Rectangular sets of probability measures

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Abstract. In this paper we consider the notion of rectangularity of a set of probability measures, introduced in Epstein and Schneider [4], from a somewhat different point of view. We define rectangularity as a property of dynamic decomposition of a distributionally robust stochastic optimization problem and show how it relates to the modern theory of coherent risk measures. Consequently we discuss robust formulations of multistage stochastic optimization problems in frameworks of Stochastic Programming, Stochastic Optimal Control and Markov Decision Processes.

Key Words. Multistage stochastic optimization, rectangularity, risk averse optimization, robust optimal control and Markov decision processes, dynamic programming, coherent risk measures, time consistency.

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1 Introduction

In an abstract form static formulation of a risk neutral stochastic optimization problem can be written as

$$\min_{x\in\mathcal{X}} \mathbb{E}_P[G(x,\xi)]. \tag{1.1}$$

Here $\mathcal{X}$ is a subset of $\mathbb{R}^n$ representing set of feasible decisions, $\xi \in \mathbb{R}^d$ is a random vector having probability distribution P and $G : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ is an objective function. In this formulation vector ξ of uncertain parameters is modeled as a random variable with a specified probability distribution P (with some abuse of the notation we use the same notation for random vector ξ and for its realization, a particular meaning will be clear from the context). Of course in reality the "true" distribution of ξ is never known exactly. In fact the concept of "probability distribution" is a mathematical abstraction and its application in some situations can be questionable.

This gave a motivation for considering the so-called *distributionally robust* approach to stochastic optimization. In that approach a family $\mathfrak{M}$ of probability measures (probability distributions) of the vector of uncertain parameters is specified and problem (1.1) is replaced by

$$\min_{x\in\mathcal{X}} \sup_{Q\in\mathfrak{M}} \mathbb{E}_Q[G(x,\xi)]. \tag{1.2}$$

Such worst-case approach has a long history, in stochastic programming it goes back at least to Žáčková [20]. There are basically two approaches to constructing the uncertainty set $\mathfrak{M}$ of probability measures which have been used in recent literature. In one approach the set $\mathfrak{M}$ is defined by moment constraints and may be by some other qualitative type properties. In another approach a reference probability measure P is specified and a set of probability measures in some sense close to P is considered. In particular, if the set $\mathfrak{M}$ consists of all probability measures supported on a set $\Xi \subset \mathbb{R}^d$, then the maximum in (1.2) is attained at a probability measure of mass one at some point of Ξ. In that case problem (1.2) becomes the min-max problem $\min_{x\in\mathcal{X}} \sup_{\xi\in\Xi} G(x,\xi)$.

The situation becomes considerably more involved in a dynamical setting when decisions are made in stages based on information available at the time of the decision. Straightforward extension of the min-max formulation (1.2) to a dynamical setting is problematic. In general it does not have a property of time consistency and is not amendable to writing dynamic programming equations. There are different approaches to a precise definition of the time consistency concept and its detailed discussion is beyond the scope of this paper. It seems that there is a general consensus that problems which are decomposable into an appropriate nested form are time consistent, we will discuss this later.

The notion of rectangularity of a set of probability measures was introduced in Epstein and Schneider [4]. In that paper a development of the concept of rectangularity was motivated by enforcing time (dynamic) consistency of updating rules of recursive Bayesian priors. Consequently the notion of rectangularity was used in formulation of robust time consistent Markov Decision Processes in Iyengar [6] and, in an implicit form, in Nilim and El Ghaoui [8], and more recently in Wiesemann, Kuhn and Rustem [19] where additional references can be found.

One of the aims of this paper is to clarify the concept of rectangularity from a somewhat different point of view. We define rectangularity as a property of dynamic decomposition of distributionally robust stochastic optimization problems, and show how it relates to the specific construction used in [4] and [6]. We also show a relation of this concept to the modern theory of coherent risk measures. On one hand this will simplify understanding of the basic property of this concept, on the other hand we will argue that a particular construction used in [4] and

[6] is just one possible way to define the relevant decomposability property. We also argue that restricting analysis to considering probability measures given by direct products of time individual probability measures is not essential.

It is possible to identify three communities; namely, Stochastic Programming, Stochastic Optimal Control and Markov Decision Processes, dealing with similar classes of problems but using somewhat different modeling approaches and different terminology. An additional goal of this paper is to present a unified framework for these approaches to multistage stochastic optimization.

This paper is organized as follows. In the next section we introduce the basic concept of rectangularity and discuss its properties. In section 3 we show its relation to the modern theory of coherent risk measures. In section 4 we briefly outline three standard approaches to formulation of multistage stochastic optimization problems, namely: Stochastic Programming (SP), Stochastic Optimal Control (SOC) and Markov Decision Processes (MDP). In section 5 we formulate distributionally robust versions of the SP, SOC and MDP approaches. Finally in the appendix section we discuss some technical details of the interchangeability principle in the spaces of continuous functions.

2 Rectangularity

In order to avoid some technical details it is tempting to restrict the analysis to a finite dimensional setting when the corresponding data process can be represented by a finite scenario tree. However, this will eliminate important examples of continuous distributions. Therefore we consider the following setting. Let $\Xi = \Xi_1 \times \cdots \times \Xi_T \subset \mathbb{R}^d$ with Ξ_t being a nonempty closed subset of $\mathbb{R}^{d_t}$, $t = 1, ..., T$. Let $\mathcal{B}$ be the Borel sigma algebra of Ξ and $\mathfrak{M}$ be a set of probability measures on $(\Xi, \mathcal{B})$. Furthermore consider a measurable function $Z : \Xi \to \mathbb{R}$. One can think about $(\xi_1, ..., \xi_T) \in \Xi$ as a data process having probability distribution Q, from the specified family $\mathfrak{M}$, and $Z = Z(\xi_1, ..., \xi_T)$ as an objective (say cost) value for a specified policy of a considered multistage optimization (minimization) problem. By $\xi_{[t]} := (\xi_1, ..., \xi_t)$ we denote history of the data process up to time t. The notation $\mathbf{1}_A$ is used for the indicator function of set A.

We assume that the objective function belongs to a vector (linear) space $\mathcal{Z}$ of measurable functions $Z : \Xi \to \mathbb{R}$. One can think about $\mathcal{Z}$ as a set of allowable objective functions. If the set Ξ is finite, then the sigma algebra $\mathcal{B}$ consists of all subsets of Ξ, and the space $\mathcal{Z}$ consists of all functions $Z : \Xi \to \mathbb{R}$ and can be identified with $\mathbb{R}^K$, where K is the cardinality of the set Ξ. As it was pointed above in that case the analysis is simplified. Wherever possible we will try to give an intuition appealing to this finite dimensional case.

In general there are two natural ways to define such relevant spaces $\mathcal{Z}$. One setting assumes existence of a reference probability measure P on $(\Xi, \mathcal{B})$ and defines $\mathcal{Z}$ to be the space of random variables $Z : \Xi \to \mathbb{R}$ with finite p-th order moments. That is, $\mathcal{Z} := L_p(\Xi, \mathcal{B}, P)$, $p \in [1, \infty)$. The space $L_p(\Xi, \mathcal{B}, P)$ equipped with norm $\|Z\|_p = \left(\int |Z|^p dP\right)^{1/p}$ becomes a Banach space with dual space $\mathcal{Z}^* = L_q(\Xi, \mathcal{B}, P)$, where $q \in (1, \infty]$ is such that $1/p + 1/q = 1$. For $Z \in \mathcal{Z}$ and $\zeta \in \mathcal{Z}^*$ the respective scalar product is defined as $\langle Z, \zeta \rangle = \int_\Xi Z\zeta dP$. In that framework the set $\mathfrak{M}$ consists of probability measures Q absolutely continuous with respect to the reference probability measure P and such that $dQ/dP \in \mathcal{Z}^*$. In that sense we identify probability measure Q with the respective density $\zeta = dQ/dP$. The corresponding scalar product can be then written as

$$\langle Z, Q \rangle = \int_\Xi Z dQ = \int_\Xi Z\zeta dP, \quad Z \in \mathcal{Z}, \ \zeta \in \mathcal{Z}^*.$$

The above framework does not cover, however, the important cases where probability measures of the uncertainty set $\mathfrak{M}$ are defined by moment constraints. In order to cover such cases we also consider the setting where the set Ξ is assumed to be compact and the space $\mathcal{Z}$ consists of continuous functions $Z : \Xi \to \mathbb{R}$, denoted $\mathcal{Z} := C(\Xi)$. Equipped with the sup-norm the space $\mathcal{Z}$ becomes a Banach space. By a representation theorem (due to Riesz) its dual space $\mathcal{Z}^*$ is formed by the linear space of finite signed Borel measures on $(\Xi, \mathcal{B})$. For $Z \in \mathcal{Z}$ and $Q \in \mathcal{Z}^*$ the corresponding scalar product is defined as $\langle Z, Q\rangle = \int_\Xi Z dQ$.

Unless stated otherwise we make the following assumptions.

(i) *We assume that the space $\mathcal{Z}$ is either $\mathcal{Z} := L_p(\Xi, \mathcal{B}, P)$, $p \in [1, \infty)$, or $\mathcal{Z} := C(\Xi)$. In both cases we have that $\langle Z, Q\rangle = \int_\Xi Z dQ$, and that $\langle Z, Q\rangle = \mathbb{E}_Q[Z]$ if $Q \in \mathcal{Z}^*$ is a probability measure.*

(ii) *We assume that the set $\mathfrak{M} \subset \mathcal{Z}^*$ is* bounded *in the norm topology of the space $\mathcal{Z}^*$.*

Note that there is natural partial order $Z \succeq Z'$ on $\mathcal{Z}$, defined pointwise, and that $\mathcal{Z}$ is a Banach lattice with $(Z \vee Z')(\cdot) = \max\{Z(\cdot), Z'(\cdot)\}$, $Z, Z' \in \mathcal{Z}$.

If the set Ξ is finite of cardinality K, then both $\mathcal{Z}$ and its dual $\mathcal{Z}^*$ can be identified with $\mathbb{R}^K$ and we can use the standard scalar product on $\mathbb{R}^K$. Also in that case a probability measure Q can be identified with vector $q \in \Delta_K$, where $\Delta_K := \{q \in \mathbb{R}^K : q_1 + ... + q_K = 1,\ q \geq 0\}$. Since the set Δ_K is bounded, it follows that $\mathfrak{M}$ is bounded. In the setting of $\mathcal{Z} = C(\Xi)$, for any probability measure $Q \in \mathcal{Z}^*$ its dual norm satisfies $\|Q\|^* = 1$ (see Section 6.1 of the Appendix). Therefore in that case any set $\mathfrak{M} \subset \mathcal{Z}^*$ of probability measures is bounded. The assumption (ii) is relevant for the case of $\mathcal{Z} := L_p(\Xi, \mathcal{B}, P)$, $p \in [1, \infty)$, we will discuss this later.

In the subsequent analysis we will have to deal with the weak* topology of the dual space $\mathcal{Z}^*$. This is the weakest topology in which all linear functionals on $\mathcal{Z}^*$ of the form $\ell(\zeta) := \langle \zeta, Z\rangle$, $Z \in \mathcal{Z}$, are continuous. We will use the following important properties of the weak* topology. By the Banach-Alaoglu theorem a set $\mathcal{A} \subset \mathcal{Z}^*$ is weakly* compact iff it is weakly* closed and bounded. If the Banach space $\mathcal{Z}$ is separable (i.e., has a countable dense subset) and a set $\mathcal{A} \subset \mathcal{Z}^*$ is bounded, then the weak* topology of $\mathcal{A}$ is metrizable and separable. Note that in the considered framework the space $\mathcal{Z}$ is separable. If the space $\mathcal{Z} = \mathbb{R}^K$ is finite dimensional, then $\mathcal{Z}^*$ can be identified with $\mathbb{R}^K$, and the weak and weak* topologies do coincide with the standard topology of $\mathbb{R}^K$.

Since $\mathfrak{M} \subset \mathcal{Z}^*$, the integral (expected value) $\mathbb{E}_Q[Z] = \int_\Xi Z dQ$ is well defined and finite valued for every $Z \in \mathcal{Z}$ and $Q \in \mathfrak{M}$. By $\mathbb{E}_{Q|\xi_{[t]}}[Z]$ we denote the conditional expectation of Z with respect to $Q \in \mathfrak{M}$ given $\xi_{[t]}$. We also can write this conditional expectation as $\mathbb{E}_{Q|\mathcal{F}_t}[Z]$, where $\mathcal{F}_t$ is the sigma algebra $\sigma(\xi_1, ..., \xi_t)$.

For $Z \in \mathcal{Z}$ consider the problem

$$\operatorname*{Max}_{Q \in \mathfrak{M}} \mathbb{E}_Q[Z]. \tag{2.1}$$

The expectation operator has the following property

$$\mathbb{E}_Q[Z] = \mathbb{E}_Q\left[\mathbb{E}_{Q|\xi_{[1]}}\left[\ \cdots\ \mathbb{E}_{Q|\xi_{[T-1]}}[Z]\right]\right]. \tag{2.2}$$

It follows that (cf., [16])

$$\sup_{Q \in \mathfrak{M}} \mathbb{E}_Q[Z] \leq \sup_{Q \in \mathfrak{M}} \mathbb{E}_Q\left[\sup_{Q \in \mathfrak{M}} \mathbb{E}_{Q|\xi_{[1]}}\left[\ \cdots \sup_{Q \in \mathfrak{M}} \mathbb{E}_{Q|\xi_{[T-1]}}[Z]\right]\right]. \tag{2.3}$$

Of course we should be careful in verifying that the right hand side of (2.3) is well defined. The last term $Y := \sup_{Q\in\mathfrak{M}} \mathbb{E}_{Q|\xi_{[T-1]}}[Z]$ in (2.3) is a function of $\xi_{[T-1]}$. In order for the next term

$\sup_{Q\in\mathfrak{M}} \mathbb{E}_{Q|\xi_{[T-2]}}[Y]$ to be well defined it should be verified that $Y = Y(\xi_{[T-1]})$ is measurable and finite valued for a.e. $\xi_{[T-1]}$, and so on moving backward in time. The left hand side of (2.3) is finite for all $Z \in \mathcal{Z}$ iff the set $\mathfrak{M} \subset \mathcal{Z}^*$ is bounded. Therefore boundedness of $\mathfrak{M}$ is a necessary condition for the right hand side of (2.3) to be finite for all $Z \in \mathcal{Z}$. For a further discussion of this issue see Section 6.2 of the Appendix. In the finite dimensional setting the measurability holds automatically. In general, since $Q \in \mathfrak{M}$ is an element of the dual space $\mathcal{Z}^*$, we have that $\mathbb{E}_Q[Z] = \langle Z, Q\rangle$ is continuous in Q with respect to the weak* topology of $\mathcal{Z}^*$. In the considered setting the space $\mathcal{Z}$ is separable, and hence the weak* topology is metrizable and separable on the bounded set $\mathfrak{M} \subset \mathcal{Z}^*$. Therefore it suffices to take the suprema in the right hand side of (2.3) over a countable subset of $\mathfrak{M}$. This takes care of the measurability question.

Note that in the term $\sup_{Q\in\mathfrak{M}} \mathbb{E}_{Q|\xi_{[T-2]}}[Y]$ it suffices to take the maximum over the corresponding set of marginal probability measures (distributions) of $\xi_{[T-1]}$, and so on going backward in time. We will discuss this further later.

Definition 2.1 *Let $\mathfrak{Z}$ be a subset of the space $\mathcal{Z}$. We say that a set $\widehat{\mathfrak{M}}$ of probability measures on $(\Xi, \mathcal{B})$ is a* rectangular *set, associated with the sets $\mathfrak{M}$ and $\mathfrak{Z}$, if*

$$\sup_{Q\in\widehat{\mathfrak{M}}} \mathbb{E}_Q[Z] = \sup_{Q\in\mathfrak{M}} \mathbb{E}_Q\left[\sup_{Q\in\mathfrak{M}} \mathbb{E}_{Q|\xi_{[1]}}\Big[\cdots \sup_{Q\in\mathfrak{M}} \mathbb{E}_{Q|\xi_{[T-1]}}[Z]\Big]\right], \;\; \forall Z \in \mathfrak{Z}. \tag{2.4}$$

In particular, if (2.4) holds for $\widehat{\mathfrak{M}} = \mathfrak{M}$ we say that the set $\mathfrak{M}$ is rectangular (with respect to $\mathfrak{Z}$).

This definition differs in several ways from the approach used in [4] and [6]. We define rectangularity by the decomposability property (2.4) rather than by a specific construction of the set $\widehat{\mathfrak{M}}$. Note the set $\widehat{\mathfrak{M}}$ is not defined uniquely by the equation (2.4). We also do not assume that the set $\mathfrak{M}$ is of a product form (see equations (2.13) and (2.14) below).

As we will see it happens that $\mathfrak{M}$, in itself, is rectangular with respect to the whole space $\mathcal{Z}$ in rather exceptional cases. The restricted set $\mathfrak{Z}$ could represent our a priori knowledge about generic form of considered objective functions. Of course, if $\widehat{\mathfrak{M}}$ is rectangular with respect to the whole set $\mathcal{Z}$, then it is rectangular with respect to its any subset $\mathfrak{Z}$.

It is not difficult to see that a sufficient condition for the set $\mathfrak{M}$ to be rectangular is that for every $Z \in \mathfrak{Z}$ all maxima in the right hand side of (2.3) are attained at some probability measure $\bar{Q} \in \mathfrak{M}$ independent of the respective $\xi_{[t]}$. Indeed in that case

$$\sup_{Q\in\mathfrak{M}} \mathbb{E}_Q[Z] \geq \mathbb{E}_{\bar{Q}}[Z] = \sup_{Q\in\mathfrak{M}} \mathbb{E}_Q\left[\sup_{Q\in\mathfrak{M}} \mathbb{E}_{Q|\xi_{[1]}}\Big[\cdots \sup_{Q\in\mathfrak{M}} \mathbb{E}_{Q|\xi_{[T-1]}}[Z]\Big]\right], \tag{2.5}$$

which together with (2.3) implies that the equality in (2.3) holds. This is the case discussed in Example 2 below.

Immediate questions are existence and uniqueness of such rectangular set $\widehat{\mathfrak{M}}$. We will show now that indeed, under quite general conditions, such rectangular set exists, although it is not unique. Also it is natural to ask that if, for a given space $\mathcal{Z}$, a set $\widehat{\mathfrak{M}}$ is a rectangular set with respect to a set $\mathfrak{M}$, then whether $\widehat{\mathfrak{M}}$ is rectangular with respect to itself. As we will see this is not true in general.

Theorem 2.1 *Suppose that the right hand side of (2.3) is finite for all $Z \in \mathcal{Z}$. Then there exists a bounded set $\widehat{\mathfrak{M}} \subset \mathcal{Z}^*$ of probability measures which is rectangular with respect to $\mathfrak{M}$ and $\mathcal{Z}$.*

Proof. Consider the following function

$$\rho(Z) := \sup_{Q\in\mathfrak{M}} \mathbb{E}_Q\left[\sup_{Q\in\mathfrak{M}} \mathbb{E}_{Q|\xi_1}\Big[\cdots \sup_{Q\in\mathfrak{M}} \mathbb{E}_{Q|\xi_{[T-1]}}[Z]\Big]\right],\ Z\in\mathcal{Z}. \tag{2.6}$$

By the assumption the function $\rho:\mathcal{Z}\to\mathbb{R}$ is finite (real) valued. It is straightforward to verify that it has the following properties.

(A1) Subadditivity: $\rho(Z+Z')\le\rho(Z)+\rho(Z')$.

(A2) Monotonicity: $\rho(Z)\ge\rho(Z')$ for any $Z,Z'\in\mathcal{Z}$ such that $Z\succeq Z'$.

(A3) Translation equivariance: $\rho(Z+a)=\rho(Z)+a$ for any $Z\in\mathcal{Z}$ and $a\in\mathbb{R}$:

(A4) Positive homogeneity: $\rho(\lambda Z)=\lambda\rho(Z)$ for any $Z\in\mathcal{Z}$ and $\lambda\ge 0$.

Functionals $\rho:\mathcal{Z}\to\mathbb{R}$ satisfying the above conditions (axioms) (A1)–(A4) are called *coherent risk measures* (Artzner, Delbaen, Eber and Heath [1]). Properties (A1) and (A4) imply that the functional ρ is convex. Moreover, convexity and monotonicity of $\rho:\mathcal{Z}\to\mathbb{R}$ imply that ρ is continuous in the norm topology of the Banach space (lattice) $\mathcal{Z}$, [13, Proposition 3.1].

It follows by convex analysis that there exists a bounded set $\mathcal{A}\subset\mathcal{Z}^*$ such that

$$\rho(Z)=\sup_{Q\in\mathcal{A}}\langle Z,Q\rangle,\ \ \forall Z\in\mathcal{Z}. \tag{2.7}$$

Furthermore, by the monotonicity and translation equivariance of ρ we have that $\mathcal{A}$ is a set of probability measures [13, Theorem 2.2]. That is, $\widehat{\mathfrak{M}}:=\mathcal{A}$ is a set of probability measures satisfying (2.4) for $\mathfrak{Z}=\mathcal{Z}$. This completes the proof. ∎

Remark 1 As it was already mentioned, different approaches to precise definition of time consistency of multistage optimization problems were suggested in recent literature . Anyway it seems that there is a general agreement that risk measure $\rho(\cdot)$ given in the nested form (2.6) is time consistent. It is argued in Iancu, Petrik and Subramanian [5, Lemma 2] that in a certain sense the risk measure $\rho(\cdot)$ represents the tightest upper bound for the left hand side of (2.3) among all possible coherent and time consistent upper bounds.

In the terminology of convex analysis, equation (2.7) means that $\rho(\cdot)$ is the support function of the set $\mathcal{A}$. The set $\mathcal{A}$ is not defined uniquely by this equation. That is, the rectangular (with respect to $\mathcal{Z}$) set $\widehat{\mathfrak{M}}$ is not unique and it can happen that the set $\mathfrak{M}$ is not a subset of $\widehat{\mathfrak{M}}$. By (2.3) we have that the set $\widehat{\mathfrak{M}}$ *can be chosen* in such a way that $\mathfrak{M}\subset\widehat{\mathfrak{M}}$. Consider

$$\mathfrak{M}^\star:=\{Q\in\mathcal{Z}^*:\rho(Z)\ge\langle Z,Q\rangle,\ \forall Z\in\mathcal{Z}\}. \tag{2.8}$$

By convex analysis we have that the set $\mathfrak{M}^\star$ is convex, weakly* compact and is the *largest* set satisfying (2.7). This can be shown, for example, by computing the conjugate of ρ and applying the Fenchel-Moreau Theorem. Also it can be noted that the set $\mathfrak{M}^\star$ is the subdifferential of $\rho(Z)$ at $Z=0$.

It follows that $\mathfrak{M}\subset\mathfrak{M}^\star$. For a weakly* compact set $\mathcal{A}$, the maximum in the right hand side of (2.7) is attained for any $Z\in\mathcal{Z}$. Of course if $\rho(Z)=\sup_{Q\in\mathcal{A}}\langle Z,Q\rangle$ holds for all $Z\in\mathcal{Z}$, then it holds for any $\mathfrak{Z}\subset\mathcal{Z}$ and $Z\in\mathfrak{Z}$. For a subset $\mathfrak{Z}$ of $\mathcal{Z}$ it suffices to take

$$\widehat{\mathfrak{M}}:=\bigcup_{Z\in\mathfrak{Z}}\arg\max_{Q\in\mathfrak{M}^\star}\langle Z,Q\rangle. \tag{2.9}$$

Theorem 2.2 *Suppose that the set $\mathfrak{M}$ is convex bounded and weakly* closed, and let $\widehat{\mathfrak{M}} \subset \mathcal{Z}^*$ be a rectangular set with respect to $\mathfrak{M}$ and $\mathcal{Z}$ such that $\mathfrak{M} \subset \widehat{\mathfrak{M}}$. Then $\mathfrak{M}$ is rectangular with respect to $\mathcal{Z}$ iff $\mathfrak{M} = \widehat{\mathfrak{M}}$.*

Proof. Since $\widehat{\mathfrak{M}}$ is rectangular with respect to $\mathfrak{M}$ and $\mathcal{Z}$, it follows that if $\mathfrak{M} = \widehat{\mathfrak{M}}$, then $\mathfrak{M}$ is rectangular with respect to $\mathcal{Z}$.

Conversely suppose that $\mathfrak{M}$ is rectangular with respect to $\mathcal{Z}$. Consider the support function $\varrho(Z) = \sup_{Q\in\mathfrak{M}}\langle Z, Q\rangle$ of the set $\mathfrak{M}$, and the support function $\rho(\cdot)$ of the set $\widehat{\mathfrak{M}}$. Since $\mathfrak{M}$ is a subset of $\widehat{\mathfrak{M}}$, it follows that $\rho(\cdot) \geq \varrho(\cdot)$. Moreover, since $\mathfrak{M}$ is convex and weakly* closed and $\mathfrak{M} \subset \widehat{\mathfrak{M}}$, it follows that $\rho(\cdot) = \varrho(\cdot)$ iff $\mathfrak{M} = \widehat{\mathfrak{M}}$, (e.g., [3, Proposition 2.116]). Since $\widehat{\mathfrak{M}}$ is rectangular, the equality (2.6) holds for its support function. Since $\mathfrak{M}$ is rectangular, $\varrho(Z)$ is equal to the right hand side of (2.6), and hence $\rho(\cdot) = \varrho(\cdot)$. It follows that $\mathfrak{M} = \widehat{\mathfrak{M}}$. ∎

It could be noted that taking weak* closure of convex hull of the set $\mathfrak{M}$ does not change optimal value of the original problem (2.1). Therefore the condition for $\mathfrak{M}$ to be convex and weakly* closed is not that restrictive.

It is a more delicate question to define a "smallest" rectangular set.

Definition 2.2 *Let $\mathfrak{K}$ be a nonempty weakly* compact subset of $\mathcal{Z}^*$. It is said that $\bar{Q} \in \mathfrak{K}$ is a weak* exposed point of $\mathfrak{K}$ if there exists $Z \in \mathcal{Z}$ such that $\phi(Q) := \langle Z, Q\rangle$ attains its maximum over $Q \in \mathfrak{K}$ at the unique point $\bar{Q}$. In that case it is said that Z exposes $\mathfrak{K}$ at $\bar{Q}$.*

In the finite dimensional setting, "weak* exposed point" simply means "exposed point". In general a result, going back to Mazur [7], says that if the space $\mathcal{Z}$ is separable, then the set of points $Z \in \mathcal{Z}$ which exposes a nonempty weakly* compact set $\mathfrak{K} \subset \mathcal{Z}^*$ is a dense subset of $\mathcal{Z}$, e.g., [18, Theorem 7.81]. Note that the considered spaces $L_p(\Xi, \mathcal{B}, P)$, $p \in [1, \infty)$, and $C(\Xi)$ are separable. This leads to the following result saying that in a sense the set of weakly* exposed points of the set $\mathfrak{M}^\star$ is the smallest rectangular set with respect to $\mathfrak{M}$ and $\mathcal{Z}$. Proof of the following theorem can be found in [9, Proposition 2.2].

Theorem 2.3 *Consider the setting of Theorem 2.1. Let $\mathrm{Exp}(\mathfrak{M}^\star)$ be the set of weakly* exposed points of the largest rectangular set $\mathfrak{M}^\star$ (defined in (2.8)). Then $\mathrm{Exp}(\mathfrak{M}^\star)$ is a rectangular set with respect to $\mathfrak{M}$ and $\mathcal{Z}$. Moreover, if $\widehat{\mathfrak{M}}$ is any weakly* closed rectangular set with respect to $\mathfrak{M}$ and $\mathcal{Z}$, then $\mathrm{Exp}(\mathfrak{M}^\star) \subset \widehat{\mathfrak{M}}$.*

By the above theorem we have the following necessary and sufficient conditions for the set $\mathfrak{M}$ to be rectangular with respect to $\mathcal{Z}$.

Corollary 2.1 *Suppose that the set $\mathfrak{M}$ is weakly* closed. Then $\mathfrak{M}$ is rectangular with respect to $\mathcal{Z}$ iff $\mathrm{Exp}(\mathfrak{M}^\star) \subset \mathfrak{M}$.*

Proof. If $\mathfrak{M}$ is rectangular with respect to $\mathcal{Z}$, then $\mathrm{Exp}(\mathfrak{M}^\star) \subset \mathfrak{M}$ by the last statement of Theorem 2.3. Conversely if $\mathrm{Exp}(\mathfrak{M}^\star) \subset \mathfrak{M}$, then $\sup_{Q\in\mathfrak{M}} \mathbb{E}_Q[Z] = \rho(Z)$ for all $Z \in \mathcal{Z}$. By Definition 2.1 this implies that the set $\mathfrak{M}$ is rectangular. ∎

For $Q \in \mathfrak{M}$ the marginal distribution of $\xi_{[T-1]} = (\xi_1, ..., \xi_{T-1})$ has the form $Q_{[T-1]}(A) = Q(A \times \Xi_T)$, where A is a Borel subset of $\Xi_1 \times \cdots \times \Xi_{T-1}$. And so on, let $\mathfrak{M}_{[T]} := \mathfrak{M}$ and

$$\mathfrak{M}_{[t]} := \left\{Q_{[t]} : Q_{[t]}(A) = Q(A \times \Xi_{t+1} \times \cdots \times \Xi_T),\ Q \in \mathfrak{M}\right\},\ t = 1, ..., T-1, \tag{2.10}$$

be the set of marginal probability distributions of $(\xi_1, ..., \xi_t)$. Then the right hand side of (2.3) can be written as

$$\sup_{Q \in \mathfrak{M}_{[1]}} \mathbb{E}_Q \left[\sup_{Q \in \mathfrak{M}_{[2]}} \mathbb{E}_{Q|\xi_{[1]}} \Big[\cdots \sup_{Q \in \mathfrak{M}_{[T]}} \mathbb{E}_{Q|\xi_{[T-1]}}[Z] \Big] \right]. \tag{2.11}$$

The analysis simplifies if we assume that the set $\mathfrak{M}$ consists of product measures. That is,

$$\mathfrak{M} := \{Q = Q_1 \times \cdots \times Q_T : Q_t \in \mathfrak{M}_t, \ t = 1, ..., T\}, \tag{2.12}$$

where $\mathfrak{M}_t$ is a set of probability measures on $(\Xi_t, \mathcal{B}_t)$, $t = 1, ..., T$. If we view $\xi_1, ..., \xi_T$ as a random process having distribution $Q \in \mathfrak{M}$, then (2.12) means that random vectors ξ_t, $t = 1, ..., T$, are mutually independent with respective marginal distributions $Q_t \in \mathfrak{M}_t$. If the uncertainty set $\mathfrak{M}$ is of the product form (2.12), then

$$\mathfrak{M}_{[t]} = \{Q_1 \times \cdots \times Q_t, \ Q_\tau \in \mathfrak{M}_\tau, \ \tau = 1, ..., t\}, \ t = 1, ..., T. \tag{2.13}$$

Moreover, for $\mathfrak{M} \ni Q = Q_1 \times \cdots \times Q_T$ we can write

$$\mathbb{E}_{Q|\xi_{[T-1]}}[Z] = \int_{\Xi_T} Z(\xi_{[T-1]}, \xi_T) dQ_T(\xi_T) =: \mathbb{E}_{Q_T|\xi_{[T-1]}}[Z].$$

Now $\mathbb{E}_{Q_T|\xi_{[T-1]}}[Z]$ is a function of $\xi_{[T-1]}$. In a similar way we can apply to it the operator $\mathbb{E}_{Q_{T-1}|\xi_{[T-2]}}$, and so on going backward in time. Consequently in that case we can write (2.3) as

$$\sup_{Q \in \mathfrak{M}} \mathbb{E}_Q[Z] \le \sup_{Q_1 \in \mathfrak{M}_1} \mathbb{E}_{Q_1} \left[\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2|\xi_{[1]}} \Big[\cdots \sup_{Q_T \in \mathfrak{M}_T} \mathbb{E}_{Q_T|\xi_{[T-1]}}[Z] \Big] \right]. \tag{2.14}$$

Example 1 Suppose that the set Ξ is compact, i.e., each Ξ_t, $t = 1, ..., T$, is compact, and let $\mathfrak{M}$ be the set of *all* probability measures on $(\Xi, \mathcal{B})$ and $\mathcal{Z} := C(\Xi)$ be the space of continuous functions $Z : \Xi \to \mathbb{R}$. For every $Z \in \mathcal{Z}$ the maximum of $\mathbb{E}_Q[Z]$ over $Q \in \mathfrak{M}$ is attained at a measure of mass one and

$$\sup_{Q \in \mathfrak{M}} \mathbb{E}_Q[Z] = \sup_{\xi_{[T]} \in \Xi} Z(\xi_{[T]}). \tag{2.15}$$

Also

$$\sup_{Q \in \mathfrak{M}} \mathbb{E}_{Q|\xi_{[T-1]}}[Z] = \sup_{\xi_T \in \Xi_T} Z(\xi_{[T-1]}, \xi_T), \tag{2.16}$$

and so on going backward in time. Here the equality (2.4) holds with $\widehat{\mathfrak{M}} = \mathfrak{M}$, and hence the set $\mathfrak{M}$ is rectangular.

Suppose now that the set $\mathfrak{M}$ is of the product form (2.12) with $\mathfrak{M}_t$, $t = 1, ..., T$, being the set of all probability measures on $(\Xi_t, \mathcal{B}_t)$. In that case the equations (2.15) and (2.16) still hold, and the set $\mathfrak{M}$ is rectangular. ■

Example 2 Let $\Xi_t := [a_t, b_t] \subset \mathbb{R}$, $t = 1, ..., T$, be bounded intervals, the set $\mathfrak{M}$ be of the product form (2.12) with $\mathfrak{M}_t := \{Q_t : \mathbb{E}_{Q_t}[Z_t] = \mu_t\}$ for some $\mu_t \in [a_t, b_t]$, $t = 1, ..., T$, $\mathcal{Z} := C(\Xi)$ and $\mathfrak{Z} \subset \mathcal{Z}$ be the set of convex functions. Consider $Z \in \mathfrak{Z}$. Because of convexity of $Z(\xi_{[T-1]}, \cdot)$, maximum of $\mathbb{E}_{Q_T|\xi_{[T-1]}}[Z]$, over $Q_T \in \mathfrak{M}_T$, is attained at a probability measure $\bar{Q}_T$, supported at two end points of the interval $[a_T, b_T]$, independent of $\xi_{[T-1]}$ (cf., [16]). The obtained function of $\xi_{[T-1]}$ is again convex, and so on. It follows here that for every $Z \in \mathfrak{Z}$ the maxima in the right hand side of (2.4) is attained at the probability measure $\bar{Q} = \bar{Q}_1 \times \cdots \times \bar{Q}_T$ which does not depend on the respective $\xi_{[t]}$. Thus the set $\mathfrak{M}$ is rectangular with respect to $\mathfrak{Z}$. Note that it is essential here that the set $\mathfrak{Z}$ consists of *convex* functions, and the set $\mathfrak{M}$ is not rectangular with respect to the space $\mathcal{Z}$. ■

Theorem 2.1 shows existence of a rectangular set, but it does not give a constructive way for verification that a given set is rectangular. We discuss below a specific, and in a sense natural, way for constructing rectangular sets. This construction is similar to the one used in Epstein and Schneider [4, Definition 3.1] (and also in [6]). As we will see the requirement for the set $\mathfrak{M}$ to be rectangular in itself, could be quite restrictive.

In order to simplify the presentation, and to avoid delicate measurability questions, let us assume for the moment that the sets Ξ_t are finite, say $\Xi_t = \{\xi_t^1, ..., \xi_t^{m_t}\}$, $t = 1, ..., T$. Then the set Ξ is also finite, we equip Ξ with the sigma algebra of all its subsets and consider the space $\mathcal{Z}$ of all functions $Z : \Xi \to \mathbb{R}$, and talk about rectangularity with respect to a set of probability measures (distributions) $\mathfrak{M}$ on Ξ and the space $\mathcal{Z}$.

Let us consider first the case of $T = 2$. Let $\mathfrak{M}_{[1]}$ be the set of marginal distributions of ξ_1 and consider a probability distribution $Q_{[1]} \in \mathfrak{M}_{[1]}$. Given ξ_1, having distribution $Q_{[1]}$, consider the set of all conditional distributions $Q \in \mathfrak{M}$ of (ξ_1, ξ_2). Consequently take the union of all these sets over distributions $Q_{[1]} \in \mathfrak{M}_{[1]}$. It follows directly from the definition that such constructed set $\widetilde{\mathfrak{M}}$ is rectangular.

This construction can be extended for $T \geq 2$ in an iterative way. For $t \in \{1, ..., T - 1\}$ consider the set $\mathfrak{M}_{[t]}$ of marginal distributions of $\xi_{[t]} = (\xi_1, ..., \xi_t)$. Let $\widetilde{\mathfrak{M}}_{[t]}$ be a rectangular set associated with $\mathfrak{M}_{[t]}$ (and the space of all real valued functions on $\Xi_1 \times \cdots \times \Xi_t$). Consider a distribution $Q_{[t]} \in \widetilde{\mathfrak{M}}_{[t]}$. Given $\xi_{[t]}$, having distribution $Q_{[t]}$, consider the set of all conditional distributions $Q \in \widetilde{\mathfrak{M}}_{[t+1]}$ of $\xi_{[t+1]} = (\xi_{[t]}, \xi_{t+1})$. Consequently take the union of all these sets over distributions $Q_{[t]} \in \mathfrak{M}_{[t]}$, to construct the corresponding rectangular set $\widetilde{\mathfrak{M}}_{[t+1]}$ of distributions of $\xi_{[t+1]}$. Continuing this process forward eventually we construct a rectangular set $\widetilde{\mathfrak{M}} = \widetilde{\mathfrak{M}}_{[T]}$ on Ξ. We refer to $\widetilde{\mathfrak{M}}$ as the *natural rectangular* set of the set $\mathfrak{M}$. Clearly $\mathfrak{M}$ is a subset of $\widetilde{\mathfrak{M}}$.

The above construction becomes more transparent if we assume that the set $\mathfrak{M}$ has the product form (2.12). Probability distribution on (finite) set Ξ_t can be identified with vector $q = (q_1, ..., q_{m_t})$ of respective probabilities of $\xi_t^1, ..., \xi_t^{m_t}$. Suppose $T = 2$, for $T > 2$ we can apply the iterative procedure discussed above. Consider $Q_1 \in \mathfrak{M}_1$ assigning probabilities p_i, $i = 1, ..., m_1$, to respective $\xi_1^i \in \Xi_1$, and a function $q : \Xi_1 \to \mathfrak{M}_2$, i.e., $q^i = q(\xi_1^i)$ is a probability distribution of ξ_2 depending on $\xi_1^i \in \Xi_1$. Conditional on $\xi_1 = \xi_1^i$ consider the distribution of (ξ_1, ξ_2) assigning to (ξ_1^i, ξ_2^j) probability $p_i q_j^i$, $i = 1, ..., m_1$, $j = 1, ..., m_2$. Consequently take the union of such distributions of (ξ_1, ξ_2) over all functions $q : \Xi_1 \to \mathfrak{M}_2$ and all $Q_1 \in \mathfrak{M}_1$. In particular suppose that $\mathfrak{M}_1 = \{Q_1\}$ is a singleton, and the set $\mathfrak{M}_2$ is finite, say of cardinality k. Then the respective natural rectangular set $\widetilde{\mathfrak{M}}$ has as many elements as there are functions $q : \Xi_1 \to \mathfrak{M}_2$, that is $\widetilde{\mathfrak{M}}$ has cardinality k^{m_1}.

If the set Ξ is finite and the set $\mathfrak{M}$ of probability distributions is finite, then the respective natural rectangular set $\widetilde{\mathfrak{M}}$ is also finite. By taking the convex hull of $\widetilde{\mathfrak{M}}$ we obtain the largest rectangular set $\mathfrak{M}^\star$. Since $\widetilde{\mathfrak{M}}$ is finite, its convex hull $\mathfrak{M}^\star$ is a polyhedral set. Moreover, vertices (extreme points) of $\mathfrak{M}^\star$ is a subset of $\widetilde{\mathfrak{M}}$ and form the set of exposed points of $\mathfrak{M}^\star$.

For not necessarily finite set Ξ we can proceed in a similar way. For the sake of simplicity suppose that $T = 2$ (for $T > 2$ we can proceed in an iterative way as it was discussed above), and that the set $\mathfrak{M}$ has the product form (2.12). In that case the functional ρ, defined in (2.6), takes the form

$$\rho(Z) = \sup_{Q_1 \in \mathfrak{M}_1} \mathbb{E}_{Q_1}\left[\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2|\xi_1}[Z]\right]. \tag{2.17}$$

Then, under some regularity conditions (cf., Rockafellar and Wets [12, Theorem 14.60, page 677]), we can interchange the sup and integral operators to write

$$\mathbb{E}_{Q_1}\left[\sup_{Q_2\in\mathfrak{M}_2}\mathbb{E}_{Q_2|\xi_1}[Z]\right]=\sup_{Q_2^{(\cdot)}\in\mathfrak{F}}\mathbb{E}_{Q_1}\left[\mathbb{E}_{Q_2^{\xi_1}|\xi_1}[Z]\right]. \tag{2.18}$$

The maximum in the right hand side of (2.18) is taken with respect to family $\mathfrak{F}$ of mappings $\chi:\Xi_1\to\mathfrak{M}_2$, such that for $Q_2^{\xi_1}=\chi(\xi_1)$ the integral

$$\mathbb{E}_{Q_1}\left[\mathbb{E}_{Q_2^{\xi_1}|\xi_1}[Z]\right]=\int_{\Xi_1}\left(\int_{\Xi_2}Z(\xi_1,\xi_2)dQ_2^{\xi_1}(\xi_2)\right)dQ_1(\xi_1) \tag{2.19}$$

is well defined. The notation $Q_2^{(\cdot)}$ emphasizes that the probability measure $Q_2^{\xi_1}$ in (2.18) is a function of ξ_1.

The right hand side of (2.19) defines a functional $\phi:\mathcal{Z}\to\mathbb{R}$. This functional is linear, monotone and $\phi(Z+a)=\phi(Z)+a$ for $Z\in\mathcal{Z}$ and $a\in\mathbb{R}$. Hence there exists a probability measure Q on $\Xi_1\times\Xi_2$, which depends on the mapping $\xi^1\mapsto Q_2^{\xi_1}$ (and also on Q_1), such that

$$\mathbb{E}_{Q_1}\left[\mathbb{E}_{Q_2^{\xi_1}|\xi_1}[Z]\right]=\mathbb{E}_Q[Z]. \tag{2.20}$$

For measurable sets $A_1\subset\Xi_1$ and $A_2\subset\Xi_2$, and $Z(\xi_1,\xi_2):=\mathbf{1}_{A_1}(\xi_1)\mathbf{1}_{A_2}(\xi_2)$, we have

$$\int_{\Xi_1}\left(\int_{\Xi_2}Z(\xi_1,\xi_2)dQ_2^{\xi_1}(\xi_2)\right)dQ_1(\xi_1)=\int_{A_1}Q_2^{\xi_1}(A_2)dQ_1(\xi_1), \tag{2.21}$$

and hence for $A:=A_1\times A_2$,

$$Q(A)=\int_{A_1}Q_2^{\xi_1}(A_2)dQ_1(\xi_1). \tag{2.22}$$

This can be compared with [14, Proposition 5.1 and formula 5.7]. We can take $\widetilde{\mathfrak{M}}$ to be the union of such measures Q taken over all mappings $\chi\in\mathfrak{F}$ and $Q_1\in\mathfrak{M}_1$. If we take the mapping $\xi^1\mapsto Q_2^{\xi_1}$ in (2.18) to be constant, i.e., independent of ξ_1, then the corresponding measure $Q=Q_1\times Q_2$.

The set $\mathfrak{M}$ is a subset of the natural rectangular set $\widetilde{\mathfrak{M}}$. Hence by Theorem 2.2 we have the following.

Corollary 2.2 *If the set $\mathfrak{M}$ is bounded convex and weakly* closed, then $\mathfrak{M}$ is rectangular with respect to $\mathcal{Z}$ iff $\mathfrak{M}=\widetilde{\mathfrak{M}}$.*

3 Coherent risk measures

Consider a functional $\varrho:\mathcal{Z}\to\mathbb{R}$ satisfying axioms (A1)–(A4) specified in the proof of Theorem 2.1. Such functionals were called *coherent risk measures* in the pioneering paper by [1] (note that we assume here that the functional ϱ is real valued). As it was pointed out in the proof of Theorem 2.1, a (real valued) coherent risk measure ϱ has the dual representation (2.7) with the corresponding set $\mathcal{A}\subset\mathcal{Z}^*$ being a set of probability measures. That is,

$$\varrho(Z)=\sup_{Q\in\mathfrak{M}}\mathbb{E}_Q[Z],\ Z\in\mathcal{Z}, \tag{3.1}$$

for some set $\mathfrak{M}\subset\mathcal{Z}^*$ of probability measures. *We say that set $\mathfrak{M}$ of probability measures satisfying* (3.1) *is the* dual set *of ϱ if it is convex and weakly* compact.*

Suppose now that $\mathcal{Z} = L_p(\Xi, \mathcal{B}, P)$, $p \in [1, \infty)$. For a process $\xi_1, ..., \xi_T$ let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T$ be the corresponding filtration with $\mathcal{F}_0 = \{\emptyset, \Xi\}$ and $\mathcal{F}_t = \sigma(\xi_1, ..., \xi_t)$, $t = 1, ..., T$. Denote by $\mathcal{Z}_t$ the subspace of $\mathcal{Z}$ formed by $\mathcal{F}_t$-measurable functions. Note that since the sigma algebra $\mathcal{F}_0$ is trivial, the space $\mathcal{Z}_0$ consists of constant functions and $\mathbb{E}_{|\mathcal{F}_0}[\,\cdot\,] = \mathbb{E}[\,\cdot\,]$. Then the inequality (2.3) can be written as

$$\varrho(Z) \le \varrho_{1|\mathcal{F}_0}\Big[\varrho_{2|\mathcal{F}_1}\big[\cdots \varrho_{T|\mathcal{F}_{T-1}}(Z)\big]\Big], \tag{3.2}$$

where

$$\varrho_{t|\mathcal{F}_{t-1}}(\cdot) := \sup_{Q \in \mathfrak{M}} \mathbb{E}_{Q|\mathcal{F}_{t-1}}[\,\cdot\,] : \mathcal{Z}_t \to \mathcal{Z}_{t-1},\ t = 1, ..., T,$$

are the corresponding conditional coherent risk mappings, cf., [18, section 6.8.6].

The functional

$$\bar{\rho}(Z) := \varrho_{1|\mathcal{F}_0}\Big[\varrho_{2|\mathcal{F}_1}\big[\cdots \varrho_{T|\mathcal{F}_{T-1}}(Z)\big]\Big] \tag{3.3}$$

is a coherent risk measure, and hence has the dual representation

$$\bar{\rho}(Z) = \sup_{Q \in \widehat{\mathfrak{M}}} \mathbb{E}_Q[Z],\ Z \in \mathcal{Z}, \tag{3.4}$$

for some set $\widehat{\mathfrak{M}} \subset \mathcal{Z}^*$ of probability measures. That is, $\widehat{\mathfrak{M}}$ is a rectangular set associated with the set $\mathfrak{M}$ of the representation (3.1). This is another way of looking at the result of Theorem 2.1.

Example 3 Let $\Xi_t = [0, 1]$, $t = 1, ..., T$, be intervals, equipped with respective uniform distribution P_t, $\Xi = \Xi_1 \times \cdots \times \Xi_T$ equipped with the uniform distribution $P = P_1 \times \cdots \times P_T$, the space $\mathcal{Z} = L_1(\Xi, \mathcal{B}, P)$ and the set $\mathfrak{M}$ be of the product form (2.12) with

$$\mathfrak{M}_t = \left\{\zeta_t \in L_\infty(\Xi_t, \mathcal{B}_t, P_t) : \zeta_t(\omega) \in [0, \alpha^{-1}] \text{ a.e. } \omega \in [0, 1],\ \mathbb{E}[\zeta] = 1\right\},$$

for some $\alpha > 0$ (the expectation $\mathbb{E}[\,\cdot\,]$ is taken here with respect to the uniform distribution on the interval [0,1]). Note that for $Z_t \in L_1(\Xi_t, \mathcal{B}_t, P_t)$,

$$\sup_{\zeta_t \in \mathfrak{M}_t} \mathbb{E}[Z_t \zeta_t] = \mathsf{AVaR}_\alpha(Z_t), \tag{3.5}$$

i.e., (3.5) is the dual representation of the so-called Average Value-at-Risk.

We have then that for $Z \in \mathcal{Z}$ (compare with (3.2))

$$\sup_{\zeta \in \mathfrak{M}} \mathbb{E}[Z\zeta] \le \mathsf{AVaR}_\alpha\Big[\mathsf{AVaR}_{\alpha|\xi_{[1]}}\big[\cdots \mathsf{AVaR}_{\alpha|\xi_{[T-1]}}(Z)\big]\Big], \tag{3.6}$$

where $\mathsf{AVaR}_{\alpha|\xi_{[t-1]}} : \mathcal{Z}_t \to \mathcal{Z}_{t-1}$, $t = 2, ..., T$, are

$$\mathsf{AVaR}_{\alpha|\xi_{[t-1]}}(Z) = \sup_{\zeta_t \in \mathfrak{M}_t} \int_0^1 Z\left(\xi_{[t-1]}, \xi_t\right) \zeta_t(\xi_t) d\xi_t,\ Z \in \mathcal{Z}_t, \tag{3.7}$$

the corresponding conditional Average Value-at-Risk mappings. With $\mathfrak{M}$ is associated rectangular set $\widehat{\mathfrak{M}} \subset L_\infty(\Xi, \mathcal{B}, P)$ such that

$$\sup_{\zeta \in \widehat{\mathfrak{M}}} \mathbb{E}[Z\zeta] = \mathsf{AVaR}_\alpha\Big[\mathsf{AVaR}_{\alpha|\xi_{[1]}}\big[\cdots \mathsf{AVaR}_{\alpha|\xi_{[T-1]}}(Z)\big]\Big],\ Z \in \mathcal{Z}. \tag{3.8}$$

Note that both $\sup_{\zeta \in \mathfrak{M}} \mathbb{E}[Z\zeta]$ and $\sup_{\zeta \in \widehat{\mathfrak{M}}} \mathbb{E}[Z\zeta]$ are not equal to $\mathsf{AVaR}_\alpha(Z)$. Therefore (3.6) does not imply that $\mathsf{AVaR}_\alpha(Z)$ is less than or equal to the nested Average Value-at-Risk given in the right hand side of (3.6). And indeed such inequality between $\mathsf{AVaR}_\alpha(Z)$ and the respective nested Average Value-at-Risk does not hold for all Z, e.g., [10, p.157], [5, Example 2]. ■

3.1 Law invariance

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$, $\mathcal{Z}^* = L_q(\Omega, \mathcal{F}, P)$ and $\varrho : \mathcal{Z} \to \mathbb{R}$ be a coherent risk measure. We say that two random variables $Z, Z' \in \mathcal{Z}$ are *distributionally equivalent*, denoted $Z \stackrel{\mathcal{D}}{\sim} Z'$, if their cumulative distribution functions are the same, i.e., $P(Z \le z) = P(Z' \le z)$ for all $z \in \mathbb{R}$. It is said that ϱ is *law invariant* (with respect to the reference probability measure P) if for all $Z, Z' \in \mathcal{Z}$ the implication $Z \stackrel{\mathcal{D}}{\sim} Z' \Rightarrow \varrho(Z) = \varrho(Z')$ holds. In case the functional $\varrho(Z)$ is law invariant, it can be considered as a function of the cdf $F_Z(z) = P(Z \le z)$.

A natural question is how law invariance of ϱ can be described in terms of a corresponding set $\mathfrak{A} \subset \mathcal{Z}^*$ in the dual representation

$$\varrho(Z) = \sup_{\zeta \in \mathfrak{A}} \int_\Omega \zeta Z dP. \tag{3.9}$$

In order to proceed we need the following concept.

Definition 3.1 *We say that a mapping $T : \Omega \to \Omega$ is a* measure preserving *transformation (with respect to the reference probability measure P) if T is measurable, one-to-one and onto, and for any $A \in \mathcal{F}$ it follows that $P(A) = P(T^{-1}(A))$. We denote by $\mathfrak{G}$ the set of measure preserving transformations.*

Since $T \in \mathfrak{G}$ is one-to-one and onto, it is invertible and $P(A) = P(T(A))$ for any $A \in \mathcal{F}$. The set $\mathfrak{G}$ forms a group of transformations, i.e., if $T_1, T_2 \in \mathfrak{G}$, then their composition $T_1 \circ T_2 \in \mathfrak{G}$, and if $T \in \mathfrak{G}$ then its inverse $T^{-1} \in \mathfrak{G}$. For measurable function $Z(\omega)$ and $T \in \mathfrak{G}$ we denote by $Z \circ T$ their composition $Z(T(\omega))$. Note that for any integrable function $h : \Omega \to \mathbb{R}$ and $T \in \mathfrak{G}$ it follows that

$$\int_\Omega h(\omega) dP(\omega) = \int_\Omega h(T(\omega)) dP(\omega).$$

If the space $\Omega = \{\omega_1, ..., \omega_m\}$ is finite with the reference probability P assigning to each elementary event ω_i equal probability $p_i = 1/m$, $i = 1, ..., m$, then $\mathfrak{G}$ coincides with the group of permutations of the set Ω.

We have the following important properties of measure preserving transformations, e.g., [18, Section 6.3.3].

Proposition 3.1 *If $Z \in \mathcal{Z}$ and $Z' = Z \circ T$ for some $T \in \mathfrak{G}$, then $Z' \in \mathcal{Z}$ and $Z \stackrel{\mathcal{D}}{\sim} Z'$. If, moreover, the space $(\Omega, \mathcal{F}, P)$ is nonatomic or $\Omega = \{\omega_1, ..., \omega_m\}$ is finite equipped with equal probabilities $p_i = 1/m$, $i = 1, ..., m$, then the converse implication holds, i.e., if $Z \in \mathcal{Z}$ and $Z' \in \mathcal{Z}$ are distributionally equivalent, then there exists a measure-preserving transformation $T \in \mathfrak{G}$ such that $Z' = Z \circ T$.*

For $T \in \mathfrak{G}$ and $Q \in \mathfrak{M}$ we denote by $Q \circ T$ the composite probability measure $Q'(A) := Q(T(A))$, $A \in \mathcal{F}$. We say that the set $\mathfrak{M}$ is *invariant with respect to measure preserving transformations* if $Q \in \mathfrak{M}$ and $T \in \mathfrak{G}$ imply that $Q \circ T \in \mathfrak{M}$. In particular, the set $\mathfrak{A} \subset \mathcal{Z}^*$ is invariant with respect to measure preserving transformations if $\zeta \in \mathfrak{A}$ and $T \in \mathfrak{G}$ imply that $\zeta \circ T \in \mathfrak{A}$.

Proposition 3.2 *The following holds.* (i) *If the set $\mathfrak{A}$ is invariant with respect to measure preserving transformations and the space $(\Omega, \mathcal{F}, P)$ is nonatomic or $\Omega = \{\omega_1, ..., \omega_m\}$ is finite equipped with equal probabilities $p_i = 1/m$, $i = 1, ..., m$, then the functional ϱ is law invariant.* (ii) *Conversely, if ϱ is law invariant and the set $\mathfrak{A}$ is convex and weakly* closed, then $\mathfrak{A}$ is invariant with respect to measure-preserving transformations.*

Proof. Consider distributionally equivalent $Z, Z' \in \mathcal{Z}$. By Proposition 3.1 there exists $T \in \mathfrak{G}$ such that $Z' = Z \circ T$. Furthermore for $\zeta \in \mathfrak{A}$ and $dQ = \zeta dP$ we have

$$\int_\Omega Z'(\omega) dQ(\omega) = \int_\Omega Z(T(\omega)) dQ(\omega) = \int_\Omega Z(\omega) dQ(T^{-1}(\omega)).$$

By invariance of $\mathfrak{M}$ with respect to $\mathfrak{G}$, we have that $Q \circ T^{-1} \in \mathfrak{G}$. This proves (i). The converse assertion (ii) can be proved by duality arguments, e.g., [18, Corollary 6.30]. ∎

Example 4 Let $\mathfrak{H}$ be a family of measurable functions $h : \Omega \to \mathbb{R}$. Consider the following semi-distance between two probability measures

$$d(Q, P) := \sup_{h \in \mathfrak{H}} \left| \int_\Omega h dQ - \int_\Omega h dP \right|. \tag{3.10}$$

Suppose that the set $\mathfrak{M}$ consists of probability measures Q such that $d(Q, P) \le c$ for some $c > 0$. If the set $\mathfrak{H}$ is invariant with respect to measure preserving transformations, then $\mathfrak{M}$ is invariant with respect to measure preserving transformations. Indeed, for $T \in \mathfrak{G}$ we have

$$\int_\Omega h(\omega) dQ(T(\omega)) - \int_\Omega h(\omega) dP(\omega) = \int_\Omega h(T^{-1}(\omega)) dQ(\omega) - \int_\Omega h(T^{-1}(\omega)) dP(\omega).$$

Since $\mathfrak{H}$ is invariant with respect to measure preserving transformations, it follows that if $h \in \mathfrak{H}$, then $h \circ T^{-1} \in \mathfrak{H}$. Hence the maximum in the right hand side of (3.10) does not change by replacing Q with $Q \circ T$, i.e., $d(Q \circ T, P) = d(Q, P)$.

For example, the set

$$\mathfrak{H} := \{h : |h(\omega)| \le 1, \ \omega \in \Omega\}$$

is invariant with respect to measure preserving transformations. In that case $d(Q, P) = \|Q - P\|^*$, where $\|\cdot\|^*$ is the total variation norm

$$\|Q - P\|^* = \sup_{A \in \mathcal{F}} (Q(A) - P(A)) - \inf_{A \in \mathcal{F}} (Q(A) - P(A)).$$

If we assume further that measures $Q \in \mathfrak{M}$ are absolutely continuous with respect to P, then for $dQ = \zeta dP$ we have

$$d(Q, P) := \sup_{h \in \mathfrak{H}} \left| \int_\Omega h(\zeta - 1) dP \right| = \int_\Omega |\zeta - 1| dP.$$

That is, the corresponding set $\mathcal{A} = \{\zeta = dQ/dP : Q \in \mathfrak{M}\}$ consists of densities ζ such that the L_1 distance from ζ to $\mathbf{1}_\Omega$ is less than or equal to c. ■

Let us consider now the following construction. Let Ξ_1 and Ξ_2 be finite sets each having $n \ge 3$ elements. Consider the set $\Xi = \Xi_1 \times \Xi_2$ equipped with the sigma algebra $\mathcal{F}$ of all its subsets and reference probability measure P assigning equal probabilities $1/n^2$ to each element of Ξ. Let $\mathcal{Z}$ be the space of real valued functions (random variables) $Z : \Xi \to \mathbb{R}$. Of course, the space $\mathcal{Z}$ has finite dimension n^2. Let $\varrho : \mathcal{Z} \to \mathbb{R}$ be a coherent risk measure defined on the space $\mathcal{Z}$.

We can think about Ξ as a scenario tree with $T = 3$ stages and $K = n^2$ scenarios of moving from a deterministic root node ξ_0 to a node $\xi_1 \in \Xi_1$, and from node ξ_1 to a node $\xi_2 \in \Xi_2$. Suppose now that risk measure ϱ can be decomposed as $\varrho(Z) = \rho_0(\rho_{|\xi_1}(Z))$, where $Z = Z(\xi_1, \xi_2) \in \mathcal{Z}$, ρ_0 is a (real valued) coherent risk measure defined on the space of random variables on Ξ_1, and $\rho_{|\xi_1}$ is a conditional on $\xi_1 \in \Xi_1$ coherent risk measure. Then it is known that if ρ_0 is law invariant and

$\rho_{|\xi_1}$ is law invariant conditional on ξ_1, then $\varrho(Z)$ is law invariant only if either ρ_0 and $\rho_{|\xi_1}$ are the expectation and conditional expectation, respectively, or both ρ_0 and $\rho_{|\xi_1}$ are max-risk measures (cf., [17]).

Using dual representations of coherent risk measures and Proposition 3.2 this result can be translated into the following. Let $\mathfrak{M}$ be a set of probability measures on Ξ. Recall that we can identify a probability measure Q on Ξ with vector $q \in \Delta_K$, where $K = n^2$. Consider the corresponding coherent risk measure $\rho(Z) := \sup_{Q\in\mathfrak{M}} \sum_{i=1}^K q_i Z_i$. Suppose that $\mathfrak{M}$ is *invariant* with respect to permutations of the set Ξ. It follows by Proposition 3.2 (i) that $\rho : \Xi \to \mathbb{R}$ is law invariant. Rectangularity of $\mathfrak{M}$ means that ρ can be decomposed into a composition of coherent risk measures, and by the invariance of $\mathfrak{M}$ these risk measures are law invariant. By the above discussion it follows that ρ is either the expectation or the max-risk measure. That is, $\mathfrak{M}$ is either the singleton $\mathfrak{M} = \{P\}$ or $\mathfrak{M}$ is such that

$$\sup_{Q\in\mathfrak{M}} \sum_{i=1}^K q_i Z_i = \max_{1\le i\le K} Z_i, \;\; \forall Z \in \mathcal{Z}. \tag{3.11}$$

If we assume further that $\mathfrak{M}$ is convex and closed, then (3.11) holds iff $\mathfrak{M}$ consists of all probability distributions.

By the above discussion we have the following result.

Proposition 3.3 *Suppose that in the considered framework of finite number of scenarios with equal probabilities, the set $\mathfrak{M}$ of probability measures on Ξ is* invariant *with respect to permutations of the set Ξ. Then $\mathfrak{M}$ can be rectangular with respect to $\mathcal{Z}$ only if either $\mathfrak{M} = \{P\}$ is the singleton or is such that* (3.11) *holds.*

This could be extended to an appropriate framework of finite number of scenarios with equal probabilities for more than three stages. Also it should be possible to extend this result to an appropriate framework with *nonatomic* distributions (this could be a subject of future research). This indicates that the requirement for the set $\mathfrak{M}$ to be rectangular with respect to the whole space $\mathcal{Z}$ could be quite restrictive in some situations.

4 Multistage optimization

As it was pointed out in the Introduction there are three somewhat different modeling approaches to formulation of multistage stochastic optimization problems; namely, Stochastic Programming (SP), Stochastic Optimal Control (SOC) and Markov Decision Process (MDP). In this section we quickly describe these formulations in a standard setting.

Stochastic Programming. In a fairly general form a T-stage stochastic programming problem can be written as (e.g., [18])

$$\begin{array}{ll} \text{Min} & \mathbb{E}\big[c_1(x_1) + c_2(x_2, \xi_2) + \cdots + c_T(x_T, \xi_T)\big], \\ \text{s.t.} & x_1 \in \mathcal{X}_1, \; x_t \in \mathcal{X}_t(x_{t-1}, \xi_t), \; t = 2, ..., T-1, \end{array} \tag{4.1}$$

driven by the random data process $\xi_1, \xi_2, \ldots, \xi_T$. Here $x_t \in \mathbb{R}^{n_t}$, $t = 1, \ldots, T$, are decision variables, $c_t : \mathbb{R}^{n_t} \times \mathbb{R}^{d_t} \to \mathbb{R}$ are continuous functions and $\mathcal{X}_t : \mathbb{R}^{n_{t-1}} \times \mathbb{R}^{d_t} \rightrightarrows \mathbb{R}^{n_t}$, $t = 2, \ldots, T$, are measurable multifunctions. The first stage data, i.e., the vector ξ_1, the function $c_1 : \mathbb{R}^{n_1} \to \mathbb{R}$, and the set $\mathcal{X}_1 \subset \mathbb{R}^{n_1}$ are deterministic. The optimization (minimization) in (4.1) is performed over (measurable) policies $x_t = x_t(\xi_{[t]})$, $t = 1, ..., T$. The feasibility constraints in (4.1) should be satisfied for a.e. realization of the random process $\xi_1, \xi_2, \dots, \xi_T$.

Problem (4.1) can be written in the following nested form

$$\operatorname*{Min}_{x_1 \in \mathcal{X}_1} c_1(x_1) + \mathbb{E}_{|\xi_1} \Big[\inf_{x_2 \in \mathcal{X}_2(x_1, \xi_2)} c_2(x_2, \xi_2) + \cdots + \mathbb{E}_{|\xi_{[T-1]}} \big[\inf_{x_T \in \mathcal{X}_T(x_{T-1}, \xi_T)} c_T(x_T, \xi_T) \big] \Big], \tag{4.2}$$

by emphasizing the conditional structure of this formulation. Recall that ξ_1 is deterministic, therefore the conditional expectation $\mathbb{E}_{|\xi_1}$ is the same as the corresponding unconditional expectation; we write $\mathbb{E}_{|\xi_1}$ for uniformity of the notation.

Nested formulation (4.2) leads to the corresponding dynamic programming equations for the cost-to-go (value) functions

$$V_t\left(x_{t-1}, \xi_{[t]}\right) = \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \Big\{ c_t(x_t, \xi_t) + \mathbb{E}_{|\xi_{[t]}} \left[V_{t+1}\left(x_t, \xi_{[t+1]}\right) \right] \Big\}, \tag{4.3}$$

$t = 1, ..., T$, with $V_{T+1}(\cdot, \cdot) \equiv 0$ by the definition, and for $t = 1$ the set $\mathcal{X}_1$ is deterministic.

Recall that the process $\xi_1, \xi_2, \dots, \xi_T$ is said to be *Markovian* if the conditional distribution of ξ_{t+1} given $\xi_{[t]}$ is the same as the conditional distribution of ξ_{t+1} given ξ_t, $t = 1, ..., T-1$; and it is *stagewise independent* if ξ_{t+1} is independent of $\xi_{[t]}$, $t = 1, ..., T-1$. For the Markovian process the cost-to-go (value) functions $V_t\left(x_{t-1}, \xi_t\right)$ depend only on ξ_t rather than the whole history $\xi_{[t]}$ of the random process, and dynamic programming equations (4.3) take the form

$$V_t\left(x_{t-1}, \xi_t\right) = \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \left\{ c_t(x_t, \xi_t) + \mathbb{E}_{|\xi_t} \left[V_{t+1}\left(x_t, \xi_{t+1}\right) \right] \right\}. \tag{4.4}$$

In that case an optimal policy $\bar{x}_t = \pi_t(\bar{x}_{t-1}, \xi_t)$ is defined by

$$\bar{x}_t \in \operatorname*{arg\,min}_{x_t \in \mathcal{X}_t(\bar{x}_{t-1}, \xi_t)} \left\{ c_t(x_t, \xi_t) + \mathbb{E}_{|\xi_t} \left[V_{t+1}\left(x_t, \xi_{t+1}\right) \right] \right\}. \tag{4.5}$$

If, moreover, the stagewise independence condition holds, then the conditional expectation in (4.4) and (4.5) becomes the corresponding unconditional expectation.

Stochastic Optimal Control, Discrete Time Case. In a fairly general form a SOC problem can be written as (e.g., [2])

$$\begin{array}{cl} \operatorname*{Min}_{y,u} & \mathbb{E}\left[\sum_{t=1}^{T} c_t(y_t, u_t, \xi_t) + c_{T+1}(y_{T+1}) \right], \\ \text{s.t.} & y_{t+1} = F_t(y_t, u_t, \xi_t), \ t = 1, ..., T, \\ & u_t \in \mathcal{U}_t(y_t), \ t = 1, ..., T. \end{array} \tag{4.6}$$

Here $y_t \in \mathbb{R}^{n_t}$, $t = 1, ..., T+1$, represent states of the system, $u_t \in \mathbb{R}^{m_t}$, $t = 1, ..., T$, are controls, $\xi_t \in \mathbb{R}^{d_t}$, $t = 1, ..., T$, are random vectors (random noise or disturbances), $c_t : \mathbb{R}^{n_t} \times \mathbb{R}^{m_t} \times \mathbb{R}^{d_t} \to \mathbb{R}$, $t = 1, ..., T$, are cost functions, $c_{T+1}(y_{T+1})$ is final cost function, $F_t : \mathbb{R}^{n_t} \times \mathbb{R}^{m_t} \times \mathbb{R}^{d_t} \to \mathbb{R}^{n_{t+1}}$ are measurable mappings, and $\mathcal{U}_t : \mathbb{R}^{n_t} \rightrightarrows \mathbb{R}^{m_t}$ are multifunctions. Values y_1 and ξ_0 are deterministic (initial conditions). The optimization in (4.6) is performed over policies satisfying the corresponding feasibility constraints w.p.1.

In this formulation there is an explicit separation between control variables u_t and state variables y_t. In principle it is possible to formulate problem (4.6) in the framework of the stochastic programming problem (4.1) by defining decision variables $x_t := (y_{t+1}, u_t)$, multifunctions

$$\mathcal{X}_t(y_t, \xi_t) := \big\{ (y_{t+1}, u_t) : y_{t+1} = F_t(y_t, u_t, \xi_t), \ u_t \in \mathcal{U}_t(y_t) \big\},$$

and constraints $(y_{t+1}, u_t) \in \mathcal{X}_t(y_t, \xi_t)$.

The corresponding dynamic programming equations here take the form (compare with (4.3)): $V_{T+1}(y_{T+1}) = c_{T+1}(y_{T+1})$ and

$$V_t(y_t, \xi_{[t-1]}) = \inf_{u_t \in \mathcal{U}_t(y_t)} \mathbb{E}_{|\xi_{[t-1]}}\big[c_t(y_t, u_t, \xi_t) + V_{t+1}(F_t(y_t, u_t, \xi_t), \xi_{[t]})\big],\ t = 1, ..., T. \tag{4.7}$$

Therefore optimization in (4.6) can be performed over policies of the form $u_t = \pi_t(y_t, \xi_{[t-1]})$, $t = 1, ..., T$.

If the process $\xi_1, \xi_2, \ldots, \xi_T$ is Markovian, then the value functions

$$V_t(y_t, \xi_{t-1}) = \inf_{u_t \in \mathcal{U}_t(y_t)} \mathbb{E}_{|\xi_{t-1}}\big[c_t(y_t, u_t, \xi_t) + V_{t+1}(F_t(y_t, u_t, \xi_t), \xi_t)\big],\ t = 1, ..., T. \tag{4.8}$$

depend only on ξ_{t-1} rather than the whole history $\xi_{[t-1]}$, and an optimal policy $\bar{u}_t = \pi_t(\bar{y}_t, \xi_{t-1})$ is defined by the optimality conditions

$$\bar{u}_t \in \arg\min_{u_t \in \mathcal{U}_t(y_t)} \mathbb{E}_{|\xi_{t-1}}\big[c_t(\bar{y}_t, u_t, \xi_t) + V_{t+1}(F_t(\bar{y}_t, u_t, \xi_t), \xi_t)\big],\ t = 1, ..., T. \tag{4.9}$$

Hence in the Markovian case we can formulate problem (4.6) as minimization over Markovian policies $\pi \in \Pi$ of the form of (measurable) functions $u_t = \pi_t(y_t, \xi_{t-1})$, $t = 1, ..., T$, satisfying the feasibility constraints

$$\begin{gathered} y_{t+1} = F_t(y_t, \pi_t(y_t, \xi_{t-1}), \xi_t),\ t = 1, ..., T, \\ \pi_t(y_t, \xi_{t-1}) \in \mathcal{U}_t(y_t),\ t = 1, ..., T. \end{gathered} \tag{4.10}$$

That is,

$$\operatorname*{Min}_{\pi \in \Pi} \mathbb{E}\left[\sum_{t=1}^{T} c_t(y_t, \pi_t(y_t, \xi_{t-1}), \xi_t) + c_{T+1}(y_{T+1})\right]. \tag{4.11}$$

If, moreover, the stagewise independence condition holds, then equations (4.7) become

$$V_t(y_t) = \inf_{u_t \in \mathcal{U}_t(y_t)} \mathbb{E}\big[c_t(y_t, u_t, \xi_t) + V_{t+1}(F_t(y_t, u_t, \xi_t))\big],\ t = 1, ..., T, \tag{4.12}$$

with the value functions $V_t(y_t)$ independent of the noise process. In the stagewise independence case, conditional expectations in the optimality conditions (4.9) become the corresponding unconditional expectations and an optimal policy $\bar{u}_t = \pi_t(\bar{y}_t)$, $t = 1, ..., T$, is a function of $\bar{y}_t$, with $\bar{y}_t = F_{t-1}(\bar{y}_{t-1}, \bar{u}_{t-1}, \xi_{t-1})$, $t = 2, ..., T$, and given initial value $\bar{y}_1$ (closed-loop, feedback policy).

So far it was assumed that probability distribution of the data process $\xi_1, ..., \xi_T$ does not depend on our decisions. Consider now the Markovian framework, with conditional distribution $Q_t(y_t, u_t, \xi_{t-1})$ of ξ_t, given ξ_{t-1}, being also a function of (y_t, u_t). Furthermore, consider optimization over Markovian policies $u_t = \pi_t(y_t, \xi_{t-1})$, $t = 1, ..., T$. Now the conditional distribution $\hat{Q}^{\pi}_t(y_t, \xi_{t-1}) = Q_t(y_t, \pi_t(y_t, \xi_{t-1}), \xi_{t-1})$ of ξ_t, given ξ_{t-1}, is also a function of π. Consequently the distribution in the objective of problem (4.11) depends on $\pi \in \Pi$. We emphasize this by writing $\mathbb{E}^{\pi}$ for the corresponding expectation in (4.11). In the corresponding dynamic programming equations (4.8) and (4.9) the expectation is taken with respect to the conditional distribution $Q_{t|\xi_{t-1}}(y_t, u_t) = Q_t(y_t, u_t, \xi_{t-1})$.

Markov Decision Processes, e.g., [11]. A finite horizon MDP is defined by decision horizon $\mathcal{T} = \{1, ..., T\}$, for each $t \in \mathcal{T}$ there are state space $\mathcal{S}_t$, action set $\mathcal{A}_t(s_t)$, $s_t \in \mathcal{S}_t$, and cost $c_t(s_t, a_t, s_{t+1})$, for the sake of simplicity we assume that the state and action spaces are discrete.

With state-action pair $(s_t, a_t) \in \mathcal{S}_t \times \mathcal{A}_t(s_t)$ is associated transition probability $p_t(s_{t+1}|s_t, a_t)$ of moving from state s_t to state s_{t+1}.

We consider deterministic Markovian policies $a_t = \pi_t(s_t)$, $t = 1, ..., T$. Then the corresponding optimization problem can be written as

$$\operatorname*{Min}_{\pi \in \Pi} \mathbb{E}^{\pi} \left[\sum_{t=1}^{T} c_t(s_t, \pi_t(s_t), s_{t+1}) + c_{T+1}(s_{T+1}) \right], \tag{4.13}$$

with initial distribution q_1 of s_1. The associated dynamic equations are: $V_{T+1}(s_{T+1}) = c_{T+1}(s_{T+1})$, and

$$V_t(s_t) = \inf_{a_t \in \mathcal{A}_t(s_t)} \mathbb{E}\big[c_t(s_t, a_t, s_{t+1}) + V_{t+1}(s_{t+1})\big], \ t \in \mathcal{T}. \tag{4.14}$$

These equations are similar to (4.8) with s_{t+1} viewed as a random variable with (conditional) probability distribution $p_t(\cdot|s_t, a_t)$ of s_{t+1} and the expectation in the right hand side of (4.14) is taken with respect to this distribution given s_t.

5 Multistage robust optimization problems

In the framework of sections 2 and 4 consider the following approaches to multistage robust optimization. Note that we can view $(\xi_1, ..., \xi_T) \in \Xi = \Xi_1 \times \cdots \times \Xi_T$ as a sequence of random vectors equipped with probability distribution $Q \in \mathfrak{M}$. As in section 2 we consider the corresponding Banach space $\mathcal{Z}$ and its dual space $\mathcal{Z}^*$. We assume that the uncertainty set $\mathfrak{M}$, of probability distributions, is a bounded subset of $\mathcal{Z}^*$. Let us start with the stochastic programming approach.

Robust Stochastic Programming. Consider the following robust counterpart of the multistage stochastic programming problem (4.1)

$$\operatorname*{Min}_{\pi \in \Pi} \sup_{Q \in \mathfrak{M}} \mathbb{E}_Q\big[Z^{\pi}\big], \tag{5.1}$$

where $\mathfrak{M} \subset \mathcal{Z}^*$ is a bounded set of probability measures, Π is a set of policies of the form $\pi = (x_1, x_2(\xi_{[2]}), ..., x_T(\xi_{[T]}))$ satisfying the feasibility constraints

$$x_1 \in \mathcal{X}_1, \ x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}), \xi_t), \ t = 2, ..., T-1, \tag{5.2}$$

and $Z^{\pi} = Z^{\pi}(\xi_{[T]})$ is defined as

$$Z^{\pi} := c_1(x_1) + c_2(x_2(\xi_{[2]}), \xi_2) + \cdots + c_T(x_T(\xi_{[T]}), \xi_T). \tag{5.3}$$

We make the following assumption.

(B) *Assume that* $Z^{\pi} \in \mathcal{Z}$ *for every* $\pi \in \Pi$.

This assumption imposes a certain type of constraints on the set Π of considered policies. If the set Ξ of possible scenarios is finite, the above assumption is rather straightforward. In case of $\mathcal{Z} := L_p(\Xi, \mathcal{B}, P)$ the considered policies should be measurable and the random variables Z^{π} should have finite p-th moments with respect to the reference distribution P. In that case the feasibility constraints (5.2) should be satisfied for a.e. $\xi_{[T]}$ with respect to the reference measure P. In case of $\mathcal{Z} := C(\Xi)$ with Ξ being compact and all involved data functions being continuous, it is natural to consider policies which are continuous functions of $\xi_1, ..., \xi_T$ (see the Appendix Section 6). In this case the feasibility constraints should be satisfied for all $\xi_{[T]} \in \Xi$.

Let $\widehat{\mathfrak{M}}$ be a rectangular set, associated with $\mathfrak{M}$ and $\mathcal{Z}$ (existence of such rectangular set is discussed in Theorem 2.1). Consider the associated problem by replacing the set $\mathfrak{M}$ in (5.1) with the set $\widehat{\mathfrak{M}}$:

$$\operatorname*{Min}_{\pi \in \Pi} \sup_{Q \in \widehat{\mathfrak{M}}} \mathbb{E}_Q\big[Z^\pi\big]. \tag{5.4}$$

By (2.4) problem (5.4) can be represented in the corresponding nested form

$$\operatorname*{Min}_{\pi \in \Pi} \sup_{Q \in \mathfrak{M}_{[1]}} \mathbb{E}_Q \left[\sup_{Q \in \mathfrak{M}_{[2]}} \mathbb{E}_{Q|\xi_{[1]}} \Big[\cdots \sup_{Q \in \mathfrak{M}_{[T]}} \mathbb{E}_{Q|\xi_{[T-1]}} [Z^\pi] \Big] \right], \tag{5.5}$$

where $\mathfrak{M}_{[t]}$ are the sets of respective marginal distributions (defined in (2.10)). We refer to (5.1) as *multistage static formulation* and to (5.5) as *multistage dynamic formulation* of robust multistage stochastic programming problems. If the set $\mathfrak{M}$ is of the product form (2.12), then the dynamic formulation (5.5) takes the form (compare with (2.14))

$$\operatorname*{Min}_{\pi \in \Pi} \sup_{Q_1 \in \mathfrak{M}_1} \mathbb{E}_{Q_1} \left[\sup_{Q_2 \in \mathfrak{M}_2} \mathbb{E}_{Q_2|\xi_{[1]}} \Big[\cdots \sup_{Q_T \in \mathfrak{M}_T} \mathbb{E}_{Q_T|\xi_{[T-1]}} [Z^\pi] \Big] \right]. \tag{5.6}$$

Remark 2 It could be noted that for coherent risk measure $\varrho(Z)$ of the form (3.1), the static problem (5.1) can be written as

$$\operatorname*{Min}_{\pi \in \Pi} \varrho(Z^\pi). \tag{5.7}$$

The corresponding dynamic problem (5.5) can be written as

$$\operatorname*{Min}_{\pi \in \Pi} \varrho_1 \left[\varrho_{2|\xi_{[1]}} \Big[\cdots \varrho_{T|\xi_{[T-1]}}(Z^\pi) \Big] \right], \tag{5.8}$$

where $\varrho_{t|\xi_{[t-1]}}$ are the respective conditional risk mappings. In such framework the static and dynamic formulations of robust multistage stochastic programming problems can be considered as risk averse counterparts of the risk neutral problem (4.1). ■

For the dynamic formulation (5.5) we can write the following dynamic programming equations for value (cost-to-go) functions (compare with (4.3))

$$V_t\left(x_{t-1}, \xi_{[t]}\right) = \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \left\{ c_t(x_t, \xi_t) + \sup_{Q \in \mathfrak{M}_{[t+1]}} \mathbb{E}_{Q|\xi_{[t]}} \big[V_{t+1}(x_t, \xi_{[t+1]})\big] \right\}, \tag{5.9}$$

$t = 1, ..., T$, with $V_{T+1}(\cdot, \cdot) \equiv 0$. If the set $\mathfrak{M}$ is of the product form (2.12), and hence the respective dynamic formulation takes the form (5.6), then the value functions $V_t(x_{t-1}, \xi_t)$ depend only on ξ_t rather than the whole history $\xi_{[t]}$, and the dynamic equations (5.9) take the form

$$V_t\left(x_{t-1}, \xi_t\right) = \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \left\{ c_t(x_t, \xi_t) + \sup_{Q_{t+1} \in \mathfrak{M}_{t+1}} \mathbb{E}_{Q_{t+1}} \big[V_{t+1}(x_t, \xi_{t+1})\big] \right\}, \tag{5.10}$$

In that case an optimal policy $\bar{x}_t = \pi_t(\bar{x}_{t-1}, \xi_t)$ is defined by the corresponding optimality conditions in a way similar to (4.5).

Robust Stochastic Optimal Control. Robust formulation of the SOC problem (4.6) can be written as

$$\begin{array}{cl} \underset{y,u}{\text{Min}} & \sup_{Q\in\mathfrak{M}} \mathbb{E}_Q\left[\sum_{t=1}^T c_t(y_t,u_t,\xi_t)+c_{T+1}(y_{T+1})\right] \\ \text{s.t.} & y_{t+1}=F_t(y_t,u_t,\xi_t),\ t=1,...,T, \\ & u_t\in\mathcal{U}_t(y_t),\ t=1,...,T. \end{array} \tag{5.11}$$

Similar to (5.4) the dynamic counterpart of (5.11) is obtained by replacing the set $\mathfrak{M}$ with $\widehat{\mathfrak{M}}$. The corresponding dynamic programming equations here take the form (compare with (4.7) and (5.9)): $V_{T+1}(y_{T+1})=c_{T+1}(y_{T+1})$ and

$$V_t(y_t,\xi_{[t-1]})=\inf_{u_t\in\mathcal{U}_t(y_t)}\ \sup_{Q\in\mathfrak{M}_{[t]}} \mathbb{E}_{Q|\xi_{[t-1]}}\left[c_t(y_t,u_t,\xi_t)+V_{t+1}(F_t(y_t,u_t,\xi_t),\xi_{[t]})\right],\ t=1,...,T. \tag{5.12}$$

Furthermore if the uncertainty set $\mathfrak{M}$ is of the product form (2.12), then equations (5.12) become (compare with (4.12) and (5.10)):

$$V_t(y_t)=\inf_{u_t\in\mathcal{U}_t(y_t)}\ \sup_{Q_t\in\mathfrak{M}_t} \mathbb{E}_{Q_t}\left[c_t(y_t,u_t,\xi_t)+V_{t+1}(F_t(y_t,u_t,\xi_t))\right],\ t=1,...,T. \tag{5.13}$$

In that case the optimality conditions are

$$\bar u_t\in\arg\min_{u_t\in\mathcal{U}_t(y_t)}\left\{\sup_{Q_t\in\mathfrak{M}_t}\mathbb{E}_{Q_t}\left[c_t(\bar y_t,u_t,\xi_t)+V_{t+1}(F_t(\bar y_t,u_t,\xi_t))\right]\right\},\ t=1,...,T. \tag{5.14}$$

That is, here an optimal policy $\bar u_t=\pi_t(\bar y_t)$, $t=1,...,T$, is a function of $\bar y_t$, with $\bar y_t=F_{t-1}(\bar y_{t-1},\bar u_{t-1},\xi_{t-1})$, $t=2,...,T$, and given initial value $\bar y_1$ (closed-loop, feedback policy).

In particular, consider the setting of Example 1 with $\mathcal{Z}=C(\Xi)$ and $\mathfrak{M}$ being the set of all probability measures on $(\Xi,\mathcal{B})$. Then $\mathfrak{M}=\widehat{\mathfrak{M}}$ and the dynamic programming equations (5.13) become

$$V_t(y_t)=\inf_{u_t\in\mathcal{U}_t(y_t)}\ \sup_{\xi_t\in\Xi_t}\left\{c_t(y_t,u_t,\xi_t)+V_{t+1}(F_t(y_t,u_t,\xi_t))\right\},\ t=1,...,T. \tag{5.15}$$

That is, the problem becomes a minimax optimal control problem, cf., [2, section 2.3.5].

So far we assumed that probability distributions of the random data process $\xi_1,...,\xi_T$ do not depend on our actions. Now consider the case where the uncertainty set is of the product form and suppose that the probability distributions $Q_t(\cdot|y_t,u_t)$, of ξ_t, could depend on y_t and u_t and belong to a specified set $\mathfrak{M}_t(y_t,u_t)$, $t=1,...,T$. Consider policies $\pi\in\Pi$ (controls) of the form $u_t=\pi_t(y_t)$, $t=1,...,T$, satisfying the feasibility constraints

$$\begin{array}{l} y_{t+1}=F_t(y_t,\pi_t(y_t),\xi_t),\ t=1,...,T, \\ \pi_t(y_t)\in\mathcal{U}_t(y_t),\ t=1,...,T. \end{array} \tag{5.16}$$

For a given policy $\pi\in\Pi$, the corresponding set $\mathfrak{M}^\pi$ of probability measures of $(\xi_1,...,\xi_T)$ is

$$\mathfrak{M}^\pi=\left\{Q=Q_1(\cdot|y_1,u_1)\times\cdots\times Q_T(\cdot|y_T,u_T): Q_t(\cdot|y_t,u_t)\in\mathfrak{M}_t(y_t,\pi(y_t)),\ t=1,...,T\right\}, \tag{5.17}$$

of conditional probability measures. We can view $y_t=y_t(\xi_{[t-1]})$, $t=2,...,T$, as a function of $\xi_{[t-1]}=(\xi_1,...,\xi_{t-1})$ and hence view $Q\in\mathfrak{M}^\pi$ as a probability measure on $(\xi_1,...,\xi_T)$ depending on $\pi\in\Pi$.

The corresponding (static) formulation, similar to (5.11), can be written as

$$\operatorname*{Min}_{\pi\in\Pi}\sup_{Q\in\mathfrak{M}^\pi}\mathbb{E}_Q\left[\sum_{t=1}^T c_t(y_t,\pi_t(y_t),\xi_t)+c_{T+1}(y_{T+1})\right]. \tag{5.18}$$

Nested formulation of the associated dynamic problem can be written as

$$\begin{aligned}\operatorname*{Min}_{\pi\in\Pi}\ \sup_{Q\in\mathfrak{M}^\pi}\mathbb{E}_{Q|y_1}\Big[c_1(y_1,\pi_1(y_1),\xi_1)+\sup_{Q\in\mathfrak{M}^\pi}\mathbb{E}_{Q|y_2}\Big[c_2\left(y_2,\pi_2(y_2),\xi_2\right)\\ +\cdots+\sup_{Q\in\mathfrak{M}^\pi}\mathbb{E}_{Q|y_T}\left[c_T\left(y_T,\pi(y_T),y_T\right)+c_{T+1}(y_{T+1})\right]\Big]\Big].\end{aligned} \tag{5.19}$$

The conditional expectations $\mathbb{E}_{Q|y_t}$ in (5.19) are understood in the natural way as expectations with respect to $Q\in\mathfrak{M}^\pi$ for given (random variable) y_t and $u_t=\pi(y_t)$, $t=1,...,T$.

By Theorem 2.1 we have that for a given $\pi\in\Pi$ there exists a set $\widehat{\mathfrak{M}}^\pi$ of probability measures on $(\Xi,\mathcal{B})$ such that the nested problem (5.19) can be equivalently written as

$$\operatorname*{Min}_{\pi\in\Pi}\sup_{Q\in\widehat{\mathfrak{M}}^\pi}\mathbb{E}_Q\left[\sum_{t=1}^T c_t(y_t,\pi_t(y_t),\xi_t)+c_{T+1}(y_{T+1})\right]. \tag{5.20}$$

For the (dynamic) problem (5.19) the corresponding dynamic programming equations take the form: $V_{T+1}(y_{T+1})=c_{T+1}(y_{T+1})$ and

$$V_t(y_t)=\inf_{u_t\in\mathcal{U}_t(y_t)}\ \sup_{Q_t\in\mathfrak{M}_t(y_t,u_t)}\mathbb{E}_{Q_t}\left[c_t(y_t,u_t,\xi_t)+V_{t+1}(F_t(y_t,u_t,\xi_t))\right],\ t=1,...,T. \tag{5.21}$$

Robust Markov Decision Processes. In definition of robust MDP we follow Iyengar [6] (see also [8]). A finite horizon MDP is defined by decision horizon $\mathcal{T}=\{1,...,T\}$, for each $t\in\mathcal{T}$ there are state space $\mathcal{S}_t$, action set $\mathcal{A}_t(s_t)$, $s_t\in\mathcal{S}_t$, and cost $c_t(s_t,a_t,c_{t+1})$, for the sake of simplicity we assume that the state and action states are discrete. With state-action pair $(s_t,a_t)\in\mathcal{S}_t\times\mathcal{A}_t(s_t)$ is associated set $\mathcal{P}_t(s_t,a_t)$ of transition probabilities $p_t(s_{t+1}|s_t,a_t)$ of moving from state s_t to state s_{t+1}. We consider the set Π of deterministic Markov policies $a_t=\pi_t(s_t)$. For a policy $\pi\in\Pi$, the set $\mathfrak{M}^\pi$ is defined as the set of transition probabilities (compare with (5.17)) $p_t(s_{t+1}|s_t,\pi_t(s_t))\in\mathcal{P}_t(s_t,\pi_t(s_t))$. That is, given state $s_1\in\mathcal{S}_1$ we consider all transition probabilities $p_1(s_2|s_1,\pi_1(s_1))\in\mathcal{P}_1(s_1,\pi_1(s_1))$. Next for given state $s_2\in\mathcal{S}_2$ and every previously chosen transition probability of moving from state s_1 to state s_2 we consider all transition probabilities $p_2(s_3|s_2,\pi_2(s_2))\in\mathcal{P}_2(s_2,\pi_2(s_2))$ of moving to state s_3, and so on.

The static formulation can be written then as the min-max problem (compare with (5.18))

$$\operatorname*{Min}_{\pi\in\Pi}\sup_{P\in\mathfrak{M}^\pi}\mathbb{E}_P\left[\sum_{t=1}^T c_t(s_t,\pi_t(s_t),s_{t+1})+c_{T+1}(s_{T+1})\right], \tag{5.22}$$

with initial distribution q_1 of s_1.

Dynamic programming equations for the respective dynamic formulation can be written as [6, Theorem 2.2]: $V_{T+1}(s_{T+1})=c_{T+1}(s_{T+1})$ and

$$V_t(s_t)=\inf_{a_t\in\mathcal{A}_t(s_t)}\left\{\sup_{p\in\mathcal{P}_t(s_t,a_t)}\mathbb{E}_p\left[c_t(s_t,a_t,s_{t+1})+V_{t+1}(s_{t+1})\right]\right\},\ t\in\mathcal{T}. \tag{5.23}$$

These equations are similar to (5.21) with s_{t+1} viewed as a random variable with (conditional) probability distribution $p \in \mathcal{P}_t(s_t, a_t)$. By the general theory we have that for every policy $\pi \in \Pi$ there exists a rectangular set $\widehat{\mathfrak{M}}^\pi$ of transition probabilities, associated with $\mathfrak{M}^\pi$, such that the dynamic formulation can be written as the min-max problem:

$$\operatorname*{Min}_{\pi \in \Pi} \sup_{P \in \widehat{\mathfrak{M}}^\pi} \mathbb{E}_P \left[\sum_{t=1}^{T} c_t(s_t, \pi_t(s_t), s_{t+1}) + c_{T+1}(s_{T+1}) \right]. \tag{5.24}$$

6 Appendix

6.1 Interchangeability principle

Let $(\Omega, \mathcal{F})$ be a measurable space. As it was mentioned before we consider two types of linear spaces $\mathcal{Z}$ of measurable functions $Z : \Omega \to \mathbb{R}$, namely $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$, and $\mathcal{Z} = C(\Omega)$. In case of $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$ interchangeability property of a convex functional $\rho : \mathcal{Z} \to \mathbb{R}$, satisfying the monotonicity property (A2) (described in the proof of Theorem 2.1), is discussed in [18, Proposition 6.60]. Such interchangeability property is needed in order to derive the corresponding dynamic programming equations. We discuss here the interchangeability property for the space $\mathcal{Z} = C(\Omega)$.

Let Ω be a compact metric space and consider the linear space of continuous functions $f : \Omega \to \mathbb{R}$ equipped with the sup-norm $\|f\| := \sup_{\omega \in \Omega} |f(\omega)|$. This is a Banach space, denoted $C(\Omega)$. By a representation theorem (due to Riesz), the dual space of $C(\Omega)$ is formed by the linear space of finite signed regular Borel measures on $(\Omega, \mathcal{B})$, with the scalar product $\langle f, \mu \rangle = \int_\Omega f(\omega) d\mu(\omega)$ and the respective dual norm given by the total variation

$$\|\mu\|^* = \sup_{A \in \mathcal{B}} \mu(A) - \inf_{B \in \mathcal{B}} \mu(B).$$

The partial order and the least upper bound are defined in the natural way. For $f, g \in C(\Omega)$ it is said that $f \succeq g$ if $f(\omega) \geq g(\omega)$ for all $\omega \in \Omega$, $(f \vee g)(\omega) := \max\{f(\omega), g(\omega)\}$. With this, $C(\Omega)$ becomes a Banach lattice.

Consider a function $f : \mathbb{R}^n \times \Omega \to \mathbb{R}$. Suppose that $f(x, \omega)$ is continuous on $\mathbb{R}^n \times \Omega$ and let X be a nonempty compact subset of $\mathbb{R}^n$. Then $\inf_{x \in X} f(x, \omega)$ is continuous in ω and hence is an element of the space $\mathcal{Z}$.

Proposition 6.1 *Suppose that $\rho : \mathcal{Z} \to \mathbb{R}$ satisfies the monotonicity condition* (A2) *and the translation equivariance condition* (A3). *Let A_X be a set of mappings $\chi : \Omega \to X$ such that:* (i) *for every $\chi \in A_X$ the function $f(\chi(\omega), \omega)$ is continuous on Ω,* (ii) *for any $\varepsilon > 0$ there exists $\chi \in A_X$ such that*

$$f(\chi(\omega), \omega) \leq \inf_{x \in X} f(x, \omega) + \varepsilon, \; \forall \omega \in \Omega. \tag{6.1}$$

Then the following interchangeability property holds

$$\inf_{\chi \in A_X} \rho\big(f(\chi(\omega), \omega)\big) = \rho\left(\inf_{x \in X} f(x, \omega)\right). \tag{6.2}$$

Proof. For any $\chi \in A_X$ we have that $\chi(\omega) \in X$ and hence $f(\chi(\omega), \omega) \geq \inf_{x \in X} f(x, \omega)$ for all $\omega \in \Omega$. By monotonicity property of ρ it follows that

$$\rho\big(f(\chi(\omega), \omega)\big) \geq \rho\left(\inf_{x \in X} f(x, \omega)\right).$$

Consequently the left hand side of (6.2) is greater than or equal to the right hand side of (6.2).

Conversely let $\chi \in A_X$ be satisfying condition (6.1). Then

$$\rho\big(f(\chi(\omega),\omega)\big) \le \rho\left(\inf_{x\in X} f(x,\omega)+\varepsilon\right) = \rho\left(\inf_{x\in X} f(x,\omega)\right)+\varepsilon. \tag{6.3}$$

Since $\varepsilon > 0$ is arbitrary it follows that the right hand side of (6.2) is greater than or equal to the left hand side of (6.2). This completes the proof. ∎

Remark 3 A natural question is whether we can take A_X to be the set of *all continuous* mappings $\chi : \Omega \to X$. Since $f(x,\omega)$ is continuous, it follows that if $\chi(\omega)$ is continuous, then $f(\chi(\omega),\omega)$ is continuous. Therefore we only need to verify condition (ii) of the above proposition. Since the set X is compact, it follows that for every $\omega \in \Omega$ the minimum of $f(x,\omega)$ over $x \in X$ is attained at a point $\bar{x} = \bar{x}(\omega) \in X$. Suppose further that the minimizer $\bar{x}(\omega)$ is *unique* for all $\omega \in \Omega$. Then $\bar{x}(\omega)$ is continuous, and we can take A_X to be the set of all continuous mappings $\chi : \Omega \to X$.

Suppose now that for every $c > 0$ the function $f(x,\omega)+c\,x^{\mathsf T}x$ attains its minimum over $x \in X$ at unique point $\bar{x}_c = \bar{x}_c(\omega)$ for all $\omega \in \Omega$. This holds, in particular, if $f(x,\omega)$ is convex in x and the set X is convex. Then

$$f(\bar{x}_c,\omega) \le f(x,\omega) + c(x^{\mathsf T}x - \bar{x}_c^{\mathsf T}\bar{x}_c),\ \forall x \in X.$$

Since the set X is bounded, the term $c(x^{\mathsf T}x - \bar{x}_c^{\mathsf T}\bar{x}_c)$ can be made smaller than $\varepsilon > 0$ for all $x \in X$ and sufficiently small $c > 0$. It follows that in this case we can take A_X to be the set of all continuous mappings $\chi : \Omega \to X$.

Remark 4 We have that a coherent risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ has the dual representation

$$\rho(Z) = \sup_{Q\in\mathfrak{A}} \int_\Omega Z(\omega)dQ(\omega), \tag{6.4}$$

for some bounded set $\mathfrak{A} \subset \mathcal{Z}^*$ of probability measures. Therefore the interchangeability formula (6.2) can be written in the form

$$\inf_{\chi\in A_X}\sup_{Q\in\mathfrak{A}} \int_\Omega f(\chi(\omega),\omega)dQ(\omega) = \sup_{Q\in\mathfrak{A}} \int_\Omega \left[\inf_{x\in X} f(x,\omega)\right] dQ(\omega). \tag{6.5}$$

It could be noted that the space of continuous mappings $\chi : \Omega \to \mathcal{X}$ is not decomposable in the sense of [12, Definition 14.59]). Therefore the interchangeability theorem of [12, Theorem 14.60] cannot be applied here to the integral in the right hand side of (6.5).

6.2 Finiteness of the max-functions

We discuss now conditions ensuring that the right hand side of (2.3) is finite. In case the set Ξ is compact and $\mathcal{Z} = C(\Xi)$, any function $Z \in C(\Xi)$ is bounded. Its conditional expectation is bounded by the maximum and minimum of $Z(\xi_{[T]})$ over $\xi_{[T]} \in \Xi$. It follows that the right hand side of (2.3) is finite for any set $\mathfrak{M} \subset \mathcal{Z}^*$ of probability measures and $Z \in \mathcal{Z}$.

Consider now the case of $\mathcal{Z} = L_p(\Xi,\mathcal{B},P)$, $p \in [1,\infty)$. Let us start with $p = 1$. Consider the last term in the right hand side of (2.3). For $Q \in \mathfrak{M}$ the corresponding density $\zeta = dQ/dP$ is an element of the dual space $\mathcal{Z}^* = L_\infty(\Xi,\mathcal{B},P)$ with the dual norm (i.e., L_∞ norm) $\|\zeta\|^*$ equal to the essential supremum of ζ. Suppose that the set $\mathfrak{M}$ is bounded, i.e., there is a constant $c > 0$

such that $\|\zeta\|^* \le c$ for all $\zeta \in \mathfrak{M}$ (with some abuse of the notation we write $\zeta \in \mathfrak{M}$ to denote that $Q \in \mathfrak{M}$ for $\zeta = dQ/dP$). Then for $Z \in \mathcal{Z}$,

$$\mathbb{E}_P \left| \sup_{Q \in \mathfrak{M}} \mathbb{E}_{Q|\xi_{[T-1]}}\big[Z\big] \right| = \mathbb{E}_P \left| \sup_{\zeta \in \mathfrak{M}} \mathbb{E}_{P|\xi_{[T-1]}}\big[Z\zeta\big] \right| \preceq c\, \mathbb{E}_P \left[\mathbb{E}_{P|\xi_{[T-1]}}\big[|Z|\big] \right] = c\, \mathbb{E}_P|Z|. \tag{6.6}$$

It follows that the L_1 norm of the last term in the right hand side of (2.3) is bounded by c times the L_1 norm of Z. An so on, we obtain that in this case the right hand side of (2.3) is finite for all $Z \in \mathcal{Z}$ iff the set $\mathfrak{M}$ is bounded.

Consider the general case of $p \in [1, \infty)$. Suppose that the set $\mathfrak{M}$ is of the product form (2.12). By (2.14) in that case we can take the supremum in the last term with respect to $\zeta_T \in \mathcal{Z}_T^* = L_q(\Xi_T, \mathcal{B}_T, P_T)$, where $\zeta_T = dQ_T/dP_T$. We have by Hölder's inequality

$$\mathbb{E}_{P|\xi_{[T-1]}}\big[|Z\zeta_T|\big] \preceq \left(\mathbb{E}_{P|\xi_{[T-1]}}\big[|Z|^p\big] \right)^{1/p} \left(\mathbb{E}_{P|\xi_{[T-1]}}\big[\zeta_T^q\big] \right)^{1/q}, \tag{6.7}$$

and because P has the product form,

$$\left(\mathbb{E}_{P|\xi_{[T-1]}}\big[\zeta_T^q\big] \right)^{1/q} = \left(\mathbb{E}_{P_T}\big[\zeta_T^q\big] \right)^{1/q}. \tag{6.8}$$

Suppose that the set $\mathfrak{M}_T$ is bounded, i.e., there is a constant $c > 0$ such that $\|\zeta_T\|_q = \left(\mathbb{E}_{P_T}\big[\zeta_T^q\big] \right)^{1/q} \le c$ for all $\zeta_T \in \mathfrak{M}_T$. Then it follows by (6.7) and (6.8) that

$$\mathbb{E}_P \left| \sup_{Q_T \in \mathfrak{M}_T} \mathbb{E}_{Q_T|\xi_{[T-1]}}\big[Z\big] \right|^p \le c^p\, \mathbb{E}_P\big[|Z|^p\big].$$

We obtain that in the case the set $\mathfrak{M}$ is of the product form (2.12), the right hand side of (2.14) is finite for all $Z \in \mathcal{Z}$ iff the set $\mathfrak{M}$ is bounded (here the set $\mathfrak{M}$ is bounded iff the sets $\mathfrak{M}_1, ..., \mathfrak{M}_T$ are bounded).

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