The Descriptor Continuous-Time Algebraic Riccati Equation. Numerical Solutions and Some Direct Applications

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***Abstract*— We investigate here the numerical solution of a special type of descriptor continuous-time Riccati equation which is involved in solving several key problems in robust control, formulated under very general hypotheses. We also give necessary and sufficient existence conditions together with computable formulas for both stabilizing and antistabilizing solutions in terms of the associated matrix pencils. In the end, analytic formulas for computing normalized coprime factorizations of an arbitrary rational matrix function are presented as a direct consequence.**

I. INTRODUCTION

The algebraic Riccati equation has been involved in solving various kinds of problems, not only in robust control theory, but in other fields starting with filtering problems, systems identification, dynamic games, to applied mathematics or economic science (see [13], [14], or [15]). Various forms of algebraic Riccati equation have been given (continuous- or discrete-time, symmetric or non-symmetric, generalized, rectangular, nonstandard, constrained, descriptor, reverse-time, etc.), each of them having its well defined scope in the above mentioned domains. Among all the methods proposed so far in the literature, the method of invariant or deflating subspaces, associated with Hamiltonian matrices or matrix pencils, respectively, has played a central role, leading to the most general numerical algorithms. To this end, in this paper, we reconsider the method of deflating subspaces associated with a regular matrix pencil in [1], and show how this theory may be applied to give elegant computable methods for computing the stabilizing and antistabilizing solutions of a general descriptor continuous-time Riccati equation. Incidentally, we show that once this problem is solved, then a normalized coprime factorization approach, in terms of state-space description, may be given, revealing at the same time the same simplicity as in the standard proper case. The paper is organised as follows. In *Section 2* we give some preliminary notions and then state, in *Section 3*, the main results. *Sectiuon 4* presents the normalized coprime factorizations, and in *Section 5* we draw some conclusions.

Key words. Riccati equations, centered realizations, descriptor systems, coprime factorizations.

II. PRELIMINARIES

We start by introducing some notations and concepts that will be intensively used in the sequel.

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By $\mathbb{C}_-$, $\mathbb{C}_+$, $j\mathbb{R}$ we denote the open left-half plane, the open right-half plane and the imaginary axis, respectively. $j\bar{\mathbb{R}}$ stands for the closure of the *imaginary* axis. By $\Lambda(A - sE)$ we denote the spectrum of the matrix pencil $A - sE$. For any *rational matrix function* (*rmf*) $G(s)$, we denote by $G^*(s)$ the adjoint of $G(s)$, i.e. $G^*(s) := \bar{G}^T(-s)$, where the bar takes the complex conjugate of the coefficients of s^k, for any $k \in \mathbb{N}$. For any square matrix $A \in \mathbb{C}^{n\times n}$, we denote by A^* its transpose-conjugate, i.e. $A^* = \bar{A}^T$.
Further, we recall the notion of centered realization of a generalized system that have been introduced to solve several control problems for this class of singular systems [9]. For more details about centered realizations see for example [10] or [5].

Let $G(s)$ be an arbitrary $p \times m$ *rmf* (possibly improper or polynomial) and let $s_0 := j\omega$ be any point in $j\mathbb{R}$, different from its poles. Then, $G(s)$ may be represented as

$$G(s) = C(sE-A)^{-1}B(s_0-s)+D =: \left[\begin{array}{c|c} A-sE & B \\ \hline C & D \end{array}\right]_{s_0}, \tag{1}$$

which is called a *centered realization* (at s_0). Here A, E, B, C, D are matrices of dimension $n\times n$, $n\times n$, $n\times m$, $p\times n$ and $p\times m$, respectively, $A-zE$ is a regular pencil, i.e., $\det(A-\lambda E) \not\equiv 0$, and n is called the order (or dimension) of the realization. The realization is called *minimal* if its order is as small as possible.

In particular, descriptor realizations are nothing but a particular instance of centered realizations, with $s_0 = \infty$. Obviously, we can center the above realization wherever in the closed complex-plane but, by arguments of symmetry, we will choose to center it on the imaginary axis.

Centered realizations (with s_0 outside the set of poles and zeroes of the *rmf*) have some nice features that make them suitable for the problems at hand: Any *minimal* realization of a *rmf* has the order equal to the total number of its poles, multiplicities counted; if $G(s)$ has a centered realization (1), where s_0 is supposed not to be a pole of $G(s)$, then $G(s_0) = D$. Finally, any realization may be normalized, i.e. by a mere equivalence transformation on state-space we can suppose that $s_0E - A = I_n$, which may be, sometimes, insightful in simplifying the computations. To this end, a natural choice of s_0 is on $j\mathbb{R}$, outside the set of poles of the *rmf*, and this assumption will be in force for the rest of the paper.

Centered realizations may be obtained from descriptor realizations (centered at infinity) and, conversely, any centered realization may be converted into a descriptor realization (for more details see [11] or [12]).

A. The descriptor continuous-time algebraic Riccati equation

We investigate below the descriptor continuous-time algebraic Riccati equation (*DCTARE*)

$$0 = Q + A^*XE + E^*XA - ((A - s_0E)^*XB + L)R^{-1}((A - s_0E)^*XB + L)^* \quad (2)$$

where $E \in \mathbb{C}^{n\times n}$, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $Q = Q^* \in \mathbb{C}^{n\times n}$, $L \in \mathbb{C}^{n\times m}$, $R = R^* \in \mathbb{C}^{m\times m}$ and R is nonsingular. The *DCTARE* has, in general, many solutions. In particular, we are interested in two types of solutions.

Definition 1: The matrix $X = X^*$ is called a *stabilizing* (*antistabilizing*) solution for the *DCTARE* above and $s_0 \in j\mathbb{R}$, if it satisfies (2), and moreover $\Lambda(A - sE + BF_s(s_0 - s)) \subset \mathbb{C}_-$ ($\Lambda(A - sE + BF_s(s_0 - s)) \subset \mathbb{C}_+$), where

$$F_s := -R^{-1}((A - s_0E)^*XB + L)^*, \quad (3)$$

is called the *stabilizing* (*antistabilizing*) feedback.

The following theorem ensures us about the uniqueness of the *stabilizing* (*antistabilizing*) solution of (2), provided one such solution exists.

Theorem 1: If (2) has a *stabilizing* (*antistabilizing*) solution, then it is unique.

Proof: We shall prove the uniqueness of the stabilizing solution only, since the other one follows in a very similar way. Let $X_s = X_s^*$ and $\tilde{X}_s = \tilde{X}_s^*$ be two stabilizing solutions of (2), and denote by F_s and $\tilde{F}_s$ the corresponding stabilizing feedbacks. Thus

$$0 = A^*X_sE + E^*X_sA - ((A - s_0E)^*X_sB + L)R^{-1}(B^*X_s(A - s_0E) + L^*) + Q, \quad (4)$$

$$0 = A^*\tilde{X}_sE + E^*\tilde{X}_sA - ((A - s_0E)^*\tilde{X}_sB + L)R^{-1}(B^*\tilde{X}_s(A - s_0E) + L^*) + Q. \quad (5)$$

Replacing the expresion of F_s in (4) and rearranging (3) we get

$$0 = A^*X_s(E+BF_s)+E^*X_s(A+s_0BF_s)+Q+LF_s, \quad (6)$$

$$0 = B^*X_s(A+s_0BF_s)+s_0^*B^*X_s(E+BF_s)+L^*+RF_s = 0, \quad (7)$$

Premultiplying (7) by $\tilde{F}_s^*$ and adding it to (6) gives

$$0 = Q + LF_s + (A + s_0B\tilde{F}_s)^*X_s(E + BF_s) + (E + B\tilde{F}_s)^*X_s(A + s_0BF_s) + \tilde{F}_s^*L^* + \tilde{F}_s^*RF_s. \quad (8)$$

Substituting now $\tilde{F}_s$ in (5) and rewriting again (3), we obtain

$$0 = A^*\tilde{X}_s(E+B\tilde{F}_s)+E^*\tilde{X}_s(A+s_0B\tilde{F}_s)+Q+L\tilde{F}_s, \quad (9)$$

$$0 = B^*\tilde{X}_s(A+s_0B\tilde{F}_s)+s_0^*B^*\tilde{X}_s(E+B\tilde{F}_s)+L^*+R\tilde{F}_s = 0. \quad (10)$$

Premultiplying (10) by F_s^*, adding it to (9) and transpose-conjugating the result leads to

$$0 = Q + LF_s + (A + s_0B\tilde{F}_s)^*\tilde{X}_s(E + BF_s) + (E + B\tilde{F}_s)^*\tilde{X}_s(A + s_0BF_s) + \tilde{F}_s^*L^* + \tilde{F}_s^*RF_s. \quad (11)$$

Finally we substract (11) from (8) to obtain

$$0 = (A + s_0B\tilde{F}_s)^*X_\Delta(E + BF_s) + (E + B\tilde{F}_s)^*X_\Delta(A + s_0BF_s), \quad (12)$$

where $X_\Delta := X_s - \tilde{X}_s$. Finally we notice that, since $\Lambda(A - sE + BF_s(s_0 - s)) \subset \mathbb{C}_-$ and $\Lambda(A - sE + B\tilde{F}_s(s_0 - s)) \subset \mathbb{C}_-$, $A + s_0BF_s$ and $A + s_0B\tilde{F}_s$ are invertible and thus $S := (E + BF_s)(A + s_0BF_s)^{-1}$ and $\tilde{S} := (A + s_0B\tilde{F}_s)^{-*}(E + B\tilde{F}_s)^*$ are stable. Hence, (12) may be written in an equivalent form as

$$X_\Delta S + \tilde{S}X_\Delta = 0, \quad (13)$$

which is a standard Sylvester equation, with the (unique) solution $X_\Delta = 0$, from where the conclusion follows. ■

B. Deflating subspaces and descriptor symplectic pencils

We start by recalling the definition of a deflating subspace (see [1]).

Definition 2: Given an arbitrary, regular matrix-pencil $sM - N$, with M and N belonging to $\mathbb{C}^{n\times n}$, and a basis-matrix $V \in \mathbb{C}^{n\times l}$, we say that the space $\gamma := Span(V)$ is a deflating subspace for $sM - N$, provided

$$dim(M\gamma + N\gamma) = dim(\gamma). \quad (14)$$

Moreover, we denote by $(sM - N)|_\gamma$ and by $\Lambda(sM - N)|_\gamma$ the map and the spectrum of the respective matrix-pencil, restricted to γ.

Invoking *Proposition 5* in [3], we say that the deflating subspace γ is stable (antistable) if $\Lambda(sM - N)|_\gamma \subset \mathbb{C}_-$ ($\Lambda(sM - N)|_\gamma \subset \mathbb{C}_+$), where

$$MVS = NVT \quad (15)$$

holds for a basis matrix V of γ.

Denote by $sM - N$ the following matrix-pencil

$$sM - N := s\begin{bmatrix} E & 0 & B \\ Q & -E^* & L \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 & s_0B \\ s_0Q & A^* & s_0L \\ L^* & -B^* & R \end{bmatrix}, \quad (16)$$

associated with the *DCTARE* (2), called the *Descriptor Symplectic Pencil* (*DSP*).

Also, denote

$$sM_R - N_R := s\begin{bmatrix} E - BR^{-1}L^* & BR^{-1}B^* \\ Q - LR^{-1}L^* & -E^* + LR^{-1}B^* \end{bmatrix} - \begin{bmatrix} A - s_0BR^{-1}L & s_0BR^{-1}B^* \\ s_0Q - s_0LR^{-1}L^* & A^* + s_0LR^{-1}B^* \end{bmatrix}. \quad (17)$$

Remark 1: It can be easily noticed that $M \in \mathbb{C}^{(2n+m)\times(2n+m)}$, $N \in \mathbb{C}^{(2n+m)\times(2n+m)}$, $M_R \in \mathbb{C}^{2n\times 2n}$ and $N_R \in \mathbb{C}^{2n\times 2n}$.

Denote by n^-, n^+ and n^0 the number of eigenvalues of any matrix-pencil, having $Re(\cdot) < 0$, $Re(\cdot) > 0$ and $Re(\cdot) = 0$ respectively, multiplicities counted.

We have the following result

Lemma 1: Let $(s_0E - A)$ and R be invertible matrices. The following statements hold

1) $GCTHP$ and $RCTHP$ are regular pencils.
2) $n^- = n^+ \leq n$, for both DSP and $sM_R - N_R$, with equality iff $n^0 = 0$.

Proof: Define

$$W := \begin{bmatrix} I & 0 & -s_0 BR^{-1} \\ 0 & I & -s_0 LR^{-1} \\ 0 & 0 & I \end{bmatrix},$$

$$Z := \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -R^{-1}L^* & R^{-1}B^* & I \end{bmatrix}.$$

It could be easily noticed that we have the following relations among $sM - N$ and $sM_R - N_R$

$$W(sM-N)Z = s\begin{bmatrix} E - BR^{-1}L^* & BR^{-1}B^* & B \\ Q - LR^{-1}L^* & -E^* + LR^{-1}B^* & L \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A - s_0 BR^{-1}L^* & BR^{-1}B^* & 0 \\ s_0 Q - s_0 LR^{-1}L^* & A^* + s_0 LR^{-1}B^* & 0 \\ 0 & 0 & R \end{bmatrix}, \quad (18)$$

and, since R was supposed to be *invertible*, we may conclude that $sM - N$ is regular iff $sM_R - N_R$ is, the *finite* generalized eigenvalues of $sM - N$ and $sM_R - N_R$ coincide and, moreover, their *infinite* generalized eigenvalues are related through

$$n_\infty = n_{R_\infty} + m, \quad (19)$$

where n_∞ and n_{R_∞} represent the number of infinite generalized eigenvalues of $sM - N$ and $sM_R - N_R$ respectively, multiplicities counted. Thus, we only need to prove the statements for $sM_R - N_R$.

The first statement is obvious if we evaluate $s_0 M_R - N_R = \begin{bmatrix} s_0 E - A & 0 \\ 0 & (s_0 E - A)^* \end{bmatrix}$ and take into account the invertibility of $s_0 E - A$. To this end, let $V := \begin{bmatrix} V_{R_1} \\ V_{R_2} \end{bmatrix}$ be a basis-matrix for a (maximal) stable deflating subspace of $sM_R - N_R$. It follows that (see [3]) there exists a regular matrix-pencil $sT_R - S_R$, with $\Lambda(sT_R - S_R) \subset \mathbb{C}_-$, such that

$$((E - BR^{-1}L^*)V_{R_1} + BR^{-1}B^*V_{R_2})S_R = ((A - s_0 BR^{-1}L^*)V_{R_1} + s_0 BR^{-1}B^*V_{R_2})T_R, \quad (20)$$

$$((Q - LR^{-1}L^*)V_{R_1} - (E^* - LR^{-1}B^*)V_{R_2})S_R = (s_0(Q - LR^{-1}L^*)V_{R_1} + (A^* + s_0 LR^{-1}B^*)V_{R_2})T_R. \quad (21)$$

Straightforward computations show that we have

$$(-N_R^*)\begin{bmatrix} -V_{R_2} \\ V_{R_1} \end{bmatrix} T_R = M_R^* \begin{bmatrix} -V_{R_2} \\ V_{R_1} \end{bmatrix} S_R, \quad (22)$$

which is equivalent to

$$N_R^* W_R(-T_R) = M_R^* W_R S_R, \quad (23)$$

where $W_R := \begin{bmatrix} -V_{R_2} \\ V_{R_1} \end{bmatrix}$. So, it follows that $(sM_R^* - N_R^*)$ has a deflating subspace υ whose basis-matrix is W_R and $\Lambda(sM_R^* - N_R^*)|_\upsilon = \Lambda((-s)T_R - S_R)$, which is, obviously, antistable. Invoking *Proposition 5* in [3], we conclude that if $sM_R - N_R$ has exactly n^- stable eigenvalues, then it must have at least n^+ antistable eigenvalues. Hence $n^- \leq n^+$. Analogously, we have that $n^+ \leq n^-$, from where we conclude that $n^- = n^+$. Notice now that $2n = n^- + n^+ + n^0$, which means that $2n = 2n^- + n^0 = 2n^+ + n^0$, with equality iff $n^0 = 0$. ∎

The following *Lemma* will be crucial in proving the conjugate-symmetry of the stabilizing (antistabilizing) solution of (2).

Lemma 2: Suppose DSP has an *n-dimensional* stable (antistable) deflating subspace, having a basis-matrix V partitioned in accordance to (16)

$$V =: \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \quad (24)$$

Then we have

$$V_2^*(s_0 E - A)V_1 = V_1^*(s_0 E - A)^* V_2. \quad (25)$$

Proof: We prove the statement only for the *stable* case, since the *antistable* one could be proved in the same way.

Let V be a basis-matrix for an *n-dimensional* stable deflating subspace of the *DSP*. Then, there exists a regular-stable matrix-pencil $(sT - S)$, such that

$$MVS = NVT. \quad (26)$$

Writing (26) component wise, we get

$$\begin{aligned} (EV_1 + BV_3)S &= (AV_1 + s_0 BV_3)T, \quad (27) \\ (QV_1 - E^*V_2 + LV_3)S &= (s_0 QV_1 + A^*V_2 + s_0 LV_3)T, \\ (L^*V_1 - B^*V_2 + RV_3)T &= 0, \end{aligned}$$

or, equivalently

$$EV_1 S_T = AV_1 + BV_3(s_0 I - S_T), \quad (28)$$

$$E^*V_2 S_T = -QV_1(s_0 I - S_T) - A^*V_2 - LV_3(s_0 I - S_T), \quad (29)$$

$$L^*V_1 - B^*V_2 + RV_3 = 0. \quad (30)$$

Multiplying (28) with V_2^* to the left we obtain

$$V_2^* EV_1 S_T = V_2^* AV_1 + V_2^* BV_3(s_0 I - S_T). \quad (31)$$

Making the transpose-conjugate of (29) and then multiplying it, to the right, with V_1, gives

$$S_T^* V_2^* EV_1 = -(s_0 I - S_T)^* V_1^* QV_1 - V_2^* AV_1 - (s_0 I - S_T)V_3^* L^* V_1. \quad (32)$$

Adding (31) and (32), and then multiplying the obtained relation with $s_0 \neq 0$ we get

$$\begin{aligned} &S_T^*(V_2^*(s_0 E)V_1) + (V_2^*(s_0 E)V_1)S_T \\ &= -(s_0 I - S_T)^* V_1 QV_1 s_0 - s_0^* V_2^* BV_3(s_0 I - S_T) \\ &- (s_0 I - S_T)^* V_3^* L^* V_1 s_0. \quad (33) \end{aligned}$$

Now, add (28) pre-multiplied by $S_T^* V_2$ to the conjugate-transposed of (29) post-multiplied by $V_1 S_T$, to arrive at

$$(S_T^*(V_2^* A V_1) + (V_2^* A V_1) S_T) + S_T^* V_2^* B V_3 (s_0 I - S_T) + (s_0 I - S_T)^* V_1^* Q V_1 S_T + (s_0 I - S_T)^* V_3^* L^* V_1 S_T. \tag{34}$$

Add (33) to (34), to finally obtain

$$\begin{aligned} 0 &= S_T^*(V_2^*(s_0 E - A) V_1) + (V_2^*(s_0 E - A) V_1) S_T \\ &+ (s_0 I - S_T)^* V_1^* Q V_1 (s_0 I - S_T) \\ &+ (s_0 I - S_T)^* V_2^* B V_3 (s_0 I - S_T) \\ &+ (s_0 I - S_T)^* V_3^* L^* V_1 (s_0 I - S_T). \end{aligned} \tag{35}$$

Take into account (30) to obtain a hermitical continuous-time Lyapunov equation

$$\begin{aligned} 0 &= S_T^*(V_2^*(s_0 E - A) V_1) + (V_2^*(s_0 E - A) V_1) S_T \\ &+ (s_0 I - S_T)^* V_1^* Q V_1 (s_0 I - S_T) \\ &+ (s_0 I - S_T)^* (V_2^* B V_3 + V_3^* B^* V_2)(s_0 I - S_T) \\ &- (s_0 I - S_T)^* V_3^* R V_3 (s_0 I - S_T). \end{aligned} \tag{36}$$

which has a (unique) hermitical solution since S_T is stable and its free term, namely $(s_0 I - S_T)^* V_1^* Q V_1 (s_0 I - S_T) + (s_0 I - S_T)^* (V_2^* B V_3 + V_3^* B^* V_2)(s_0 I - S_T) - (s_0 I - S_T)^* V_3^* R V_3 (s_0 I - S_T)$ is a hermitical matrix (for further details, see [1], *Theorem 1.5.2.*), which concludes our proof. ■

C. Normalized coprime factorizations

We recall below the notion of (stable) coprime factorization of a general *rmf* (for more details see [4], or [12]).

Let $G(s)$ be a general (possibly improper or polynomial) *rmf*, having a realization

$$G(s) =: \left[\begin{array}{c|c} A - sE & B \\ \hline C & D \end{array}\right]_{s_0}, \tag{37}$$

where $s_0 =: j\omega$ is not a pole of G.

Definition 3: We say that the pair $(N(s); M(s)) \in \mathbb{RH}_\infty \times \mathbb{RH}_\infty$ is a *right* coprime factorization (*rcf*) of $G(s)$ given in (37), over $\mathbb{RH}_\infty$, if there is a pair of matrices $(X(s); Y(s)) \in \mathbb{RH}_\infty \times \mathbb{RH}_\infty$ such that $X(s)N(s) + Y(s)M(s) \equiv I$ and, moreover, $G(s) = N(s)M^{-1}(s)$ holds. In a very similar fashion, the pair $(\tilde{N}(s); \tilde{M}(s)) \in \mathbb{RH}_\infty \times \mathbb{RH}_\infty$ is said to be a *left* coprime factorization (*lcf*) of $G(s)$, over $\mathbb{RH}_\infty$, if there is a pair of matrices $(\tilde{X}(s); \tilde{Y}(s)) \in \mathbb{RH}_\infty \times \mathbb{RH}_\infty$ such that $\tilde{N}(s)\tilde{X}(s) + \tilde{M}(s)\tilde{Y}(s) \equiv I$ and, moreover, $G(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$ holds.

Definition 4: Let $(N(s); M(s))$ be a *rcf* of $G(s)$ over $\mathbb{RH}_\infty$. We say that $(N(s); M(s))$ is also *normalized*, and we call it a *normalized right coprime factorization*, provided

$$M^*(s)M(s) + N^*(s)N(s) \equiv I. \tag{38}$$

Similarly, let $(\tilde{N}(s); \tilde{M}(s))$ be a *lcf* of $G(s)$ over $\mathbb{RH}_\infty$. We say that $(\tilde{N}(s); \tilde{M}(s))$ is *normalized*, and we call it a *normalized left coprime factorization*, if

$$\tilde{M}(s)\tilde{M}^*(s) + \tilde{N}(s)\tilde{N}^*(s) \equiv I. \tag{39}$$

III. Main results

In this section we state and prove the main results of this paper.

The first *theorem* gives necessary and sufficient conditions for the existence of a stabilizing (antistabilizing) solution of a *DCTARE*, as defined in (2), together with two computable formulas for both the stabilizing (antistabilizing) solution (provided it exists) and the associated Riccati feedback.

Theorem 2: Suppose $s_0 E - A$ and R are invertible. Then, the following two assertions are equivalent

1) The *DCTARE*

$$\begin{aligned} 0 &= Q + A^* X E + E^* X A \\ &- ((A - s_0 E)^* X B + L) R^{-1} ((A - s_0 E)^* X B + L)^* \end{aligned} \tag{40}$$

has a hermitical stabilizing (antistabilizing) solution $(X_s; F_s)$, with $F_s := -R^{-1}(B^* X_s (A - s_0 E) + L)$.

2) The *DSP* (16) has a (maximal) n-dimensional stable (antistable) deflating subspace, having a basis matrix V, partitioned as in (24), with V_1 invertible.

Moreover, the stabilizing (antistabilizing) solutions may be computed from

$$X_s := -V_2 V_1^{-1}(A - s_0 E), \tag{41}$$

while the associated feedback from

$$F_s := V_3 V_1^{-1}. \tag{42}$$

Proof: Notice that, since $(s_0 E - A)$ and R are invertible matrices, the *DSP* is regular.

We prove first that 1 implies 2. To this end, let X_s be the hermitic stabilizing solution of (40), and let F_s be the asociated feedback. It follows that $(A - sE + BF_s(s_0 - s))$ is a regular pencil, and $\Lambda(A - sE + BF_s(s_0 - s)) \subset \mathbb{C}_-$. It follows now from [3] that there exists a regular matrix-pencil $(sT - S)$, having its spectrum in $\mathbb{C}_-$ such that

$$(E + BF_s)S = (A + s_0 BF_s)T. \tag{43}$$

Moreover, from the expression of the stabilizing feedback F_s, we obtain

$$L^* - B^* X_s (s_0 E - A) + F_s R = 0. \tag{44}$$

Equation (40) could be written now in an equivalent form, as

$$A^* X_s E + E^* X_s A + Q + ((A - s_0 E) X_s B + L) F_s = 0,$$

which, post-multiplied by $(s_0 T - S)$, leads to

$$\begin{aligned} &A^* X_s E (s_0 T - S) + E^* X_s A (s_0 T - S) + Q (s_0 T - S) \\ &+ (A - s_0 E)^* X_s B F_s (s_0 T - S) + L F_s (s_0 T - S) = 0. \end{aligned} \tag{45}$$

Straightforward computations give

$$\begin{aligned} &A^* X_s E (s_0 T - S) + E^* X_s A (s_0 T - S) \\ &+ (A - s_0 E)^* X_s B F_s (s_0 T - S) \\ &= -E^* X_s (A - s_0 E) S - A^* X_s (A - s_0 E) T. \end{aligned} \tag{46}$$

Comparing (45) and (46), we conclude

$$(Q + E^*X_s(A - s_0E) + LF_s)S = (s_0Q - A^*X_s(A - s_0E) + s_0LF_s)T. \quad (47)$$

Taking into account equations (43), (44) and (47), we obtain

$$\begin{bmatrix} E & 0 & B \\ Q & -E^* & L \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ -X_s(A - s_0E) \\ F_s \end{bmatrix} S = \begin{bmatrix} A & 0 & s_0B \\ s_0Q & A^* & s_0L \\ L^* & -B^* & R \end{bmatrix} \begin{bmatrix} I \\ -X_s(A - s_0E) \\ F_s \end{bmatrix} T, \quad (48)$$

which shows that $V := \begin{bmatrix} I \\ -X_s(A - s_0E) \\ F_s \end{bmatrix}$ is a stable deflating subspace of dimension n, with $V_1 = I_n$. But $n^- \leq n$ from where the conclusion follows.

Now we prove the next implication, namely, the existence of an n-dimensional, stable deflating subspace V, with V_1 invertible, for the *DSP* implies the existence of a hermitic stabilizing solution of $DCTARE$, which will conclude the proof of the theorem.

Hence, consider a basis-matrix $V := \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$ for a (maximal) n-dimensional deflating subspace of *DSP*, with V_1 invertible. Then, invoking *Proposition 5* in [3], there is a regular matrix-pencil $(sT - S)$, having its spectrum in the open-left half plane, such that

$$MVS = NVT. \quad (49)$$

Denote

$$s\hat{T} - \hat{S} := V_1(sT - S)V_1^{-1}, \quad X := -V_2V_1^{-1}(A - s_0E)^{-1}, \quad (50)$$

and

$$F := V_3V_1^{-1}. \quad (51)$$

From *Lemma 2*, we conclude that X, given in (50) is a hermitical matrix. Further, we have

$$(EV_1 + BV_3)S = (AV_1 + s_0BV_3)T,$$

or, with the new notations

$$(E + BF)\hat{S} = (A + s_0BF)\hat{T}. \quad (52)$$

From

$$(L^* - B^*V_2 + RV_3)T = 0,$$

we obtain

$$(L^* + B^*X(A - s_0E) + RF)T = 0. \quad (53)$$

Also, straightforward computations show that, starting from the relation

$$(QV_1 - E^*V_2 + LV_3)S = (s_0QV_1 + A^*V_2 + s_0LV_3)T,$$

we get

$$A^*XE(s_0\hat{T} - \hat{S}) + E^*XA(s_0\hat{T} - \hat{S}) + Q(s_0\hat{T} - \hat{S}) + ((A - s_0E)^*XB + L)F(s_0\hat{T} - \hat{S}) = 0,$$

which is exactly *DCTARE* (40), if we notice that, as $sT - S$ is stable and $s_0 \notin \mathbb{C}_-$, then $s_0\hat{T} - \hat{S} := V_1(s_0T - S)V_1^{-1}$ is invertible. But, relation (52) shows that the solution of the above Riccati equation is, indeed, stabilizing.

Finally the relations for the stabilizing solution and the associated feedback follow from equations (50) and (51), respectively. ∎

Suppose, further $Q := C^*C$, $L := 0$ and $R := I_m$. Then, the corresponding Riccati equation

$$0 = A^*XE + E^*XA - (A - s_0E)^*XBB^*X(A - s_0E) + C^*C \quad (54)$$

will be called the Riccati equation for *control*.

$(C; A - sE)$ will be called $j\bar{\mathbb{R}}$-controllable provided $rank \begin{bmatrix} A - j\omega E \\ C \end{bmatrix} = n$, for all $j\omega \in j\mathbb{R}$ and $rank \begin{bmatrix} E \\ C \end{bmatrix} = n$.

We state now the second assertion of this section, which is related to the above introduced Riccati equation.

Corollary 1: Let $(A - zE; B)$ be stabilizable and $(C; A - sE)$ $j\bar{\mathbb{R}}$-controllable. Then, the Riccati equation for *control* (54) has a hermitic stabilizing solution.

Proof: Let us denote $\hat{C} := \begin{bmatrix} C \\ 0 \end{bmatrix}$ and $\hat{D} := \begin{bmatrix} 0 \\ I \end{bmatrix}$. Also let $S_H(s) := \begin{bmatrix} A - sE & B \\ \hat{C} & \hat{D} \end{bmatrix}$ and $H(s) := \left[\begin{array}{c|c} A - sE & B \\ \hline \hat{C} & \hat{D} \end{array}\right]_{s_0}$, where $s_0 := j\omega \notin \Lambda(A - zE)$.

As we supposed $(C; A - sE)$ is $j\bar{\mathbb{R}}$-controllable, S_H does not have zeros on the extended imaginary axis, from where we conclude that H (whose system pencil is S_H) does not have zeros on the extended imaginary axis. It is not difficult to see that $H^*(s)H(s) = \left[\begin{array}{cc|c} A - sE & 0 & B \\ C^*C(s_0 - s) & A^* - sE^* & 0 \\ \hline 0 & -B^* & I \end{array}\right]_{s_0} =: \Pi_{s_0}(s)$. Thus, Π_{s_0} does not have zeros on the imaginary axis and, as its realization is stabilizable and $j\bar{\mathbb{R}}$-controllable, it follows that its system pencil, which is in fact the *DSP* associated with our Riccati equation for *control*, does not have eigenvalues on the extended imaginary axis. This shows that, for our *DSP*, we have $n^0 = 0$, from where we conclude that the *DSP* has an n-dimensional (maximal) stable deflating subspace. Using *Theorem 2*, we only need to show that the n-dimensional stable deflating subspace above has a basis matrix $V := \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$ with V_1 invertible. In order to simplify our computations, we shall assume $s_0E - A = I$.

To this end, suppose the contrary, i.e. V_1 is singular. This implies that $Ker(V_1) \neq 0$. Now, let K be any basis matrix for $Ker(V_1)$ and choose any $x \in \mathbb{C}^n$ such that $0 \neq x \in Span(K)$. Note that we can always choose such an x, since we have assumed that the V_1 matrix is singular. From *Lemma 2*, we have that $V_2^*V_1 = V_1^*V_2$, from where we obtain

$$0 = S_T^*(V_2^*V_1) + (V_2^*V_1)S_T + (s_0I - S_T)^*[V_1^*QV_1 + V_2^*BV_3 + V_3^*B^*V_2 - V_3^*RV_3](s_0I - S_T), \tag{55}$$

where $\Lambda(S_T) \subset \mathbb{C}_-$. But, $\Lambda(S_T) \subset \mathbb{C}_-$ implies that $s_0I - S_T$ is nonsingular. $V =: \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$ being a (maximal) stable deflating subspace of *DSP*, we have $B^*V_2 = V_3$. So, V becomes

$$V =: \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ B^*V_2 \end{bmatrix}. \tag{56}$$

Equation (55) premultiplied by x^* and postmultiplied by x gives

$$0 = x^*(s_0I - S_T)^*[V_1^*C^*CV_1 + V_2^*BB^*V_2 + V_2^*BB^*V_2 - V_2^*BB^*V_2](s_0I - S_T)x, \tag{57}$$

or

$$0 = x^*(s_0I - S_T)^*[V_1^*C^*CV_1 + V_2^*BB^*V_2](s_0I - S_T)x, \tag{58}$$

where we took into account $V_1x = 0$, which, finally, gives:

$$\begin{bmatrix} CV_1 \\ B^*V_2 \end{bmatrix} (s_0I - S_T)x = 0. \tag{59}$$

Using the definition of the K-matrix we have

$$\begin{bmatrix} CV_1 \\ B^*V_2 \end{bmatrix} (s_0I - S_T)K = 0. \tag{60}$$

Further, lengthy but straightforward computations show that we have the following relations

$$V_2S_TK = 0, \tag{61}$$

$$A^*V_2(s_0I - S_T)K = 0. \tag{62}$$

Finally, we obtain

$$A^*V_2(s_0I - S_T)K = 0, \tag{63}$$

$$B^*V_2(s_0I - S_T)K = 0, \tag{64}$$

or equivalently

$$\begin{bmatrix} A^* \\ B^* \end{bmatrix} V_2(s_0I - S_T)K = 0. \tag{65}$$

Thus, taking into account that $\begin{bmatrix} A^* \\ B^* \end{bmatrix}$ is a full column rank matrix, which is a mere consequence of the fact that $s = 0$ is a controllable mode of $(A - sE; B)$ (for further details see also [4]), and equation (61), we arrive at

$$V_2K = 0. \tag{66}$$

In the end, we notice that we have obtained

$$VK = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} K = \begin{bmatrix} V_1K \\ V_2K \\ B^*V_2K \end{bmatrix} = 0, \tag{67}$$

which is, obviously, a contradiction since both V and K were chosen to have full column rank. This ends the whole proof. ■

IV. Applications to robust control problems

In what follows, we present a solution to the normalized coprime factorization problem, which plays a key role in solving various kinds of problems in robust control theory.

To start with, consider the Riccati equation for *control* as defined in (54)

$$0 = C^*C + A^*XE + E^*XA - (A - s_0E)^*XBB^*X(A - s_0E). \tag{68}$$

Further, consider the following *rmf*

$$G(s) := \left[\begin{array}{c|c} A - sE & B \\ \hline C & D \end{array}\right]_{s_0}, \tag{69}$$

where $s_0 \in (j\bar{\mathbb{R}} - \Lambda(A - sE))$, whose realization is supposed to be *stabilizable* and $j\bar{\mathbb{R}}$-controllable.

We have the following result

Theorem 3: A normalized right coprime factorization (*nrcf*) of $G(s)$ over $\mathbb{RH}_\infty$ is given by

$$\begin{bmatrix} N(s) \\ M(s) \end{bmatrix} = \left[\begin{array}{c|c} A_F - sE_F & B \\ \hline C + DF_s & D \\ F_s & I \end{array}\right]_{s_0}, \tag{70}$$

where $A_F - sE_F := A - sE + BF_s(s_0 - s)$, $F_s := -B^*X_s(A - s_0E)$ and $X_s = X_s^*$ is the (unique) stabilizing solution of (54).

Proof: In order to simplify our computations, we suppose that $s_0E - A = I$.

The fact that $\Lambda(A_F - sE_F) \in \mathbb{C}_-$ results by a simple observation on the hypothesis made for the realization (69) of G and invoking *Corollary 1*. Also, from *Theorem 3* in [12] we deduce taht $(N(s); M(s))$ has exactly the form indicated in (70). Thus, it only remains to prove that (70) is indeed *normalized*, i.e.

$$M^*(s)M(s) + N^*(s)N(s) \equiv I. \tag{71}$$

Let

$$\begin{bmatrix} M(s) \\ N(s) \end{bmatrix} = \left[\begin{array}{c|c} A_F - sE_F & B \\ \hline F_s & I \\ C + DF_s & D \end{array}\right]_{s_0}, \tag{72}$$

where X_s is the (unique) stabilizing solution of (54) and $F_s := -B^*X_s$ is its corresponding feedback.
Now define

$$S(s) := \begin{bmatrix} A_F - sE_F & B(s_0 - s) \\ F_s & I \\ C + DF_s & D \end{bmatrix}. \tag{73}$$

It could be easily noticed that $S(s)$ is the system pencil associated with $\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}$. Denote $V := \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -D & I \end{bmatrix}$, which is, obviously, invertible. We have

$$VS(s) = \begin{bmatrix} A_F - sE_F & B(s_0 - s) \\ F_s & I \\ C & 0 \end{bmatrix}, \tag{74}$$

that is actually a *Rosenbrook* transformation which leaves unchanged the corresponding transfer function (for further details see for example [5]). Hence

$$\begin{bmatrix} M(s) \\ N(s) \end{bmatrix} = \left[\begin{array}{c|c} A_F - sE_F & B \\ \hline F_s & I \\ C + DF_s & D \end{array}\right]_{s_0} = \left[\begin{array}{c|c} A_F - sE_F & B \\ \hline F_s & I \\ C & 0 \end{array}\right]_{s_0} =: \left[\begin{array}{c|c} A_F - sE_F & B \\ \hline C_F & D_F \end{array}\right]_{s_0}. \quad (75)$$

Straightforward computations show that we have

$$A_F^* X_s E_F + E_F^* X_s A_F + C_F^* C_F = A^* X_s E + E^* X_s A - X_s B B^* X_s + C^* C = 0. \quad (76)$$

Now, the result follows by noticing that the *isometry* property is satisfied, provided we have

- $D^* D = I$,
- $C^* D = -(A - s_0 E)^* X_s B$,
- $C^* C + A^* X_s E + E^* X_s A = 0$,

which are simple consequences of the fact that $F_s = -B^* X_s (A - s_0 E)$ is the stabilizing feedback of (54); (69) and (76). ■

Remark 2: We have given above a normalized *right* coprime factorization of $G(s)$ over $\mathbb{RH}_\infty$. One may notice that, starting with $G^T(s)$, computing a normalized right coprime factorizatiom for it (as described in the theorem above), and then transposing back the result, a normalized *left* coprime factorization of $G(s)$ over $\mathbb{RH}_\infty$ is thus obtained.

Remark 3: Also, a normalized *right* (*left*) coprime factorization of $G(s)$ over $\mathbb{RH}_\infty^\perp$ (i.e. over the antistable subspace of $\mathbb{L}_\infty$) may be computed in the very similar way, except that, instead of X_s (the *stabilizing* solution) and F_s (the *stabilizing* feedback), we have X_{as} (the *antistabilizing* solution) and F_{as} (the *antistabilizing* feedback), of (54).

V. Conclusions

We have managed to give analytical formulas, that may be implemented by using numerically sound algorithms, for computing the *stabilizing* (*antistabilizing*) solution of a generalized continuous-time algebraic Riccati equation (in the case in which it *exists*), as well as its associated feedback. Moreover, we proved that (as in the standard proper case) the Riccati equation for *control* has always a *stabilizing* (*antistabilizing*) solution and gave an algorithm for computing a normalized *right* (*left*) coprime factorization of an arbitrary *rmf* over $\mathbb{RH}_\infty$ ($\mathbb{RH}_\infty^\perp$), which is a direct application of the Riccati theory developed in *Section 3*.

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