bmatrix} w = w^* \begin{bmatrix} I & h^* \end{bmatrix} \begin{bmatrix} P_+ I & 0 \\ 0 & P_+ I \end{bmatrix} \mathcal{R}_{\Sigma_e} \begin{bmatrix} P_+ I & 0 \\ 0 & P_+ I \end{bmatrix} \begin{bmatrix} I \\ h \end{bmatrix} w$$

$$' = w^* \begin{bmatrix} I & h^* \end{bmatrix} \mathcal{R}_{\Sigma_e} \begin{bmatrix} I \\ h \end{bmatrix} w. \tag{4.25}$$

Combining the last two equations above, together with the coercivity of $-\mathcal{R}^X_{\Sigma_{11}}$, we get

$$\langle w; \begin{bmatrix} I & h^* \end{bmatrix} (-\mathcal{R}_{\Sigma_e}) \begin{bmatrix} I \\ h \end{bmatrix} w \rangle_{|l^{2;m_1}_+} > C\|w\|^2_{|l^{2;m_1}_+}, \qquad (4.26)$$

where $C > 0$ is a real constant. Finally, we shall prove that for any m_1-vector $\hat{w}$ we have

$$\hat{w}^*(e^{j\theta}) \begin{bmatrix} I & H^*(e^{j\theta}) \end{bmatrix} (-\Pi_\Sigma(e^{j\theta})) \begin{bmatrix} I \\ H(e^{j\theta}) \end{bmatrix} \hat{w}(e^{j\theta}) \geq \mu\|w(e^{j\theta})\|^2_2, \quad (4.27)$$

where $\hat{w} := \sum_{k=0}^{\infty} w_k e^{jk\theta}$ is the Fourier series of w, $\theta \in [0; 2\pi)$, and $\mu > 0$ is a real constant. Suppose by contradiction there is a $\hat{w}_0$ and a $\theta_0 \in [0; 2\pi)$ such that

$$\hat{w}_0^*(e^{j\theta_0}) \begin{bmatrix} I & H^*(e^{j\theta_0}) \end{bmatrix} (-\Pi_\Sigma(e^{j\theta_0})) \begin{bmatrix} I \\ H(e^{j\theta_0}) \end{bmatrix} \hat{w}_0(e^{j\theta_0}) < \mu\| \begin{bmatrix} I \\ H(e^{j\theta_0}) \end{bmatrix} \hat{w}_0(e^{j\theta_0})\|^2_2,$$
$$(4.28)$$

where $\mu > 0$ is to be determined. By continuity, there is an $\epsilon > 0$ such that

$$\langle \hat{w}_0^*(e^{j\theta_0}) \begin{bmatrix} I & H^*(e^{j\theta}) \end{bmatrix} (-\Pi_\Sigma(e^{j\theta})) \begin{bmatrix} I \\ H(e^{j\theta}) \end{bmatrix} \hat{w}_0(e^{j\theta_0}) \rangle_{|\mathbb{C}^m} < \mu\|\hat{w}_0(e^{j\theta_0})\|^2_2,$$
$$(4.29)$$

for $\theta \in (\theta_0 - \epsilon/2; \theta_0 + \epsilon/2)$. Take $\mu = C/\epsilon$ and denote $\hat{w}_0^\epsilon(e^{j\theta}) := \hat{w}_0(e^{j\theta_0})$ for any $\theta \in (\theta_0 - \epsilon/2; \theta_0 + \epsilon/2)$ and 0 otherwise. Also let w_0^ϵ be its inverse Fourier transform. We have thus obtained

$$\langle w_0^\epsilon; \begin{bmatrix} I & h^* \end{bmatrix} (-\mathcal{R}_{\Sigma_e}) \begin{bmatrix} I \\ h \end{bmatrix} w_0^\epsilon \rangle_{|l^{2;m_1}_+} \geq C\|w_0^\epsilon\|^2_{|l^{2;m_1}_+} > 1/2\pi \langle \hat{w}_0^\epsilon; \begin{bmatrix} I & H^* \end{bmatrix} (-\Pi_\Sigma) \begin{bmatrix} I \\ H \end{bmatrix} \rangle_{|2;m_1}$$
$$(4.30)$$

which is obviously a contradiction.

□

Consider again the system (4.13) with the assumptions $A.1 : A.4$ and the algebraic quintet $\Sigma := (A - zE, B; Q, L, R)$ given in (4.14).

We infer from [8] (proof of Theorem 12) that the Riccati equation

$$A^*XA - E^*XE - ((\bar{\alpha}A - \alpha E)^*XB_2 + L_2)R_{22}^{-1}((\bar{\alpha}A - \alpha E)^*XB_2 + L_2)^* + Q = 0 \quad (4.31)$$

has a stabilizing solution $X_{s_2} = X_{s_2}^* \geq 0$. Let $F_{s_2} := -R_{22}^{-1}((\bar{\alpha}A - \alpha E)^*XB_2 + L_2)^*$ be the associated beedback, and denote $\bar{\Sigma} := (\bar{A} - z\bar{E}, \bar{B}; \bar{Q}, \bar{L}, \bar{R})$ where

$$\begin{aligned}
\bar{A} &:= A + \alpha B_2 F_{s_2}, \\
\bar{E} &:= E + \bar{\alpha} B_2 F_{s_2}, \\
\bar{B} &:= B, \\
\bar{Q} &:= \bar{C}_1^* \bar{C}_1 := (C_1 + D_{12} F_{s_2})^*(C_1 + D_{12} F_{s_2}), \\
\bar{L} &:= \bar{C}_1 \begin{bmatrix} D_{11} & D_{12} \end{bmatrix}, \\
\bar{R} &:= R.
\end{aligned} \qquad (4.32)$$

Also let $\Pi_{\bar{\Sigma}}$ be the Popov function associated with $\bar{\Sigma}$. Immediate computations using relation 3. in [3] lead to

$$\Pi_{\bar{\Sigma}}(z) = \begin{bmatrix} \bar{T}_{11}^*(z) \\ \bar{T}_{12}^*(z) \end{bmatrix} \begin{bmatrix} \bar{T}_{11}(z) & \bar{T}_{12}(z) \end{bmatrix} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad (4.33)$$

where

$$\begin{bmatrix} \bar{T}_{11}(z) & \bar{T}_{12}(z) \end{bmatrix} := \left[\begin{array}{c|cc} \bar{A} - z\bar{E} & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \end{array} \right]_{z_0}. \tag{4.34}$$

The following result gives necessary and sufficient conditions for the existence of a solution to the γ-suboptimal H^∞ problem in therms of $\mathcal{R}_{11}^X$. We shall see that, using this result, we obtain a value of the maximum stability margin σ_{max} of G in operatorial form. Consequently, this formula will be used further for computing σ_{max} in (3.6).

THEOREM 4.11. *Consider the system T biven by (4.13) and suppose $A.1 : A.4$ are also in force. Let*

$$\mathcal{R}_{\bar{\Sigma}} =: \begin{bmatrix} \bar{\mathcal{R}}_{11} & \bar{\mathcal{R}}_{12} \\ \bar{\mathcal{R}}_{12}^* & \bar{\mathcal{R}}_{22} \end{bmatrix} \tag{4.35}$$

be the $\bar{\Sigma}$ Toeplitz operator partitioned in accordance with $\Pi_{\bar{\Sigma}}$. Then we have

    i The γ-suboptimal H^∞ control problem formulated for $T(z)$ in (4.13) has a solution iff

$$-\mathcal{R}_{11}^X >> 0. \tag{4.36}$$

    ii The last achievable $\gamma = \gamma_{min}$ (the optimal value) such that the γ-suboptimal H^∞ problem still has a solution is given by

$$\gamma_{min} = \rho(\mathbb{H}_{\bar{T}_{11}^* \hat{T}_{12}} \mathbb{H}_{\bar{T}_{11}^* \hat{T}_{12}}^* + \mathbb{T}_{(\bar{T}_{11}^* \hat{T}_{12}^\perp)(\bar{T}_{11}^* \hat{T}_{12}^\perp)^*})^{1/2}, \tag{4.37}$$

where $\hat{T}_{12}^\perp \in \mathbb{RL}^\infty$ is the orthogonal completion of the (inner) rmf $\hat{T}_{12}(z) := \bar{T}_{12} V^{-1}$ in such manner that $\hat{T}_{12}^a(z) := \begin{bmatrix} \hat{T}_{12} & \hat{T}_{12}^\perp \end{bmatrix}$ is all pass.

Proof. We prove first that $\hat{T}_{12}$ is inner. To do this, notice that $\hat{T}_{12}$ is stable since F_{s_2} is the stabilizing feedback of (4.31). It only remains to show that $\hat{T}_{12}^*(z)\hat{T}_{12}(z) = I_{m_2}$. We have

$$\hat{T}_{12}(z) = \left[\begin{array}{c|c} \bar{A} - z\bar{E} & B_2 D_{12}^{-1} \\ \hline C_1 & I_{m_2} \end{array} \right]_{z_0}. \tag{4.38}$$

It can be easily checked that we have the following relations

$$\bar{C}_1 - D_{12}^{-*} B_2^* X(\alpha E - \bar{\alpha} A) = 0, \tag{4.39}$$
$$\bar{E}^* X \bar{E} - \bar{A}^* X \bar{A} - \bar{C}^* \bar{C} = 0, \tag{4.40}$$

where X is the (unique) stabilizing solution of (4.31). Invoking relation (4.38) and Theorem A.2. in [4] the conclusion follows. The fact that we can always find an orthogonal completion of $\hat{T}_{12}$ such that $\hat{T}_{12}^a$ is all pass follows in a straightforward way from some standard arguments regarding the Hilbert spaces. Moreover, since $\hat{T}_{12}$ is inner, we get that $\Pi_{\bar{\Sigma}_{2,2}}(z) = \bar{T}_{12}^*(z)\bar{T}_{12}(z) = D_{12}^* D_{12} =: V^* V > 0 \ \forall z \in \bar{\mathbb{C}}$, where V is the *Cholesky* factor of $D_{12}^* D_{12}$. So, it follows that $\mathcal{R}_{22} >> 0$.

It is worth remarking that X_s is the stabilizing solution of (4.15) (with the associated stabilizing feedback F_s) iff $\bar{X}_s = X_s$ is the stabilizing solution of

$$\bar{A}^* \bar{X} \bar{A} - \bar{E}^* \bar{X} \bar{E} - ((\bar{\alpha}\bar{A} - \alpha\bar{E})^* \bar{X} \bar{B} + \bar{L})\bar{R}^{-1}((\bar{\alpha}\bar{A} - \alpha\bar{E})^* \bar{X} \bar{B} + \bar{L})^* + \bar{Q} = 0 \tag{4.41}$$

(with the stabilizing feedback $\bar{F}_s = F_s - F_{s_2}$).

i Based on Lemma 4.6 and the last relation (4.32), all it remains now is to show that the Riccati equation above has a stabilizing solution $\bar{X}_s = \bar{X}_s^* \geq 0$ and $sgn(\bar{R}) = J$ iff condition (4.36) holds true.

$\Longrightarrow$ In order to prove that relation (4.36) is fulfilled, we shall show that (4.20) holds for $\Pi_{\bar{\Sigma}}$, which in turn will involve by Lemma 4.10 (also using the fact that $\Pi_{\bar{\Sigma}_{2,2}}(z) > 0 \ \forall z \in C_{o;1}$) that $-\mathcal{R}_{11}^X$ is indeed coercive. Since we supposed that the γ-suboptimal H^∞ control problem for T in (4.13) has a solution (say $K(z)$), we may take $H(z) := K(z)(I_{p_2} - T_{22}(z)K(z))^{-1}T_{21}(z) \in \mathbb{R}\mathbb{H}^\infty$ and notice that we have

$$\begin{bmatrix} I_{m_2} & H^*(z) \end{bmatrix} \Pi_\Sigma(z) \begin{bmatrix} I_{m_2} \\ H(z) \end{bmatrix} = T_{cl}^*(z)T_{cl}(z) - \gamma^2 I_{m_1} < 0, \qquad (4.42)$$

for all $z \in C_{o;1}$ since the controller K is γ-contracting (i.e. $\|T_{cl}\|_\infty < \gamma$). Using now relations 6.10 and 6.11 in [9] (for $H(z) \leftarrow M(z)$ and $\hat{H}(z) \leftarrow \tilde{M}(z)$), the conclusion follows.

$\Longleftarrow$ As $\mathcal{R}_{22} >> 0$, (4.36) hold, and R is invertible, we deduce from *Lemma* 4.10 and *Lemma* 15 in [9] that the Riccati equation (4.41) has a stabilizing solution $\bar{X}_s = \bar{X}_s^* \geq 0$. In order to complete our proof, we need to show further that $sgn(R) = J$. We have $\Pi_{\Sigma_{22}} > 0$ and its Schur complement

$$\Pi_{\bar{\Sigma}_{11}}^X = \bar{T}_{11}^*\bar{T}_{11} - \gamma^2 I_{m_1} - \bar{T}_{11}^*\hat{T}_{12}^*\hat{T}_{12}\bar{T}_{11} = -\gamma^2 I_{m_1} < 0. \qquad (4.43)$$

But $R = \Pi_\Sigma(z_0)$ which concludes the proof of the first part.

ii Taking into account that $\hat{T}_{12}^a$ is all pass and the following identity

$$\mathbb{T}_{GG^*} = \mathbb{T}_G\mathbb{T}_G^* + \mathbb{H}_G\mathbb{H}_G^* \qquad (4.44)$$

which holds for any rmf (linear bounded input output operator) G, we have

$$\mathcal{R}_{11}^X = \mathbb{H}_{\bar{T}_{11}^*\hat{T}_{12}}\mathbb{H}_{\bar{T}_{11}^*\hat{T}_{12}}^* + \mathbb{T}_{(\bar{T}_{11}^*\hat{T}_{12}^\perp)(\bar{T}_{11}^*\hat{T}_{12}^\perp)^*} - \gamma^2 I_{m_1}. \qquad (4.45)$$

Hence, according to $i.$, the above condition is equivalent to (4.37).

$\square$

We are now in position to prove the main results of this paper.

4.4. Proof of Theorem 3.2.

i It follows straight from *Lemma* 2.4.

ii Let G be a general rmf given by the following stabilizable and detectable realization

$$G(z) =: \left[\begin{array}{c|c} A - zE & B \\ \hline C & D \end{array} \right]_{z_0}. \qquad (4.46)$$

As one may notice that the *maximum stability margin* does not depend on the D-matrix it is no loss of generality in supposing here $D = 0$.

Let $F_s := -B^*X(\bar{\alpha}A - \alpha E)$ and $K_s := -(\bar{\alpha}A - \alpha E)YC^*$ be the stabilizing feedbacks associated with the Riccati equations (3.4) and (3.5) respectively. Let $T^{cf}(z)$ be given by (4.10). We have

$$T^{cf}(z) := \left[\begin{array}{c|cc} A - zE & -K_s & B \\ \hline \begin{bmatrix} 0 \\ C \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} I \\ 0 \end{bmatrix} \\ C & I & 0 \end{array} \right]_{z_0}, \qquad (4.47)$$

which satisfies all the assumptions A1:A4. Compute now

$$
\begin{bmatrix} \bar{T}_{11}(z) & \bar{T}_{12}(z) \end{bmatrix} = \left[\begin{array}{c|cc} A_F - zE_F & -K_s & B \\ \hline \begin{matrix} F_c \\ C \end{matrix} & \begin{matrix} 0 \\ I \end{matrix} & \begin{matrix} I \\ 0 \end{matrix} \end{array} \right]_{z_0}, \tag{4.48}
$$

where $A_F - zE_F := A - zE + BF_s(\alpha - \bar{\alpha}z)$ is stable.

The ideea is to use now *Theorem* 4.11 to compute the maximum stability margin. To this end, notice first that we can take $V = I$ which means that $\bar{T}_{12} = \hat{T}_{12}$. Denote

$$
\left[\begin{array}{c|cc} A_F - zE_F & -K_s & B \\ \hline F_c & 0 & I \\ C & I & 0 \end{array} \right]_{z_0} = \begin{bmatrix} -W(z) & M(z) \\ U(z) & N(z) \end{bmatrix} \tag{4.49}
$$

and let the pairs $(N; M)$ and $(\bar{N}; \bar{M})$ be normalized right and left coprime factorization of G, respectively. It follows that $\begin{bmatrix} M & -\bar{N}^* \\ N & \bar{M}^* \end{bmatrix}$ is all-pass, and thus $\hat{T}_{12}^{\perp} = \begin{bmatrix} -\bar{N}^* \\ \bar{M}^* \end{bmatrix}$ is an orthogonal completion of $\hat{T}_{12}$. Further, we compute

$$
\bar{T}_{11}^*(z)\hat{T}_{12}(z) = -W^*(z)M(z) + U(z)N(z) =: \Omega(z), \tag{4.50}
$$

as well as

$$
\bar{T}_{11}^*(z)\hat{T}_{12}^{\perp}(z) \equiv I. \tag{4.51}
$$

Using now relations (4.50), (4.51) and (4.37) we conclude that

$$
\sigma_{max} = (1 + \rho(\mathbb{H}_\Omega \mathbb{H}_\Omega^*))^{-\frac{1}{2}}, \tag{4.52}
$$

where

$$
\Omega = \left[\begin{array}{cc|c} E_F^* - zA_F^* & (F_s^*F_s + C^*C)(\alpha - \bar{\alpha}z) & F_s^* \\ 0 & A_F - zE_F & B \\ \hline K_s^* & C & 0 \end{array} \right]_{z_0}. \tag{4.53}
$$

Compute now

$$
\begin{aligned}
& A_F^* X_s A_F - E_F^* X_s E_F + F_s^* F_s + C^* C \\
&= A^* X_s A - E^* X_s E - (\bar{\alpha}A - \alpha E)^* X_s B B^* X_s (\bar{\alpha}A - \alpha E) + C^* C = 0.
\end{aligned} \tag{4.54}
$$

Provided we write the state space equations for Ω, under zero initial conditions, we obtain

$$
\begin{aligned}
E_F x_{[k+1]} &= A_F x_{[k]} + B(\alpha u_{[k]} - \bar{\alpha}u_{[k+1]}), \\
-E_F^* \xi_{[k]} &= -A_F^* \xi_{[k+1]} + (F_s^*F_s + C^*C)(\alpha x_{[k]} - \bar{\alpha}x_{[k+1]}) + F_s^*(\alpha u_{[k]} - \bar{\alpha}u_{[k+1]}), \\
y_{[k]} &= K_s^* \xi_{[k]} + C x_{[k]}.
\end{aligned} \tag{4.55}
$$

It is now a matter of simple algebraic manipulations, using the above results, to show that

$$
\mathbb{H}_\Omega = \mathbb{H}_{\Omega_r}, \tag{4.56}
$$

for

$$\Omega_r := \left[\begin{array}{c|c} A_F - zE_F & B \\ \hline C_r & 0 \end{array}\right]_{z_0}, \qquad (4.57)$$

where $C_r := C(I + Y_s(\bar{\alpha}A - \alpha E)^*X_s(\bar{\alpha}A - \alpha E))$. Further, denote by Ψ_r and Φ_r the causal direct-time controlability operator and the causal direct-time observability operator respectively, both of them associated with the stable system Ω_r. Using relations 2.194 and 2.195 in [2] we can compute their adjoints, and after some lengthy calculations, also invoking relation 2.196 in [2], we finally obtain

$$\mathbb{H}_\Omega \mathbb{H}_\Omega^* = \mathbb{H}_{\Omega_r} \mathbb{H}_{\Omega_r}^* = (\bar{\alpha}A - \alpha E)P(\bar{\alpha}A - \alpha E)^*Q, \qquad (4.58)$$

where the constant matrices P and Q are the (unique) solutions of the Stein equations

$$0 = A_F^* Q A_F - E_F^* Q E_F + C_r^* C_r, \qquad (4.59)$$
$$0 = A_F P A_F^* - E_F P E_F^* + BB^*. \qquad (4.60)$$

Using the last two equations above, one may notice that

$$(\bar{\alpha}A - \alpha E)P(\bar{\alpha}A - \alpha E)^*Q = (\bar{\alpha}A - \alpha E)^*X_s(\bar{\alpha}A - \alpha E)Y_s, \qquad (4.61)$$

which ends our proof.

Before proving the next result, we need the following lemma.

LEMMA 4.12. *Let X_s and Y_s be the (unique) stabilizing solutions of* (3.4) *and* (3.5). *Also let* $\Sigma_\gamma := (A - zE, \begin{bmatrix} -K_s & B \end{bmatrix}; C^*C, \begin{bmatrix} C^* & 0 \end{bmatrix}, \begin{bmatrix} (1-\gamma)I & 0 \end{bmatrix}^* \begin{bmatrix} (1-\gamma)I & 0 \end{bmatrix})$ *and consider the Riccati equation*

$$0 = C^*C + A^*X_\gamma A - E^*X_\gamma E$$
$$- \begin{bmatrix} C^* - (\bar{\alpha}A - \alpha E)^*X_\gamma K_s & (\bar{\alpha}A - \alpha E)^*X_\gamma B \end{bmatrix} \begin{bmatrix} (1-\gamma)^{-2}I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} C - K_s^*X_\gamma(\bar{\alpha}A - \alpha E) \\ B^*X_\gamma(\bar{\alpha}A - \alpha E) \end{bmatrix},$$
$$(4.62)$$

associated with Σ_γ.

We have:

$$X_\gamma = \gamma^2 X_s[(\gamma^2 - 1)I + (\bar{\alpha}A - \alpha E)^*X_s(\bar{\alpha}A - \alpha E)Y_s]^{-1}, \qquad (4.63)$$

for all $\gamma > \gamma_{min}$.

Proof. In order to simplify our computations, we shall assume without loss of generality that $(\alpha E - \bar{\alpha}A) = I$. This means that realtion (4.63) becomes

$$X_\gamma = \gamma^2 X_s[(\gamma^2 - 1)I + X_s Y_s]^{-1}, \qquad (4.64)$$

$\forall \gamma > \gamma_{min}$. Provided $(\alpha E - \bar{\alpha}A) \neq I$, we have $X_s \leftarrow (\bar{\alpha}A - \alpha E)^*X_s(\bar{\alpha}A - \alpha E)$ and Y_s remains the same. The ideea is to link the *descriptor symplectic pencils* (DSP) associated with the Riccati equations (3.4) and (4.63) (for further details regarding the DSPs, see [8]).

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Denote

$$zM_\Sigma - N_\Sigma := z \begin{bmatrix} E & 0 & \bar{\alpha}Y_sC^* & \bar{\alpha}B \\ \bar{\alpha}C^*C & -A^* & \bar{\alpha}C^* & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 & \alpha Y_sC^* & \alpha B \\ \alpha C^*C & -E^* & \alpha C^* & 0 \\ C & -CY_s & -(1-\gamma^2)I_n & 0 \\ 0 & -B^* & 0 & I_m \end{bmatrix}.$$
(4.65)

Since X_γ is a stabilizing solution of (4.62), we have

$$\begin{bmatrix} E & 0 & \bar{\alpha}Y_sC^* & \bar{\alpha}B \\ \bar{\alpha}C^*C & -A^* & \bar{\alpha}C^* & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_n \\ X_\gamma \\ F_1 \\ F_2 \end{bmatrix} S_c = \begin{bmatrix} A & 0 & \alpha Y_sC^* & \alpha B \\ \alpha C^*C & -E^* & \alpha C^* & 0 \\ C & -CY_s & -(1-\gamma^2)I_n & 0 \\ 0 & -B^* & 0 & I_m \end{bmatrix} \begin{bmatrix} I_n \\ X_\gamma \\ F_1 \\ F_2 \end{bmatrix}$$
(4.66)

holds for a stable matrix S_c. It follows that

$$(E + \bar{\alpha}Y_sC^*F_1 + \bar{\alpha}BF_2)S_c = A + \alpha Y_sC^*F_1 + \alpha BF_2, \tag{4.67}$$
$$(\bar{\alpha}C^*C - A^*X_\gamma + \bar{\alpha}C^*F_1)S_c = \alpha C^*C - E^*X_\gamma + \alpha C^*F_1, \tag{4.68}$$
$$C - CY_sX_\gamma + (1-\gamma^2)F_1 = 0, \tag{4.69}$$
$$- B^*X_\gamma + F_2 = 0. \tag{4.70}$$

Note that the last two equations may be written, equivalently, as

$$F_1 = -(1-\gamma^2)^{-1}C(I_n - Y_sX_\gamma), \tag{4.71}$$

$$F_2 = B^*X_\gamma. \tag{4.72}$$

Denote now

$$\bar{M} := (1-\gamma^2)^{-1}(I_n - Y_sX_\gamma). \tag{4.73}$$

Relation (4.71) becomes

$$F_1 = -C\bar{M}. \tag{4.74}$$

We compute

$$\bar{\alpha}Y_sE^* = EY_sE^* - \bar{\alpha}^2AY_sE^* = EY_sE^* - AY_sA^* - \bar{\alpha}AY_s,$$

or, taking into account the Lyapunov equation for Y_s, we get

$$\bar{\alpha}Y_sE^* = Y_sC^*CY_s - BB^* - \bar{\alpha}AY_s. \tag{4.75}$$

Analogously

$$\alpha Y_sA^* = Y_sC^*CY_s - BB^* - \alpha EY_s, \tag{4.76}$$

from where we obtain

$$(\bar{\alpha}C^*C(I_n - \bar{M}) - A^*X_\gamma)S_c = \alpha C^*C(I_n - \bar{M}) - E^*X_\gamma. \tag{4.77}$$

Equations (4.67), (4.77), (4.74) and (4.72) give

$$\begin{bmatrix} E & 0 & \bar\alpha Y_s C^* & \bar\alpha B \\ \bar\alpha C^* C(I_n - \bar M) & -A^* & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_n \\ X_\gamma \\ F_1 \\ F_2 \end{bmatrix} S_c = \begin{bmatrix} A & 0 & \alpha Y_s C^* & \alpha B \\ \alpha C^* C(I_n - \bar M) & -E^* & 0 & 0 \\ C\bar M & 0 & I_p & 0 \\ 0 & -B^* & 0 & I_m \end{bmatrix} \begin{bmatrix} I_n \\ X_\gamma \\ F_1 \\ F_2 \end{bmatrix},$$

from where we have

$$\left(\begin{bmatrix} E & 0 & \bar\alpha B \\ \bar\alpha C^* C(I_n - \bar M) & -A^* & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_n \\ X_\gamma \\ F_2 \end{bmatrix} + \begin{bmatrix} \bar\alpha Y_s C^* F_1 \\ 0 \\ 0 \end{bmatrix} \right) S_c =$$
$$\begin{bmatrix} A & 0 & \alpha B \\ \alpha C^* C(I_n - \bar M) & -E^* & 0 \\ 0 & -B^* & I_m \end{bmatrix} \begin{bmatrix} I_n \\ X_\gamma \\ F_2 \end{bmatrix} + \begin{bmatrix} \alpha Y_s C^* F_1 \\ 0 \\ 0 \end{bmatrix}. \tag{4.78}$$

Premultiplying (4.78) with $\begin{bmatrix} I_n & -Y_s & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_m \end{bmatrix}$, taking into account (4.74) and extracting the first equality gives

$$(E - \bar\alpha Y_s C^* C\bar M + \bar\alpha BB^* X_\gamma) S_c = A - \alpha Y_s C^* C\bar M + \alpha BB^* X_\gamma. \tag{4.79}$$

Straightforward computations show that

$$E(I_n - \bar M) + \bar\alpha BB^* X_\gamma = -\gamma^2 (1 - \gamma^2)^{-1} (E - \bar\alpha Y_s C^* C)$$
$$- (1 - \gamma^2)^{-1} Y_s A^* X_\gamma + (1 - \gamma^2)^{-1} \bar\alpha Y_s C^* C(-\gamma^2 I_n + Y_s C^* C) - \gamma^2 (1 - \gamma^2)^{-1} (\bar\alpha BB^* X_\gamma), \tag{4.80}$$

$$A(I_n - \bar M) + \alpha BB^* X_\gamma = -\gamma^2 (1 - \gamma^2)^{-1} (A - \alpha Y_s C^* C)$$
$$- (1 - \gamma^2)^{-1} Y_s E^* X_\gamma + (1 - \gamma^2)^{-1} \alpha Y_s C^* C(-\gamma^2 I_n + Y_s C^* C) - \gamma^2 (1 - \gamma^2)^{-1} (\alpha BB^* X_\gamma). \tag{4.81}$$

Hence, we get

$$\begin{bmatrix} E - \bar\alpha Y_s C^* C & Y_s A^* & \bar\alpha B \\ \bar\alpha C^* C(I_n - \bar M) & -A^* & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_n \\ X_\gamma \\ F_2 \end{bmatrix} S_c = \begin{bmatrix} A - \alpha Y_s C^* C & Y_s A^* & \alpha B \\ \alpha C^* C(I_n - \bar M) & -E^* & 0 \\ 0 & -B^* & 0 \end{bmatrix} \begin{bmatrix} I_n \\ X_\gamma \\ F_2 \end{bmatrix}. \tag{4.82}$$

Now, premultiply the above relation with $\begin{bmatrix} -\gamma^2 (1 - \gamma^2)^{-1} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_m \end{bmatrix}$ and taking into account (4.72) we get

$$\begin{bmatrix} E(I_n - \bar M) & 0 & \bar\alpha B \\ \bar\alpha C^* C(I_n - \bar M) & -A^* & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_n - \bar M \\ X_\gamma \\ F_2 \end{bmatrix} S_c = \begin{bmatrix} A(I_n - \bar M) & 0 & \alpha B \\ \alpha C^* C(I_n - \bar M) & -E^* & 0 \\ 0 & -B^* & I_m \end{bmatrix} \begin{bmatrix} I_n - \bar M \\ X_\gamma \\ F_2 \end{bmatrix}, \tag{4.83}$$

where, additionaly, we have used the following relations obtained by straightforward but lengthy computations

$$\gamma^2 (1 - \gamma^2)^{-1} + 1 = (1 - \gamma^2)^{-1},$$

$$(1 - \gamma^2)^{-1}(-\gamma^2(E - \bar{\alpha}Y_sC^*C) - Y_sA^*X_\gamma - \gamma^2\bar{\alpha}BB^*X_\gamma)+$$
$$\bar{\alpha}Y_sC^*C(I_n - \bar{M}) = E(I_n - \bar{M}) + \bar{\alpha}BB^*X_\gamma,$$
$$(1 - \gamma^2)^{-1}(-\gamma^2(A - \alpha Y_sC^*C) - Y_sE^*X_\gamma - \gamma^2\alpha BB^*X_\gamma)+$$
$$\alpha Y_sC^*C(I_n - \bar{M}) = A(I_n - \bar{M}) + \alpha BB^*X_\gamma.$$

Invocking now *Theorem* 4 and *Proposition* 21 in [9] we conclude that $I_n - \bar{M}$ is actually invertible, from where we deduce (also looking at (4.83)) that

$$X_s := X_\gamma(I_n - \bar{M})^{-1} \tag{4.84}$$

is the (unique) stabilizing solution of (3.4). Thus, the conclusion follows. □

We are now ready to prove *Theorem* 3.3.

4.5. Proof of Theorem 3.3. We shall assume $D = 0$ since, for $D \neq 0$, we have the following identity

$$K(z) = K_0(z)(I + DK_0(z))^{-1}, \tag{4.85}$$

where K_0 is the optimal controller for $D = 0$ and K is the optimal controller for $D \neq 0$.

Denote

$$\epsilon := \gamma^2 - \gamma_{min}^2 \geq 0 \tag{4.86}$$

For any $\epsilon > 0$ there exists a (suboptimal) robustly stabilizing controller, denoted

$$K_\epsilon(z) = \left[\begin{array}{c|c} A_\epsilon - zE_\epsilon & B_\epsilon \\ \hline C_\epsilon & D_\epsilon \end{array}\right]_{z_0}, \tag{4.87}$$

where

$$A_\epsilon := A + \alpha(K_sC - BB^*X_\gamma(\bar{\alpha}A - \alpha E)),$$
$$E_\epsilon := E + \bar{\alpha}(K_sC - BB^*X_\gamma(\bar{\alpha}A - \alpha E)),$$
$$B_\epsilon := -K_s,$$
$$C_\epsilon := -B^*X_\gamma(\bar{\alpha}A - \alpha E),$$
$$D_\epsilon := 0;$$

the X_γ matrix is given in (4.63).

Compute now

$$T_\epsilon^{cl}(z) := LLFT(T^{cf}(z); K_\epsilon(z)), \tag{4.88}$$

where $LLFT(;)$ denotes the lower linear fractional transformation (for further details see Chap. 2.1.5 in [2]), which gives after an equivalence transformation on state space and removing the uncontrollable part

$$T_\epsilon^{cl} = \left[\begin{array}{c|c} A_{R_\epsilon} - zE_{R_\epsilon} & B_{R_\epsilon} \\ \hline C_{R_\epsilon} & D_{R_\epsilon} \end{array}\right]_{z_0}$$
$$:= \left[\begin{array}{c|c} A - \alpha BB^*X_\gamma(\bar{\alpha}A - \alpha E) - z(E - \bar{\alpha}BB^*X_\gamma(\bar{\alpha}A - \alpha E)) & -K_s \\ \hline \begin{bmatrix} -B^*X_\gamma(\bar{\alpha}A - \alpha E) \\ C \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \end{array}\right]_{z_0} \tag{4.89}$$

It is easy to notice that, since $K_\epsilon(z)$ is a (suboptimal) robustly stabilizing controller, we have $\Lambda(A - zE - BB^*X_\gamma(\alpha - \bar{\alpha})) \in \mathbb{D}$, for any $\epsilon > 0$.

Rearrange now (4.62) to obtain

$$A^*X_\gamma A - E^*X_\gamma E - (\bar{\alpha}A - \alpha E)^*X_\gamma)BB^*X_\gamma(\bar{\alpha}A - \alpha E) + C_\epsilon^*C_\epsilon, \qquad (4.90)$$

where

$$C_\epsilon := \begin{bmatrix} (C - K_s^*X_\gamma(\bar{\alpha}A - \alpha E))(\gamma^2 - 1)^{-\frac{1}{2}} \\ C \end{bmatrix}. \qquad (4.91)$$

or, equivalently

$$A_{R_\epsilon}^*X_\gamma A_{R_\epsilon} - E_{R_\epsilon}^*X_\gamma E_{R_\epsilon} + \hat{C}_\epsilon^*\hat{C}_\epsilon = 0, \qquad (4.92)$$

where A_{R_ϵ} and E_{R_ϵ} were defined in (4.89) and

$$\hat{C}_\epsilon := \begin{bmatrix} B^*X_\gamma(\bar{\alpha}A - \alpha E) \\ C \\ (C - K_s^*X_\gamma(\bar{\alpha}A - \alpha E))(\gamma^2 - 1)^{-\frac{1}{2}} \end{bmatrix}. \qquad (4.93)$$

Getting $\epsilon \to 0$, we get from (4.63) and (4.90) that $X_\gamma \to X_0$, $A_{R_\epsilon} \to A_{R_0}$, $E_{R_\epsilon} \to E_{R_0}$, $\gamma \to \gamma_{min}$, $\hat{C}_\epsilon \to \hat{C}_0$ and

$$A_{R_0}^*X_0 A_{R_0} - E_{R_0}^*X_0 E_{R_0} + \hat{C}_0^*\hat{C}_0 = 0, \qquad (4.94)$$

where

$$\begin{aligned} X_0 &= \gamma_{min}^2 X_s((\gamma_{min}^2 - 1)I + X_s Y_s)^{-1}, \\ A_{R_0} &:= A - \alpha BB^*X_0, \\ E_{R_0} &:= E - \bar{\alpha}BB^*X_0, \\ \hat{C}_0 &:= \begin{bmatrix} B^*(-X_0) \\ C \\ (-CY_s(-X_0) + C)(\gamma^2 - 1)^{-\frac{1}{2}} \end{bmatrix}, \end{aligned} \qquad (4.95)$$

and $X_0 = X_0^* \geq 0$ is a solution to (4.94).

It is easy to notice that, by continuity, $det(E_{R_0}) \neq 0$ and the stability of $A_{R_0} - zE_{R_0}$ follows from *Theorem* 1.5.5 in [2] provided we show that $(C; A - zE)$ is detectable. But, this follows straight from

$$\begin{aligned} n = rank \begin{bmatrix} A_{R_0} - zE_{R_0} \\ \hat{C}_0 \end{bmatrix} &= rank \begin{bmatrix} A - zE + BB^*(-X_0)(\alpha - \bar{\alpha}z) \\ B^*(-X_0) \\ C \\ (-CY_s(-X_0) + C)(\gamma^2 - 1)^{-\frac{1}{2}} \end{bmatrix} \\ &= rank \begin{bmatrix} I & B(\alpha - \bar{\alpha}z) & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} A - zE \\ B^*(-X_0) \\ C \\ (-CY_s(-X_0) + C)(\gamma^2 - 1)^{-\frac{1}{2}} \end{bmatrix} = rank \begin{bmatrix} A - zE \\ B^*(-X_0) \\ C \\ (-CY_s(-X_0) + C)(\gamma^2 - 1)^{-\frac{1}{2}} \end{bmatrix}, \end{aligned} \qquad (4.96)$$

$\forall z \in \mathbb{C}$.

The final step is to show that relations (3.8)-(3.12) are indeed satisfied. To show this we only need to show the structure of the *closed-loop system* (4.89) , as $\epsilon \to 0$. To do this, denote

$$T_0^{cl}(z) := \left[\begin{array}{c|c} A_{R_0} - zE_{R_0} & B_{R_0} \\ \hline C_{R_0} & D_{R_0} \end{array} \right]_{z_0},$$

$$\hat{T}_\epsilon^{cl}(z) := T_\epsilon^{cl}(z) - T_0^{cl}(z)$$

$$= \left[\begin{array}{cc|c} A_{R_\epsilon} - zE_{R_\epsilon} & 0 & B_{R_\epsilon} \\ 0 & A_{R_0} - zE_{R_0} & -B_{R_0} \\ \hline C_{R_\epsilon} & C_{R_0} & D_{R_\epsilon} - D_{R_0} \end{array} \right]_{z_0} =: \left[\begin{array}{c|c} \hat{A}_\epsilon - z\hat{E}_\epsilon & \hat{B}_\epsilon \\ \hline \hat{C}_\epsilon & \hat{D}_\epsilon \end{array} \right]_{z_0}.$$

$$(4.97)$$

We have that $\Lambda(\hat{A}_\epsilon - z\hat{E}_\epsilon) \in \mathbb{D}$, $\forall \epsilon > 0$. Furthermore

$$\lim_{\epsilon \to 0} \hat{A}_\epsilon - z\hat{E}_\epsilon =: \hat{A}_0 - z\hat{E}_0$$

is stable, $\hat{D}_0 := \lim_{\epsilon \to 0} \hat{D}_\epsilon = 0$ and $\hat{T}_0^{cl}(z) := \left[\begin{array}{c|c} \hat{A}_0 - z\hat{E}_0 & \hat{B}_0 \\ \hline \hat{C}_0 & \hat{D}_0 \end{array} \right]_{z_0} \equiv 0$, where we have denoted

$$\hat{B}_0 := \lim_{\epsilon \to 0} \hat{B}_\epsilon,$$

$$\hat{C}_0 := \lim_{\epsilon \to 0} \hat{C}_\epsilon.$$

As we proved that $\Lambda(A_{R_0} - zE_{R0}) \in \mathbb{D}$, we may apply *Theorem* 2.5.1 in [2] to obtain

$$\lim_{\epsilon \to 0} \|\hat{T}_\epsilon^{cl}\|_\infty = \lim_{\epsilon \to 0} \|T_\epsilon^{cl} - T_0^{cl}\|_\infty = 0, \qquad (4.98)$$

from where we conclude (recalling the properties of any norm) that

$$\lim_{\epsilon \to 0} T_\epsilon^{cl} - T_0^{cl} = 0$$

, or equivalently

$$\lim_{\epsilon \to 0} T_\epsilon^{cl} = T_0^{cl}, \qquad (4.99)$$

which concludes the whole proof.

5. Conclusions. We have managed to provide analytical formulas for computing *the maximum stability margin* as well as *the optimal controller*, both of them formulated for a completely general rational matrix function (possibly improper or polynomial). Also, reliable computational formulas are given in therms of the original data, thus regaining the same elegance and simplicity of the standard proper case.

REFERENCES

[1] K. Zhou, J. C. Doyle and K. Glover, *Robust and Optimal Control*, Prentice Hall, 1995.

[2] V. Ionescu, C. Oară and M. Weiss, *Generalized Riccati Theory and Robust Control: A Popov Function Approach*, John Wiley and Sons, 1996.

[3] C. Oară and Şabău, *All Doubly Coprime Factorizations of a General Rational Matrix*, Automatica, 2009, pp. 1960-1964.

[4] C. Oară, A. Varga, *Minimal Degree Coprime Factorization of Rational Matrices*, Siam J. Matrix Anal. Appl., 1999.

[5] C. Dinicu and C. Oară, *The Descriptor Continuous-Time Algebraic Riccati Equation: Numerical Solution and Application*, ECC 2015, 2015.

[6] I. Gohberg, M. A. Kaashock and A. C. M. Ram *Factorization of and Extensions to J-unitary Rational Matrix Functions on the Unit Circle*.

[7] B. A. Francis, *A Course in H_∞ Control Theory*, Springer-Verlag, Berlin, 1987.

[8] C. Oară and R. Andrei, *Numerical Solution to a Descriptor Discrete-Time Algebraic Riccati Equation*, Systems and Control Letters, 2013, pp. 201-208.

[9] S. F. Tudor and C. Oară, *H_∞ Control Problem for Discrete-Time Algebraic Dynamical Systems*, Siam J. Matrix Anal. Appl., 2014.

[10] F. S. Tudor and C. Oară, *H_2 Optimal Control for Generalized Discrete-Time Systems*, Automatica, 2014.