SUFFICIENT WEIGHTED COMPLEMENTARITY PROBLEMS

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Abstract. This paper presents some fundamental results about sufficient linear weighted complementarity problems. Such a problem depends on a nonnegative weight vector. If the weight vector is zero, the problem reduces to a sufficient linear complementarity problem that has been extensively studied. The introduction of the more general notion of a weighted complementarity problem (wCP) was motivated the fact that wCP can model more general equilibrium problems than the classical complementarity problem (CP). The introduction of a nonzero weight vector makes the theory of wCP more complicated than the theory of CP. The paper gives a characterization of sufficient linear wCP and proposes a corrector–predictor interior-point method for its numerical solution. While the proposed algorithm does not depend on the handicap κ of the problem its computational complexity is proportional with $1+\kappa$. If the weight vector is zero and the starting point is relatively well centered, then the computational complexity of our algorithm is the same as the best known computational complexity for solving sufficient linear CP.

Key words. weighted complementarity, interior-point, path-following, sufficient matrix

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1. Introduction. In a previous paper [14] we introduced the notion of a weighted complementarity problem (wCP) that significantly extends the notion of a complementarity problem (CP). The motivation for introducing this notion lies in the fact that wCP can be used for modeling a larger class of problems from science and engineering than CP. In the same paper we showed that the Fisher market equilibrium problem with linear utilities reduces to a monotone linear wCP. In doing so we used the Eisenberg-Gale formulation of the Fisher problem [7]. Moreover we showed that the Linear Programming and Weighted Centering (LPWC) problem introduced by Anstreicher [3] also reduces to a monotone linear wCP. We note that LPWC generalizes linear programming (LP), the problem of finding the weighted analytic center of a polytope [4, 8] as well as the Eisenberg-Gale formulation of the Fisher problem. Furthermore, we proposed and analyzed two interior-point methods for solving monotone linear wCPs. When applied to the Fisher market equilibrium problem both interior-point methods achieve the best known computational complexity for solving this problem. Similar computational results had been previously obtained by Ye [22] who used a two phase algorithm. The notion of a sufficient linear complementary problem was introduced in 1989 by Cottle et al. [6]. Two years later Kojima et al. [11] introduced the notion of a P_* linear complementarity problem and showed that a P_*-matrix is column sufficient. Subsequently, Guu and Cottle [10] proved that a P_*-matrix is also row sufficient and therefore the class P_* is included in the class of sufficient matrices. Soon after that Väliaho [21] proved the reverse inclusion. Therefore P_* coincides with the class of sufficient matrices. Since $P_* = \cup_{\kappa \geq 0} P_*(\kappa)$, it follows that a matrix is sufficient if and only if it is a $P_*(\kappa)$-matrix for some $\kappa \geq 0$. The smallest κ with this property is called the handicap of the matrix. Kojima et al. introduced the class P_* because many interior-point methods, originally developed for linear programming, can be extended in a natural way to P_* linear complementarity problems. For example they proved that the primal-dual potential reduction method can solve

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a $P_*(\kappa)$ linear complementarity problem in at most $O((1+\kappa)\sqrt{n}L)$ iterations, where n is the dimension of the problem and $L = \log(\varepsilon_0/\varepsilon)$, with ε the required precision (duality gap) and ε_0 the duality gap at the starting point. This is still the best complexity result for solving $P_*(\kappa)$ linear complementarity problems. However, like the vast majority of interior-point methods for solving sufficient linear complementarity problems that appeared subsequently in the literature, the above primal-dual potential reduction method makes explicit use of the parameter κ, so that in order to use it for solving a given sufficient linear complementarity problem one has first to find an upper bound of the handicap of that problem. Or it is well known that such an upper bound is very difficult to estimate. Therefore, it is very important to develop interior-point methods for solving sufficient linear complementarity problems that do not depend on the handicap of the problem (see [16, 12, 9].

The purpose of the present paper is to introduce the notion of a sufficient linear wCP, to study its properties, and to propose an interior-point method for its numerical solution. This notion extends both the notion of a monotone linear wCP and that of a sufficient linear complementarity problem. We associate with each sufficient linear wCP an appropriate optimization problem and we show that a linear wCP is row sufficient if and only if every KKT point of that optimization problem is a solution of the wCP. We prove that every column sufficient linear wCP has a convex (perhaps empty) solution set. If the weight vector is zero then the reverse implication is also true. This generalizes the well known characterization of sufficient linear complementarity problems (see [5], [18]). We also show that if a sufficient linear wCP is strictly feasible then it is solvable (i.e., its solution set is not empty). We then use the notion of the central path of the wCP introduced in [14] to develop a path-following algorithm. Since neither of the two interior-point methods from [14] can be extended to sufficient wCP without explicitly using the parameter κ we use a corrector-predictor approach. This approach was first used in [13] for devising efficient interior-point methods for solving monotone linear complementarity problems in large neighborhoods of the central path, and then in [12, 9] for constructing interior-point methods for sufficient linear complementarity problems that do not depend on the handicap of the problem. We show that the proposed corrector-predictor method is well defined and we find upper bounds on its computational complexity in terms of the handicap of the problem and a measure of the quality of the starting point.

Conventions. We denote by $I\!N$ the set of all nonnegative integers. $I\!R$, $I\!R_+$ and $I\!R_{++}$ denote the set of real, nonnegative real, and positive real numbers respectively. The symbol e represents the vector of all ones, with dimension given by the context. We denote by $\log t$ the natural logarithm of t.

We denote component-wise operations on vectors by the usual notations for real numbers. Thus, given two vectors u, v of the same dimension we denote by uv the vector with components $u_i v_i$. This notation is consistent as long as component-wise operations always have precedence in relation to matrix operations. Note that $Auv = A(uv) \neq (Au)v$. We convene to denote by u/v the vector whose ith component is equal to u_i/v_i if $u_i v_i \neq 0$, and to 0 if $u_i v_i = 0$. Also, if f is a scalar function and v is a vector, then $f(v)$ denotes the vector with components $f(v_i)$. For example, if $v \in I\!R^n_+$, then $\sqrt{v}$ denotes the vector with components $\sqrt{v_i}$. For a vector $v \in I\!R^n$ we denote $\max v = \max\{v_i \ : \ i = 1, \ldots, n\}$ and $\min v = \min\{v_i \ : \ i = 1, \ldots, n\}$. If $\| \, . \, \|$ is a vector norm on $I\!R^n$ and A is a matrix, then the operator norm induced by $\| \, . \, \|$ is defined by $\| A \| = \max\{\| Ax \| \ ; \ \| x \| = 1\}$. As a particular case we note that if U is the diagonal matrix defined by the vector u, then $\| U \|_2 = \| u \|_\infty$.

Throughout this paper we use the following MATLAB-like notation. If A, B, C are matrices that have the same number of rows then $[A\,B\,C]$ or $[A,\, B,\, C]$ denotes the matrix obtained by concatenating their rows. If they have the same number of columns then $[\,A\,;\,B\,;\,C\,]$ denotes the matrix obtained by their column concatenation. In particular if u, v, w are (column) vectors then $[\,u\,;\,v\,]$ and $[\,u\,;\,v\,;\,w\,]$ to denote the column vectors $[u^T v^T]^T$ and $[u^T v^T w^T]^T$ respectively. Given a matrix P, we denote by $\operatorname{Ran} P$ its range (or column space) and by $\operatorname{Ker} P$ its kernel (or null space).

2. Weighted linear complementarity problems. A mixed weighted linear complementarity problem consists in finding vectors $x \in I\!R^n$, $s \in I\!R^n, y \in I\!R^m$ such that

$$\begin{aligned} xs &= w \\ Ax + Bs + Cy &= d \\ x, s &\geq 0\,. \end{aligned} \tag{2.1}$$

Here $A \in I\!R^{(n+m)\times n}$, $B \in I\!R^{(n+m)\times n}$, $C \in I\!R^{(n+m)\times m}$ are given matrices, $d \in I\!R^{n+m}$ is a given vector, and $w \in I\!R^n_+$ is a given weight vector (the data of the problem). The matrix C is assumed to have full column rank. For any matrix K whose columns form a basis of $\operatorname{Ker} C^T$ (2.1) is equivalent to finding x, s such that

$$\begin{aligned} xs &= w \\ Qx + Rs &= a \\ x, s &\geq 0\,, \end{aligned} \tag{2.2}$$

where,

$$Q = K^T A \in I\!R^{n\times n}, \quad R = K^T B \in I\!R^{n\times n}, \quad a = K^T d \in I\!R^n. \tag{2.3}$$

Indeed, if $[\,x\,;\,s\,;\,y\,]$ is a solution of (2.1) then $[\,x\,;\,s\,]$ satisfies (2.2). Conversely, if $[\,x\,;\,s\,]$ is a solution of (2.2), then there is a unique $y \in I\!R^m$ such that $[\,x\,;\,s\,;\,y\,]$ is a solution of (2.1). In the remainder of this paper, problem (2.2) will be called a horizontal weighted complementarity problem.

If the weight vector w is chosen to be the zero vector, then (2.1) reduces to a mixed horizontal linear complementarity problem and (2.2) reduces to a horizontal linear complementarity problem .

2.1. Associated optimization problems. The mixed weighted linear complementarity problem (2.1) is closely related to the following optimization problem

$$\begin{aligned} \underset{x,s,y}{\text{minimize}} \quad & x^T s - \sum_{i=1}^{n} w_i \log x_i s_i \\ \text{subject to} \quad & Ax + Bs + Cy = d \\ & x \geq 0, \quad s \geq 0 \end{aligned} \tag{2.4}$$

Similarly, (2.2) is closely related to

$$\begin{aligned} \underset{x,s}{\text{minimize}} \quad & x^T s - \sum_{i=1}^{n} w_i \log x_i s_i \\ \text{subject to} \quad & Qx + Rs = a \\ & x \geq 0, \quad s \geq 0 \end{aligned} \tag{2.5}$$

The two optimization problems above are equivalent in the sense that if $[\,x\,;\,s\,;\,y\,]$ is an optimal solution of (2.4) then $[\,x\,;\,s\,]$ is an optimal solution of (2.5), and, conversely, if $[\,x\,;\,s\,]$ is an optimal solution of (2.5) then there is a unique y such that $[\,x\,;\,s\,;\,y\,]$ is an optimal solution of (2.4). Let consider the following index sets

$$\overline{\mathcal{W}} = \{i \in \{1,\ldots,n\} \;:\; w_i > 0\}\,, \quad \widehat{\mathcal{W}} = \{i \in \{1,\ldots,n\} \;:\; w_i = 0\}\,. \tag{2.6}$$

For any vector $u \in I\!R^n$ we denote by $u_{\overline{\mathcal{W}}}$ the vector formed by those components of u that correspond to indices in $\overline{\mathcal{W}}$, and by $u_{\widehat{\mathcal{W}}}$ the vector formed by those components of u that correspond to indices in $\widehat{\mathcal{W}}$. We note that if $[\,x\,;\,s\,]$ is in the domain of definition of the objective function in (2.4) then $x_{\overline{\mathcal{W}}}s_{\overline{\mathcal{W}}} > 0$. Using the inequality constraints $x \geq 0\,,\ s \geq 0$ it follows that $x_{\overline{\mathcal{W}}} > 0\,,\ s_{\overline{\mathcal{W}}} > 0$. Using the notation u/v described at the end of the Introduction, the KKT conditions for (2.4) can be written as

$$\begin{aligned} &Ax + Bs + Cy = d\,,\ x_{\overline{\mathcal{W}}} > 0\,,\ s_{\overline{\mathcal{W}}} > 0\,,\\ &[\,s - w/x - p\,;\, x - w/s - q\,;\, 0\,] \in \operatorname{Ran}[A^T; B^T; C^T]\,,\\ &x \geq 0\,,\ s \geq 0\,,\ p \geq 0\,,\ q \geq 0\,,\ p^T x = 0\,,\ q^T s = 0\,. \end{aligned} \tag{2.7}$$

Clearly if $[\,x\,;\,s\,;\,y\,]$ satisfies the above KKT conditions then $[\,x\,;\,s\,]$ satisfies

$$\begin{aligned} &Qx + Rs = a\,,\ x_{\overline{\mathcal{W}}} > 0\,,\ s_{\overline{\mathcal{W}}} > 0\,,\\ &[\,s - w/x - p\,;\, x - w/s - q\,] \in \operatorname{Ran}[Q^T; R^T]\,,\\ &x \geq 0\,,\ s \geq 0\,,\ p \geq 0\,,\ q \geq 0\,,\ p^T x = 0\,,\ q^T s = 0\,, \end{aligned} \tag{2.8}$$

where we have used the notation in (2.3). Let us remark that (2.8) are exactly the KKT conditions for the optimization problem (2.5). Conversely, if $[\,x\,;\,s\,]$ satisfies (2.8) then there exists a unique $y \in I\!R^m$ so that $[\,x\,;\,s\,;\,y\,]$ satisfies (2.7).

2.2. Sufficiency. The triple (A, B, C) is called column sufficient if

$$[\,x\,;\,s\,;\,y\,] \in \operatorname{Ker}[A, B, C] \text{ and } xs \leq 0 \quad \text{imply } xs = 0, \tag{2.9}$$

and it is called row sufficient if

$$[\,x\,;\,s\,;\,0\,] \in \operatorname{Ran}[\,A^T\,;\,B^T\,;\,C^T\,] \text{ and } xs \geq 0 \quad \text{imply } xs = 0\,. \tag{2.10}$$

The triple (A, B, C) is called sufficient if it is both column sufficient and row sufficient.

Similarly, the pair (Q, R) is called column sufficient if

$$[\,x\,;\,s\,] \in \operatorname{Ker}[Q, R] \text{ and } xs \leq 0 \quad \text{imply } xs = 0, \tag{2.11}$$

and it is called row sufficient if

$$[\,x\,;\,s\,] \in \operatorname{Ran}\left[\,Q^T\,;\,R^T\,\right] \text{ and } xs \geq 0 \quad \text{imply } xs = 0\,. \tag{2.12}$$

The pair (Q, R) is called sufficient if it is both column sufficient and row sufficient. If Q and R are given by (2.3), then it can be easily shown that the triplet (A, B, C) is column (row) sufficient if and only if the pair (Q, R) is column (row) sufficient. We note that the pair $(Q, -I)$ is column (row) sufficient if and only if the matrix Q is column (row) sufficient in the sense of Cottle et al. [6] (see also [5]).

Let $\kappa \geq 0$ be a given constant. We say that the triplet (A, B, C) has the $P_*(\kappa)$ property if

$$(1 + 4\kappa) \sum_{i \in \mathcal{I}^+} u_i v_i + \sum_{i \in \mathcal{I}^-} u_i v_i \geq 0, \ \forall [u; v; w] \in \operatorname{Ker}[A, B, C]\,, \tag{2.13}$$

where

$$\mathcal{I}^+ = \{i : u_i v_i > 0\} \text{ and } \mathcal{I}^- = \{i : u_i v_i < 0\}. \tag{2.14}$$

If $\kappa = 0$ then the triplet (A, B, C) is called monotone. We say that the triplet (A, B, C) has the P_* property if it has the $P_*(\kappa)$ property for some $\kappa \geq 0$. Using Väliaho's result [21] it follows that triplet (A, B, C) has the P_* property if and only if it is sufficient. If the triplet (A, B, C) is sufficient then the smallest κ for which (A, B, C) has the $P_*(\kappa)$ property is called the handicap of the problem. The corresponding notions for a pair (Q, R) are defined in a similar manner. In particular we say that (Q, R) has the $P_*(\kappa)$ property if

$$(1 + 4\kappa) \sum_{i \in \mathcal{I}^+} u_i v_i + \sum_{i \in \mathcal{I}^-} u_i v_i \geq 0, \ \forall [u; v] \in \operatorname{Ker}[Q, R]\,, \tag{2.15}$$

where $\mathcal{I}^+$ and $\mathcal{I}^-$ are defined in (2.14). The notions of column and row sufficient matrices are defined in [6]. The $P_*(\kappa)$ property for matrices is defined in the 1991 monograph of Kojima et al. [11]. The appropriate generalization to pairs a was introduced in [19, 20].

We say that the mixed weighted linear complementarity problem (2.1) is column sufficient, row sufficient, sufficient, or $P_*(\kappa)$, if the triplet (A, B, C) has the respective property. The same convention applies to the horizontal mixed weighted linear complementarity problem (2.1). For more results on the equivalence between different formulations see [**?**, 2].

2.3. Properties of the mixed weighted linear complementarity problem. We denote by

$$\mathcal{F} = \left\{ z = [\, x\,;\, s\,;\, y\,] \in I\!R^n_+ \times I\!R^n_+ \times I\!R^m \ : \ Ax + Bs + Cy = d \right\}, \tag{2.16}$$

the set of all feasible points for (2.1) and by

$$\mathcal{F}^* = \{ z = [\, x\,;\, s\,;\, y\,] \in \mathcal{F} \ : \ xs = w \} \tag{2.17}$$

the solution set of (2.1). The relative interior of $\mathcal{F}$, or the set of all strictly feasible points of (2.2), is given by

$$\mathcal{F}^0 = \{ z = [\, x\,;\, s\,;\, y\,] \in \mathcal{F} \ \ x > 0, \ \ s > 0 \}\,. \tag{2.18}$$

THEOREM 2.1. *The following statements are equivalent:*

(i) The triplet (A, B, C) is row sufficient.

(ii) There is a weight vector $w \in I\!R^n_+$ such that for all vectors $d \in I\!R^n$ any KKT point of (2.4) is a solution of (2.1).

(iii) For all weight vectors $w \in I\!R^n_+$ and all vectors $d \in I\!R^n$ any KKT point of (2.4) is a solution of (2.1).

Proof. $(i) \Rightarrow (iii)$. Assume that (i) is satisfied and let x, s, y, p, q satisfy the KKT conditions (2.7). Denote

$$\begin{aligned} &\overline{x} = x_{\overline{\mathcal{W}}}\,,\ \overline{s} = s_{\overline{\mathcal{W}}}\,,\ \overline{p} = p_{\overline{\mathcal{W}}}\,,\ \overline{q} = q_{\overline{\mathcal{W}}}\,,\ \widehat{x} = x_{\widehat{\mathcal{W}}}\,,\ \widehat{s} = s_{\widehat{\mathcal{W}}}\,,\ \widehat{p} = p_{\widehat{\mathcal{W}}}\,,\ \widehat{q} = q_{\widehat{\mathcal{W}}}\,, \\ &r = (s - w/x - p)(x - w/s - q)\,,\ \overline{r} = r_{\overline{\mathcal{W}}}\,,\ \widehat{r} = r_{\widehat{\mathcal{W}}}\,. \end{aligned}$$

From the third row of (2.7) it follows that $px = qs = 0$, so that we have

$$\begin{aligned} &r = xs - 2w + w^2/(xs) + q(w/x) + p(w/s) + pq\,, \\ &\overline{r} = (\overline{x}\,\overline{s})^{-1}\left((\overline{x}\,\overline{s} - \overline{w})^2 + \overline{p}\,\overline{q}\right) \ge 0\,,\ \widehat{r} = \widehat{x}\widehat{s} + \widehat{p}\widehat{q} \ge 0\,. \end{aligned}$$

According to the row sufficiency of (A, B, C) we must have $\overline{r} = 0$, $\widehat{r} = 0$. Because $xs \ge 0$ and $pq \ge 0$, this implies

$$\overline{x}\,\overline{s} = \overline{w}\,, \quad \widehat{x}\,\widehat{s} = 0\,, \quad \overline{p}\,\overline{q} = 0\,, \quad \widehat{p}\,\widehat{q} = 0\,.$$

The first two equations above together with (2.7) show that (2.1) is satisfied.

The implication $(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$. Let $w \in I\!R^n_+$ be given. Assume that (i) is not satisfied. Then there are vectors $u, v \in I\!R^n$ and an index $j \in \{1, 2, \ldots, n\}$ such that

$$[\,u\,;\,v\,;\,0\,] \in \operatorname{Ran}[\,A^T\,;\,B^T\,;\,C^T\,],\ uv \ge 0,\ u_j v_j > 0\,.$$

Without loss of generality we may assume that $u_j > 0$ and $v_j > 0$ (otherwise we replace $[\,u\,;\,v\,]$ by $-[\,u\,;\,v\,]$). We will determine vectors $a, p, q, x, s \in I\!R^n$ such that (2.8) is satisfied but (2.2) is not satisfied. If $w_i = 0$ then we take $p_i = u_i^-$, $q_i = v_i^-$ $s_i = u_i^+$, $x_i = v_i^+$, so that

$$[\,s_i - p_i\,;\,x_i - q_i\,] = [\,u_i\,;\,v_i\,]\ \forall i \in \widehat{\mathcal{W}}.$$

Let us choose an arbitrary $i \in \overline{\mathcal{W}}$. The components p_i, q_i, x_i and s_i will be defined such that

$$[\,s_i - w_i/x_i - p_i\,;\,x_i - w_i/s_i - q_i\,] = [\,u_i\,;\,v_i\,]\,. \tag{2.19}$$

We have to consider several possible cases.

1) If $u_i > 0,\ v_i > 0$ we set $p_i = q_i = 0$ and define $x_i > 0$, $s_i > 0$ such that

$$s_i = u_i + w_i/x_i,\ x_i = v_i + w_i/s_i.$$

By substituting the second equation above into the first one we get

$$s_i = u_i + \frac{w_i}{v_i + w_i/s_i} = \frac{u_i v_i s_i + u_i w_i + s_i w_i}{s_i v_i + w_i}\,.$$

Therefore

$$s_i = \frac{u_i v_i + \sqrt{u_i^2 v_i^2 + 4 u_i v_i w_i}}{2 v_i} > 0, \quad x_i = v_i + w_i/s_i > 0. \tag{2.20}$$

2) If $u_i > 0,\ v_i = 0$, we set $p_i = 0,\ q_i = 1$ and define $x_i > 0$, $s_i > 0$ such that

$$s_i = u_i + w_i/x_i,\ x_i = 1 + w_i/s_i.$$

Proceeding as in case 1) we obtain the following explicit solutions

$$s_i = \frac{u_i + \sqrt{u_i^2 + 4u_i w_i}}{2} > 0, \quad x_i = 1 + w_i/s_i > 0.$$

3) If $u_i = 0,\ v_i > 0$, we set $p_i = 1,\ q_i = 0$ and

$$x_i = \frac{v_i + \sqrt{v_i^2 + 4v_i w_i}}{2} > 0, \quad s_i = 1 + w_i/x_i > 0.$$

4) If $u_i = 0,\ v_i \le 0$, we set $p_i = 0,\ q_i = -v_i \ge 0,\ s_i = w_i,\ x_i = 1$.

5) If $u_i < 0,\ v_i = 0$, we set $p_i = 0,\ q_i = 1$ and

$$s_i = \frac{-u_i + \sqrt{u_i^2 - 4u_i w_i}}{2} > 0, \quad x_i = 1 + w_i/s_i > 0.$$

6) If $u_i < 0,\ v_i < 0$, we set $p_i = -u_i,\ q_i = -v_i,\ s_i = w_i,\ x_i = 1$.

With the vectors x and s obtained above we define $a = Qx + Rs$. It follows that the KKT conditions (2.8) are satisfied. However $[\,x\,;\,s\,]$ is not a solution of (2.2) because $x_j s_j > w_j$. Indeed, if $j \in \widehat{\mathcal{W}}$ then $x_j s_j = v_j u_j > 0$, while if $j \in \overline{\mathcal{W}}$, then according to (2.20) $x_j s_j = s_j v_j + w_j > w_j$.

∎

Theorem 2.2. *If the triplet (A, B, C) is column sufficient then for all weight vectors $w \in I\!R^n_+$ and all vectors $d \in I\!R^n$ the mixed linear weighted complementarity problem (2.1) has a convex (perhaps empty) solution set $\mathcal{F}^*$. Moreover, $x_{\overline{\mathcal{W}}}$ and $s_{\overline{\mathcal{W}}}$ are uniquely defined for any $[\,x\,;\,s\,;\,y\,] \in \mathcal{F}^*$.*

Proof. Consider some arbitrary $w \in I\!R^n_+$, $d \in I\!R^n$. If $\mathcal{F}^*$ does not contain at least two distinct solutions there is nothing to prove. Let $[\,x\,;\,s\,;\,y\,], [\,u\,;\,v\,;\,h\,] \in \mathcal{F}^*$, and denote

$$\overline{x} = x_{\overline{\mathcal{W}}},\ \overline{s} = s_{\overline{\mathcal{W}}},\ \overline{u} = u_{\overline{\mathcal{W}}},\ \overline{v} = v_{\overline{\mathcal{W}}},\ \widehat{x} = x_{\widehat{\mathcal{W}}},\ \widehat{s} = s_{\widehat{\mathcal{W}}},\ \widehat{u} = u_{\widehat{\mathcal{W}}},\ \widehat{v} = v_{\widehat{\mathcal{W}}},$$

Since for any $i \in \overline{\mathcal{W}}$ we have $x_i s_i = u_i v_i = w_i > 0$ it follows that $x_i \ge u_i$ if and only if $s_i \le v_i$. This shows that $(\overline{x} - \overline{u})\,(\overline{s} - \overline{v}) \le 0$. Therefore

$$\begin{aligned}
(\overline{x} - \overline{u})\,(\overline{s} - \overline{v}) &= \overline{x}\,\overline{s} + \overline{u}\,\overline{v} - \overline{x}\,\overline{v} - \overline{u}\,\overline{s} = 2\overline{w} - \overline{x}\,\overline{v} - \overline{u}\,\overline{s} \le 0\,,\\
(\widehat{x} - \widehat{u})\,(\widehat{s} - \widehat{v}) &= \widehat{x}\,\widehat{s} + \widehat{u}\,\widehat{v} - \widehat{x}\,\widehat{v} - \widehat{u}\,\widehat{s} = -\widehat{x}\,\widehat{v} - \widehat{u}\,\widehat{s} \le 0\,.
\end{aligned}$$

It follows that $(x - u)(s - v) \le 0$. Using the fact that $[\,x - u\,;\,s - v\,] \in \operatorname{Ker}[Q, R]$ and the column sufficiency, we deduce that $(\overline{x} - \overline{u})\,(\overline{s} - \overline{v}) = 0$ and $(\widehat{x} - \widehat{u})\,(\widehat{s} - \widehat{v}) = 0$. Therefore, $\overline{x}\,\overline{v} + \overline{u}\,\overline{s} = 2\overline{w}$ and $\widehat{x}\,\widehat{v} + \widehat{u}\,\widehat{s} = 0$. In condensed form, we have $xv + us = 2w$. Hence for any $\varsigma \in I\!R$ we can write

$$\begin{aligned}
(\varsigma x + (1-\varsigma)u)(\varsigma s + (1-\varsigma)v) &= \varsigma^2 xs + \varsigma(1-\varsigma)(xv + us) + (1-\varsigma)^2 uv\\
&= \varsigma^2 w + 2\varsigma(1-\varsigma)w + (1-\varsigma)^2 w \;=\; w.
\end{aligned} \tag{2.21}$$

If $0 \le \varsigma \le 1$, then we have clearly $\varsigma[\,x\,;\,s\,;\,y\,] + (1-\varsigma)[\,u\,;\,v\,;\,h\,] \in \mathcal{F}$. It follows that $\varsigma[\,x\,;\,s\,;\,y\,] + (1-\varsigma)[\,u\,;\,v\,;\,h\,] \in \mathcal{F}^*$, for any $\varsigma \in [0, 1]$, which proves the convexity of the solution set.

We will prove now the last statement of our theorem. Consider as above two solutions $[\,x\,;\,s\,;\,y\,]\,,[\,u\,;\,v\,;\,h\,]\in\mathcal{F}^*$. We have shown above that $(\overline{x}-\overline{u})\,(\overline{s}-\overline{v})=0$. It is easily seen that this relation implies $\overline{x}=\overline{u}\,,\ \overline{s}=\overline{v}$. Indeed for any $i\in\overline{\mathcal{W}}$, we have $x_is_i=u_iv_i=w_i>0$. Therefore $x_i\neq u_i$ if and only if $s_i\neq v_i$. This observation together with the fact that $(x_i-u_i)\,(u_i-v_i)=0$ shows that $x_i=u_i$ and $s_i=v_i$. ∎

We note that the existence of a weight vector $w\in I\!R^n_+-\{0\}$ and a vector $a\in I\!R^m$ such that the solution set of (2.2) is convex does not imply that the pair (Q,R) is row sufficient. For example, in the two dimensional case, let us take

$$Q=\begin{pmatrix}-1&0\\0&0\end{pmatrix},R=\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\ a=\begin{pmatrix}-3\\0\end{pmatrix},w=\begin{pmatrix}2\\0\end{pmatrix},u=-v=\begin{pmatrix}1\\0\end{pmatrix}.$$

Then the solution set $\{[\,[\,1\,;\,x_2\,]\,;\,[\,2\,;\,0\,]\,]\ :\ x_2\in I\!R_+\}$ of (2.2) is convex and we have $[\,u\,;\,v\,]\in\operatorname{Ker}(Q,R),\ uv\leq 0,\ u_1v_1=-1<0$, which shows that the pair (Q,R) is not column sufficient.

However if $w=0$ we have the following well known result from the theory of linear complementarity problems.

THEOREM 2.3. *The triplet (A,B,C) is column sufficient if and only if the mixed linear weighted complementarity problem (2.1) with $w=0$ and arbitrary $d\in I\!R^n$ has a convex (perhaps empty) solution set $\mathcal{F}^*$.*

Proof. From Theorem 2.1 of [12] it follows that the matrix $[Q,R]$ is full rank. Therefore using the techniques from [2] we can show that (2.1) is equivalent to a standard linear complementarity problem of the form $s=Mx+q\geq 0,x\geq 0,xs=0$, where M is a column sufficient matrix . The proof of our theorem follows then from the proof Theorem 3.5.8 of [5].

For the sake of completeness we give the following direct proof. One implication follows from Theorem 2.2. Assume now that the pair (Q,R) is not column sufficient. Then there are vectors $u,v\in I\!R^n$ and an index j such that

$$[\,u\,;\,v\,]\in\operatorname{Ker}\,[\,Q\,;\,R\,]\,,\ uv\leq 0,\ u_jv_j<0.$$

Without loss of generality we may assume that $u_j>0$ and $v_j<0$. (otherwise we replace $[\,u\,;\,v\,]$ by $-[\,u\,;\,v\,]$). Define $x=u^+,s=v^+,\widetilde{x}=u^-,\widetilde{s}=v^-$ and $a=Qx+Rs=Q\widetilde{x}+R\widetilde{s}$. It is easily seen that $[\,x\,;\,s\,]$ and $[\,\widetilde{x}\,;\,\widetilde{s}\,]$ are distinct solutions of (2.2), but $0.5\,[\,x\,;\,s\,]+0.5\,[\,\widetilde{x}\,;\,\widetilde{s}\,]$ is not a solution of (2.2). Indeed, let i be an arbitrary index. If $u_i=0$ or $v_i=0$ then $u_i^+v_i^+=u_i^-v_i^-=0$. On the other hand if $u_i\neq 0$ then we must have either $u_i>0,v_i\leq 0$ or $u_i<0,v_i\geq 0$. In both cases $u_i^+v_i^+=u_i^-v_i^-=0$. The case $v_i\neq 0$ implies in a similar manner that $u_i^+v_i^+=u_i^-v_i^-=0$. Also we have $u_j^+>0$ and $u_j^-=0$. Therefore $[\,x\,;\,s\,]$ and $[\,\widetilde{x}\,;\,\widetilde{s}\,]$ are distinct solutions of (2.2) as stated. On the other hand $(u_j^++u_j^-)(u_j^-+v_j^-)=-u_jv_j>0$, which shows that $0.5\,[\,x\,;\,s\,]+0.5\,[\,\widetilde{x}\,;\,\widetilde{s}\,]$ is not a solution of (2.2) with $w=0$. ∎

We end this subsection by giving a simple sufficient condition for the solvability of (2.1).

THEOREM 2.4. *If the weighted complementarity problem (2.1) is sufficient and strictly feasible, then it is solvable.*

Proof. The strict feasibility assumption implies that there is $[\,\check{x}\,;\,\check{s}\,]\in I\!R^{2n}_{++}$ such that $Q\check{x}+R\check{s}=b$. We denote by

$$f_w(x,s)=x^Ts-\sum_{i=1}^{n}w_i\log x_is_i \tag{2.22}$$

the objective function in (2.5), and we consider the sublevel set

$$\mathcal{L}_w(\check{x}, \check{s}) = \{[\,x\,;\,s\,] \in I\!R^{2n}_+ \;:\; Qx + Rs = a,\; f_w(x,s) \leq f_w(\check{x}, \check{s})\}\,. \tag{2.23}$$

The following implication holds

$$[\,x\,;\,s\,] \in \mathcal{L}_w(\check{x}, \check{s}) \Rightarrow x_i s_i > 0,\; \forall i \in \overline{\mathcal{W}}\,.$$

Otherwise, if $x_i s_i = 0$ for some $i \in \overline{\mathcal{W}}$, then $f_w(x,s) = \infty > f_w(\check{x}, \check{s})$. Using the inequality

$$\rho - \sigma \log \rho \geq \sigma - \sigma \log \sigma,\; \forall \rho, \sigma > 0\,, \tag{2.24}$$

we deduce that for any $[\,x\,;\,s\,] \in \mathcal{L}_w(\check{x}, \check{s})$ there holds

$$\sum_{i \in \widehat{\mathcal{W}}} x_i s_i = f_w(x,s) - \sum_{i \in \overline{\mathcal{W}}} (x_i s_i - w_i \log x_i s_i) \leq f_w(\check{x}, \check{s}) - \sum_{i \in \overline{\mathcal{W}}} (w_i - w_i \log w_i)\,. \tag{2.25}$$

For any $j \in \overline{\mathcal{W}}$ we have

$$\begin{aligned} x_j s_j - w_j log(x_j s_j) &= f_w(x,s) - \sum_{i \in \widehat{\mathcal{W}}} x_i s_i - \sum_{i \in \overline{\mathcal{W}} - \{j\}} (x_i s_i - w_i \log x_i s_i) \\ &\leq f_w(\check{x}, \check{s}) - \sum_{i \in \overline{\mathcal{W}} - \{j\}} (w_i - w_i \log w_i). \end{aligned}$$

It follows that there are constants λ and Λ such that

$$0 < \lambda \leq x_j s_j \leq \Lambda, \quad \forall j \in \overline{\mathcal{W}}. \tag{2.26}$$

From (2.25) and (2.26) we deduce that there is a constant $\eta > 1$ such that $x^T s \leq \eta$. For any $[\,x\,;\,s\,] \in \mathcal{L}_w(\check{x}, \check{s})$ we have $[\,\check{x} - x\,;\,\check{s} - s\,] \in \mathrm{Ker}[Q\;R]$, so that

$$(\check{x} - x)^T (\check{s} - s) \geq -4\kappa \sum_{i \in I^+} (\check{x} - x)_i (\check{s} - s)_i\,,$$

where κ is the handicap of the sufficient pair (Q, R), and

$$I^+ = \{i \in \{1, \ldots, n\} \;:\; (\check{x} - x)_i (\check{s} - s)_i > 0\,\}\,.$$

It follows that

$$\begin{aligned} \check{s}^T x + \check{x}^T s &\leq \check{x}^T \check{s} + x^T s + 4\kappa \sum_{i \in I^+} (\check{x}_i \check{s}_i + x_i s_i - \check{x} s_i - \check{s} x_i) \\ &\leq 2\eta + 4\kappa \sum_{i \in I^+} (\check{x}_i \check{s}_i + x_i s_i) \leq 2(1 + 4\kappa)\eta\,. \end{aligned}$$

Therefore,

$$\|\,[\,x\,;\,s\,]\,\|_1 \leq \frac{2(1+4\kappa)\eta}{\min\,[\,\check{x}\,;\,\check{s}\,]}\,, \quad \forall\,[\,x\,;\,s\,] \in \mathcal{L}_w(\check{x}, \check{s})\,.$$

The above relation shows that $\mathcal{L}_w(\check{x}, \check{s})$ is bounded. It is also closed because of the continuity of f_w. Therefore $\mathcal{L}_w(\check{x}, \check{s})$ is compact and consequently f_w has a minimum on $\mathcal{L}_w(\check{x}, \check{s})$. According to Theorem 2.1 this implies the existence of a solution of (2.2). ∎

2.4. Maximal complementarity. Throughout this subsection we assume that (2.1) is strictly feasible, and therefore solvable. Besides the index sets from (2.6), let us also consider the following index sets

$$\begin{aligned} \mathcal{I} &= \{i \in \{1,\ldots,n\} \ : \ \exists [\,x\,;\,s\,;\,y\,] \in \mathcal{F}^*,\, x_i > 0,\, s_i = 0\}\,, \\ \mathcal{J} &= \{i \in \{1,\ldots,n\} \ : \ \exists [\,x\,;\,s\,;\,y\,] \in \mathcal{F}^*,\, x_i = 0,\, s_i > 0\}\,, \\ \mathcal{K} &= \{i \in \{1,\ldots,n\} \ : \ \forall [\,x\,;\,s\,;\,y\,] \in \mathcal{F}^*,\, x_i = 0,\, s_i = 0\}\,, \end{aligned} \tag{2.27}$$

PROPOSITION 2.5. *If (2.1) is row sufficient then the index sets $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ are disjoint and form a partition of $\widehat{\mathcal{W}}$. Moreover $x_{\mathcal{J}} = 0, s_{\mathcal{I}} = 0$, $\forall [\,x\,;\,s\,;\,y\,] \in \mathcal{F}^*$.*

Proof. We have obviously $\mathcal{I} \cap \mathcal{K} = \mathcal{J} \cap \mathcal{K} = \emptyset$. If $i \in \mathcal{I} \cap \mathcal{J}$ then there are $[\,x\,;\,s\,;\,y\,], [\,u\,;\,v\,;\,h\,] \in \mathcal{F}^*$ such that $x_i > 0$, $s_i = 0$, $u_i = 0$, $v_i > 0$. Therefore we have $x_i + u_i > 0$, $s_i + v_i > 0$, which contradicts the fact that $.5[\,x\,;\,s\,;\,y\,] + .5[\,u\,;\,v\,;\,h\,] \in \mathcal{F}^*$. According to Theorem 2.2 this contradicts the row sufficiency of (A, B, C). Hence $\mathcal{I} \cap \mathcal{J} = \emptyset$. The equality $\mathcal{I} \cup \mathcal{J} \cup \mathcal{K} = \overline{\mathcal{W}}$ follows from the definition of the index sets (2.27). Let us consider now an arbitrary $[\,x\,;\,s\,;\,y\,] \in \mathcal{F}^*$ and an index $i \in \mathcal{I}$. If $s_i > 0$, then we must have $x_i = 0$ so that $i \in \mathcal{J}$ which contradicts the fact that $\mathcal{I} \cap \mathcal{J} = \emptyset$. Therefore $s_{\mathcal{I}} = 0$. In a similar manner it follows that $x_{\mathcal{J}} = 0$. ∎

DEFINITION 2.6. *A solution $[\,x\,;\,s\,;\,y\,] \in \mathcal{F}^*$ is said to be a maximal complementarity solution if $x_{\mathcal{I}} > 0$ and $s_{\mathcal{J}} > 0$.*

A maximal complementarity solution has the maximum possible number of nonzero entries in $[\,x\,;\,s\,]$ among all solutions of (2.1). This number is obviously equal to $2\,\text{cardinal}(\overline{\mathcal{W}}) + \text{cardinal}(\mathcal{I}) + \text{cardinal}(\mathcal{J})$.

PROPOSITION 2.7. *If (2.1) is column sufficient and solvable then the set of its maximal complementarity solutions is a nonempty convex set that coincides with the relative interior of $\mathcal{F}^*$.*

Proof. According to Theorem 2.2, $\mathcal{F}^*$ is convex. For any $i \in \mathcal{I}$ consider a solution $[\,x^i\,;\,s^i\,;\,y^i\,] \in \mathcal{F}^*$ with $x^i_i > 0$, and for any $j \in \mathcal{J}$ consider a solution $[\,u^j\,;\,v^j\,;\,h^j\,] \in \mathcal{F}^*$ with $v^j_j > 0$. It follows that

$$[\,x\,;\,s\,;\,y\,] = (\text{cardinal}(\mathcal{I} \cup \mathcal{J}))^{-1} \left(\sum_{i \in \mathcal{I}} [\,x^i\,;\,s^i\,;\,y^i\,] + \sum_{j \in \mathcal{J}} [\,u^j\,;\,v^j\,;\,y^j\,] \right)$$

is a maximal solution of (2.1). Since $\mathcal{F}^*$ is convex then its relative interior is given by

$$\text{ri}(\mathcal{F}^*) = \{z \in \mathcal{F}^* \ : \ \forall \tilde{z} \in \mathcal{F}^* \exists \lambda > 1 \ : \ \lambda z + (1-\lambda)\tilde{z} \in \mathcal{F}^*\}.$$

See [17]. Let $[\,x\,;\,s\,;\,y\,]$ and $[\,u\,;\,v\,;\,h\,]$ be two arbitrary solutions of (2.1). Then, according to (2.21) we have

$$[A, B, C]\,(\varsigma[\,x\,;\,s\,;\,y\,] + (1-\varsigma)[\,u\,;\,v\,;\,h\,]) = d, \tag{2.28}$$

$$(\varsigma x + (1-\varsigma)u)(\varsigma s + (1-\varsigma)v) \ = \ w, \ \forall \varsigma \in I\!R. \tag{2.29}$$

If $[\,x\,;\,s\,;\,y\,]$ is a maximal complementarity and $[\,u\,;\,v\,;\,h\,]$ an arbitrary solution of (2.1), then the following implications hold

$$u_i > 0 \Rightarrow x_i > 0, \quad v_i > 0 \Rightarrow s_i > 0, \quad i = 1, 2, \ldots, n. \tag{2.30}$$

For any $\lambda > 1$ and any i we have

$$\lambda x_i + (1-\lambda)u_i \geq x_i + (1-\lambda)u_i, \quad \lambda s_i + (1-\lambda)v_i \geq s_i + (1-\lambda)v_i.$$

By taking

$$\eta = \min\left\{\min_i\{x_i/u_i \ : \ u_i > 0\},\, \min_i\{s_i/v_i \ : \ v_i > 0\}\right\}, \quad \lambda = 1 + \eta,$$

we deduce that if $[\,x\,;\,s\,;\,y\,]$ is a maximal complementarity solution of (2.1) then $\lambda\,[\,x\,;\,s\,] + (1-\lambda)\,[\,u\,;\,v\,] \geq 0$. Therefore, according to (2.28) and (2.29), $\lambda[\,x\,;\,s\,;\,y\,] + (1-\lambda)[\,u\,;\,v\,;\,h\,] \in \mathcal{F}^*$. This shows that $[\,x\,;\,s\,;\,y\,] \in \mathrm{ri}(\mathcal{F}^*)$. Conversely, assume that $[\,x\,;\,s\,;\,y\,] \in \mathrm{ri}(\mathcal{F}^*)$ and let $[\,u\,;\,v\,;\,h\,]$ be any solution in (2.1). Then there is $\lambda > 1$ such that

$$\lambda x_i + (1-\lambda)u_i \geq 0, \quad \lambda s_i + (1-\lambda)v_i \geq 0, \quad i = 1, 2, \ldots, n.$$

It follows that the implications (2.30) hold for all $[\,u\,;\,v\,;\,h\,] \in \mathcal{F}^*$. Hence $[\,x\,;\,s\,;\,y\,]$ is a maximal complementarity solution of (2.1) ∎

If $\mathcal{K}$ is empty then any maximal complementarity solution is called a strict complementarity solution. It is easily seen that the set of strict complementarity solutions is given by

$$\mathcal{F}^c = \{z = [\,x\,;\,s\,;\,y\,] \in \mathcal{F}^* \ : \ x + s > 0\}\,. \tag{2.31}$$

Since $z = [\,x\,;\,s\,;\,y\,] \in \mathcal{F}^*$ implies $x_{\overline{\mathcal{W}}} > 0$ and $s_{\overline{\mathcal{W}}} > 0$, it follows that $\mathcal{F}^c$ is nonempty if and only if $\mathcal{K}$ is empty.

3. A corrector-predictor algorithm for solving sufficient wCP's .

The interior-point method presented in this section is a path-following method. The path is similar to the one considered in [14] for monotone mixed linear wCP. However, the present interior-point method differs from the two path-following methods from [14] because the natural extensions of those methods to the sufficient case would need explicit use of he handicap of the problem.

3.1. The central path. Let us consider the notations from (2.1), (2.16) and (2.18). Given a strictly feasible starting point $z^0 = \left[\,x^0\,;\,s^0\,\right] y^0 \in \mathcal{F}^0$, we denote

$$t_0 = \frac{x^{0\,T} s^0}{n},\ c = x^0 s^0,\ \gamma = \frac{\min\ c}{t_0},\ \ w(t) = (1 - t/t_0)w + (t/t_0)c,\ t \in (0, t_0]. \tag{3.1}$$

We define the central path of wCP (2.1) emanating from z^0 as the set $\mathcal{C}$ of all points $[\,t\,;\,z\,]] = [t; x; s; y]$, with $t \in (0, t_0]$, satisfying

$$\begin{array}{rl} xs & = w(t) \\ Ax + Bs + Cy & = d \\ x > 0,\ s > 0 & \end{array}\quad . \tag{3.2}$$

First, we show that this path is well defined. The following proposition follows from well known results from the theory of sufficient linear complementarity problems, but we give a simple proof for the sake of completeness.

Proposition 3.1. *If the triplet (A, B, C) is sufficient and $\mathcal{F}^0$ is nonempty then (3.2) has a unique solution for any $t \in (0, t_0]$.*

Proof. We first note that $w(t) > 0,\ \forall t \in (0, t_0]$. Let $f_{w(t)}$ be defined as in (2.22), with w replaced by $w(t)$, and let $\mathcal{L}_{w(t)} = \mathcal{L}_{w(t)}(x^0, s^0)$ be the sublevel set corresponding to (2.23). Using the same techniques as in the proof of Proposition 2.4 we can show that $\mathcal{L}_{w(t)}$ is compact. Hence $f_{w(t)}$ attains its minimum on $\mathcal{L}_{w(t)}$ at a point $z(t,w,z^0) = [\,x(t,w,z^0)\,;\, s(t,w,z^0)\,;\, y(t,w,z^0)\,]$. As in the proof of Theorem 2.1 it can be shown that this point satisfies (3.2). Since $w(t) > 0$, the index set corresponding to $\overline{\mathcal{W}}$ from (2.6) becomes $\{1, 2, \ldots, n\}$, and by obvious modifications of the arguments from the proof of Theorem 2.2 it follows that $z(t,w,z^0)$ is the unique solution of (3.2). ∎

By construction $[t_0; z^0]$ belongs to the path $\mathcal{C}$. The proximity of a point $[\,t\,;\,z\,] = [t;x;s;y]$ to this central path can be measured by the function

$$\delta(t,z) = t^{-1}\,\|\,xs - w(t)\,\|_2\,. \tag{3.3}$$

Obviously, a point $[\,t\,;\,z\,] \in (0,t_0] \times \mathcal{F}^0$ belongs to the central path if and only if $\delta(t,z) = 0$. Given a parameter α such that

$$0 \le \frac{\gamma}{3} \le \alpha \le \frac{2\gamma}{3}\,, \tag{3.4}$$

we define the following neighborhood of the above central path:

$$\mathcal{N}_2(w,c,\alpha) = \{[t;z] = [t;x;s;y] \in (0,t_0] \times \mathcal{F}^0 \;:\; \delta(t,z) \le \alpha t\}. \tag{3.5}$$

For our starting point we have $x^0 s^0 = c = w(t_0)$, so that $\left[\,t_0\,;\,z^0\,\right] \in \mathcal{N}_2(w,c,\alpha)$.

3.2. The Algorithm. At a typical iteration of our algorithm we have a point $[\,t\,;\,z\,] \in \mathcal{N}_2(w,c,\alpha)$, for some $t \le t_0$.

The corrector. The role of the corrector step is to produce a point $[\,t\,;\,\overline{z}\,] = [t;\overline{x}\,;\overline{s}\,;\overline{y}\,]$ in $(0,t_0] \times \mathcal{F}^0$ with a smaller proximity measure, i.e., $\delta(t,\overline{z}\,) < \delta(t,z)$. Since the starting point is perfectly centered, i.e., $\delta(t_0,z^0) = 0$, no corrector is needed at the first iteration. Formally, at the first iteration we can take $\overline{z} = z^0$. For all subsequent iterations the corrector direction $[u;v;h]$ is obtained as the solution of the following linear system

$$\left\{ \begin{array}{ccc} su + xv & = & w(t) - xs \\ Au + Bv + Ch & = & 0 \end{array} \right. . \tag{3.6}$$

Let us denote

$$z(\theta) = [\,x(\theta)\,;\,s(\theta)\,;\,y(\theta)\,] = z + \theta[u;v;h] = [\,x + \theta u\,;\,s + \theta v\,;\,y + \theta h\,] \tag{3.7}$$

and define the steplength $\overline{\theta}$ of the corrector as the minimizer of the proximity measure $\delta(t,z(\theta))$. It turns out that minimizing $\delta(t,z(\theta))$ is equivalent to minimizing a quartic, which can be done very efficiently. With

$$\overline{\theta} = \operatorname*{argmin}_{\theta \ge 0}\,(1-\theta)^2\,\|\,xs - w(t)\,\|_2^2 + 2\theta^2(1-\theta)(uv)^T(xs - w(t)) + \theta^4\,\|\,uv\,\|_2^2\,, \tag{3.8}$$

the output of the corrector step is the point

$$\overline{z} = [\,\overline{x}\,;\,\overline{s}\,;\,\overline{y}\,] = z(\overline{\theta}\,)\,. \tag{3.9}$$

We will prove that

$$[\,t\,;\,\overline{z}\,] \in \mathcal{N}_2(w,c,\overline{\alpha}\,)\,, \quad \overline{\alpha} := \delta(t,\overline{z}\,) < \left(1 - \frac{1}{4(1+2\kappa)}\right)\alpha\,. \tag{3.10}$$

The predictor. The predictor step follows the corrector step, so that the point $[\,t\,;\,\overline{z}\,] \in \mathcal{N}_2(w,c,\overline{\alpha}\,)$ is available. The purpose of the predictor step is to produce a new point $[\,t^+\,;\,z^+\,] \in \mathcal{N}_2(w,c,\alpha)$ with $t^+ < t$. Given the point (3.9) we compute the predictor direction $[\,\overline{u}\,;\,\overline{v}\,;\,\overline{h}\,]$ by solving the linear system

$$\left\{ \begin{array}{rcl} \overline{s}\,\overline{u} + \overline{x}\,\overline{v} & = & w - \overline{x}\,\overline{s} \\ A\overline{u} + B\overline{v} + C\overline{h} & = & 0 \end{array} \right. . \tag{3.11}$$

We define

$$\begin{array}{c} \overline{x}\,(\theta) = \overline{x} + \theta\overline{u}\,,\ \overline{s}\,(\theta) = \overline{s} + \theta\overline{v}\,,\ \overline{y}\,(\theta) = \overline{y} + \theta\overline{h}\,,\ t(\theta) = (1-\theta)t, \\ \overline{z}\,(\theta) = [\,\overline{x}\,(\theta)\,;\,\overline{s}\,(\theta)\,;\,\overline{y}\,(\theta)\,]\,. \end{array} \tag{3.12}$$

The stepsize θ^+ along this direction is defined as the largest θ for which $z(\theta)$ belongs to the neighborhood $\mathcal{N}_2(w,c,\alpha)$. It turns out that this stepsize is given by the following explicit formula

$$\theta^+ = \left\{ \begin{array}{ll} 1 & \text{if } \overline{u}\,\overline{v} = 0 \\ \dfrac{2\overline{\phi}}{\overline{\phi} + \sqrt{\overline{\phi} + \overline{\phi}^{\,2}}} & \text{if } \overline{u}\,\overline{v} \neq 0 \end{array} \right. , \tag{3.13}$$

where,

$$\overline{\phi} = \frac{-\beta_0}{\beta_1 + \sqrt{\beta_1^2 - \beta_0\beta_2}}\,, \tag{3.14}$$

with

$$\beta_0 = \frac{\|\overline{x}\,\overline{s} - w(t)\|_2^2}{t^2} - \alpha^2,\ \beta_1 = \frac{(\overline{u}\,\overline{v}\,)^T(\overline{x}\,\overline{s} - w(t))}{t^2},\ \beta_2 = \frac{\|\overline{u}\,\overline{v}\,\|_2^2}{t^2}\,. \tag{3.15}$$

Having computed this steplength we obtain the predicted point

$$\left[\,t^+\,;\,z^+\,\right] = [\,(1-\theta^+)t\,;\,\overline{x}\,(\theta^+)\,;\,\overline{s}\,(\theta^+)\,;\,y(\theta^+)\,] \in \mathcal{N}_2(w,c,\alpha). \tag{3.16}$$

Hence we can start a new corrector step from this point followed by the corresponding predictor step. We obtain the following iterative scheme:

Corrector-Predictor Algorithm
Given a starting point $z^0 = [\,x^0\,;\,s^0\,;\,y^0\,] \in \mathcal{F}^0$:
    Consider the notation from (3.1);
    Choose a parameter α satisfying (3.4);
    Set $k \leftarrow 0$;
    repeat
        Set $z = [\,x\,;\,s\,;\,y\,] \leftarrow z^k\,,\ t \leftarrow t_k$;

        Corrector
        If $k = 0$ set $\overline{z} = z^0$, $\overline{\theta} = 0$ and go to *Predictor*;
        Solve the linear system (3.6);
        Compute steplength $\overline{\theta}$ from (3.8) ;
        Compute $\overline{z}$ from (3.7) and (3.9);

Predictor
Solve the linear system (3.11);
Compute steplength θ^+ from (3.13)-(3.15) ;
Compute t^+, z^+ from (3.12) and (3.16);

Set $\theta_k \leftarrow \theta^+$, $t_{k+1} \leftarrow t^+$, $\overline{z}^{\,k} \leftarrow \overline{z}$, $z^{k+1} \leftarrow z^+$;
Set $k \leftarrow k+1$.
continue

3.3. The computational complexity of the algorithm. In the proof of the main theorem of this subsection we will use the following two technical lemmas. The first one contains a well known result that goes back to the monograph [11]. See also [9].

LEMMA 3.2. *If the triplet (A, B, C) has the $P_*(\kappa)$ property then for any $z = [\,x\,;\,s\,;\,y\,] \in I\!R^{2n}_{++} \times I\!R^m$ and $g \in I\!R^n$ the linear system*

$$\left\{ \begin{array}{rcl} su + xv & = & g \\ Au + Bv + Ch & = & 0 \end{array} \right. , \tag{3.17}$$

has a unique solution $[\,u\,;\,v\,;\,h\,]$ for which the following estimates hold

$$\|\,uv\,\|_2 \le \left(\frac{1}{\sqrt{8}} + \kappa\right) \|\,\tilde{g}\,\|_2^2\,, \; -\kappa\,\|\,\tilde{g}\,\|_2^2 \le u^T v \le \frac{1}{4}\,\|\,\tilde{g}\,\|_2^2\,,$$

where $\tilde{g} = (xs)^{-1/2} g$.

The second lemma is proved in [1] (see also [15, Proposition 3.1]).

LEMMA 3.3. *If*

$$[\,x\,;\,s\,] \in I\!R^{2n}_{++},\ [\,u\,;\,v\,] \in I\!R^{2n},\ \tau \in (0,1],\ su + xv + xs \ge 0,\ (x+\tau u)(s+\tau v) > 0,$$

then $x + \theta u > 0,\ s + \theta v > 0,\ \forall \theta \in (0, \tau]$.

THEOREM 3.4. *If wCP (2.1) is sufficient, then the Corrector-Predictor Algorithm is well defined and generates two iteration sequences satisfying the following properties*

$$[t_k; z^k] \in \mathcal{N}_2(w, c, \alpha)\,,\ [t_k; \overline{z}^{\,k}] \in \mathcal{N}_2(w, c, \overline{\alpha}\,)\,,\ \overline{\alpha} < \alpha\,,$$
$$t_{k+1} = (1-\theta_k) t_k\,,\ \theta_k \ge \frac{\alpha}{2\rho(1+2\kappa)}\ ,\ \rho = 1 + \frac{\|c - w\|_2}{t_0}\,,$$

for $k = 0, 1, \ldots$.

Proof. We drop the index k and use instead the notation from the Corrector-Predictor Algorithm. We first analyze the corrector step. From (3.6) and (3.7) it follows that

$$x(\theta)s(\theta) - w(t) = xs + \theta(su + xv) + \theta^2 uv - w(t) = (1-\theta)(xs - w(t)) + \theta^2 uv\,,$$

which shows that $\overline{\theta}$ from (3.8) is the minimizer of the proximity measure $\delta(t, z(\theta))$. The inequalities $\delta(t, z) \le \alpha$, $t \le t_0$ and $c \ge \gamma t_0\, e$ imply that

$$xs \ge w(t) - \alpha te \ge (t/t_0)c - \alpha te \ge (\gamma - \alpha)te = \beta te, \quad \beta = \gamma - \alpha \ge \frac{\gamma}{3} \ge \frac{\alpha}{2}\,\cdot \tag{3.18}$$

For any $\theta \in [0,1]$ we have

$$\delta(z(t, z(\theta))) \le (1-\theta)\delta(t, z) + t^{-1}\theta^2\,\|\,uv\,\|_2 < \phi_1(\theta) := (1-\theta)\alpha + (1+2\kappa)\alpha\theta^2\,.$$

In establishing the last inequality above we used (3.18) and Lemma 3.2 to obtain

$$\| uv \|_2 \le \left(\frac{1}{\sqrt{8}} + \kappa\right) \left\| (xs)^{-1/2}(xs - w(t) \right\|_2^2 \le \left(\frac{1}{\sqrt{8}} + \kappa\right) 2\alpha t < (1 + 2\kappa)\alpha t\,.$$

The stepsize $\overline{\theta}$ is the minimizer of $\delta(t, z(\theta))$, so that we have

$$\overline{\alpha} := \delta(z(t, z(\overline{\theta}\,)) < \phi_1 \left(\frac{1}{2(1+2\kappa)}\right) = \left(1 - \frac{1}{4(1+2\kappa)}\right)\alpha\,.$$

On the other hand from (3.6) we have $su + xv + xs = w(t) \ge 0$, and similarly to (3.18) we deduce that $\overline{x}\,\overline{s} \ge (\gamma - \overline{\alpha}\,)te > 0$. Therefore according to Lemma 3.3 we have $\overline{x} > 0$ and $\overline{s} > 0$. From (3.6) we have $[\,u\,;\,v\,;\,h\,] \in \mathrm{Ker}\,[A, B, C]$ which implies $Ax + Bs + Cy = d$. This completes the proof of the fact that the corrector step produces a point satisfying (3.10) .

In the predictor step we have

$$\begin{aligned}
&\overline{x}\,(\theta)\overline{s}\,(\theta) = (\overline{x} + \theta u)(\overline{s} + \theta v) = (1-\theta)\overline{x}\,\overline{s} + \theta w + \theta^2 \overline{u}\,\overline{v}\ ,\\
&w(t(\theta)) = w + (1-\theta)t(c - w)\,,\\
&\overline{x}\,(\theta)\overline{s}\,(\theta) - w(t(\theta)) = (1-\theta)(\overline{x}\,\overline{s} - w(t)) + \theta^2 \overline{u}\,\overline{v}\ .
\end{aligned}$$

Therefore the inequality $\delta(t(\theta), \overline{z}\,(\theta)) \le \alpha$ can be written as

$$\beta_0(1-\theta)^2 + 2\beta_1(1-\theta)\theta^2 + \beta_2\theta^4 \le 0, \tag{3.19}$$

where β_0, β_1, and β_2 are given by (3.15).

Since $[\,t\,;\,\overline{z}\,] \in \mathcal{N}(w, c, \overline{\alpha}\,)$ we have $\beta_0 \le \overline{\alpha}^{\,2} - \alpha^2 < 0$. If $\overline{u}\,\overline{v} = 0$ then $\beta_1 = \beta_2 = 0$ and $\theta^+ = 1$. Therefore, in what follows we assume $\beta_2 > 0$. By using the substitution $\phi = (1-\theta)/\theta^2$, (3.19) can be reduced to the following quadratic inequality

$$\beta_0 + 2\beta_1\phi + \beta_2\phi^2 \le 0. \tag{3.20}$$

The left-hand-side of the above inequality is strictly negative for $\phi = 0$ (since $\beta_0 < 0$), and strictly positive for ϕ sufficiently large (since $\beta_2 > 0$). Therefore the above inequality holds for all $\phi \in [0, \overline{\phi}\,]$, where $\overline{\phi}$ is given by (3.14). Thus we have proved that $\delta(t(\theta^+), \overline{z}\,(\theta^+)) \le \alpha$. As in (3.18) we deduce that $x^+s^+ \ge \beta(1-\theta^+)te > 0$. From (3.11) we have $\overline{s}\,\overline{u} + \overline{x}\,\overline{v} + \overline{x}\,\overline{s} = w \ge 0$, and by applying Lemma 3.3 we deduce that $x^+ > 0$ and $s^+ > 0$. From (3.11) we have $[\,\overline{u}\,;\,\overline{v}\,;\,\hbar\,] \in \mathrm{Ker}\,[A, B, C]$ which implies $Ax^+ + Bs^+ + Cy^+ = d$. Hence $[\,t^+\,;\,z^+\,] \in \mathcal{N}_2(w, c, \alpha)$.

In order to find a lower bound for θ^+ we note that

$$|\beta_1| \le \frac{\|\overline{x}\,\overline{s} - w(t)\|_2 \|\overline{u}\,\overline{v}\,\|_2}{t^2} = \sqrt{\beta_0 + \alpha^2}\sqrt{\beta_2}, \quad 0 \le \beta_0 + \alpha^2 \le \overline{\alpha}^{\,2},$$

and therefore

$$\begin{aligned}
\overline{\phi} &\ge \frac{-\beta_0}{|\beta_1| + \sqrt{\beta_1^2 - \beta_0\beta_2}} \ge \frac{-\beta_0}{\left(\sqrt{\beta_0 + \alpha^2} + \alpha\right)\sqrt{\beta_2}}\\
&\ge \frac{-\beta_0}{(\alpha + \overline{\alpha}\,)\sqrt{\beta_2}} \ge \frac{\alpha - \overline{\alpha}}{\sqrt{\beta_2}}\,.
\end{aligned}$$

On the other by using Lemma 3.2 and (3.18) we obtain

$$\|\overline{x}\,\overline{s} - w\|_2 \le \|\overline{x}\,\overline{s} - w(t)\|_2 + \|w(t) - w\|_2 \le (\overline{\alpha} + t_0^{-1}\|c - w\|_2)t < \rho t\,, \tag{3.21}$$

$$\|\overline{u}\,\overline{v}\,\|_2 \le \left(\kappa + \frac{1}{\sqrt{8}}\right) \frac{\|\overline{x}\,\overline{s} - w\|_2^2}{\min \overline{x}\,\overline{s}} < \frac{(2\kappa+1)\,\rho^2 t}{\alpha}\,, \tag{3.22}$$

which, by virtue of (3.10), leads to the lower bound

$$\overline{\phi} \ge \frac{\alpha(\alpha - \overline{\alpha}\,)}{(1+2\kappa)\rho^2} > \frac{\alpha^2}{4(1+2\kappa)^2\rho^2} =: \widehat{\phi}\,. \tag{3.23}$$

Since $\alpha < 2\gamma/3 \le 2/3$, and $\rho > 1$, we have $\widehat{\phi} < 1/9$. Using (3.13) and the fact that

$$\frac{2\phi}{\phi + \sqrt{\phi + \phi^2}} \ge \sqrt{\phi}, \quad \forall \phi \in [0,\, 0.5],$$

we obtain

$$\theta^+ > \frac{2\widehat{\phi}}{\widehat{\phi} + \sqrt{\widehat{\phi} + \widehat{\phi}^2}} \ge \sqrt{\widehat{\phi}} = \frac{\alpha}{2\rho(1+2\kappa)}\,.$$

∎

Corollary 3.5. *If wCP (2.1) is sufficient, then the Corrector-Predictor Algorithm finds an ε-approximate solution for this problem, i.e., a point $z = [\,x\,;\,s\,;\,y\,] \in \mathcal{F}^0$ such that $\|\,xs - w\,\|_2 \le \varepsilon$, in at most*

$$O\left(\frac{x^{0\,T}s^0/n + \|\,x^0s^0 - w\,\|_2}{\min x^0 s^0}\,(1+\kappa)\log \frac{x^{0\,T}s^0/n + \|\,x^0s^0 - w\,\|_2}{\varepsilon}\right)$$

iterations.

Proof. Let us denote by $[\,t\,;\,z\,]$ the point $\left[\,t_k\,;\,z^k\,\right]$ produced at the kth iteration of our algorithm. Since $[\,t\,;\,z\,] \in \mathcal{N}_2(w,c,\alpha)$ we deduce as in (3.21) that

$$\|\,xs - w\,\|_2 \le \rho t \le \rho t_0 \left(1 - \frac{\alpha}{2\rho(1+2\kappa)}\right)^k \le (t_0 + \|\,c - w\,\|_2)\,\left(1 - \frac{\gamma}{6\rho(1+2\kappa)}\right)^k.$$

Given the definition of γ from (3.1) it follows that $\|\,xs - w\,\|_2 \le \varepsilon$ is satisfied for all $k \in I\!N$ such that

$$k \ge \frac{6(t_0 + \|\,c - w\,\|_2)(1+2\kappa)}{\min c}\log\frac{t_0 + \|\,c - w\,\|_2}{\varepsilon}\,.$$

∎

4. Conclusions. In this paper we introduced the notion of a sufficient linear wCP. Such a problem is defined by a triple of matrices (A, B, C), a right-hand side d, and a weight vector w. The definition of sufficiency involves only the triple (A, B, C). When $w = 0$ the problem reduces to a sufficient linear complementarity problem. We proved that if (A, B, C) is row sufficient then, for all weight vectors w and all right hand sides d, any KKT point of an associated optimization problem is a solution of the wCP. Conversely we proved that if there is a weight vector w such that, for all right hand sides d, any KKT point of the associated optimization problem is a solution of the wCP, then (A, B, C) is row sufficient. This gives a new characterization of row sufficiency. We also proved that if (A, B, C) is column sufficient then, for all weight vectors w and all right hand sides d, the wCP has a convex (perhaps empty) solution set. We gave a simple example of a wCP having a convex solution set for which the triple (A, B, C) is not column sufficient. Our results generalize some well known results from the theory of linear complementarity problems. The generalizations are not trivial as shown by the above mentioned counter example, and by the methods used for proving of our results.

We showed that if a sufficient linear wCP is strictly feasible then it is solvable. Moreover we showed that any strictly feasible sufficient linear wCP has a maximal complementarity solution and that the set of all such solutions form the relative interior of the solution set. Finally we proposed an interior-point method for solving sufficient linear wCPs. The algorithm does not depends on the handicap of the problem. The computational complexity is proportional with $1+\kappa$, where κ is the handicap of the problem. If $w = 0$ and the starting point is relatively well centered, in the sense that γ is bounded below by a positive constant, then the computational complexity of our algorithm is the same as the best known computational complexity for solving sufficient linear CP. For $\kappa = 0$, i.e. in the case of a monotone linear wCP, our algorithm has the same computational complexity as the two algorithms developed in [14] for solving such problems.

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