Finding Shortest Path in a Combined Exponential -Gamma-Normal Probability Distribution Arc Length

Mohammad Hessam Olya[1], Hamed Fazlollahtabar[2,*]
[1]Department of Industrial Engineering, Mazandaran University of Science and Technology, Babol, Iran
[2]Faculty of Industrial Engineering, Iran University of Science and Technology, Tehran, Iran

Abstract

We propose a dynamic program to find the shortest path in a network having exponential, gamma and normal probability distributions as arc lengths. Two operators of sum and comparison need to be adapted for the proposed dynamic program. Convolution approach is used to sum probability distributions being employed in the dynamic program.

Keywords: Shortest path; Dynamic program; Convolution

1. Introduction

The shortest path problem has been widely studied in the fields of operations research, computer science, and transportation engineering. There are several methods to find the shortest path from the source node to the sink node based on dynamic programming, zero-one programming and also network flows theory when the arc lengths are constant. Some of these algorithms can be found in Bazaraa (1990). If the arc lengths are constant, there are several efficient algorithms have been developed by (Bellman, 1958; Dijkstra, 1959; Dreyfus, 1969). Cook and Halsey (1958) have extended Bellman's principle of optimality for dynamic programming (1958) to this case and Dreyfus (1969) has suggested the use of Dijkstra's algorithm (1959) for determining time-dependent shortest paths. Halpern (1977) noted the limitations of the approach of Dreyfus (1969). It should be noted that the standard shortest path algorithms also have been found to be applicable to compute shortest paths in time-dependent but not stochastic networks (Orda and Rom, 1990; Kaufman et al., 1993; Ziliaskopoulos and Mahmassani, 1993; Chabini, 1997).

Malandraki (1989) analyzed the time-dependent shortest path problem and extended Halpern's result for the special case of differentiable link delay functions and showed that the consistency assumption would be satisfied by verifying that the first derivative of the link delay function did not exceed negative unity.

Ziliaskopoulos and Mahmassani (1996) noted that turning movements of vehicles in congested urban networks contribute significantly to the travel time. The authors have prescribed an efficient label-correcting procedure that uses an extended forward-star structure to represent the network including intersection movements and movement prohibitions.

Mirchandani (1976) presented another method for obtaining the distribution function of shortest path in stochastic networks. It is not required to solve multiple integrals in this paper, but this method can only be used for the special case where arc lengths are discrete random variables.

Haquari and Dejax (1997) have analyzed a similar problem, considering time-varying costs and knapsack-like constraints.

However, due to failure, maintenance or other reasons, different kinds of uncertainties are frequently encountered in practice, and must be taken into account. For example, the lengths of

* Corresponding author's email : hfazl@iust.ac.ir

the arcs are assumed to represent transportation time or cost rather than the geographical distances, as time or cost fluctuate with traffic or weather conditions, payload and so on, it is not practical to consider each arc as a deterministic value. In these cases, probability theory has been used to attack randomness, and many researchers have done lots of work on stochastic shortest path problem.

When arc lengths are random variables, the problem will become more difficult. Frank (1969) computed the probability that the time of the shortest path of the network is smaller than a specific value where link travel times are random variables but not time dependent.

Loui (1983), Mirchandani and Soroush (1986), and Murthy and Sarkar (1996) showed that for identifying the expected shortest path if the random link travel times are replaced by their expected values, then the problem simply reduces to a deterministic shortest path problem and standard shortest path algorithms still can be used to find the expected shortest paths in a network.

Wijerante et al. (1993) presented a method to find the set of non-dominated paths from the source node to the sink node, in which each arc includes several criteria that some of them might be stochastic. Martins (1984) provided set of efficient paths for bicriteria shortest path problem by dynamic programming.

The more general case of the stochastic, time-dependent shortest path problem was first studied by Hall (1986). He introduced the problem of finding least expected travel time path between two nodes in a network with travel times that are both random and time-dependent.

Fu (1998) identified a set of relationships between the mean and variance of the travel time of a given path and the mean and variance of the dynamic and stochastic link travel times on networks with intelligent transportation system capabilities.

Fan et al. (2005) minimize expected travel time from any origin to a specific destination in a congestible network with correlated link costs.

Bertsekas and Tsitsiklis (1991) considered a stochastic version of the classical shortest path problem whereby for each node of a graph, we must choose a probability distribution over the set of successor nodes such that we reach a certain destination node with minimum expected cost.

Koutsopoulos and Xu (1994) noted the need for realistic traffic link delay functions in order for time-dependent shortest path algorithms to be effective and prescribed the use of a link-delay function that gives weightage for both real-time and historic delay components. Miller (1994) noted the time-dependent nature of risk associated with links in a transportation networks and proposed the application of time-dependent shortest paths to finding dynamic, minimum risk routes for the shipment of hazardous materials. Cai et al. (1997) have analyzed the problem of finding a least cost path on a network with time-dependent delays and costs such that the total delay is less than or equal to a pre-specified value.

Some researchers focused on problems that include waiting at nodes. Orda and Rom (1990), studied various types of waiting-at-nodes scenarios, and proposed algorithms for these different cases. They showed that if waiting is allowed at nodes, then the consistency assumption is not required, and they prescribed an algorithm for identifying optimal waiting times at the source node if waiting is not allowed elsewhere in the network.

The shortest path problems have been studied under the situations that arc costs are functions of environmental variables in network. Friesz et al. (1986) analyzed the source waiting case in the context of the traffic equilibrium problem. They postulated that route choice is not independent of departure times and they calculated optimal source waiting times based on minimizing the total cost to the user. Azaron and Kianfar (2003), applied the stochastic dynamic programming to find the dynamic shortest path from the source node to the sink node in stochastic dynamic networks, in which the arc lengths were independent random variables with exponential distributions. They defined an environmental variable in each node, which evolved in accordance with a continuous time Markov process. They considered move or wait option in each node and assumed that upon arriving at each node, they know the state of its environmental variable and also the states of the environmental variables of its adjacent nodes.

Thompson and Cluett (2005) presented a new stochastic dynamic programming algorithm that used a Monte Carlo approach to circumvent the need for numerical integration, thereby dramatically reduced computational requirements.

2. Problem definition and modelling

Consider a network as shown in Figure 1 consisting of a finite set of nodes and arcs of the directed acyclic network. We assume that the admissible paths are always continuous and always move toward the right and the length of each arc is an exponential random variable with parameter λ_i or a gamma random variable with shape parameter α and rate parameter λ_j or a normal random variable with mean μ and variance σ^2. We want to find the shortest path from the source node 1 to the sink node N using the backward dynamic programming approach.

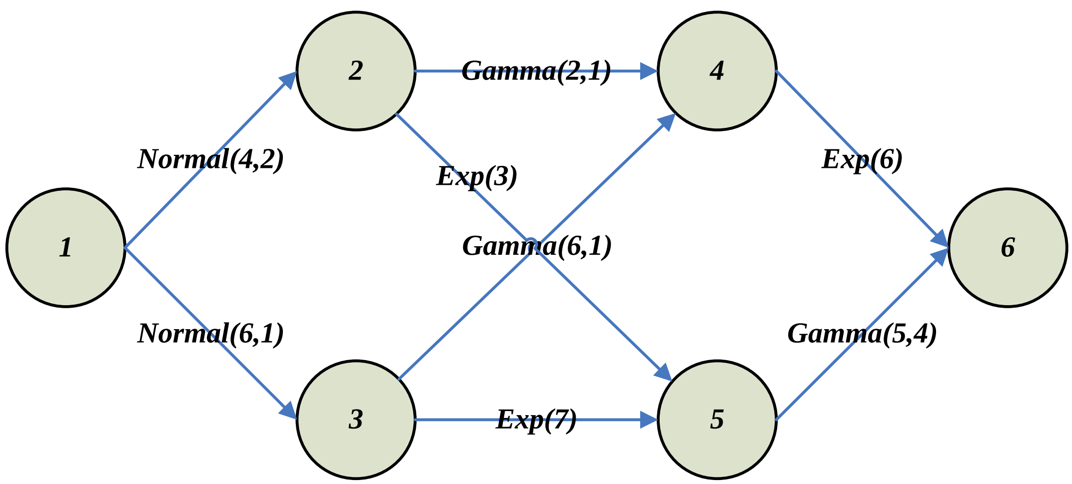

Figure 1. Acyclic network with exponential, gamma and normal arcs

The optimal value function S_i can be defined by

S_i = the distribution of the shortest path from node i to node N.

Then the recurrence relation can be stated as

$$S_i = \min_{j>i}\left\lfloor d_{ij} + S_j \right\rfloor \qquad \text{For} \quad i = N-1,\dots 1 \qquad (1)$$

And the boundary condition is

$$S_N = 0.$$

In this paper we use convolution to find distribution of sum of two distributions in each stage. And for comparison in each stage we find the probability that a random variable with first distribution become smaller than another random variable with second distribution.

Definition 1. Let X and Y be two continuous random variables with density functions f(x) and g(y), respectively. Assume that both f(x) and g(y) are defined for all real numbers. Then the convolution f *g of f and g is the function given by

$$(f * g)(z) = \int_{-\infty}^{+\infty} f(x)\, g(z - x)\; dx$$
$$= \int_{-\infty}^{+\infty} f\,(z - y)\, g(y)\, dy$$

Theorem 1. Let X and Y be two independent random variables with density functions $f_X(x)$ and $f_Y(y)$ defined for all x. Then the sum Z = X +Y is a random variable with density function $f_Z(z)$, where f_Z is the convolution of f_X and f_Y.

$$f_z(z) = \int_{-\infty}^{+\infty} f_{X,Y}\,(x, z - x)\, dx$$
$$= \int_{-\infty}^{+\infty} f_{X,Y}\,(z - y, y)\, dy \tag{2}$$

Proof: as we knew the joint density function of independent variables is equal to the products of their density functions therefore to find density function of Z = X + Y we apply cumulative distribution function technique.

$$P(Z \leq z) = P(X + Y \leq z) = \int_{-\infty}^{+\infty} P(X + Y \leq z | X = x)\, f_X(x)$$
$$= \int_{-\infty}^{+\infty} P(x + y \leq z)\, f_X(x)dx = \int_{-\infty}^{+\infty} F_Y(z - x)\, f_X(x)dx$$

Now, we set partial derivative to obtain the summation density function

$$f_Z(z) = \frac{dF_Z\,(z)}{dz} = \frac{d}{dz}\left[\int_{-\infty}^{+\infty} F_Y(z - x\,)\, f_X(x)dx\right]$$
$$= \int_{-\infty}^{+\infty} \frac{dF_Y(z - x\,)}{dz} f_X(x)dx = \int_{-\infty}^{+\infty} f_Y(z - x)\; f_X(x)dx$$

Now we illustrate the method that we use to find minimum between two random variables. In order to find the minimum random variable we compute the probability that the first random variable X_1 became smaller than the second random variable X_2.

$$P(X_1 < X_2) = \int_0^{\infty} P(X_1 < X_2 | X_1 = x_1) \,.\, f_{x_1}(x_1)\, dx_1$$

3. Numerical example

Consider the network depicted in Figure 1. We want to obtain the shortest path from node 1 to node 6 where arcs have exponential, gamma or normal distribution.
Boundary condition is $S_6 = 0$. Using the recurrence relation (1) we have

$$S_5 = \mathrm{Gamma}(5{,}4), S_4 = \mathrm{Exp}(6)$$

For each arc doesn't exist in network we replace infinity for d_{ij}

$$S_3 = \min\begin{bmatrix}\mathrm{Gamma}(6{,}1) + S_4 \\ \mathrm{Exp}(7) + S_5\end{bmatrix} = \min\begin{bmatrix}\mathrm{Gamma}(6{,}1) + \mathrm{Exp}(6) \\ \mathrm{Exp}(7) + \mathrm{Gamma}(5{,}4)\end{bmatrix}$$

For convolution between Gamma(6,1) and Exp(6) we have

$$f_X(x) = \frac{1}{5!} x^5 e^{-x} \qquad x \geq 0\,,$$
$$f_Y(y) = 6e^{-6y} \qquad y \geq 0\,,$$
$$f_Z(z) = \int_{-\infty}^{+\infty} f_Y(z-x)\, f_X(x)\, dx$$
$$= \int_0^z \frac{1}{5!} x^5 e^{-x} \,.\, 6e^{-6(z-x)}\, dx$$
$$= \frac{6}{15625} e^{-6z} - \frac{6}{15625} e^{-z} + \frac{6}{3125} e^{-z} z - \frac{3}{625} e^{-z} z^2 + \frac{1}{125} e^{-z} z^3 - \frac{1}{100} e^{-z} z^4 + \frac{1}{100} e^{-z} z^5 \quad , z \geq 0$$

And we can convolute Exp(7), Gamma(5,4) as follows

$$f_X(x) = \frac{4^5}{4!} x^4 e^{-4x} \qquad x \geq 0\,,$$
$$f_Y(y) = 7e^{-7y} \qquad y \geq 0\,,$$
$$f_Z(z) = \int_{-\infty}^{+\infty} f_Y(z-x)\, f_X(x)\, dx$$

$$= \int_0^z \frac{4^5}{4!} x^4 e^{-4x} .\ 7e^{-7(z-x)}\, dx$$

$$= \frac{-7168}{243} e^{-7z} + \frac{7168}{243} e^{-4z} - \frac{7168}{81} e^{-4z} z + \frac{3584}{27} e^{-4z} z^2 - \frac{3584}{27} e^{-4z} z^3 + \frac{896}{9} e^{-4z} z^4$$

$$, z \geq 0$$

We find the minimum value between two density function as follows

$$P(X_1 < X_2) = \int_0^\infty \int_{x_1}^\infty f_{x_2}(x_2) .\ f_{x_1}(x_1)\, dx_2 dx_1$$

$$= \int_0^\infty f_{x_1}(x_1) \left(\int_{x_1}^\infty f_{x_2}(x_2) dx_2 \right) dx_1$$

$$= \int_0^\infty \Big(\frac{6}{15625} e^{-6x_1} - \frac{6}{15625} e^{-x_1} + \frac{6}{3125} e^{-x_1} x_1 - \frac{3}{625} e^{-x_1} {x_1}^2 + \frac{1}{125} e^{-x_1} {x_1}^3 - \frac{1}{100} e^{-x_1} {x_1}^4 + \frac{1}{100} e^{-x_1} {x_1}^5 \Big) \left(\int_{x_1}^\infty \frac{-7168}{243} e^{-7x_2} + \frac{7168}{243} e^{-4x_2} - \frac{7168}{81} e^{-4x_2} x_2 + \frac{3584}{27} e^{-4x_2} {x_2}^2 - \frac{3584}{27} e^{-4x_2} {x_2}^3 + \frac{896}{9} e^{-4x_2} {x_2}^4 dx_2 \right) dx_1 = \frac{536460723}{81250000000}$$

$$= 0.0066$$

So the second density function is minimal,

$$S_3 = \min \begin{bmatrix} \text{Gamma}(6,1) + \text{Exp}(6) \\ \text{Exp}(7) + \text{Gamma}(5,4) \end{bmatrix} = \text{Exp}(7) + \text{Gamma}(5,4)$$

We illustrate the operation of node 2 as follows

$$S_2 = \min \begin{bmatrix} \text{Gamma}(2,1) + S_4 \\ \text{Exp}(3) + S_5 \end{bmatrix} = \min \begin{bmatrix} \text{Gamma}(2,1) + \text{Exp}(6) \\ \text{Exp}(3) + \text{Gamma}(5,4) \end{bmatrix}$$

We can state convolution between Exp(6) and gamma(2,1) as follows

$$f_X(x) = 6e^{-6x} \qquad x \geq 0,$$

$$f_Y(y) = \frac{1^2}{1!} y^1 e^{-y} \qquad y \geq 0,$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_Y(z-x)\, f_X(x)\, dx$$

$$= \int_0^z 6e^{-6x} \ (z-x)^1 e^{-(z-x)} \, dx \ = \frac{6}{5} e^{-z} z - \frac{6}{25} e^{-z} + \frac{6}{25} e^{-6z}$$

We can convolute exp(3) and gamma(5,4) as follows

$$f_X\,(x) = \frac{4^5}{4!} x^4 e^{-4x} \quad x \geq 0\,,$$
$$f_Y\,(y) = 3e^{-3y} \quad\quad y \geq 0\,,$$
$$f_Z(z) = \int_{-\infty}^{+\infty} f_Y(z-x)\, f_X(x)\, dx$$
$$= \int_0^z \frac{4^5}{4!} x^4 e^{-4x} \,.\, 3e^{-3(z-x)}\, dx$$
$$= 3072e^{-3z} - 3072e^{-4z} - 3072e^{-4z}z - 1536e^{-4z}z^2 - 512e^{-4z}z^3 - 128e^{-4z}z^4$$
$$z \geq 0$$

To find minimum density, we make use of the following integral,

$$P(X_1 < X_2) = \int_0^\infty \int_{x_1}^\infty f_{x_2}(x_2)\,.\ f_{x_1}(x_1)\, dx_2 dx_1$$
$$= \int_0^\infty \left(\frac{6}{5} e^{-x_1} x_1 - \frac{6}{25} e^{-x_1} + \frac{6}{25} e^{-6x_1}\right)\left(\int_{x_1}^\infty 3072e^{-3x_2} - 3072e^{-4x_2} - 3072e^{-4x_2}x_2 - 1536e^{-4x_2}{x_2}^2 - 512e^{-4x_2}{x_2}^3 - 128e^{-4x_2}{x_2}^4 dx_2\right) dx_1 \ = \frac{92647}{234375} = 0.395$$

So with probability 0.605 we choose second density function as minimum.

$$S_2 = \min \begin{bmatrix} \text{Gamma}(2,1) + \text{Exp}(6) \\ \text{Exp}(3) + \text{Gamma}(5,4) \end{bmatrix} = \text{Exp}(3) + \text{Gamma}(5,4)$$

We illustrate the operation of node 1 as follows

$$S_1 = \min \begin{bmatrix} \text{Normal}(4,2) + S_2 \\ \text{Normal}(6,1) + S_3 \end{bmatrix} = \min \begin{bmatrix} \text{Normal}(4,2) + \text{Exp}(3) + \text{Gamma}(5,4) \\ \text{Normal}(6,1) + \ \text{Exp}(7) + \text{Gamma}(5,4) \end{bmatrix}$$

Considering the convolution, we can obtain the density function of the three different density functions by convoluting the convolution of the first and second distribution with the third one as follows.

So we have these relations for node 1

$$f_Z(z) = \int_{-\infty}^{+\infty} f_Y(z-x)\, f_X(x)\, dx$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2*2\pi}} e^{-\frac{(z-x-4)^2}{2*2}} .\ (3072e^{-3x} - 3072e^{-4x} - 3072e^{-4x}x - 1536e^{-4x}x^2 - 512e^{-4x}x^3 - 128e^{-4x}x^4)\, dx$$

$$= \frac{1}{\sqrt{\pi}} \left(64\left(-744\sqrt{\pi}\,\mathrm{erf}\left(-6+\frac{1}{2}z\right)e^{32}z^2 + 5712\sqrt{\pi}\,\mathrm{erf}\left(-6+\frac{1}{2}z\right)e^{32}z - 16764\sqrt{\pi}\,\mathrm{erf}\left(-6+\frac{1}{2}z\right)e^{32} + 44\sqrt{\pi}\,\mathrm{erf}\left(-6+\frac{1}{2}z\right)e^{32}z^3 - \sqrt{\pi}\,\mathrm{erf}\left(-6+\frac{1}{2}z\right)e^{32}z^4 + 24\sqrt{\pi}\,\mathrm{erf}\left(-5+\frac{1}{2}z\right)e^{21+z} - 716ze^{\frac{-1}{4}z^2+6z-4} + 64z^2e^{\frac{-1}{4}z^2+6z-4} - 2z^3e^{\frac{-1}{4}z^2+6z-4} + 2752e^{\frac{-1}{4}z^2+6z-4} + 24\sqrt{\pi}e^{21+z} - 16764\sqrt{\pi}e^{32} + 5712\sqrt{\pi}e^{32}z - 744\sqrt{\pi}e^{32}z^2 + 44\sqrt{\pi}e^{32}z^3 - \sqrt{\pi}e^{32}z^4\right) e^{-4z}\right)$$

$$-\infty \le z \le \infty$$

And for convolution of Normal(6,1), Exp(7) and Gamma(5,4) we have

$$f_Z(z) = \int_{-\infty}^{+\infty} f_Y(z-x)\, f_X(x)\, dx$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2*1\pi}} e^{-\frac{(z-x-6)^2}{2*1}} .\ \left(\frac{-7168}{243}e^{-7x} + \frac{7168}{243}e^{-4x} - \frac{7168}{81}e^{-4x}x + \frac{3584}{27}e^{-4x}x^2 - \frac{3584}{27}e^{-4x}x^3 + \frac{896}{9}e^{-4x}x^4\right) dx$$

$$= \frac{448}{243}\frac{1}{\sqrt{\pi}}\left(\left(8991\sqrt{2}\,z\,e^{-\frac{1}{2}z^2+13z-18} - 846\sqrt{2}\,z^2\,e^{-\frac{1}{2}z^2+13z-18} - 1116\,\mathrm{erf}\left(-5\sqrt{2}+\frac{1}{2}\sqrt{2}z\right)z^3\sqrt{\pi}e^{32+3z} + 27\mathrm{erf}\left(-5\sqrt{2}+\frac{1}{2}\sqrt{2}z\right)z^4\sqrt{\pi}e^{32+3z} + 27\sqrt{2}z^3e^{-\frac{1}{2}z^2+13z-18} - 32406\sqrt{2}e^{-\frac{1}{2}z^2+13z-18} + 327245\,\mathrm{erf}\left(-5\sqrt{2}+\frac{1}{2}\sqrt{2}z\right)\sqrt{\pi}e^{32+3z} - 122892\,\mathrm{erf}\left(-5\sqrt{2}+\frac{1}{2}\sqrt{2}z\right)z\sqrt{\pi}e^{32+3z} + 17478\,\mathrm{erf}\left(-5\sqrt{2}+\frac{1}{2}\sqrt{2}z\right)z^2\sqrt{\pi}e^{32+3z} - 8e^{\frac{133}{2}}\mathrm{erf}\left(-\frac{13}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}z\right)\sqrt{\pi} - 8\sqrt{\pi}e^{\frac{133}{2}} + 327245\sqrt{\pi}e^{32+3z} - 122892\sqrt{\pi}e^{32+3z}z + 17478\sqrt{\pi}e^{32+3z}z^2 - 1116\sqrt{\pi}e^{32+3z}z^3 + 27\sqrt{\pi}e^{32+3z}z^4\right)e^{-7z}\right)$$

$$-\infty \leq z \leq \infty$$

We can obtain the minimum density as follows

$$P(X_1 < X_2) = \int_{-\infty}^{\infty}\int_{x_1}^{\infty} f_{x_2}(x_2)\,.\,f_{x_1}(x_1)\,dx_2dx_1 =$$

$$= \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\Bigg(\frac{448}{243}\frac{1}{\sqrt{\pi}}\Bigg(\Bigg(8991\sqrt{2}\,x_2\,e^{-\frac{1}{2}x_2{}^2+13x_2-18} - 846\sqrt{2}\,x_2{}^2\,e^{-\frac{1}{2}x_2{}^2+13x_2-18}$$
$$- 1116\,\mathrm{erf}\left(-5\sqrt{2}+\frac{1}{2}\sqrt{2}x_2\right)x_2{}^3\sqrt{\pi}e^{32+3x_2}$$
$$+ 27\mathrm{erf}\left(-5\sqrt{2}+\frac{1}{2}\sqrt{2}x_2\right)x_2{}^4\sqrt{\pi}e^{32+3x_2} + 27\sqrt{2}x_2{}^3e^{-\frac{1}{2}x_2{}^2+13x_2-18}$$
$$- 32406\sqrt{2}e^{-\frac{1}{2}x_2{}^2+13x_2-18} + 327245\,\mathrm{erf}\left(-5\sqrt{2}+\frac{1}{2}\sqrt{2}x_2\right)\sqrt{\pi}e^{32+3x_2}$$
$$- 122892\,\mathrm{erf}\left(-5\sqrt{2}+\frac{1}{2}\sqrt{2}x_2\right)x_2\sqrt{\pi}e^{32+3x_2}$$
$$+ 17478\,\mathrm{erf}\left(-5\sqrt{2}+\frac{1}{2}\sqrt{2}x_2\right)x_2{}^2\sqrt{\pi}e^{32+3x_2}$$
$$- 8e^{\frac{133}{2}}\,\mathrm{erf}\left(-\frac{13}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}x_2\right)\sqrt{\pi} - 8\sqrt{\pi}e^{\frac{133}{2}} + 327245\sqrt{\pi}e^{32+3x_2}$$
$$- 122892\sqrt{\pi}e^{32+3x_2}x_2 + 17478\sqrt{\pi}e^{32+3x_2}x_2{}^2 - 1116\sqrt{\pi}e^{32+3x_2}x_2{}^3$$
$$+ 27\sqrt{\pi}e^{32+3x_2}x_2{}^4\Bigg)e^{-7x_2}\Bigg)\Bigg)\,.\Bigg(\frac{1}{\sqrt{\pi}}\Bigg(64\left(-744\sqrt{\pi}\,\mathrm{erf}\left(-6+\frac{1}{2}x_1\right)e^{32}x_1{}^2\right.$$
$$+ 5712\sqrt{\pi}\,\mathrm{erf}\left(-6+\frac{1}{2}x_1\right)e^{32}x_1 - 16764\sqrt{\pi}\,\mathrm{erf}\left(-6+\frac{1}{2}x_1\right)e^{32}$$
$$+ 44\sqrt{\pi}\,\mathrm{erf}\left(-6+\frac{1}{2}x_1\right)e^{32}x_1{}^3 - \sqrt{\pi}\,\mathrm{erf}\left(-6+\frac{1}{2}x_1\right)e^{32}x_1{}^4$$
$$+ 24\sqrt{\pi}\,\mathrm{erf}\left(-5+\frac{1}{2}x_1\right)e^{21+x_1} - 716x_1e^{\frac{-1}{4}x_1{}^2+6x_1-4} + 64x_1{}^2e^{\frac{-1}{4}x_1{}^2+6x_1-4}$$
$$- 2x_1{}^3e^{\frac{-1}{4}x_1{}^2+6x_1-4} + 2752e^{\frac{-1}{4}x_1{}^2+6x_1-4} + 24\sqrt{\pi}e^{21+x_1} - 16764\sqrt{\pi}e^{32}$$
$$+ 5712\sqrt{\pi}e^{32}x_1 - 744\sqrt{\pi}e^{32}x_1{}^2 + 44\sqrt{\pi}e^{32}x_1{}^3$$
$$\left.- \sqrt{\pi}e^{32}x_1{}^4\right)e^{-4x_1}\Bigg)\Bigg)\,dx_2dx_1$$

To solve this integral we used the mathematical software Maple 15 but it was not able to compute because of the computational complexity, so we recommended a method based on simulation.

4. Solving by simulation

Here to solve the proposed model, we model paths depicted in figure 2 with simulation software, ARENA 13.5. We run the simulation model for hundred replications for each path and obtain the following results in Table 1.

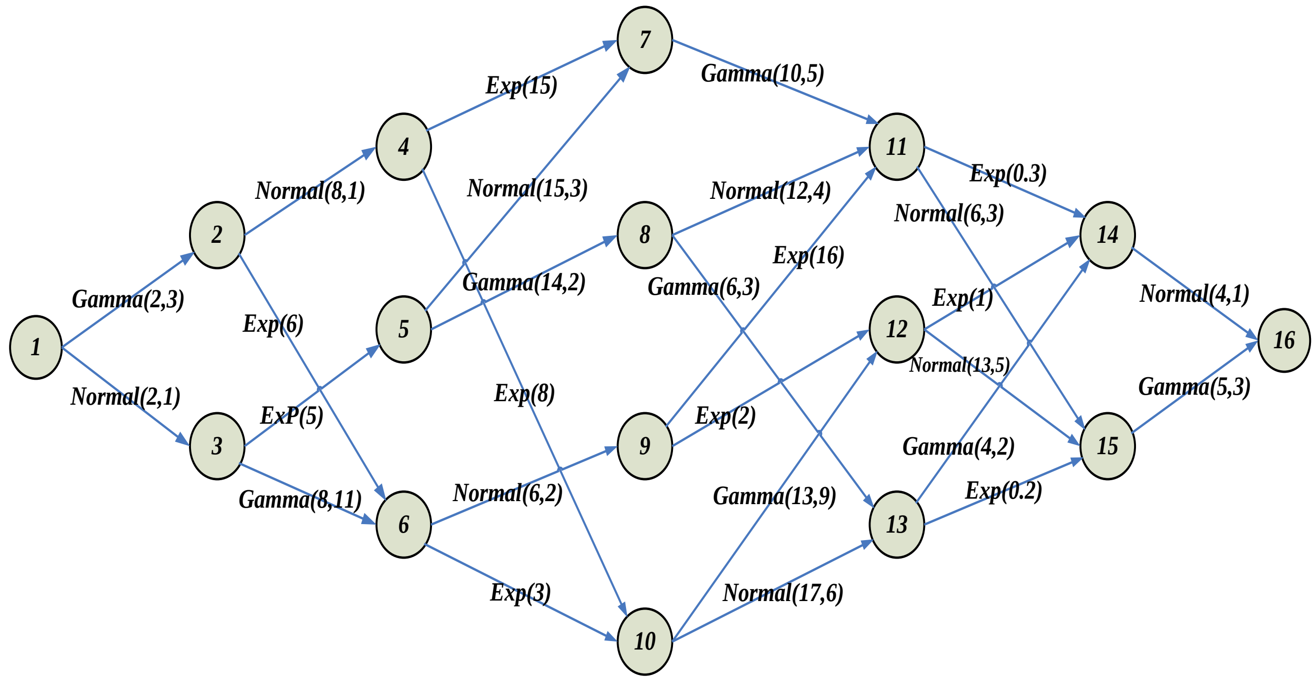

Figure 2. A stochastic network with 16 nodes

Table 1. Obtained results for hundred replications						
Path	Average	Half Width	Minimum Average	Maximum Average	Minimum Value	Maximum Value
1-2-4-10-12-14-16	14.9716	0.39	10.4994	20.3940	10.4994	20.3940
1-2-4-10-12-15-16	24.9342	0.50	18.2201	30.6844	18.2201	30.6844
1-2-4-10-13-14-16	32.2663	0.63	25.6364	38.7478	25.6364	38.7478
1-2-4-10-13-15-16	32.8803	1.32	20.5640	56.8021	20.5640	56.8021
1-2-4-7-11-14-16	18.3650	0.89	13.1925	43.3847	13.1925	43.3847
1-2-4-7-11-15-16	18.4270	0.49	12.6676	25.4876	12.6676	25.4876
1-2-6-10-12-14-16	7.7556	0.30	4.6035	13.7727	4.6035	13.7727
1-2-6-10-12-15-16	17.1577	0.55	8.6838	24.0782	8.6838	24.0782
1-2-6-10-13-14-16	23.6346	0.65	15.2272	33.1837	15.2272	33.1837
1-2-6-10-13-15-16	24.2699	1.00	16.1411	41.9964	16.1411	41.9964
1-2-6-9-11-14-16	13.7539	0.73	7.4160	28.5965	7.4160	28.5965
1-2-6-9-11-15-16	14.1605	0.46	8.6469	20.9370	8.6469	20.9370
1-2-6-9-12-14-16	12.0591	0.39	7.3846	16.0609	7.3846	16.0609

1-2-6-9-12-15-16	21.8832	0.62	13.8247	29.6838	13.8247	29.6838
1-3-5-7-11-14-16	27.2517	0.75	19.5298	41.2170	19.5298	41.2170
1-3-5-7-11-15-16	27.1453	0.63	19.9355	35.8553	19.9355	35.8553
1-3-5-8-11-14-16	28.3663	0.95	19.6513	45.8492	19.6513	45.8492
1-3-5-8-11-15-16	29.1721	0.57	23.4243	37.2251	23.4243	37.2251
1-3-5-8-13-14-16	17.2458	0.87	9.6648	34.2593	9.6648	34.2593
1-3-5-8-13-15-16	18.6861	1.24	7.5693	46.1012	7.5693	46.1012
1-3-6-10-12-14-16	9.6244	0.37	6.2589	15.7634	6.2589	15.7634
1-3-6-10-12-15-16	18.9690	0.54	11.7268	24.5663	11.7268	24.5663
1-3-6-10-13-14-16	26.4478	0.58	18.9323	33.0969	18.9323	33.0969
1-3-6-10-13-15-16	25.8481	0.92	15.9543	43.1245	15.9543	43.1245
1-3-6-9-11-14-16	16.4056	0.81	9.4629	31.2910	9.4629	31.2910
1-3-6-9-11-15-16	16.7100	0.57	11.4708	23.6385	11.4708	23.6385
1-3-6-9-12-14-16	13.8863	0.45	8.4222	18.3395	8.4222	18.3395
1-3-6-9-12-15-16	24.2398	0.55	17.6289	32.5769	17.6289	32.5769

It is illustrated in Table 1 that the shortest path in network is 1-3-5-6, and this result can prove the efficiency of our method in such problems.

4. Conclusions

This paper proposed a dynamic program for determining the shortest path in a network having exponential, gamma and normal probability distributions as arc lengths. Since the definite values of the dynamic program were turned into exponential, gamma or normal random variables, two modifications were performed on sum and comparison operators. Convolution technique was employed for summing probability distributions. Numerical example via a six node network showed the performance of the proposed methodology for the shortest path. Since the problem was difficult to solve with analytical computational methods, simulation experiments were developed to obtain the numerical results for the shortest path in 16 nodes network.

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