

Totally Unimodular Multistage Stochastic Programs

Ruichen (Richard) Sun^a, Oleg V. Shylo^b, Andrew J. Schaefer^{a,*}

^a*Department of Industrial Engineering, University of Pittsburgh, Pittsburgh, PA 15261*

^b*Department of Industrial & Systems Engineering, University of Tennessee, Knoxville, TN 37996-2315*

Abstract

We consider totally unimodular multistage stochastic programs, that is, multistage stochastic programs whose extensive-form constraint matrices are totally unimodular. We establish several sufficient conditions and identify examples that have arisen in the literature.

Keywords: Stochastic mixed integer programming; Total unimodularity; Multistage optimization

1. Introduction

We consider a class of multistage stochastic programs (MSP) whose extensive-form constraint matrices are totally unimodular (TU). TU matrices have been well studied in deterministic mixed integer programming and combinatorial optimization. When the constraint matrix of a mixed integer program (MIP) is TU and the right-hand side is integral, the polyhedron described by the linear relaxation of the MIP is integral [7].

Kong et al. [10] provided several sufficient conditions for two-stage totally unimodular stochastic programs. These conditions used a generalization of TU matrices that we revisit in Section 2. This paper can be viewed as a multistage generalization of Kong et al. [10]. Romeijnnders et al. [12] studied two-stage stochastic mixed integer programs in which the only uncertainty is on the right-hand side. They established that if the probability distribution over the right-hand side is independent and uniform and the recourse matrices are totally unimodular, a certain approximation of the integer recourse

*Corresponding author

Email address: `schaefer@pitt.edu` (Andrew J. Schaefer)

function is precisely its convex hull. Huang [8] reformulated the stochastic single item, uncapacitated dynamic lot-sizing problem without setup costs and proved that the reformulation is totally unimodular. We revisit the problem and show that the total unimodularity of Huang [8]’s original formulation follows from the characterizations in Section 3.

2. Preliminaries

Consider a multistage stochastic mixed-integer program with recourse. For notational convenience and without loss of generality, we assume that the number of continuous decision variables, l , the number of integer decision variables, $n-l$, and the number of constraints, m , are the same in every stage by introducing zero row and column vectors if necessary. The deterministic equivalent program (DEP) of an MSP is

$$(DEP) \quad \min c^1{}^\top x^1 + \mathcal{Q}^2(x^1) \tag{1a}$$

$$\text{s.t. } W^1 x^1 \geq h^1, \tag{1b}$$

$$x^1 \in \mathbb{R}_+^l \times \mathbb{Z}_+^{n-l},$$

where $\mathcal{Q}^2(x^1) = \mathbb{E}_{\xi^2(\omega)} [Q^2(x^1, \xi^2(\omega))]$, and $\mathcal{Q}^\tau(x^{\tau-1}, \xi^{[\tau-1]}(\omega)) = \mathbb{E}_{\xi^\tau(\omega) | \xi^{[\tau-1]}(\omega)} [Q^\tau(x^{\tau-1}, \xi^{[\tau]}(\omega))]$ for $2 < \tau \leq H$, with

$$Q^{\tau'}(x^{\tau'-1}, \xi^{[\tau']}(\omega)) = \min \{ c^{\tau'}(\omega)^\top x^{\tau'}(\omega) + \mathcal{Q}^{\tau'+1}(x^{\tau'}, \xi^{[\tau'+1]}(\omega)) : W^{\tau'}(\omega) x^{\tau'}(\omega) \geq h^{\tau'}(\omega) - T^{\tau'-1}(\omega) x^{\tau'-1}, x^{\tau'}(\omega) \in \mathbb{R}_+^l \times \mathbb{Z}_+^{n-l} \} \text{ for } 2 \leq \tau' \leq H-1, \text{ and}$$

$$Q^H(x^{H-1}, \xi^{[H]}(\omega)) = \min \{ c^H(\omega)^\top x^H(\omega) : W^H(\omega) x^H(\omega) \geq h^H(\omega) - T^{H-1}(\omega) x^{H-1}, x^H(\omega) \in \mathbb{R}_+^l \times \mathbb{Z}_+^{n-l} \}.$$

The vectors $c^1 \in \mathbb{R}^n$, $h^1 \in \mathbb{R}^m$, and the matrix $W^1 \in \mathbb{R}^{m \times n}$ are known. For each $\tau = 2, \dots, H$ and for all ω , $W^\tau(\omega)$ is an $m \times n$ matrix, and $T^{\tau-1}(\omega)$ is an $m \times n$ matrix. $\xi^\tau(\omega)^\top = [c^\tau(\omega)^\top, h^\tau(\omega)^\top, T_{1,\bullet}^{\tau-1}(\omega), \dots, T_{m,\bullet}^{\tau-1}(\omega), W_{1,\bullet}^\tau(\omega), \dots, W_{m,\bullet}^\tau(\omega)]$ is a random $(n+m+2mn)$ -vector, and $\xi^{[\tau]}(\omega) = (\xi^2(\omega), \dots, \xi^\tau(\omega))$.

In the following discussion, we assume that $\xi = (\xi^2, \dots, \xi^H)$ follows a discrete distribution with a finite support Ξ with $|\Xi| = K$. The justification of this assumption was provided by Schultz [14], while a more thorough treatment of multistage stochastic integer programs can be found in Römisch and Schultz [13]. We call $\xi_i = (\xi_i^2, \dots, \xi_i^H) \in \Xi$ the scenario indexed by $i \in \mathbb{S} = \{1, \dots, K\}$. Each path from the root node to a leaf node at level

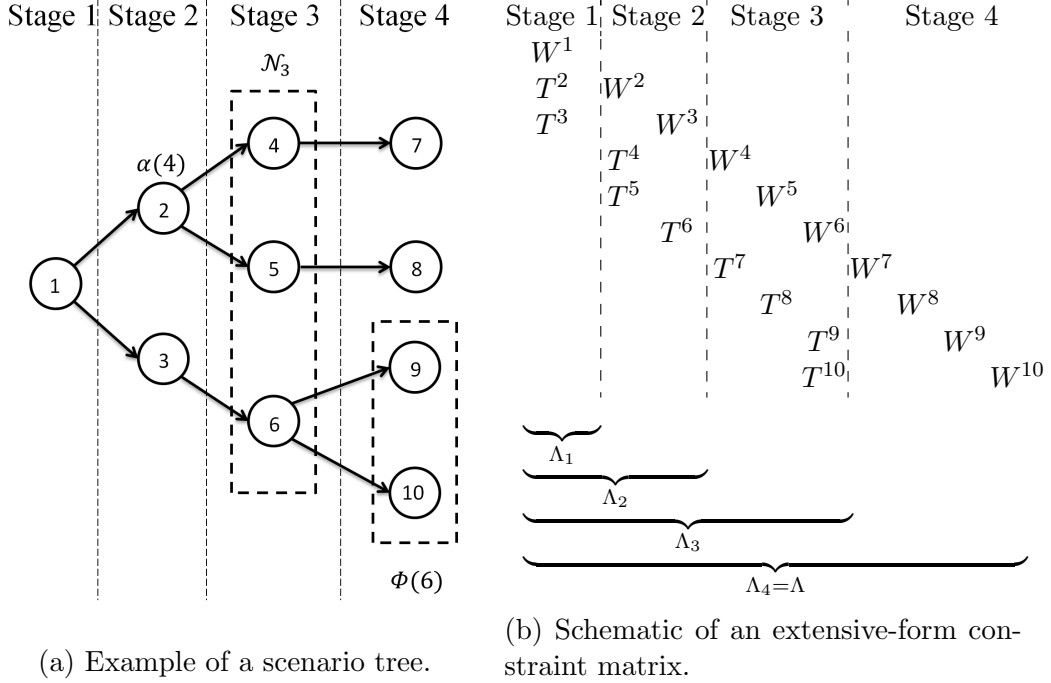


Figure 1: A scenario tree and a schematic of the corresponding extensive-form constraint matrix.

H in the scenario tree corresponds to one scenario $i \in \mathbb{S}$. An example of a scenario tree is illustrated in Figure 1a.

For a scenario tree $\mathcal{T} = \{\mathcal{N}, \mathcal{A}\}$, let Node 1 be the root node, and \mathcal{N}_τ be the set of nodes on level $1 \leq \tau \leq H$, so $\mathcal{N}_1 = \{1\}$. Let $\alpha(k) \in \mathcal{N}$ be the immediate ancestor (or parent) of a non-root node $k \in \mathcal{N} \setminus \{1\}$, $\Phi(k) \subseteq \mathcal{N}$ be the set of immediate children of a node $k \in \mathcal{N}$, and $\rho(k) = \tau$ if $k \in \mathcal{N}_\tau$. Note that $\Phi(k) = \emptyset$ if $\rho(k) = H$. Then the extensive form of (DEP) (also called the arborescent form by Dupačová et al. [3]) based on the scenario tree is given by:

$$\min \quad \sum_{k \in \mathcal{N}} p_k c_k^\top x_k \quad (2a)$$

$$\text{s.t.} \quad W^1 x_1 \geq h^1, \quad (2b)$$

$$T^k x_{\alpha(k)} + W^k x_k \geq h^k, \quad \forall k \in \mathcal{N} \setminus \mathcal{N}_1, \quad (2c)$$

$$x_k \in \mathbb{R}_+^l \times \mathbb{Z}_+^{n-l}, \quad \forall k \in \mathcal{N}.$$

A schematic of the extensive-form constraint matrix corresponding to the MSP with the scenario tree in Figure 1a is shown in Figure 1b. Let Λ denote the extensive-form constraint matrix, and Λ_τ denote the submatrix of Λ up to stage τ as illustrated in Figure 1b. Note that $\Lambda = \Lambda_H$. For every $k \in \mathcal{N}$, let A^k denote the submatrix of Λ formed only by W^k and $T^{k'}$ for all $k' \in \Phi(k)$. In particular, we have $A^k = W^k$ if $k \in \mathcal{N}_H$. We are interested in sufficient conditions for the extensive-form constraint matrix of an MSP to be TU. When the right-hand sides are integral, such stochastic programs may be solved as multistage stochastic linear programs, even if there are integrality restrictions.

Definition 1. An $m \times n$ matrix A is *totally unimodular* (TU) if and only if every square submatrix of A has determinant in $\{0, \pm 1\}$.

Theorem 1 (Hoffman and Kruskal [7]). *An integral matrix A is totally unimodular if and only if the polyhedron defined by $\{x : Ax \leq b, x \geq 0\}$ is integral for all integral b for which it is nonempty, i.e., the extreme points of the polyhedron are integral.*

Theorem 2 (Ghouila-Houri [6]). *A $m \times n$ matrix A is TU if and only if for any column subset $J \subseteq \{1, \dots, n\}$, there exists a partition (J^1, J^2) of J such that*

$$\sum_{j \in J^1} a_{ij} - \sum_{j \in J^2} a_{ij} \in \{0, \pm 1\} \quad \text{for } i = 1, \dots, m. \quad (3)$$

Definition 2 (Kong et al. [10]). Let $\mathcal{A} = \{A_1, \dots, A_T\}$ be a set of $m \times n$ matrices, and let $v \in \{0, \pm 1\}^m$. The set \mathcal{A} is *TU with respect to v* , denoted by $TU(v)$, if for any column subset $J \subseteq \{1, \dots, n\}$, there exist partitions (J_t^1, J_t^2) , $1 \leq t \leq T$, such that for $i = 1, \dots, m$,

$$\sum_{j \in J_t^1} a_{ij}^t - \sum_{j \in J_t^2} a_{ij}^t \in \{0, 1\}, \quad t = 1, \dots, T, \quad \text{if } v_i = -1, \quad (4)$$

$$\sum_{j \in J_t^1} a_{ij}^t - \sum_{j \in J_t^2} a_{ij}^t \in \{0, \pm 1\}, \quad t = 1, \dots, T, \quad \text{if } v_i = 0, \quad (5)$$

and

$$\sum_{j \in J_t^1} a_{ij}^t - \sum_{j \in J_t^2} a_{ij}^t \in \{0, -1\}, \quad t = 1, \dots, T, \quad \text{if } v_i = 1. \quad (6)$$

3. Characterizations of Total Unimodular Multistage Stochastic Programs

Applying Theorem 2 to (2) yields the following:

Proposition 1. *Let J be a subset of the columns of the extensive-form constraint matrix of an MSP, and for each $k \in \mathcal{N}$, let J_k be the set of the columns in J corresponding to A^k in Λ so that $J = \{J_k\}_{k \in \mathcal{N}}$. Then the MSP is TU if and only if for any J , there exists a partition $(J^1, J^2) := (\{J_k^1\}_{k \in \mathcal{N}}, \{J_k^2\}_{k \in \mathcal{N}})$ such that for $i = 1, \dots, m$,*

$$\left| \sum_{j \in J_1^1} w_{ij}^1 - \sum_{j \in J_1^2} w_{ij}^1 \right| \leq 1, \quad (7)$$

and

$$\left| \sum_{j \in J_{\alpha(k)}^1} t_{ij}^k + \sum_{j \in J_k^1} w_{ij}^k - \sum_{j \in J_{\alpha(k)}^2} t_{ij}^k - \sum_{j \in J_k^2} w_{ij}^k \right| \leq 1, \quad \forall k \in \mathcal{N} \setminus \mathcal{N}_1. \quad (8)$$

Corollary 1. *An MSP is TU if for every $k \in \mathcal{N}$, W^k is TU, and for every $k' \in \mathcal{N} \setminus \mathcal{N}_1$, $T^{k'}$ is a $\{0, \pm 1\}$ matrix with a single nonzero entry.*

A more interesting question is determining when an MSP is TU based on its submatrices A^k .

Theorem 3. *An MSP is TU if*

- (i) A^1 is TU, and
- (ii) For every $k \in \mathcal{N} \setminus \mathcal{N}_1$, $\{A^k\}$ is $TU(v)$ for all $v \in \{0, \pm 1\}^m \times \{0\}^{|\Phi(k)|m}$.

Proof. Λ_1 is TU by condition (i). Suppose $\Lambda_{\tau-1}$ is TU for some $\tau - 1$, $2 \leq \tau \leq H$. Consider any column subset $J = \{J_k\}_{k \in \bigcup_{t=1}^{\tau} \mathcal{N}_t}$ of Λ_{τ} . By the induction hypothesis, $\{J_k\}_{k \in \bigcup_{t=1}^{\tau-1} \mathcal{N}_t}$ can be partitioned into $(\{J_k^1\}_{k \in \bigcup_{t=1}^{\tau-1} \mathcal{N}_t}, \{J_k^2\}_{k \in \bigcup_{t=1}^{\tau-1} \mathcal{N}_t})$ satisfying (7) and (8) for all $k \in \bigcup_{t=2}^{\tau-1} \mathcal{N}_t$, and for every $k \in \mathcal{N}_{\tau}$, the vector v^k defined by

$$v_i^k = \sum_{j \in J_{\alpha(k)}^1} t_{ij}^k - \sum_{j \in J_{\alpha(k)}^2} t_{ij}^k, \quad (9)$$

for $i = 1, \dots, m$, satisfies $v^k \in \{0, \pm 1\}^m$. Using condition (ii), $\{A^k\}$ is $\text{TU}(v^k \times 0^{|\Phi(k)|m})$, implying that there exists a partition (J_k^1, J_k^2) of J_k satisfying (8) for every $k \in \mathcal{N}_\tau$, and

$$\left| \sum_{j \in J_k^1} t_{ij}^{k'} - \sum_{j \in J_k^2} t_{ij}^{k'} \right| \leq 1, \quad i = 1, \dots, m, \quad \forall k' \in \Phi(k). \quad (10)$$

Thus, $\Lambda = \Lambda_H$ is TU. \square

Corollary 2. *An MSP is TU if A^1 is TU, and for all $k \in \mathcal{N} \setminus \mathcal{N}_1, k' \in \mathcal{N} \setminus \{\mathcal{N}_1 \cup \mathcal{N}_2\}$, $W^{k'}$ and $T^{k'}$ are $(0, \pm 1)$ matrices with at most one nonzero entry in every row.*

Corollary 3. *An MSP is TU if A^1 is TU, and for every $k \in \mathcal{N} \setminus \mathcal{N}_1$, W^k are $(0, \pm 1)$ matrices with at most one nonzero entry in every column, and for all $k' \in \Phi(k)$, $T^{k'}$ are the same matrices of which only one row contains more than one nonzero entry, and all but one of those nonzero entries' corresponding columns in W^k are $\mathbf{0}$.*

Remark 1. A simple recourse matrix $W = [I \ -I]$ satisfies the recourse conditions of Corollary 2 and Corollary 3.

Theorem 4. *An MSP is TU if*

(i) W^1 is TU, and

(ii) For every $k \in \mathcal{N} \setminus \mathcal{N}_1$, W^k can be written as $\begin{bmatrix} \hat{W}_k^1 \\ \hat{W}_k^2 \end{bmatrix}$ and T^k can be

written as $\begin{bmatrix} \hat{T}_k \\ \mathbf{0} \end{bmatrix}$ satisfying

- (a) \hat{W}_k^1 and \hat{T}_k have the same dimensions,
- (b) \hat{W}_k^2 is a matrix with at most one nonzero entry in every row, and
- (c) \hat{W}_k^1 and \hat{T}_k satisfy the conditions for W^k and $T^{k'}$ in Corollary 2 or Corollary 3.

Proof. By condition (i), (7) is satisfied. Let M_1 and M_2 be the union of the row subsets corresponding to \hat{W}_k^1 and \hat{W}_k^2 for all $k \in \mathcal{N} \setminus \mathcal{N}_1$, respectively. By Corollary 2 (or Corollary 3), (8) is satisfied for all the rows in M_1 . Meanwhile, (8) also holds for all the rows in M_2 . \square

Theorem 5. An MSP is TU if for all $k \in \mathcal{N}$, $\{A^k\}$ is $TU(\{1\}^{m+|\Phi(k)|m})$.

Proof. For every $k \in \mathcal{N}$ and any column subset J_k of A_k , there exists a partition (J_k^1, J_k^2) such that for all $i = 1, \dots, m$,

$$\sum_{j \in J_k^1} w_{ij}^k - \sum_{j \in J_k^2} w_{ij}^k \in \{-1, 0\}, \quad (11)$$

$$\sum_{j \in J_k^1} t_{ij}^{k'} - \sum_{j \in J_k^2} t_{ij}^{k'} \in \{-1, 0\}, \quad \forall k' \in \Phi(k). \quad (12)$$

Hence, for any column subset $J = \{J_k\}_{k \in \mathcal{N}}$, we can construct a partition (J^1, J^2) that satisfies (7) and (8), by setting $J^1 = J_{\text{odd}}^1 \cup J_{\text{even}}^2$, and $J^2 = J_{\text{odd}}^2 \cup J_{\text{even}}^1$, where for $i = 1, 2$, $J_{\text{odd}}^i := \bigcup_{k \in \{k: \rho(k) \text{ is odd}\}} J_k^i$, $J_{\text{even}}^i := \bigcup_{k \in \{k: \rho(k) \text{ is even}\}} J_k^i$. \square

Remark 2. An MSP is TU if for all $k \in \mathcal{N}$, $\{A^k\}$ is $TU(\{-1\}^{m+|\Phi(k)|m})$.

Definition 3 (Fulkerson and Gross [5], Nemhauser and Wolsey [11]). An $m \times n$ $(0, 1)$ matrix A is called an *interval matrix* if in each column the 1's appear consecutively; that is, if $a_{ij} = a_{kj} = 1$ and $k > i + 1$, then $a_{\ell j} = 1$ for all ℓ with $i < \ell < k$.

Given an interval matrix A , for each column $j = 1, \dots, n$, let $\alpha_j(\beta_j)$ index the first (last) row in which a non-zero element appears in column j .

Definition 4 (Kong et al. [10]). An interval matrix is a *nested interval matrix* if there exists an ordering of the columns such that $\alpha_{j'} \leq \alpha_{j''}$ and $\beta_{j'} \geq \beta_{j''}$, for all $1 \leq j' \leq j'' \leq n$.

Corollary 4. An MSP is TU if for all $k \in \mathcal{N}$ and $k' \in \mathcal{N} \setminus \mathcal{N}_1$, W^k and $T^{k'}$ are nested interval matrices.

Proof. For any column subset J_k of A^k , $k \in \mathcal{N}$, we assign the g^{th} column of J_k to J_k^1 if g is an odd number. Otherwise, we assign it to J_k^2 . We have for $i = 1, \dots, m$,

$$\sum_{j \in J_k^1} w_{ij}^k - \sum_{j \in J_k^2} w_{ij}^k \in \{0, 1\}, \quad (13)$$

and

$$\sum_{j \in J_k^1} t_{ij}^{k'} - \sum_{j \in J_k^2} t_{ij}^{k'} \in \{0, 1\}, \quad \forall k' \in \Phi(k). \quad (14)$$

Therefore, $\{A^k\}$ is TU($\{-1\}^{m+|\Phi(k)|m}$) for all $k \in \mathcal{N}$. \square

Proposition 2. *An MSP is TU if for every $k \in \mathcal{N}$, $\{A^k\}$ is TU($\{1\}^m \times \{-1\}^{|\Phi(k)|m}$).*

Proof. For every $k \in \mathcal{N}$, by the hypothesis, there exists a partition (J_k^1, J_k^2) of any column subset J_k of A^k , such that for $i = 1, \dots, m$,

$$\sum_{j \in J_k^1} w_{ij}^k - \sum_{j \in J_k^2} w_{ij}^k \in \{-1, 0\}, \quad (15)$$

$$\sum_{j \in J_k^1} t_{ij}^{k'} - \sum_{j \in J_k^2} t_{ij}^{k'} \in \{0, 1\}, \quad \forall k' \in \Phi(k). \quad (16)$$

Hence, for any column subset $J = \{J_k\}_{k \in \mathcal{N}}$, we can construct a partition (J^1, J^2) satisfying (7) and (8), by setting $J^1 = \bigcup_{k \in \mathcal{N}} J_k^1$, and $J^2 = \bigcup_{k \in \mathcal{N}} J_k^2$. \square

Remark 3. An MSP is also TU if $\{A^k\}$ is TU($\{-1\}^m \times \{1\}^{|\Phi(k)|m}$).

4. Applications of Totally Unimodular Multistage Stochastic Programs

4.1. Dynamic Lot-Sizing Problems without Setup Costs

El Agizy [4] introduced a stochastic programming extension of the deterministic single item, finite-horizon, uncapacitated dynamic lot-sizing problem without setup costs, where the production volume must be determined before the demand is realized. Let $\mathcal{T} = \{\mathcal{N}, \mathcal{A}\}$ be the corresponding scenario tree. For node $k \in \mathcal{N}$, let the unit production cost, the unit shortage cost, the unit holding cost and the demand be c_k, γ_k, β_k and δ_k , respectively. Let M_k^x, M_k^I and M_k^u be the capacities of the production level, the inventory level and the shortage for node k , respectively, which are infinity in the uncapacitated case. The decision variables $x_{\alpha(k)}, I_k$ and u_k represent the production level for node k decided at period $\rho(k) - 1$, the inventory level at the end of period $\rho(k)$ and the shortage amount for node k , respectively. We create a dummy ancestor (Node 0) for the root node (Node 1) and assign initial inventory

level I_0 to it. The extensive form is:

$$\min \sum_{k \in \mathcal{N}} (c_k x_{\alpha(k)} + \beta_k I_k + \gamma_k u_k) \quad (17a)$$

$$s.t. \quad I_{\alpha(k)} + x_{\alpha(k)} + u_k - I_k \geq \delta_k, \quad k \in \mathcal{N}, \quad (17b)$$

$$x_{\alpha(k)} \leq M_k^x, \quad k \in \mathcal{N}, \quad (17c)$$

$$I_k \leq M_k^I, \quad k \in \mathcal{N}, \quad (17d)$$

$$u_k \leq M_k^u, \quad k \in \mathcal{N}, \quad (17e)$$

$$x_{\alpha(k)}, I_k, u_k \in \mathbb{Z}_+, \quad k \in \mathcal{N}.$$

El Agizy [4] studied the uncapacitated version of this problem, and showed that it can be reduced to a network flow problem. Theorem 4 indicates that both the uncapacitated and capacitated versions of the original formulation (17) are TU. Note that because the production decision for k must be made at period $\rho(k) - 1$, the coefficient of the left-hand side of (17c) appears in $W^{\alpha(k)}$ instead of T^k .

Huang [8], extending El Agizy [4], developed a similar dynamic inventory model, where the production level decisions are postponed until the beginning of the current period after the demand becomes known. Note that the production level variable for node k is x_k instead of $x_{\alpha(k)}$ in this case. Using Corollary 1, the extensive form constraint matrix under such circumstances is also TU, since for every $k \in \mathcal{N} \setminus \mathcal{N}_1$, the inventory level variable, $I_{\alpha(k)}$, corresponds to the only nonzero entry in T^k .

The special case without shortages and without capacities, i.e., $u_k = 0$ and $M_k^x = M_k^I = \infty$, was also considered by Huang [8], who reformulated the problem by aggregating the demands and summing production along a path in the scenario tree, and proved that it is TU. In fact, both Corollary 1 and Corollary 2 imply that the original formulation by Huang [8] is TU.

4.2. Stochastic Machine Setup Problem

The classical single-day machine setup problem [1] requires several tasks to be accomplished according to their predefined start and end times. Each task requires a single worker, and there is a task-specific setup time for workers who transition between tasks. The problem can be modeled as a minimum network flow problem to find the minimum number of workers needed to process all the tasks. Suppose there are p tasks, and for $i = 1, \dots, p$, the start time and the end time for task i are $\pi(i)$ and $\pi'(i)$, respectively.

For $i \neq j$, the setup time from task i to task j is $\Pi(i, j)$. Then we can construct a directed graph $G = (V, E)$, where $V = V^1 \cup V^2 \cup V^3 \cup V^4$ with $V^1 = \{s_1 \text{ (the source)}\}$, $V^2 = \{i \mid i = 1, \dots, p\}$, $V^3 = \{i' \mid i' = 1, \dots, p\}$, $V^4 = \{s_2 \text{ (the sink)}\}$, and $E = E^1 \cup E^2 \cup E^3 \cup E^4$ with $E^1 = \{(s_1, i) \mid i \in V^2\}$, $E^2 = \{(i, i') \mid i \in V^2, i' \in V^3, i = i'\}$, $E^3 = \{(i', j) \mid j \in V^2, i' \in V^3, \pi'(i') + \Pi(i', j) \leq \pi(j)\}$, $E^4 = \{(i', s_2) \mid i' \in V^3\}$. The capacities of all the arcs in E^2 have lower bounds of 1 and upper bounds of 1, enforcing the requirement to complete all the tasks, while others have lower bounds of 0 and upper bounds of 1 [2]. An optimal solution to the minimum network flow problem on this graph provides an optimal solution to the corresponding machine setup problem.

We consider a stochastic and dynamic multi-period extension of the machine setup problem, where the number of the tasks, the labor costs and the setup times are unknown until the beginning of the day. The problem can be formulated as an MSP. Let $\mathcal{T} = \{\mathcal{N}, \mathcal{A}\}$ be the scenario tree. Consider a node $k \in \mathcal{N}$, representing an instance of the single-day machine setup problem. As discussed above, we can define an equivalent instance of the network flow problem on some graph G . Let N_k be the node-arc incidence matrix of this graph, while vectors u_k and l_k denote the upper and lower capacities of its arcs. Let x_k^{ij} be the flow on the arc (i, j) , where $x_k^\top = [x_k^{ij}]_{ij}$, and let y_k be the workforce level with a cost of c_k per worker. The number of workers can be changed daily, either by hiring Δ_k^+ workers or laying off Δ_k^- workers with a nonnegative cost of b_k^+ and b_k^- per worker, respectively. The extensive form can be written as:

$$\begin{aligned}
\min \quad & \sum_{k \in \mathcal{N}} p_k (c_k y_k + b_k^+ \Delta_k^+ + b_k^- \Delta_k^-) \\
\text{s.t.} \quad & y_{\alpha(k)} + \Delta_k^+ - \Delta_k^- - y_k \geq 0, \quad \forall k \in \mathcal{N} \setminus \mathcal{N}_1, \\
& \begin{bmatrix} y_k \\ \mathbf{0} \end{bmatrix} + N_k x_k \geq \mathbf{0}, \quad \forall k \in \mathcal{N}, \\
& l_k \leq I x_k \leq u_k, \quad \forall k \in \mathcal{N}, \\
& x_k, y_k, \Delta_{k'}^+, \Delta_{k'}^- \in \mathbb{Z}_+, \quad \forall k \in \mathcal{N}, k' \in \mathcal{N} \setminus \mathcal{N}_1.
\end{aligned}$$

It follows from Corollary 1 that the extensive-form constraint matrix Λ is TU, where for every $k \in \mathcal{N} \setminus \mathcal{N}_1$, the only nonzero entry in T^k corresponds to the variable of the previous workforce level $y_{\alpha(k)}$.

Remark 4. A multistage extension of the nursing staff scheduling prob-

lem proposed by Khan and Lewis [9], which assigned nurses to different shifts across different departments with minimum and maximum number constraints, can be also formulated as a stochastic machine setup problem.

4.3. Dynamic Outside Sourcing Network Flow Problems

Consider a directed network flow problem in which only one node receives goods from an external supplier to fulfill the demand of the network. The network structure may change over time. For example, the delivery over some arcs may not be available for some periods. The demand of each node, the purchasing cost and the delivery cost are also stochastic. There is a linear penalty associated with changing the order quantity. Moreover, emergency purchases may be allowed if the demands are not satisfied for some nodes. Accordingly, each node in the scenario tree $\mathcal{T} = \{\mathcal{N}, \mathcal{A}\}$ corresponds to a realization of the network structure, the demand and the costs in a specific period. For any node $k \in \mathcal{N}$, let N_k be the node-arc incidence matrix for the realized directed network whose first row corresponds to the node receiving goods from the outside source. Let y_k be the order quantity from outside with a cost of c_k per unit, Δ_k^+ , Δ_k^- be the amount of increase and decrease in order quantity with a nonnegative cost of b_k^+ and b_k^- per unit, respectively, x_k^{ij} be the flow amount on arc (i, j) in the network with a transportation cost of a_k^{ij} per unit, d_k^ℓ be the demand for node ℓ in the network, and z_k^ℓ be the amount of emergency purchase for node ℓ with a unit-price q_k^ℓ . Let $x_k^\top = [x_k^{ij}]_{ij}$, $a_k^\top = [a_k^{ij}]_{ij}$, $d_k^\top = [d_k^\ell]_\ell$, $z_k^\top = [z_k^\ell]_\ell$ and $q_k^\top = [q_k^\ell]_\ell$. The extensive form, which is TU from Corollary 1, is as follows:

$$\begin{aligned} \min \quad & \sum_{k \in \mathcal{N}} p_k (c_k y_k + b_k^+ \Delta_k^+ + b_k^- \Delta_k^- + a_k^\top x_k + q_k^\top z_k) \\ \text{s.t.} \quad & y_{\alpha(k)} + \Delta_k^+ - \Delta_k^- - y_k \geq 0, \quad \forall k \in \mathcal{N} \setminus \mathcal{N}_1, \\ & \begin{bmatrix} y_k \\ \mathbf{0} \end{bmatrix} + N_k x_k + I z_k \geq d_k, \quad \forall k \in \mathcal{N}, \\ & x_k, y_k, z_k, \Delta_{k'}^+, \Delta_{k'}^- \in \mathbb{Z}_+, \quad \forall k \in \mathcal{N}, k' \in \mathcal{N} \setminus \mathcal{N}_1. \end{aligned}$$

Acknowledgments

The authors thank Rüdiger Schultz and an anonymous referee for their insight. This research of the authors was partially supported by National Science Foundation grants CMMI-1100082, CMMI-1333758 and CMMI-1400009.

References

- [1] R.K. Ahuja, T.L. Magnanti, and J.B. Orlin. *Network Flows: Theory, Algorithms, and Applications*. Prentice Hall, 1993.
- [2] E. Ciurea and L. Ciupală. Sequential and parallel algorithms for minimum flows. *Journal of Applied Mathematics and Computing*, 15(1-2):53–75, 2004.
- [3] J. Dupačová, G. Consigli, and S.W. Wallace. Scenarios for multistage stochastic programs. *Annals of Operations Research*, 100(1-4):25–53, 2000.
- [4] M.N. El Agizy. Dynamic inventory models and stochastic programming. *IBM Journal of Research and Development*, 13(4):351–356, 1969.
- [5] D.R. Fulkerson and O.A. Gross. Incidence matrices and interval graphs. *Pacific Journal of Mathematics*, 15(3):835–855, 1965.
- [6] A. Ghouila-Houri. Caractérisation des matrices totalement unimodulaires. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, 254(7):1192, 1962.
- [7] A.J. Hoffman and J.B. Kruskal. Integral boundary points of convex polyhedra. In H.W. Kuhn and A. W. Tucker, editors, *Linear Inequalities and Related Systems*, pages 49–76. Princeton University Press, Princeton, 2010.
- [8] K. Huang. *Multi-stage stochastic programming models in production planning*. PhD thesis, Georgia Institute of Technology, 2005.
- [9] M.R. Khan and D.A. Lewis. A network model for nursing staff scheduling. *Zeitschrift für Operations Research*, 31(6):B161–B171, 1987.
- [10] N. Kong, A.J. Schaefer, and S. Ahmed. Totally unimodular stochastic programs. *Mathematical Programming*, 138(1-2):1–13, 2013.
- [11] G.L. Nemhauser and L.A. Wolsey. *Integer and Combinatorial Optimization*, volume 18. Wiley New York, 1988.

- [12] W. Romeijnders and M.H. van der Vlerk. Convex hull approximation of TU integer recourse models: Counterexamples, sufficient conditions, and special cases. Stochastic Programming E-Print Series. Institut für Mathematik, 2012. Available: <http://edoc.hu-berlin.de/docviews/abstract.php?id=39817>; accessed 15-August-2014.
- [13] W. Römisches and R. Schultz. Multistage stochastic integer programs: An introduction. In M. Grötschel, S.O. Krumke, and J. Rambau, editors, *Online Optimization of Large Scale Systems*, pages 581–600. Springer Berlin Heidelberg, 2001.
- [14] R. Schultz. On structure and stability in stochastic programs with random technology matrix and complete integer recourse. *Mathematical Programming*, 70(1-3):73–89, 1995.