Improvement of Kalai-Kleitman bound for the diameter of a polyhedron

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Abstract

Recently, Todd got a new bound on the diameter of a polyhedron using an analysis due to Kalai and Kleitman in 1992. In this short note, we prove that the bound by Todd can further be improved. Although our bound is not valid when the dimension is 1 or 2, it fits better for a high-dimensional polyhedron with a large number of facets.

Keyword: Polytopes, Diameter, Kalai and Kleitman bound.

1 Introduction

Let $P \subseteq \mathbb{R}^d$ be a d-dimensional polyhedron. We say that F is a *facet* of P if there is a half-space $H = \{x \in \mathbb{R}^d : a^\top x \leq \beta\}$ with $F = H \cap P$, and the dimension of F is $d-1$. A *vertex* of P is a point $v \in \mathbb{R}^d$ such that $\{v\} = H \cap P$ holds for some half-space H. For a pair u, v of vertices of P, define $[u, v] := \{(1-\lambda)u + \lambda v : \ 0 \leq \lambda \leq 1\}$. If $[u, v] = H \cap P$ holds for some half-space H, we say that u and v are *adjacent*.

The 1-skeleton of P is an undirected graph $G = (V, E)$ such that V is a set of vertices of P and $\{u, v\} \in E$ if and only if u and v are adjacent on P. Let $\rho_P(u, v)$ denote the length of the shortest path from u to v in G. The *diameter* of P is defined as

$$\delta(P) = \max\{\rho_P(u, v) : u \text{ and } v \text{ are vertices of } P\}.$$

Let $\Delta(d, n)$ be the maximum of $\delta(P)$ over all d-dimensional polyhedra with n facets. It is known that the maximum $\Delta(d, n)$ can be attained by a

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simple polyhedron, that is, each vertex lies in exactly d facets (see Klee and Kleinschmidt [5] or Ziegler [9]). The well-known Hirsch conjecture states that $\Delta_b(d,n) \le n-d$, where $\Delta_b(d,n)$ is the value $\Delta(d,n)$ for bounded polyhedra.

In 1992, Kalai and Kleitman [3] showed that $\Delta(d,n) \le n^{\log d+2}$. Later, this bound $n^{\log d+2}$ is slightly improved to $n^{\log d+1}$ in Kalai [2]. Recently, Todd [8] got a new upper bound $(n-d)^{\log d}$ utilizing an analysis in Kalai and Kleitman [3]. Here, all logarithms are to base 2. For the history concerning the value $\Delta(d,n)$, see Santos [7] for instance.

In this short note, we prove the following.

Theorem 1. $\Delta(d,n) \le (n-d)^{\log(d-1)}$ *for* $n \ge d \ge 3$.

It should be noted that our bound $(n-d)^{\log(d-1)}$ in Theorem 1 is not necessarily valid when $d=1$ and 2, while the bound $(n-d)^{\log d}$ by Todd [8] is valid and tight when $d=1$ and 2 in a sense. However, our bound $(n-d)^{\log(d-1)}$ is valid and tight when $d=3$ in a sense as our bound coincides with the true value $\Delta(3,n) = \lfloor (2/3)n \rfloor - 1$ at $n=6$, see Klee [4] for the details.

Also, when n is slightly larger than d, we could further improve the above bound in Theorem 1.

Theorem 2. $\Delta(d,n) \le (n-d-1)^{\log(d-1)}$ *for* $d=3$ *and* $n \ge 7$, $d=4$ *and* $n \ge 9$, *and for* $d \ge 5$ *and* $n \ge d+3$ *in general.*

It is not difficult to see that this bound $(n-d-1)^{\log(d-1)}$ is not valid when $d \le n \le d+2$ for any $d \ge 4$ in general. On the other hand, we observe that the bound $(n-d-1)^{\log(d-1)}$ keeps the symmetry between d and $n-d$.

2 Proof of Theorem 1

As in Todd [8], the following lemma will be the key of our proof.

Lemma 1 (Kalai and Kleitman [3])**.** *For* $2 \le d \le \lfloor n/2 \rfloor$, *where* $\lfloor n/2 \rfloor$ *is the largest integer at most* $n/2$, *we have*

$$\Delta(d,n) \le \Delta(d-1,n-1) + 2\Delta(d,\lfloor n/2 \rfloor) + 2. \tag{1}$$

Bellow, assume without loss of generality that the polyhedra is simple. In order to show the validity of our bound $(n-d)^{\log(d-1)}$ for general d and n with $n \ge d \ge 3$, we show an inductive step for $d \ge 5$ and $n-d \ge 8$.

To this end, we need to check the validity of the bound $(d-1)^{\log(n-d)}$ on the base cases when (a) $d=4$ with $4 \le n$, (b) $d=5$ with $10 \le n \le 12$, (c) $d=6$ with $12 \le n \le 13$, and (d) $d=7$ with $n=14$. Here, we utilize the fact that $\Delta(d-1,n-1)$ is an upper bound on the diameter of d-polytopes with $n < 2d$ facets, since in this case, any two vertices lie on the same facet.

Table 1: Comparison of the four bounds for small n and d, where values after the decimal point are omitted.

	$d=4$						
n	6	7	8	9	10	11	12
$\tilde{\Delta}(d,n)$	2	3	6	7	10	11	14
$(n-d-1)^{\log(d-1)}$	-	-	-	9	12	17	21
$(n-d)^{\log(d-1)}$	3	5	9	12	17	21	27
$(n-d)^{\log(d)}$	4	9	16	25	36	49	64
$n^{\log(d)+2}$	1296	2401	4096	6561	10000	14641	20736

	$d=5$					
n	8	9	10	11	12	13
$\tilde{\Delta}(d,n)$	3	6	9	12	15	18
$(n-d-1)^{\log(d-1)}$	4	9	16	25	36	49
$(n-d)^{\log(d-1)}$	9	16	25	36	49	64
$(n-d)^{\log(d)}$	12	25	41	64	91	125
$n^{\log(d)+2}$	7999	13309	20985	31682	46167	65220

	$d=6$			$d=7$		$d=8$
n	12	13	14	14	15	16
$\tilde{\Delta}(d,n)$	14	17	22	19	24	26
$(n-d-1)^{\log(d-1)}$	41	64	91	102	152	235
$(n-d)^{\log(d-1)}$	64	91	125	152	216	343
$(n-d)^{\log(d)}$	102	152	216	235	343	512
$n^{\log(d)+2}$	88716	128052	179868	323477	450700	1048576

For comparison, in Table 1, we show the values of an upper bound $\tilde{\Delta}(d,n)$, the Kalai-Kleitman bound $n^{\log(d)+2}$, the Todd's bound $(n-d)^{\log(d)}$, and our bound $(n-d)^{\log(d-1)}$ for small d and n. The upper bound $\tilde{\Delta}(d,n)$ for $\Delta(d,n)$ is computed from

- The true bound $\Delta(3,n) \leq n-3$,
- Kalai-Kleitman inequality (1)
- $\Delta(d,n) \leq \Delta(d-1,n-1),\ n < 2d$, and
- $\Delta(d,d) = 0$ for any d.

We observe that (b), (c), and (d) are immediate from Table 1. To see (a), one could use the fact that $\Delta(d,n) \leq n2^{d-3}$ for $n > d \geq 3$ by Larman [6]. It is easy to see that $2n \leq (n-4)^{\log 3}$ for $n \geq 13$. On the other hand, we observe that $\Delta(4,n) \leq (n-4)^{\log 3}$ for $6 \leq n \leq 12$ from Table 1.

Since $(n-d)^{\log(d-1)} = (d-1)^{\log(n-d)}$, below, we deal with $(d-1)^{\log(n-d)}$ instead of $(n-d)^{\log(d-1)}$. Now, suppose that $d \geq 5$ and $n-d \geq 8$, which implies $\log(n-d) \geq 3$. Then, from Lemma 1, $\Delta(d,n)$ is bounded above by

$$
\begin{aligned}
& \Delta(d-1,n-1) + 2\Delta(d,\lfloor n/2 \rfloor) + 2 \\
\leq\ & (d-2)^{\log(n-d)} + 2(d-1)^{\log(n/2-d)} + 2 \\
\leq\ & \left(\frac{d-2}{d-1}\right)^{\log(n-d)} (d-1)^{\log(n-d)} + 2(d-1)^{\log((n-d)/2)} + 2 \\
\leq\ & \left(\frac{d-2}{d-1}\right)^{3} (d-1)^{\log(n-d)} + \frac{2}{d-1}(d-1)^{\log(n-d)} + 2 \\
=\ & \left[1 - \frac{3}{d-1} + \frac{3}{(d-1)^2} - \frac{1}{(d-1)^3} + \frac{2}{d-1}\right] (d-1)^{\log(n-d)} + 2 \\
\leq\ & \left[1 - \frac{12}{4(d-1)} + \frac{3}{4(d-1)} + \frac{8}{4(d-1)} - \frac{1}{(d-1)^3}\right] (d-1)^{\log(n-d)} + 2 \\
\leq\ & (d-1)^{\log(n-d)} - \frac{1}{4(d-1)}(d-1)^{\log(n-d)} - \frac{1}{(d-1)^3}(d-1)^{\log(n-d)} + 2 \\
\leq\ & (d-1)^{\log(n-d)},
\end{aligned}
$$

which completes the proof of Theorem 1.

3 Proof of Theorem 2

First of all, when $d=3$, we observe that the bound $(n-d-1)^{\log(d-1)}$ in Theorem 2 is valid for $n \geq 7$ as $\Delta(3,n) = \lfloor (2/3)n \rfloor - 1$ [4]. Also, when $d=4$, using the bound $\Delta(4,n) \leq (4/3)n$ by Barnette [1] and the values in Table 1, we observe that the bound $(n-d-1)^{\log(d-1)}$ in Theorem 2 is valid for $n \geq 9$. Now, assume that $d \geq 5$, $n \geq d+3$, and $n-d-1 \geq 8$, which implies $\log(n-d-1) \geq 3$. Slightly modifying the argument in the proof in Theorem 1 above, we have the following.

$$
\begin{aligned}
& \Delta(d-1,n-1) + 2\Delta(d,\lfloor n/2 \rfloor) + 2 \\
\leq\ & (d-2)^{\log(n-d-1)} + 2(d-1)^{\log(n/2-d)} + 2 \\
\leq\ & \left(\frac{d-2}{d-1}\right)^{\log(n-d-1)} (d-1)^{\log(n-d-1)} + 2(d-1)^{\log((n-d-1)/2)} + 2 \\
\leq\ & \left(\frac{d-2}{d-1}\right)^{3} (d-1)^{\log(n-d-1)} + \frac{2}{d-1}(d-1)^{\log(n-d-1)} + 2 \\
=\ & \left[1 - \frac{3}{d-1} + \frac{3}{(d-1)^2} - \frac{1}{(d-1)^3} + \frac{2}{d-1}\right] (d-1)^{\log(n-d-1)} + 2 \\
\leq\ & \left[1 - \frac{12}{4(d-1)} + \frac{3}{4(d-1)} + \frac{8}{4(d-1)} - \frac{1}{(d-1)^3}\right] (d-1)^{\log(n-d-1)} + 2 \\
\leq\ & (d-1)^{\log(n-d-1)} - \frac{1}{4(d-1)}(d-1)^{\log(n-d-1)} - \frac{1}{(d-1)^3}(d-1)^{\log(n-d-1)} + 2 \\
\leq\ & (d-1)^{\log(n-d-1)}.
\end{aligned}
$$

A few explanations are in order. To prove the first inequality, we use the fact that $n \geq d+3$ implies $(n-1) \geq (d-1)+3$ for the first term. From this fact and the induction assumption, we can use the inequality in Theorem 2. We also use Theorem 1 for the second term. As in Theorem 1, the correctness of the rest of the base cases can be confirmed from Table 1. Thus the proof is completed.

4 Discussion

Although our improvement may seem trivial, it should be mentioned that for fixed d, $(n-d)^{\log(d)}/(n-d)^{\log(d-1)} \to \infty$ as $n \to \infty$. However, as you see from Table 1, there still remains a large gap between $\tilde{\Delta}(d,n)$ and $(n-d)^{\log(d-1)}$. We observe this tendency for various values of d and n. We believe that by refining the analysis of the Kalai-Kleitman inequality (1), a better bound for the diameter of the polyhedron could be obtained.

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