New Discoveries of Domination between Traffic Matrices

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Abstract

A traffic matrix D_1 dominates a traffic matrix D_2 if any capacity reservation supporting D_1 supports D_2 as well. We prove that D_3 dominates $D_3 + \lambda(D_2 - D_1)$ for any $\lambda \geqslant 0$ if D_1 dominates D_2 . By the property , it is pointed out that the domains supported by different traffic matrices are isomorphic on the extended concept of support. Besides, we offer an absolutely different way based on domination to help solve the Robust Network Design problem. Furthermore, we give the generalization to integral flows.

Keywords: robust network design, multicommodity flows, domination

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1. Introduction

Network design problem is a classical problem. Given a graph with capacity installation costs for the edges and a set of pairwise traffic demands-a traffic matrix, the problem consists of choosing minimum cost capacities such that all the demands can be routed simultaneously on the network.

However, it is too restrictive for the assumption that there is only one traffic matrix to be considered, and that this matrix is known in advance or can be reliably estimated. Unfortunately, demands can be difficult to predict. Moreover, in several applications, demands change over time. Therefore, for the sake of obtain a more flexible and robust network, it is necessary to take the demand uncertainty into consideration in the design process. A way to do that is to consider, instead of a single traffic matrix, a set of traffic matrices to be served non-simultaneously. This version of the problem is known as the

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Robust Network Design problem(RND) ,which has attracted a great deal of attention [1, 2, 3, 4, 5, 6].

In [7] the author introduces the concept of domination. It is said D_1 dominates D_2 if any capacity reservation supporting D_1 also supports D_2. The author points out that D_1 dominates D_2 if and only if D_1,regarded as a capacity reservation, supports D_2.

We extend the concept of domination by allowing the elements of the traffic matrices to have negative values. One of our main results is a good property for domination: Let D_1 ,D_2 and D_3 be three matrices, if D_1 dominates D_2 , then D_3 dominates $D_3 + \lambda(D_2 - D_1)$ for any $\lambda \geq 0$ (Theorem 3.1). To the best of our knowledge this simple but surprising result has not been observed before. The direct application of this property leads to another equally surprising result. Let $\mathcal{D}(U)$ be the set of all traffic matrices that dominated by a capacity reservation matrix U. We have proven that $\mathcal{D}(U_1)$ and $\mathcal{D}(U_2)$ are isomorphic.

Another result of this article is that we show the relationship between domination and the RND problem. It is pointed out that the RND problem is equivalent to problem to find a minimum cost capacity reservation matrix which dominates all the required traffic matrices. Moreover, by giving an algorithm we state briefly the potential value of domination to solve the RND problem.

We close by showing the generalization to integral flows.

2. Preliminaries

Though the results can directly extend to the directed case, we assume that an undirected network $G(V, E)$ is given throughout the whole paper for sake of simplicity. Without loss of generality, assume that $G(V, E)$ is simple and $|V| = n$.

A capacity reservation matrix U is a $n \times n$, non-negative, symmetric, real matrix, where $u(i, j)$ is the amount of capacity we install on each edge $e(i, j)$.

A traffic matrix D is a symmetric, real matrix of size $n \times n$, where $d(i, j)$is the amount of demand that is routed from node i and j; we assume $d(i, i) = 0$ for any i.

Given their symmetry, it is convenient to regard D and U as vectors in corresponding dimensions.

A routing F is a $n \times n \times n \times n$ non-negative matrix where f_{ijhk} is the fraction of a unit flow from i to j that is routed on the edge $e(h, k)$.If demand

$d(i,j)$ need be routed, it is routed by multiplying the unit flow by $d(i,j)$.

A capacity reservation matrix U and a routing F support a traffic matrix D if D can be routed on G with capacity U by F, that is, for each edge $e \in E$, $\sum_{i,j} f_{ij}(e)d(i,j) \leq u(e)$. It is said that U support D if there exists a routing F such that U and F support a traffic matrix .

Before giving our main result, a well-known theorem is given as follows:

Definition 2.1. *Suppose $G(V,E)$ be a graph, a function $\mu : E \to R$ is a metric on G if and only if:*

(i) $\mu \geqslant 0$ *for all* $e \in E$;
(ii) $\mu(l_e) \geqslant \mu$ *for each* $e \in E$ *, where* $\mu(l_e)$ *is the length of the shortest path between the endpoints of edge e by using* μ *as weights;*
(iii) $\sum_{e \in E} \mu(e) = 1$

Let Met_G be the polytope generated by all metrics on G.

Theorem 2.1 ([8, 9]). *Let $\hat{D}$ be a matrix of which all the elements are non-negative. A capacity matrix U supports a traffic matrix $\hat{D}$ if and only if* $\sum_{i,j} \mu(ij)u(i,j) \geq \sum_{i,j} \mu(ij)\hat{d}(i,j)$ *for any* $\mu \in Met_G$.

3. Domination

Definition 3.1. *Let D_1 and D_2 be two matrices of which every element of both is non-negative. D_1 dominates D_2 if any capacity reservation matrix U supporting D_1 also supports D_2 .*

In order to keep consistency and completeness, we allow the elements of the traffic matrices to have the negative value. The definitions of domination and support are extended as following:

Definition 3.2. *Let D_1 and D_2 be two matrices. D_1 dominates D_2 if $(D_1^+ - D_2^-)$ dominates $(D_2^+ - D_1^-)$; D_1 ,regarded as capacity reservation, supports D_2 if $(D_1^+ - D_2^-)$ supports $(D_2^+ - D_1^-)$ where*

$$D_1^+ : d_1^+(i,j) = \begin{cases} 0 & if\ d_1(i,j) \leqslant 0, \\ d_1(i,j) & if\ d_1(i,j) > 0. \end{cases}$$

$$D_1^- : d_1^-(i,j) = \begin{cases} 0 & if\ d_1(i,j) \geqslant 0, \\ d_1(i,j) & if\ d_1(i,j) < 0. \end{cases}$$

$$D_2^+ : d_2^+(i,j) = \begin{cases} 0 & if\ d_2(i,j) \leqslant 0, \\ d_2(i,j) & if\ d_2(i,j) > 0. \end{cases}$$

$$D_2^- : d_2^-(i,j) = \begin{cases} 0 & if\ d_2(i,j) \geqslant 0, \\ d_2(i,j) & if\ d_2(i,j) < 0. \end{cases}$$

In [7] a beautiful result is given as following:

Lemma 3.1. *U dominates D if and only if U ,regarded as capacity reservation matrix, supports D .*

Lemma 3.1 is also true under the extension of the definitions of domination and support. The proof is similar to that in [7]. We omit it here. Following is one of the main results in this paper.

Theorem 3.1. *Let D_1 , D_2 and D_3 be three matrices. If D_1 dominates D_2, then D_3 dominates $D_3 + \lambda(D_2 - D_1)$ for any $\lambda \geqslant 0$.*

Proof. The proof is firstly given for the case in which all the elements of D_1 , D_2 , D_3 and $D_3 + \lambda(D_2 - D_1)$ are non-negative.

Trivially, D_1 as a capacity reservation matrix supports D_1 as the traffic matrix. Hence D_1 must support D_2 too. By Theorem 2.1,

$$\sum_{i,j} \mu(ij) d_1(i,j) \geq \sum_{i,j} \mu(ij) d_2(i,j) \qquad \text{for any } \mu \in Met_G \tag{1}$$

that is,

$$\sum_{i,j} \mu(ij) d_1(i,j) - \sum_{i,j} \mu(ij) d_2(i,j) \geq 0 \qquad \text{for any } \mu \in Met_G \tag{2}$$

Let U be a capacity reservation matrix supporting D_3 .Again, by Theorem 2.1,

$$\sum_{i,j} \mu(ij) u(i,j) \geq \sum_{i,j} \mu(ij) d_3(i,j) \qquad \text{for any } \mu \in Met_G \tag{3}$$

therefore, based on (2) ,(3) and $\lambda \geq 0$,

$$\sum_{i,j} \mu(ij)u(i,j) \geq \sum_{i,j} \mu(ij)d_3(i,j) + \lambda(\sum_{i,j} \mu(ij)d_2(i,j) - \sum_{i,j} \mu(ij)d_1(i,j))$$

that is, U supports $D_3 + \lambda(D_2 - D_1)$ too.

Hence, D_3 dominates $D_3 + \lambda(D_2 - D_1)$ for any $\lambda \geqslant 0$ in this case.

For the case that the elements of D_1 ,D_2 ,D_3 and $D_3 + \lambda(D_2 - D_1)$ may be negative, it can transform into the case above. For convenience let $D_4 = D_3 + \lambda(D_2 - D_1)$ .By Definition 3.2, that D_3 dominates D_4 is equivalent to that $(D_3^+ - D_4^-)$ dominates $(D_4^+ - D_3^-)$. And we have

$$\begin{aligned} & D_4^+ - D_3^- = D_4 - D_4^- - D_3^- = D_3 + \lambda(D_2 - D_1) - D_4^- - D_3^- \\ & = D_3^+ + \lambda(D_2 - D_1) - D_4^- = D_3^+ - D_4^- + \lambda((D_2^+ + D_2^-) - (D_1^+ + D_1^-)) \\ & = D_3^+ - D_4^- + \lambda((D_2^+ - D_1^-) - (D_1^+ - D_2^-)) \end{aligned}$$

That D_3 dominates D_4 is equivalent to that $(D_3^+ - D_4^-)$ dominates $(D_3^+ - D_4^- + \lambda((D_2^+ - D_1^-) - (D_1^+ - D_2^-)))$. By Definition 3.2, that D_1 dominates D_2 is equivalent to that $(D_1^+ - D_2^-)$ dominates $(D_2^+ - D_1^-)$ .And all the elements of $(D_3^+ - D_4^-)$,$(D_2^+ - D_1^-)$ and $(D_1^+ - D_2^-)$ are non-negative, just as the first case.

Above all, the statement follows immediately. □

Denote that $\mathcal{D}(U)$ is the set of all traffic matrices that dominated by the capacity reservation matrix U . It is said that $\mathcal{D}(U_1)$ and $\mathcal{D}(U_2)$ are isomorphic if and only if there exists a vector $\overrightarrow{p}$ such that $\mathcal{D}(U_1) + \overrightarrow{p} \subseteq \mathcal{D}(U_2)$and $\mathcal{D}(U_2) - \overrightarrow{p} \subseteq \mathcal{D}(U_1)$. In other words, if $\mathcal{D}(U_1)$ can overlap with $\mathcal{D}(U_2)$completely by moving along a certain vector, they are said to be isomorphic.

Considering Theorem 3.1 we have the following corollary:

Corollary 3.1. *Let U_1 and U_2 be two capacity reservation matrices , then $\mathcal{D}(U_1)$ and $\mathcal{D}(U_2)$ are isomorphic.*

Proof. Let $\overrightarrow{p}$ be $(U_2 - U_1)$. Supposing$\bar{D} \in \mathcal{D}(U_1)$, we have that U_1 dominates $\bar{D}$. According to Theorem 3.1, U_2 dominates $U_2 + \bar{D} - U_1$, that is, $\bar{D} + \overrightarrow{p} \in \mathcal{D}(U_2)$. $\mathcal{D}(U_1) + \overrightarrow{p} \subseteq \mathcal{D}(U_2)$ follows immediately. Similarly, $\mathcal{D}(U_2) - \overrightarrow{p} \subseteq \mathcal{D}(U_1)$. □

By Lemma 3.1 and Corollary 3.1 , we have the following result:

Corollary 3.2. *Let $\mathfrak{D}(U)$ be the set of all traffic matrices that supported by the capacity reservation matrix U . U_1 and U_2 are two capacity reservation matrices , then $\mathfrak{D}(U_1)$ and $\mathfrak{D}(U_2)$ are isomorphic.*

In fact, $\mathfrak{D}(U)$ is polyhedral cone . According to Corollary 3.2, U ,as capacity reservation matrix, decides only the position of $\mathfrak{D}(U)$. Independent from U , the structure of $\mathfrak{D}(U)$ relies only on the topological structure of graph G. This result is unexpected and amazing; nevertheless, it is true.

4. An algorithm based on domination for RDN problem

Firstly we will give an introduction to the Robust Network Design problem. Given a graph with capacity installation costs for the edges and a set of traffic matrices, the problem includes choosing minimum cost capacities such that all the traffic matrices can be routed on the network non-simultaneously. Usually we assume the set of traffic matrices is a polyhedron, i.e. $\mathcal{D} = \{d \in R^{V\times V} \mid Ad \leq b\}$. By Lemma 3.1 we have the following lemma:

Lemma 4.1. *The RND problem is equivalent to problem to find a minimum cost capacity reservation matrix U such that $\mathcal{D} \subseteq \mathcal{D}(U)$.*

By Corollary 3.1, $\mathcal{D}(U_1)$ and $\mathcal{D}(U_2)$ are isomorphic for any two capacity reservation matrices. Suppose that we have known the mathematical expression of $\mathcal{D}(\bar{U}_1)$ for a certain capacity reservation matrix $\bar{U}_1$. By Lemma 4.1, we can move $\mathcal{D}(\bar{U}_1)$ along a certain vector $\overrightarrow{p}$ such that $\mathcal{D} \subseteq \mathcal{D}(\bar{U}_1 + \overrightarrow{p})$ to solve RND problem. The algorithm below offers a way to find $(\bar{U}_1 + \overrightarrow{p})$ such that $\mathcal{D} \subseteq \mathcal{D}(\bar{U}_1 + \overrightarrow{p})$.

Before giving the algorithm, another lemma is needed.

Lemma 4.2. *Suppose that for a certain capacity reservation matrix U_1 ,*

$$\mathcal{D}(U_1) = \{d \in R^{V\times V} \mid \bar{a}_i d \leq \bar{b}_i, i = 1, \cdots, K\},$$

then for any capacity reservation matrix U

$$\mathcal{D}(U) = \{d \in R^{V\times V} \mid \bar{a}_i d \leq \bar{a}_i U, i = 1, \cdots, K\}.$$

Proof. From the proof of Corollary 3.1, we can deduce that $\mathcal{D}(U) = \mathcal{D}(U_1) + U - U_1$. That is, $\mathcal{D}(U) = \{d \in R^{V\times V} \mid \bar{a}_i d \leq \bar{b}_i + \bar{a}_i(U - U_1), i = 1, \cdots, K\}$. In fact, $\bar{a}_i U_1 = \bar{b}_i, i = 1, \cdots, K$ for the reason that U_1 is the conical point of $\mathcal{D}(U_1)$ and it is on the hyperplane $\bar{a}_i d = \bar{b}_i, i = 1, \cdots, K$. The statement follows immediately. □

Let $c \in R^{V \times V}$ be the capacity installation costs for the edges where $c(i,j)$ represents the capacity installation cost on edge (i,j). Following an algorithm will be given to solve the RND problem.

Algorithm 1:

(i) Compute linear program $\Gamma_i, i = 1, \cdots, K$ where

$$\Gamma_i = \max \ \bar{a}_i d \tag{4}$$

$$s.t. \ \ Ad \leq b \tag{5}$$

(ii) Compute linear program $\Gamma(U)$ where

$$\Gamma(U) = \min \ cU \tag{6}$$

$$s.t. \ \ \bar{a}_i U \geq \Gamma_i \tag{7}$$

Theorem 4.1. $\Gamma(U)$*is the solution of the RND problem.*

Proof. According to the definition, $\mathcal{D} = \{d \in R^{V \times V} \mid Ad \leq b\}$. Based on Lemma 4.2, $\mathcal{D}(U) = \{d \in R^{V \times V} \mid \bar{a}_i d \leq \bar{a}_i U, i = 1, \cdots, K\}$ .And (4) and (7) guarantee that $\forall d \in \mathcal{D}, \bar{a}_i d \leq \Gamma_i \leq \bar{a}_i U$, that is, $\mathcal{D} \subseteq \mathcal{D}(U)$. By Lemma 4.1, the statement follows immediately. □

Algorithm 1 is based on the assumption in Lemma 4.2 that we have known the mathematical expression of $\mathcal{D}(U_1)$ for a certain capacity reservation matrix $\bar{U}_1$. However, the size of the systems describing $\mathcal{D}(\bar{U}_1)$ are not polynomially bounded in the size of the input. In fact, the RND problem is coNP-hard[3]. One way to solve this dilemma is to find a proper approximation of $\mathcal{D}(\bar{U}_1)$, thus by Algorithm 1 we can get a an approximation of the optimal solution of the RND problem . How to find such an approximation is beyond the scope of the article and it need further discussion. Despite all this, this character of domination offers an alternative way to help solve the RND problem .

5. Integral flows

It is said that a capacity reservation U (integral) supports a (integral) traffic demand D with integral flows if there exists a routing F such that F and U support D and $f_{ij}(a)d(ij)$ is integral, for each pair demand (i, j) and each edge $e \in E$.

If D_1 andD_2 are two integral traffic matrices, it is said that D_1 dominates D_2 with respect to integral flows if any capacity reservation U (integral)supporting D_1 with integral flows also supports D_2 with integral flows.

Similarly, we extend the concept by allowing the elements of the traffic matrices to have the negative value. The idea is analogous to Definition 2.2, so we omit it here. Without loss of generality, we assume $G(V, E)$ is simple and complete here. In [7], the following lemma is given:

Lemma 5.1. *D_1 dominates D_2 with respect to integral flows if and only if D_1 , regarded as a capacity reservation, supports D_2 with integral flows.*

For $h = 1, 2$, define $I(D_h) = \{(i, j) : d_h(ij) > 0\}$. Also let:

$$\bar{D} = \begin{cases} 0 & if\ (i,j) \notin I(D_1) \cap I(D_2), \\ \min(d_1(i,j), d_2(i,j)) & if\ (i,j) \in I(D_1) \cap I(D_2). \end{cases}$$

By Lemma 5.1, if D_1 dominates D_2 with respect to integral flows, D_1 , regarded as a capacity reservation, supports traffic matrix D_2 with integral flows. For $(i, j) \in I(D_1) \cap I(D_2)$, when $d_1(i, j)$ (capacity on edge (i, j)) and $d_2(i, j)$ (demand between i and j) reduce by a certain number(integral) simultaneously, the reduced capacity reservation still supports the reduced traffic matrix with integral flows. Then we have the following lemma:

Lemma 5.2. *If D_1 dominates D_2 with respect to integral flows, $D_1 - \bar{D}$ dominates $D_2 - \bar{D}$ with respect to integral flows*

We will give the following result:

Theorem 5.1. *Let D_1 , D_2 and D_3 be three matrices. If D_1 dominates D_2 with respect to integral flows , then D_3 dominates $D_3 + \lambda(D_2 - D_1)$ with respect to integral flows for any $\lambda \in Z^+$.*

Proof. For convenience let $D_4 = D_3+\lambda(D_2-D_1), D_5 = D_3-\lambda D_1+\lambda\bar{D}, D_6 = \lambda(D_1-\bar{D}), D_7 = \lambda(D_2-\bar{D})$ .The proof is firstly given for the case in which all the elements of D_1 , D_2 , $D_3$and D_4 are non-negative. We have the following identical equations:

$$D_3 = D_3 - \lambda D_1 + \lambda \bar{D} + \lambda(D_1 - \bar{D}) = D_5 + D_6 \quad (8)$$

$$D_4 = D_3 - \lambda D_1 + \lambda \bar{D} + \lambda(D_2 - \bar{D}) = D_5 + D_7 \quad (9)$$

We will prove the claim: all the elements of D_5 are non-negative. Suppose not, there exists (i, j) such that $d_5(i, j) < 0$. If $(i, j) \notin I(D_1) \cap I(D_2)$, there is at least one of $d_1(i, j)$ and $d_2(i, j)$ which is zero .That is, there is at least one of $d_6(i, j)$ and $d_7(i, j)$ which is zero. If $(i, j) \in I(D_1) \cap I(D_2)$, according to the definition of $\bar{D}$, there is also at least one of $d_6(i, j)$ and $d_7(i, j)$ which is zero. When one of $d_6(i, j)$ and $d_7(i, j)$ is zero,by (8),(9) ,there is at least one of $d_3(i, j)$ and $d_4(i, j)$ which is less than zero. This contradicts with the assumption that all the elements of D_1, D_2, D_3 and D_4 are non-negative.

By Lemma 5.2, D_6 dominates D_7. That is, D_6 , regarded as capacity reservation, supports D_7 with respect to integral flows. By formulation (8)and (9), D_3 , regarded as capacity reservation, supports D_4 with respect to integral flows (Just let demand $d_5(i, j)$ rout directly on edge (i, j)). By Lemma 5.1,the statement follows immediately.

It can be proved for other cases by applying the same transforming technique as shown in the proof of Theorem 3.1. □

6. Conclusions

We show a vital property of domination that if D_1 dominates D_2 , then D_3 dominates $D_3 + \lambda(D_2 - D_1)$ for any $\lambda \geq 0$. And it is firstly pointed out that the RND problem is equivalent to problem to find a minimum cost capacity reservation matrix U which dominates all the required traffic matrices. Furthermore, by giving a simple but significant algorithm, we offer an entirely different way to solve the RND problem. It need further research for the algorithm in order to apply in practice. We also show the generalization to integral flows.

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