Directional Hölder metric subregularity and application to tangent cones*

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Abstract

In this work, we study directional versions of the Hölderian/Lipschitzian metric subregularity of multifunctions. Firstly, we establish variational characterizations of the Hölderian/Lipschitzian directional metric subregularity by means of the strong slopes and next of mixed tangency-coderivative objects . By product, we give second-order conditions for the directional Lipschitzian metric subregularity and for the directional metric subregularity of demi order. An application of the directional metric subregularity to study the tangent cone is discussed.

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1 Introduction

Solving equations of the form:

$$F(x) = y, \quad \text{for } \ y \in Y \tag{1}$$

where $F : X \to Y$ is a single mapping between, in general, two metric spaces X and Y is one of the most important problems of Mathematics. When the parameter y varies, under some conditions ensuring the uniqueness of solution, a central issue is to investigate the behavior of the solution mapping $x(y)$ of equation (1). The classical implicit function theorems tell us on the existence and the uniqueness of solutions, as well as the differentiability of the solution mapping. When the mapping defines the equation is *multi-valued,* instead of (1), we consider generalized equations (in the sense of Ronbinson) of the form:

$$\text{Find } \ x \in X \ \text{ such that } \ y \in F(x), \tag{2}$$

where $F : X \rightrightarrows Y$ is a set-valued mapping, i.e., a mapping assigns to every $x \in X$ a subset (possibly empty) $F(x)$ of Y.
As usual, we use the notations gph $F := \{(x, y) \in X \times Y \ : \ y \in F(x)\}$ for the graph of F, Dom $F :=$

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$\{x \in X \ : \ F(x) \neq \emptyset\}$ for the domain of F and $F^{-1} : Y \rightrightarrows X$ for the inverse map of F. This inverse map is defined by $F^{-1}(y) := \{x \in X \ : \ y \in F(x)\}, \ y \in Y$ and satisfies

$$(x, y) \in \text{gph } F \iff (y, x) \in \text{gph } F^{-1}.$$

In practice, we can only find out an approximate solution of (2). When an approximate solution is available, it is crucial to estimate the distance $d(x, F^{-1}(y))$, from an approximate solution x to the solution set $F^{-1}(y))$, regarded as an error of the approximation. A quantity which is used naturally to estimate the distance $d(x, F^{-1}(y))$ is $d(y, F(x))$, and this leads to the concept of the *metric regularity*: Recall that a set-valued mapping F is said to be *metrically regular* at $(\bar{x}, \bar{y})$ with $\bar{y} \in F(\bar{x})$ with modulus $\tau > 0$ if there exists a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \quad \text{for all} \quad (x, y) \in U \times V, \tag{3}$$

where, $d(x, C)$ denotes, as usual, the distance from x to a set C and is defined by $d(x, C) = \inf_{z \in C} d(x, z)$, with the convention that $d(x, S) = +\infty$ whenever S is empty.

The metric regularity of set-valued mapping is a central and crucial concept in modern variational analysis and it has many applications in optimization, control theory, game theory, etc. For a detailed account the reader is referred to the books or contributions of many researchers (e.g., [8, 16, 17, 19, 21, 22, 23, 24, 25, 36, 37, 38, 39, 45, 47, 46, 52, 53, 54, 59, 62, 65, 68] and the references given therein) for many theoretical results on metric regularity as well as for its various applications.

By fixing $y = \bar{y}$ in (3) in the definition of the metric regularity, we obtain a weaker property called *metric subregularity*: The mapping F is said to be *metrically subregular* at $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{y} \in F(\bar{x})$ with modulus $\tau > 0$ if there exists a neighborhood U of $\bar{x}$ such that

$$d(x, F^{-1}(\bar{y})) \leq \tau d(\bar{y}, F(x)) \quad \text{for all} \quad x \in U. \tag{4}$$

We also refer to the references ([32, 33, 34, 47, 52, 53, 60, 65, 67]) for the recent studies of the metric subregularity.

The Hölderian version of the metric subregularity is defined as follows: The set-valued mapping F is said to be *Hölder metrically subregular* of order $\gamma \in (0, 1]$ at $(\bar{x}, \bar{y})$ with $\bar{y} \in F(\bar{x})$ with modulus $\tau > 0$ if there exists a neighborhood U of $\bar{x}$ such that

$$d(x, F^{-1}(\bar{y})) \leq \tau [d(\bar{y}, F(x))]^{\gamma} \quad \text{for all} \quad x \in U. \tag{5}$$

When the inequality above holds for all y near $\bar{y}$, we say that F is *Hölder metrically regular* of order $\gamma \in (0, 1]$ at $(\bar{x}, \bar{y})$. The regular/subregular properties of Höder type were studied initially in late 80s of the last century by Borwein-Zuang [16], Frankowska [28], Penot [62]. Recently, it attracted a lot of interest of researchers, due to a broad range of applications of the nonlinear regularity models (see. e.g., [29], [40], [30], [49] and the references given therein). In such works, the authors have established characterizations for the Hölder metric subregularity/regularity of multifunctions by using derivative-like objects in some different ways, as well as the applications to study the stability of variational systems and the convergence analysis of algorithms.

In some situations in Optimization, e.g., in the study of sensitivity analysis; in the theory of necessary optimality conditions in Mathematical Programming, one only needs a regular behavior with respect to some directions (see [13], [19]). Due to this, several directional versions of the regular notions were considered. In [1, 3], Arutyunov et al have introduced and studied a notion of directional metric regularity. This notion is an extension of an earlier notion used by Bonnan & Shapiro ([13]) to study sensitivity analysis. Later, Ioffe ([41]) has introduced and investigated an extension called *relative metric regularity* which covers many notions of metric regularity in the literature. Recently, an other version of directional metric regularity/subregularity has been introduced and extensively studied by Gfrerer in [33], [34]. This author has established some variational characterizations of this directional metric regularity/subregularity, and it has been succesfully applied this directional regular properties to study optimality conditions for mathematical programs. In fact, this directional regular property has been earlier used by Penot ([63]) to study second order optimality conditions

In this paper, we consider the following directional version of the Hölderian metric subregularity. This notion is a natural extention of the directional metric regularity of Lipschitz type introduced by Gfrerer in [33], [34]. As usual, in a normed space X, for $x \in X$ and $r > 0$, the open and closed balls with center x and radius $r > 0$ is denoted by $B(x, r), \bar{B}(x, r)$, respectively, while cone A stands for the conic hull of $A \subseteq X$, i.e., cone $A = \cup_{\lambda \geq 0} \lambda A$.

Definition 1 *Let X be a normed space and let Y be a metric space. Let $\gamma \in (0, 1]$ and $u \in X$ be given. A mapping $F : X \rightrightarrows Y$ is said to be directionally metrically γ-subregular or directionally Hölder metrically subregular of order γ at $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{y} \in F(\bar{x})$ in direction u with modulus $\tau > 0$ if there exist a neighborhood U of $\bar{x}$ and positive real numbers c, δ such that*

$$d(x, F^{-1}(\bar{y})) \leq \tau [d(\bar{y}, F(x))]^{\gamma} \tag{6}$$

for all $x \in U \cap (\bar{x} + \text{cone } B(u, \delta))$. When $\gamma = 1$, we say simply that F is directionally metrically subregular at $(\bar{x}, \bar{y})$ in direction u.

Note that the directional Hölderian metric subregularity at $(\bar{x}, \bar{y})$ in direction $u = 0$ coincides with the Hölderian metric subregularity of the same order at $(\bar{x}, \bar{y})$.

The main objective of this paper is to characterize the directional Höderian metric subregularity by means of the strong slopes as well as of generalized derivatives or coderivatives. Such characterizations allow us to establish the second order conditions for the directional metric regularities of Lipschitz type and of order 1/2 of the mixed smooth-convex inclusions of the form:

$$0 \in g(x) - F(x),$$

where, $g : X \to Y$ is a sufficiently smooth function and $F : X \rightrightarrows Y$ is a convex multifunction. Such inclusions play an important role in many optimization and control models. As an application, we show the effectivity of the directional Hölderian metric subregularity to examine the tangent vectors to a zero set.

The paper is organized as follows. In Section 2, in counterpart of the directional Hölderian metric regularities, we introduce the directional Höderian error bound property of lower semicontinuous functions. We give a characterization of the directional error bound by means of strong slopes. Using this characterization, we establish in Section 2 a sufficient condition for the directional Hölder metric subregularity of closed set-valued mappings on Banach spaces by using mixed tangency-coderivative objects. In Section 3, the second-order characterizations for the directional Lipschitzian metric subregularity and directional $\frac{1}{2}$-metric subregularity are investigated. In the final section, we apply the directional Hölderian metric subregularity to examine the tangent cone to the solution set of equations or systems of inequalities/equalities.

2 Directional error bounds

Let X be a metric space. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function. As usual, $\mathrm{dom} f := \{x \in X : f(x) < +\infty\}$ denotes the domain of f. We set

$$S := \{x \in X : \quad f(x) \le 0\}. \tag{7}$$

We use the symbol $[f(x)]_+$ to denote $\max(f(x), 0)$. For given $\gamma \in (0, 1]$, we shall say that the system (7) admits *an error bound of order* γ around $x_0 \in X$ if there exist reals $c, \varepsilon > 0$ such that

$$d(x, S) \le c\big[f(x)\big]_+^\gamma \quad \text{for all} \quad x \in B(x_0, \varepsilon). \tag{8}$$

Several characterizations of error bounds have been established in the literature (see, e.g., [11], [58], [61]). The following characterization of the error bound in terms the strong slope is due to Azé-Corvellec in [11].

Recall from [20], [11] that the strong slope $|\nabla f|(x)$ of a lower semicontinuous function f at $x \in \mathrm{dom} f$ is the quantity defined by $|\nabla f|(x) = 0$ if x is a local minimum of f; otherwise

$$|\nabla f|(x) = \limsup_{y \to x,\, y \neq x} \frac{f(x) - f(y)}{d(x, y)}.$$

For $x \notin \mathrm{dom} f$, we set $|\nabla f|(x) = +\infty$.

Theorem 2 *([11], [59]) Let X be a complete metric space. Suppose that $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous and $\bar{x} \in S$. If there exist a neighborhood U of $\bar{x}$ and reals $m, \mu > 0$ such that $|\nabla f|(x) \ge m$ for all $x \in U$ with $f(x) \in (0, \mu)$ then there exists a neighborhood V of $\bar{x}$ such that*

$$md(x, S) \le [f(x)]_+ \quad \textit{for all} \quad x \in V.$$

Fact 3 *([59], Corollary 2.5) Let a real $\alpha > 0$ be given. Then for all $x \in X$ with $f(x) > 0$, one has*

$$|\nabla f^\alpha|(x) = \alpha f^{\alpha-1}(x)|\nabla f|(x).$$

Here, $f^\alpha(x) = (f(x))^\alpha$.

From this fact, Theorem 2 yields the following characterization of the Hölderian error bound of some order α.

Theorem 4 *([59], Corollary 2.5) Let X be a complete metric space and let a real $\alpha \in (0,1]$. Suppose that $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous and $\bar{x} \in S$. If there exist a neighborhood U of $\bar{x}$ and reals $m, \mu > 0$ such that*

$$\alpha f^{\alpha-1}(x)|\nabla f|(x) \geq m \quad \textit{for all } x \in U \textit{ with } f(x) \in (0, \mu)$$

then there exists a neighborhood V of $\bar{x}$ such that

$$md(x,S) \leq [f(x)]_+^\alpha \quad \textit{for all} \quad x \in V.$$

We introduce the directional version of the error bound.

Definition 5 *Let X be a normed space. For given $\gamma \in (0,1]$ and $u \in X$, we say that the system (7) admits an error bound of order γ around $x_0 \in S$ in direction u if there exist $c, \delta > 0$ such that*

$$d(x,S) \leq c[f(x)]_+^\gamma \quad \textit{for all } x \in B(x_0,\delta) \cap (x_0 + \text{cone } B(u,\delta)).$$

Obviously, the error bound in direction $u = 0$ coincides the usual error bound (at the same point with the same order).

The following theorem gives a slope characterization of the directional error bound.

Theorem 6 *Let X be a Banach space. Consider the system (7) associated to a lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$. For given $\bar{x} \in S$, if*

$$\liminf_{\substack{x \underset{u}{\to} \bar{x},\, x \notin S \\ \frac{f(x)}{\|x-\bar{x}\|} \to 0}} |\nabla f|(x) > 0, \tag{9}$$

where $x \underset{u}{\to} \bar{x}$ means that $x \to \bar{x}$ if $u = 0$ and $\frac{x-\bar{x}}{\|x-\bar{x}\|} \to \frac{u}{\|u\|}$ as well as $x \to \bar{x}$ if $u \neq 0$, then there exist reals $\tau, \varepsilon, \delta > 0$ such that

$$d(x,S) \leq \tau[f(x)]_+ \quad \textit{for all } x \in B(\bar{x},\varepsilon) \cap [\bar{x} + \text{cone } B(u,\delta)]\,.$$

That is, the system S admits an error bound at $\bar{x}$ in direction u with modulus τ.

Proof. The case $u = 0$ was proved in ([60], Theorem 1). For $u \neq 0$, we prove the theorem by contradiction. Suppose on the contrary that S has not an error bound at $\bar{x}$ in direction u. Then, there exists a sequence $\{x_n\} \subseteq X$ such that

$$0 < \|x_n - \bar{x}\| < \frac{1}{n}, \left\| \frac{x_n - \bar{x}}{\|x_n - \bar{x}\|} - \frac{u}{\|u\|} \right\| < \frac{1}{n},$$

such that

$$f_+(x_n) < \frac{1}{n^2} d(x_n, S).$$

By virtue of Ekeland variational principle, one gets a point $z_n \in X$, satisfying the following conditions:

$$\|z_n - x_n\| < \frac{1}{n} d(x_n, S), f_+(z_n) \leq f_+(x_n),$$

and

$$f_+(z_n) \le f_+(x) + \frac{1}{n}\|x - z_n\|, \ \forall\, x \in X.$$

Thus $z_n \notin S$ and it deduces that

$$|\nabla f|(z_n) \le \frac{1}{n},\ \frac{f(z_n)}{\|z_n - \bar{x}\|} \le \frac{f_+(z_n)}{\|z_n - \bar{x}\|} \le \frac{1}{n},\ \text{and } z_n \to \bar{x}. \tag{10}$$

Note that

$$\|z_n - \bar{x}\| \le \|x_n - z_n\| + \|x_n - \bar{x}\| \le \frac{n+1}{n}\|x_n - \bar{x}\|$$

and,

$$\|z_n - \bar{x}\| \ge \|x_n - \bar{x}\| - \|x_n - z_n\| \ge \frac{n-1}{n}\|x_n - \bar{x}\|.$$

Then,

$$\left\|\frac{z_n - \bar{x}}{\|z_n - \bar{x}\|} - \frac{u}{\|u\|}\right\| = \left\|\frac{x_n - \bar{x} + z_n - x_n}{\|z_n - \bar{x}\|} - \frac{u}{\|u\|}\right\| = \left\|\frac{\frac{x_n - \bar{x}}{\|x_n - \bar{x}\|} + \frac{z_n - x_n}{\|x_n - \bar{x}\|}}{\frac{\|z_n - \bar{x}\|}{\|x_n - \bar{x}\|}} - \frac{u}{\|u\|}\right\| \to 0, \tag{11}$$

since

$$0 \le \frac{\|z_n - x_n\|}{\|x_n - \bar{x}\|} \le \frac{1/n d(x_n, S)}{\|x_n - \bar{x}\|} \le \frac{1}{n}\frac{\|x_n - \bar{x}\|}{\|x_n - \bar{x}\|} = \frac{1}{n} \to 0,\ \text{as } n \to \infty$$

and,

$$\frac{n-1}{n} \le \frac{\|z_n - \bar{x}\|}{\|x_n - \bar{x}\|} \le \frac{n+1}{n}.$$

From (10) and (11) one see that the condition (9) does not happen. The proof is completed. □

By applying Theorem 6 for the function f_+^γ with $\gamma > 0$, one has the following sufficient condition ensuring the directional error bound of order γ.

Theorem 7 *Let X be a Banach space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Consider the system (7). Given $\bar{x} \in S$, $u \in X$, and a real $\gamma \in (0, 1]$. If*

$$\liminf_{\substack{x \xrightarrow{u} \bar{x},\ x \notin S \\ \frac{f^\gamma(x)}{\|x - \bar{x}\|} \to 0}} |\nabla f^\gamma|(x) = \liminf_{\substack{x \xrightarrow{u} \bar{x},\ x \notin S \\ \frac{f^\gamma(x)}{\|x - \bar{x}\|} \to 0}} \gamma f^{\gamma - 1}(x)|\nabla f|(x) := m_\gamma > 0, \tag{12}$$

then there exist reals $\tau, \varepsilon, \delta > 0$ such that

$$d(x, S) \le \tau[f(x)]_+^\gamma \quad \text{for all } \ x \in B(\bar{x}, \varepsilon) \cap [\bar{x} + \text{cone } B(u, \delta)]\,.$$

3 Directional Hölder metric subregularity

3.1 Directional metric subregularity via directional error bound

Let X be a normed spacce and let Y be metric space. Let $F : X \rightrightarrows Y$ be a multifunction, $(\bar{x}, \bar{y}) \in \text{gph } F$ and given $u \in X$. Recall that the lower semicontinuous envelope function of the function $x \mapsto d(\bar{y}, F(x))$ is defined by for every $x \in X$:

$$\varphi(x) := \liminf_{u \to x} d(\bar{y}, F(u)).$$

In [55], [59], this function has been effectively used to study the metric regularity/subregularity of multifunctions. The following proposition allows us to transform equivalently the directional metric regularity of the multifunction F to the directional error bound of the function φ.

Proposition 8 *Let X be a normed space and Y be a metric space. Suppose that the multifunction $F : X \rightrightarrows Y$ has a closed graph and a point $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{y} \in F(\bar{x})$. One has*

$$S := F^{-1}(\bar{y}) = \{x \in X : \varphi(x) = 0\}. \tag{13}$$

Moreover, for given $\gamma > 0$ and $u \in X$, F is directional Holder metrically subregular of order γ in direction u with modulus $\tau > 0$ if and only if φ admits an error bound of order γ in the direction u at $\bar{x}$ with the same modulus τ, i.e., there exist $\tau, \delta, \varepsilon > 0$ such that

$$d(x, S) \leq \tau \varphi^{\gamma}(x) \tag{14}$$

for all $x \in B(\bar{x}, \varepsilon) \cap (\bar{x} + \text{cone } B(u, \delta))$.

Proof. Relation (13) is obvious. Suppose now F is directional metrically $\gamma-$subregular at $(\bar{x}, \bar{y})$ in direction u. Let $\delta > 0$ be such that

$$d(x, S) \leq \tau [d(\bar{y}, F(x))]^{\gamma} \quad \forall x \in B(\bar{x}, \delta) \cap (\bar{x} + \text{cone } B(u, \delta)). \tag{15}$$

For any $x \in B(\bar{x}, \delta) \cap (\bar{x} + \text{cone } B(u, \delta))$ with $x \neq \bar{x}$, let a sequence $u_n \neq x, u_n \to x$ such that $\lim_{n\to\infty} d(\bar{y}, F(u_n)) = \varphi(x)$. Since $u_n \to x$, then when n is sufficiently large, $u_n \in B(\bar{x}, \varepsilon) \cap [\bar{x} + \text{cone } B(u, \delta)]$. Thus, by (15), one has

$$d(u_n, S) \leq \tau [d(\bar{y}, F(u_n))]^{\gamma}.$$

By letting $n \to \infty$, one obtains the desired inequality:

$$d(x, S) \leq \tau \varphi^{\gamma}(x).$$

The inverse implication is obvious. □

By virtue of this proposition, Theorem 7 yields directly the slope characterization of the Hölderian directional metric subregularity.

Theorem 9 *Let X be a Banach space and Y be a metric space. Suppose that the multifunction $F : X \rightrightarrows Y$ has a closed graph and a point $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{y} \in F(\bar{x})$. Given $u \in X$, and a real $\gamma \in (0, 1]$. If*

$$\liminf_{\substack{x \xrightarrow{u} \bar{x},\, x \notin S \\ \frac{\varphi^{\gamma}(x)}{\|x-\bar{x}\|} \to 0}} |\nabla \varphi^{\gamma}|(x) = \liminf_{\substack{x \xrightarrow{u} \bar{x},\, x \notin S \\ \frac{\varphi^{\gamma}(x)}{\|x-\bar{x}\|} \to 0}} \gamma \varphi^{\gamma-1}(x) |\nabla \varphi|(x) > 0, \tag{16}$$

then there exist reals $\tau, \varepsilon, \delta > 0$ such that

$$d(x, F^{-1}(\bar{y})) \leq \tau [d(\bar{y}, F(x))]^{\gamma} \quad \text{for all } x \in B(\bar{x}, \varepsilon) \cap [\bar{x} + \text{cone } B(u, \delta)].$$

That is, F is directional metrically $\gamma-$subregular at $(\bar{x}, \bar{y})$ in direction u with modulus τ.

3.2 Mixed tangency-coderivative conditions for Directional Hölder metric subregularity

In this subsection, we make use of the abstract subdifferential ∂ on a Banach space X,which satisfies the following conditions:

(C1) If $f : X \to \mathbb{R}$ is a convex function which is continuous around $\bar{x} \in X$ and $\beta : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable at $t = f(x)$, then

$$\partial(\beta \circ f)(x) \subseteq \{\beta'(f(x))x^* \in X^* : \langle x^*, y - x\rangle \leq f(y) - f(x) \quad \forall y \in X\}.$$

(C2) $\partial f(x) = \partial g(x)$ if $f(y) = g(y)$ for all y in a neighborhood of x.

(C3) Let $f_1 : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $f_2, ..., f_n : X \to \mathbb{R}$ be Lipschitz functions. If $f_1 + f_2 + ... + f_n$ attains a local minimum at x_0, then for any $\varepsilon > 0$, there exist $x_i \in x_0+\varepsilon B_X$, $x_i^* \in \partial f_i(x_i)$, $i \in \overline{1,n}$, such that $|f_i(x_i)-f_i(x_0)| < \varepsilon$, $i \in \overline{1,n}$, and $\|x_1^*+x_2^*+...+x_n^*\| < \varepsilon$.

For a closed subset C of X, the normal cone to C with respect to a subdifferential operator ∂ at $x \in C$ is defined by $N(C, x) = \partial\delta_C(x)$, where δ_C is the indicator function of C given by $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$, otherwise and we assume here that $\partial\delta_C(x)$ is *a cone* for all closed subset C of X.

Let X, Y be Banach spaces, and let ∂ be a subdifferential on $X \times Y$. Let $F : X \rightrightarrows Y$ be a closed multifunction and let $(\bar{x}, \bar{y}) \in \text{gph}F$. The multifunction $D^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ defined by

$$D^*F(\bar{x}, \bar{y})(y^*) = \{x^* \in X^* : (x^*, -y^*) \in N(\text{gph}F, (\bar{x}, \bar{y}))\}$$

is called *the* $\partial-$*coderivative* of F at $(\bar{x}, \bar{y})$. In the following theorem, we assume further that ∂ is a subdifferential operator on $X \times Y$ which satisfies the separable property in the following sense:
(C4) If $f(x, y) := f_1(x) + f_2(y)$, $(x, y) \in X \times Y$, where $f_1 : X \to \mathbb{R} \cup \{+\infty\}$, $f_2 : X \to \mathbb{R} \cup \{+\infty\}$, is a separable function defined on $X \times Y$, then

$$\partial f(x, y) = \partial f_1(x) \times \partial f_2(y), \quad \text{for all} \quad (x, y) \in X \times Y.$$

It is well known that the proximal subdifferential on Hilbert spaces; the Fréchet subdifferential in Asplund spaces; the viscosity subdifferentials in Smooth spaces as well as the Ioffe and the Clarke-Rockafellar subdifferentials in the setting of general Banach spaces are subdifferentials satisfying the conditions (C1)-(C4).

Let us introduce the following notion of *the directional strict limit set critical* for metric $\gamma-$subregularity. This is a directional version with some positive order of the strict limit set critical introduced in [60] as a refinement of the one by Gfrerer ([32], [33]).

Definition 10 *For a closed multifunction $F : X \rightrightarrows Y$, a given direction $u \in X$, a real $\gamma > 0$ and $(\bar{x}, \bar{y}) \in$ gph F, the directional strict limit set critical for metric $\gamma-$subregularity of F at $(\bar{x}, \bar{y})$ in direction u denoted by $SCr^\gamma F(\bar{x}, \bar{y})(u)$ is defined as the set of all $(v, x^*) \in Y \times X^*$ such that there exist sequences $\{t_n\} \downarrow 0$, $\{\varepsilon_n\} \downarrow 0$, $u_n \in \text{cone } B(u, \varepsilon_n)$, $(v_n, t_n^{\frac{\gamma-1}{\gamma}} \|v_n\|^{\gamma-1} x_n^*) \to (v, x^*)$, $(u_n, y_n^*) \in \mathcal{S}_X \times \mathcal{S}_{Y^*}$ with $x_n^* \in D^*F(\bar{x} + t_n u_n, \bar{y} + t_n^{\frac{1}{\gamma}} v_n)(y_n^*)$, $\bar{y} \notin F(\bar{x} + t_n u_n)$ $(\forall n)$, and $\frac{\langle y_n^*, v_n\rangle}{\|v_n\|} \to 1$.*

When $\gamma = 1$, we write and say simply $\text{SCr}^1F(\bar{x}, \bar{y})(u) := \text{SCr}F(\bar{x}, \bar{y})(u)$: the *directional strict limit set critical* for metric subregularity of F at $(\bar{x}, \bar{y})$ in direction u. In the case $u = 0$, we denote $\text{SCr}^\gamma F(\bar{x}, \bar{y})(0) := \text{SCr}F(\bar{x}, \bar{y})$: the *strict limit set critical* for metric $\gamma-$subregularity of F at $(\bar{x}, \bar{y})$.

The following theorem provides a sufficient condition for the directional metric $\gamma-$subregularity of closed multifunctions in terms of the abstract coderivative in the setting of Banach spaces.

Theorem 11 *Let X, Y be Banach spaces and let ∂ be a subdifferential operator on $X \times Y$. Let $F : X \rightrightarrows Y$ be a closed multifunction between X and Y with $(\bar{x}, \bar{y}) \in \text{gph } F$. For $\gamma \in (0,1]$ and a given direction $u \in X$. If $(0,0) \notin SCr^{\gamma}F(\bar{x},\bar{y})(u)$ then F is metrically $\gamma-$subregular at $(\bar{x},\bar{y})$ in direction u.*

Proof. Suppose to the contrary that F is not metrically $\gamma-$subregular at $(\bar{x},\bar{y})$ in direction u. In view of Theorem 9, there exist a sequence $\{x_n\} \subseteq X$ and a sequence of positive reals $\{\delta_n\}$ such that

$$x_n \notin F^{-1}(\bar{y}),\ \delta_n \downarrow 0, \|x_n - \bar{x}\| < \delta_n,\ x_n \in \bar{x} + \text{cone } B(u, \delta_n),$$

$$\lim_{n\to\infty} \frac{\varphi^{\gamma}(x_n)}{\|x_n - \bar{x}\|} = 0, \text{ and } \lim_{n\to\infty} |\nabla \varphi^{\gamma}|(x_n) = 0.$$

Since $\lim_{n\to\infty} \frac{\varphi^{\gamma}(x_n)}{\|x_n - \bar{x}\|} = 0$, so we can assume that $\varphi^{\gamma}(x_n) \in (0,1)$.
Without loss of generality, we choose $\{\delta_n\}$ such that $\delta_n \in (0, \varphi^{\gamma}(x_n))$ and $\delta_n/\varphi^{\gamma}(x_n) \to 0$. Then for each n, there is $\eta_n \in (0, \delta_n)$, with $2\eta_n + \delta_n < \varphi(x_n)$ such that $d(\bar{y}, F(z)) \geq \varphi(x_n)(1 - \delta_n)$, $\forall z \in B(x_n, 4\eta_n)$ and

$$\delta_n \geq \frac{\varphi^{\gamma}(x_n) - \varphi^{\gamma}(z)}{\|x_n - z\|} \quad \text{for all } z \in \overline{B}(x_n, \eta_n).$$

Equivalently,

$$\varphi^{\gamma}(x_n) \leq \varphi^{\gamma}(z) + \delta_n \|z - x_n\| \quad \text{for all } z \in \overline{B}(x_n, \eta_n).$$

Take $z_n \in B(x_n, \eta_n^2/4)$, $w_n \in F(z_n)$ such that $\|\bar{y} - w_n\| \leq \varphi(x_n) + \eta_n^{2/\gamma}/4$. Then,

$$\|\bar{y} - w_n\|^{\gamma} \leq \varphi^{\gamma}(x_n) + \eta_n^2/4,$$

and

$$\|\bar{y} - w_n\|^{\gamma} \leq \varphi^{\gamma}(z) + \delta_n \|z - x_n\| + \eta_n^2/4 \ \forall z \in \overline{B}(x_n, \eta_n).$$

Therefore,

$$\|\bar{y} - w_n\|^{\gamma} \leq \|\bar{y} - w\|^{\gamma} + \delta_{\text{gph}F}(z, w) + \delta_n \|z - z_n\| + (\delta_n + 1)\eta_n^2/4$$
$$\forall (z, w) \in \overline{B}(x_n, \eta_n) \times Y.$$

Applying the Ekeland variational principle to the function

$$(z, w) \mapsto \|\bar{y} - w\|^{\gamma} + \delta_{\text{gph}F}(z, w) + \delta_n \|z - z_n\|$$

on $\overline{B}(x_n, \eta_n) \times Y$, we can select $(z_n^1, w_n^1) \in (z_n, w_n) + \frac{\eta_n}{4} B_{X\times Y}$ with $(z_n^1, w_n^1) \in \text{gph}F$ such that

$$\|\bar{y} - w_n^1\|^{\gamma} \leq \|\bar{y} - w_n\|^{\gamma} (\leq \varphi^{\gamma}(x_n) + \eta_n^2/4); \tag{17}$$

and that the function

$$(z, w) \mapsto \|\bar{y} - w\|^{\gamma} + \delta_{\text{gph}F}(z, w) + \delta_n \|z - z_n\| + (\delta_n + 1)\eta_n \|(z, w) - (z_n^1, w_n^1)\|$$

attains a minimum on $\overline{B}(x_n, \eta_n) \times Y$ at (z_n^1, w_n^1). As the functions $\|\bar{y} - w\|^{\gamma}$, $\|z - z_n\|$, $\|(z, w) - (z_n^1, w_n^1)\|$ are locally Lipschitz around (z_n^1, w_n^1), by (C3), we can find

$$w_n^2 \in B_Y(w_n^1, \eta_n);\ (z_n^3, w_n^3) \in B_{X\times Y}((z_n^1, w_n^1), \eta_n) \cap \text{gph}F;$$

$$(z_n^4, w_n^4) \in B_{X\times Y}((z_n^1, w_n^1), \eta_n);$$
$$w_n^{2*} \in \partial(\|\bar y - w_n\|^{\gamma-1}\|\bar y - \cdot\|)(w_n^2);\ (z_n^{3*}, -w_n^{3*}) \in N(\mathrm{gph}F, (z_n^3, w_n^3))$$
$$(z_n^{4*}, -w_n^{4*}) \in \partial(\|(\cdot,\cdot) - (z_n^1, w_n^1)\|)(z_n^4, w_n^4)$$

satisfying

$$\begin{array}{l} w_n^{3*} = w_n^{2*} + (\delta_n + 2)\eta_n w_n^{4*}, \\ \|w_n^{2*} - w_n^{3*}\| < (\delta_n + 2)\eta_n \quad \text{and} \quad \|z_n^{3*}\| \le \delta_n + (\delta_n + 2)\eta_n. \end{array} \tag{18}$$

Since $w_n^{2*} \in \partial(\|\bar y - \cdot\|^\gamma)(w_n^2) = \gamma\|\bar y - w_n^2\|^{\gamma-1}\partial(\|\bar y - \cdot\|)(w_n^2)$ (note that $\|\bar y - w_n^2\| \ge \|\bar y - w_n\| - \|w_n^2 - w_n\| \ge \varphi(x_n) - \delta_n - 2\eta_n > 0$), then $w_n^{2*} = \gamma\|\bar y - w_n^2\|^{\gamma-1}e_n$ with $\|e_n\| = 1$ and $\langle e_n, w_n^2 - \bar y\rangle = \|\bar y - w_n^2\|$. Thus, from the second relation in (18), it follows that

$$\|w_n^{3*}\| \ge \|w_n^{2*}\| - (\delta_n + 2)\eta_n = \gamma\|\bar y - w_n^2\|^{\gamma-1} - (\delta_n + 2)\eta_n > 0,$$

as well as

$$\|w_n^{3*}\| \le \|w_n^{2*}\| + (\delta_n + 2)\eta_n = \gamma\|\bar y - w_n^2\|^{\gamma-1} + (\delta_n + 2)\eta_n.$$

Set

$$t_n = \|z_n^3 - \bar x\|;\ \ u_n = (z_n^3 - \bar x)/t_n;\ \ v_n = (w_n^3 - \bar y)/t_n^{\frac{1}{\gamma}},$$

and

$$y_n^* = w_n^{3*}/\|w_n^{3*}\|;\ \ x_n^* = z_n^{3*}/\|w_n^{3*}\|.$$

Since

$$\varphi(x_n)(1 - \delta_n) \le d(\bar y, F(\bar x + t_n u_n)) \le t_n^{\frac{1}{\gamma}}\|v_n\| \le \|\bar y - w_n^1\| + \eta_n \le \varphi(x_n) + \eta_n^2/4 + \eta_n;$$

and

$$t_n = \|z_n^3 - \bar x\| \ge \|x_n - \bar x\| - \|z_n^3 - x_n\| \ge \|x_n - \bar x\| - \eta_n^2/4 - 5\eta_n/4,$$
$$t_n \le \|x_n - \bar x\| + \|z_n^3 - x_n\| \le \|x_n - \bar x\| + \eta_n^2/4 + 5\eta_n/4,$$

(Here, note that since $\|x_n - \bar x\| \to 0,\ \eta_n \to 0$ as $n \to \infty$, so we can assume that $1 > \|x_n - \bar x\| - \eta_n^2/4 - 5\eta_n/4 > 0$ for n sufficiently large.) then

$$\|v_n\| \le \frac{\varphi(x_n) + \eta_n^2/4 + \eta_n}{(\|x_n - \bar x\| - \eta_n^2/4 - 5\eta_n/4)^{\frac{1}{\gamma}}}.$$

As $\varphi^\gamma(x_n)/\|x_n - \bar x\| \to 0$ as well as $\eta_n/\|x_n - \bar x\| \to 0$, one obtains

$$\lim_{n\to\infty} v_n = 0. \tag{19}$$

Next one has $x_n^* \in D^*F(\bar x + t_n u_n, \bar y + t_n^{\frac{1}{\gamma}} v_n)(y_n^*)$ with $\|y_n^*\| = 1$ and by the second relation of (18), one derives that

$$\|x_n^*\| = \|z_n^{3*}\|/\|w_n^{3*}\| \le \frac{\delta_n + (\delta_n + 2)\eta_n}{\gamma\|\bar y - w_n^2\|^{\gamma-1} - (\delta_n + 2)\eta_n}. \tag{20}$$

Note that

$$\|\bar y - w_n^3\| - \eta_n \le \|\bar y - w_n^2\| \le \|\bar y - w_n^3\| + \eta_n.$$

That is,

$$t_n^{\frac{1}{\gamma}}\|v_n\| - \eta_n \le \|\bar{y} - w_n^2\| \le t_n^{\frac{1}{\gamma}}\|v_n\| + \eta_n.$$

Hence,

$$\|x_n^*\| t_n^{\frac{\gamma-1}{\gamma}}\|v_n\|^{\gamma-1} \le \frac{t_n^{\frac{\gamma-1}{\gamma}}\|v_n\|^{\gamma-1}(\delta_n + (\delta_n+2)\eta_n)}{\gamma\|\bar{y} - w_n^2\|^{\gamma-1} - (\delta_n+2)\eta_n} \to 0 \text{ as } n \to \infty. \tag{21}$$

One has the following estimates:

$$\begin{aligned}
t_n^{\frac{1}{\gamma}}\langle y_n^*, v_n\rangle &= \tfrac{\langle w_n^{2*}, w_n^2-\bar{y}\rangle + \langle w_n^{2*}, w_n^3-w_n^2\rangle + \langle w_n^{3*}-w_n^{2*}, w_n^3-\bar{y}\rangle}{\|w^{3*}\|} \\
&\ge \tfrac{\gamma\|\bar{y}-w_n^2\|^{\gamma} - 2\eta_n\gamma\|\bar{y}-w_n^2\|^{\gamma-1} - (\delta_n+2)\eta_n\|\bar{y}-w_n^3\|}{\gamma\|\bar{y}-w_n^2\|^{\gamma-1} + (\delta_n+2)\eta_n} \\
&\ge \tfrac{\gamma\|\bar{y}-w_n^2\|^{\gamma} - 2\eta_n\gamma\|\bar{y}-w_n^2\|^{\gamma-1} - (\delta_n+2)\eta_n t_n^{\frac{1}{\gamma}}\|v_n\|}{\gamma\|\bar{y}-w_n^2\|^{\gamma-1} + (\delta_n+2)\eta_n} \\
&= \frac{\|\bar{y}-w_n^2\| - 2\eta_n - \frac{(\delta_n+2)\eta_n t_n^{\frac{1}{\gamma}}\|v_n\|}{\gamma\|\bar{y}-w_n^2\|^{\gamma-1}}}{1 + \frac{(\delta_n+2)\eta_n}{\gamma\|\bar{y}-w_n^2\|^{\gamma-1}}} \\
&\ge \frac{\|\bar{y}-w_n^3\| - 4\eta_n - \frac{(\delta_n+2)\eta_n t_n^{\frac{1}{\gamma}}\|v_n\|}{\gamma\|\bar{y}-w_n^2\|^{\gamma-1}}}{1 + \frac{(\varepsilon_n+2)\eta_n}{\gamma\|\bar{y}-w_n^2\|^{\gamma-1}}} \\
&= \frac{t_n^{\frac{1}{\gamma}}\|v_n\| - 4\eta_n - \frac{(m+\delta_n+2)\eta_n t_n^{\frac{1}{\gamma}}\|v_n\|}{\gamma\|\bar{y}-w_n^2\|^{\gamma-1}}}{1 + \frac{(\delta_n+2)\eta_n}{\gamma\|\bar{y}-w_n^2\|^{\gamma-1}}} \\
&= \frac{(1 - \frac{(\delta_n+2)\eta_n}{\gamma\|\bar{y}-w_n^2\|^{\gamma-1}}) t_n^{\frac{1}{\gamma}}\|v_n\| - 4\eta_n}{1 + \frac{(\delta_n+2)\eta_n}{\gamma\|\bar{y}-w_n^2\|^{\gamma-1}}}.
\end{aligned}$$

Hence,

$$0 \le 1 - \frac{\langle y_n^*, v_n\rangle}{\|v_n\|} \le \frac{\frac{2(\delta_n+2)\eta_n}{\gamma\|\bar{y}-w_n^2\|^{\gamma-1}} + \frac{4\eta_n}{t_n^{\frac{1}{\gamma}}\|v_n\|}}{1 + \frac{(\delta_n+2)\eta_n}{\gamma\|\bar{y}-w_n^2\|^{\gamma-1}}}. \tag{22}$$

Since $\delta_n/\varphi^{\gamma}(x_n) \to 0$ and $\eta_n \in (0, \delta_n)$, one has

$$\frac{\eta_n}{t_n^{\frac{1}{\gamma}}\|v_n\|} \le \frac{\eta_n}{\varphi(x_n)(1-\delta_n)} \le \frac{\eta_n}{\varphi^{\gamma}(x_n)(1-\delta_n)} \to 0 \ \text{ as } \ n \to \infty.$$

Therefore, one obtains

$$\lim_{n\to\infty} \frac{\langle y_n^*, v_n\rangle}{\|v_n\|} = 1. \tag{23}$$

Furthermore, for the case of $u \ne 0$, one has $z_n^3 \to \bar{x}$ and

$$\frac{z_n^3 - \bar{x}}{\|z_n^3 - \bar{x}\|} - \frac{u}{\|u\|} = \frac{\frac{x_n-\bar{x}}{\|x_n-\bar{x}\|} + \frac{z_n^3-x_n}{\|x_n-\bar{x}\|}}{\frac{\|z_n^3-\bar{x}\|}{\|x_n-\bar{x}\|}} - \frac{u}{\|u\|}.$$

As

$$\|z_n^3 - \bar{x}\| \ge \|x_n - \bar{x}\| - \|z_n^3 - x_n\| \ge \|x_n - \bar{x}\| - \eta_n^2/4 - \eta_n,$$

then

$$\frac{\|x_n - \bar{x}\| - \eta_n^2/4 - \eta_n}{\|x_n - \bar{x}\|} \leq \frac{\|z_n^3 - \bar{x}\|}{\|x_n - \bar{x}\|} \leq \frac{\|x_n - \bar{x}\| + \eta_n^2/4 + \eta_n}{\|x_n - \bar{x}\|}.$$

It follows that

$$\frac{\|z_n^3 - \bar{x}\|}{\|x_n - \bar{x}\|} \to 1 \text{ by } \frac{\eta_n}{\|x_n - \bar{x}\|} \to 0, \text{ as } n \to \infty.$$

On the other hand,

$$\frac{\|z_n^3 - x_n\|}{\|x_n - \bar{x}\|} \leq \frac{5\eta_n/4 + \eta_n^2/4}{\|x_n - \bar{x}\|} \to 0, \text{ as } n \to \infty.$$

So,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{z_n^3 - \bar{x}}{\|z_n^3 - \bar{x}\|} = \frac{u}{\|u\|}.$$

By this relation and (19), (21) and (23), we derive $(0,0) \in \mathrm{SCr}^{\gamma} F(\bar{x}, \bar{y})(u)$, a contradiction. □

Remark 12 The sufficient condition established in the above theorem is given in terms of a combination of coderivatives and tangency. Even when $u = 0$, i.e., for the usual Höderian metric regularity, it is sharper than some conditions established by Li and Mordukhovich in [49]. When $\gamma = 1$, Theorem 11 subsumes a sufficient condition for the metric subregularity that are sharper than some conditions established recently in [33].

Example 13 Let consider $F : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(x_1, x_2) = \begin{cases} (x_1 - x_2)(x_1^2 + (x_1 - x_2)^6) \sin \frac{1}{x_1 - x_2} & \text{if} \quad x_1 \neq x_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$F^{-1}(0) = \{(t,t) : \ t \in \mathbb{R}\} \cup \{(t, t + 1/(k\pi)) : \ \ t \in \mathbb{R}, \ k \in \mathbb{Z} \setminus \{0\}\}.$$

For $x = (x_1, x_2) \notin F^{-1}(0)$; $y^* \in \mathbb{R}$, one has

$$D^*F(x)(y^*) = y^* \left(\frac{\partial F}{\partial x_1}(x), \frac{\partial F}{\partial x_2}(x) \right),$$

with

$$\frac{\partial F}{\partial x_1}(x) = (x_1^2 + (x_1 - x_2)^6 + (x_1 - x_2)(2x_1 + 6(x_1 - x_2)^5) \sin \frac{1}{x_1 - x_2} - \frac{x_1^2 + (x_1 - x_2)^6}{x_1 - x_2} \cos \frac{1}{x_1 - x_2};$$

$$\frac{\partial F}{\partial x_2}(x) = (x_1^2 + (x_1 - x_2)^6 + 6(x_2 - x_1)^6) \sin \frac{1}{x_2 - x_1} - \frac{x_1^2 + (x_1 - x_2)^6}{x_2 - x_1} \cos \frac{1}{x_2 - x_1}.$$

Let $(t_n) \to 0^+$, $(y_n^*) \subseteq \mathbb{R}$ with $y_n^* = 1$ or -1; $(u_n) := (u_{1,n}, u_{2,n}) \to (1, 0)$ with $\|u_n\|_{\mathbb{R}^2} = 1$; $x_n = t_n u_n$; $y_n = t_n^3 v_n$ with $(x_n, y_n) \in \mathrm{gph}\, F$. Then,

$$v_n = (u_{1,n} - u_{2,n})(u_{1,n}^2 + t_n^4 (u_{1,n} - u_{2,n})^6) \sin \frac{1}{t_n(u_{1,n} - u_{2,n})}, \ n = 1, 2, \ldots.$$

If $v_n \to 0$, then $\sin \frac{1}{t_n(u_{1,n}-u_{2,n})} \to 0$. Hence,

$$t_n^{-2}|v_n|^{-2/3}\|D^*F(x_n)(y_n^*)\| = t_n^{-2}|v_n|^{-2/3}\left\|\left(\frac{\partial F(x_n)}{\partial x_1}, \frac{\partial F(x_n)}{\partial x_2}\right)\right\| \to +\infty,$$

which shows that $(0, 0_{\mathbb{R}^2}) \notin \mathrm{SCr}^{1/3}F(0,0)(1,0)$. Thus, by virtue of Corollary **??**, F is directionally metric 1/3-subregular at $(0,0)$ in direction $(1,0)$.

However, F is not directionally metrically 1/3-subregular in direction $(0,1)$. To see this, let us take the sequences

$$x_{1,n} := 0;\; x_{2,n} := 1/(n\pi + 1/n^{-1/2});\; x_n = (x_{1,n}, x_{2,n}).$$

Then $x_n \underset{(0,1)}{\to} (0,0)$, and

$$d(x_n, F^{-1}(0)) = \frac{n^{-1/2}}{\sqrt{2}n\pi(n\pi + 1/n^{-1/2})} \text{ (when } n \text{ is sufficiently large);}$$

$$F(x_n) = (-1)^n(n\pi + 1/n^{-1/2})^{-7}\sin(n^{-1/2}).$$

It implies that

$$\lim_{n\to\infty} \frac{|F(x_n)|}{d(x_n, F^{-1}(0))^3} = 0,$$

and therefore, F not directionally metrically 1/3-subregular in direction $(0,1)$.

When F is a convex multifunction, the sufficient condition $(0,0) \notin \mathrm{SCr}^\gamma F(\bar{x},\bar{y})(u)$ in Theorem 11 is also necessary for the directional metric $\gamma-$subregularity of F as shown in the following proposition.

Proposition 14 *Suppose that $F : X \rightrightarrows Y$ be a convex closed multifunction. Let $(\bar{x},\bar{y}) \in \mathrm{gph}\, F$ and $\gamma \in (0,1]$, $u \in X$ be given. If F is directional metrically $\gamma-$subregular at $(\bar{x},0)$ in u then $(0,0) \notin SCr^\gamma F(\bar{x},\bar{y})(u)$.*

Proof. By consider the multifunction $F(x) - \bar{y}$ instead of F, we can assume that $\bar{y} = 0$. Suppose that F is metrically $\gamma-$subregular at $(\bar{x},0)$ in direction u. There are $\tau > 0$, $\delta > 0$ such that

$$d(x, F^{-1}(0)) \le \tau d(0, F(x))^\gamma \quad \forall x \in B(\bar{x},\delta) \cap (\bar{x} + \text{cone } B(u,\delta)). \tag{24}$$

Let sequences (t_n), (u_n), (v_n), (x_n^*), (y_n^*) such that $(t_n) \downarrow 0$; $(u_n) \to u$, $(u_n, y_n^*) \in \mathcal{S}_X \times \mathcal{S}_{Y^*}$; $x_n^* \in D^*F(\bar{x} + t_n u_n, t_n^{1/\gamma} v_n)(y_n^*)$; $0 \notin F(\bar{x} + t_n u_n)\,(\forall n)$; $(v_n) \to 0$ and $\frac{\langle y_n^*, v_n\rangle}{\|v_n\|} \to 1$. We will prove that $(t_n^{(\gamma-1)/\gamma}\|v_n\|^{\gamma-1}x_n^*)$ does not converge to 0. Indeed, pick a sequence $(\varepsilon_n) \downarrow 0$ by assuming without loss of generality $\tau(1+\varepsilon_n)t_n\|v_n\|^\gamma \le t_n < \delta/2$, $u_n \in B(u,\delta), t_n\|u_n\| < \delta/2$ and $\bar{x} + t_n u_n \in (B(\bar{x},\delta) \cap [\bar{x} + \text{cone } B(u,\delta)])$ for each n, there exists $z_n \in F^{-1}(0)$ such that

$$\|z_n - \bar{x} - t_n u_n\| \le \tau(1+\varepsilon_n)\,[d(0, F(\bar{x} + t_n u_n))]^\gamma \le \tau(1+\varepsilon_n)t_n\|v_n\|^\gamma. \tag{25}$$

Consequently, $\|z_n - \bar{x}\| \le t_n\|u_n\| + \tau(1+\varepsilon_n)t_n\|v_n\|^\gamma < \delta/2 + \delta/2 = \delta$, i.e.,$z_n \in B(\bar{x},\delta)$, for all n. Since F is a convex multifunction, then

$$\langle x_n^*, z - \bar{x} - t_n u_n\rangle - \langle y_n^*, w - t_n^{1/\gamma} v_n\rangle \le 0, \quad \forall (z,w) \in \mathrm{gph}\, F.$$

By taking $(z, w) := (z_n, 0)$ into account, one has

$$\langle x_n^*, \bar{x} + t_n u_n - z_n \rangle \geq t_n^{1/\gamma} \langle y_n^*, v_n \rangle.$$

Therefore, from relation (25), one obtains

$$\tau(1 + \varepsilon_n) t_n \|v_n\|^\gamma \|x_n^*\| \geq \langle x_n^*, z_n - \bar{x} - t_n u_n \rangle \geq t_n^{1/\gamma} \langle y_n^*, v_n \rangle.$$

This implies that $\liminf_{n\to\infty} t_n^{(\gamma-1)/\gamma} \|v_n\|^{\gamma-1} \|x_n^*\| \geq 1/\tau > 0$, which ends the proof. $\square$

In the case $\gamma = 1$, with a similar proof, the conclusion of the preceding proposition also holds for the the mixed smooth-convex inclusion of the form:

$$0 \in g(x) - F(x) := G(x), \tag{26}$$

where $g : X \to Y$ is a mapping of C^1 class, i.e., the class of continuously Fréchet differentiable mappings, around $\bar{x} \in G^{-1}(0)$; $F : X \rightrightarrows Y$ is a closed convex multifunction. The following proposition is the directional version of Proposition 1 in [60].

Proposition 15 *With the assumptions as above, for given $u \in X$, the multifunction $G := g - F$ is metrically subregular at $(\bar{x}, 0)$ in direction u if and only if $(0, 0) \notin SCrG(\bar{x}, 0)(u)$.*

4 Second order characterizations of the directional metric subregularity and 1/2−subregularity

Let X, Y be normed spaces, $S \subset X$ be nonempty and $\bar{x} \in S$. The *tangent cone* $T(S, \bar{x})$ of S at $\bar{x}$ is defined by

$$T(S, \bar{x}) := \{v \in X : \ \exists (t_n) \downarrow 0, \ \exists (x_n) \subseteq S, \ x_n \to \bar{x}, \ v = \lim_{n\to\infty} (x_n - \bar{x})/t_n\}.$$

We recall that the contingent derivative of a multifunction $F : X \rightrightarrows Y$ at $(x, y) \in \text{gph } F$, denoted by $\mathrm{C}F(x, y)$, is a set valued map from X to Y defined by

$$\mathrm{C}F(x, y)(u) := \{v \in Y \ : \ (u, v) \in T(\text{gph } F, (x, y))\}.$$

We introduce the notion of the contingent derivative of high order.

Definition 16 *The contingent derivative of positive order α of a multifunction $F : X \rightrightarrows Y$ at $(x, y) \in$ gph F, denoted by $CF^\alpha(x, y)$, is a set valued map from X to Y defined by*

$$\forall u \in X, v \in CF^\alpha(x, y)(u) \Leftrightarrow \exists t_n \downarrow 0, (u_n, v_n) \to (u, v) \text{ such that } (x + t_n u_n, y + {t_n}^\alpha v_n) \in \text{gph } F.$$

The following proposition shows that the directional metric $\gamma-$subregularity at $(\bar{x}, \bar{y}) \in \text{gph } F$ is always valid in any direction $u \notin CF^{1/\gamma}(\bar{x}, \bar{y})^{-1}(0)$.

Proposition 17 *Let $F : X \rightrightarrows Y$ be a closed multifunction and let $(\bar{x}, \bar{y}) \in$ gph F, $\gamma \in (0, 1]$ and $u_0 \in X$ be given. If $u_0 \notin CF^{1/\gamma}(\bar{x}, \bar{y})^{-1}(0)$ then F is metrically $\gamma-$subregular at $(\bar{x}, \bar{y})$ in direction u_0.*

Proof. Assume on contrary that F is not metrically $\gamma-$subregular at $(\bar{x},\bar{y})$ in direction u_0. Then, there exists a sequence $(x_n) \to_{u_0} \bar{x}$ such that

$$d(x_n, F^{-1}(\bar{y})) > nd(\bar{y}, F(x_n))^\gamma, \quad \forall n \in \mathbb{N}.$$

It implies that there is a sequence (y_n) with $y_n \in F(x_n)$ such that

$$n\|y_n - \bar{y}\|^\gamma < \|x_n - \bar{x}\|, \quad \forall n \in \mathbb{N}.$$

Since $x_n \to_{u_0} \bar{x}$, there exist $t_n \to 0^+, u_n \to u_0$ with $x_n = \bar{x} + t_n u_n,\ \forall n$. By setting

$$v_n := \frac{y_n - \bar{y}}{t_n^{1/\gamma}},$$

one has $(u_n, v_n) \to (u_0, 0)$, which implies $u_0 \in CF^{1/\gamma}(\bar{x},\bar{y})^{-1}(0)$. □

Recall that (see [26]) a subset $S \subseteq X$ is said to be *first-order tangentiable* at $\bar{x}$ if for every $\varepsilon > 0$, there is a neighborhood U of the origin such that

$$(S - \bar{x}) \cap U \subset [T(S;\bar{x})]_\varepsilon$$

where $[T(S;\bar{x})]_\varepsilon := \{x \in X \ : \ d(x/\|x\|, T(S;\bar{x})) < \varepsilon\} \cup \{0\}$ is the ε-conic neighborhood of $T(S;\bar{x})$. It should note that in a finite dimensional space, every nonempty set is tangentiable at any point (see [26]).

Lemma 18 *[26] Let $S \subset X$ be nonempty, $\bar{x} \in S$ and $\{x_n\} \subset S \setminus \{\bar{x}\}$. Assume that S is tangentiable at $\bar{x}$, $T(S,\bar{x})$ is locally compact at the origin and $\{x_n\}$ converges to $\bar{x}$. Then the sequence $\left\{\frac{x_n - \bar{x}}{\|x_n - \bar{x}\|}\right\}$ has a convergent subsequence.*

As usual, the closed unit ball in X is denoted by B_X.

Proposition 19 *Let $G : X \rightrightarrows Y$ be a set-valued map from X to another normed space Y. Assume that G is directional Hölderian metrically γ-subregular at $(\bar{x},\bar{y}) \in \operatorname{gph} G$ in direction $u_0 \in X$ with some modulus κ. If $G^{-1}(\bar{y})$ is tangentiable at $\bar{x}$ and $T(G^{-1}(\bar{y}),\bar{x})$ is locally compact at the origin then the $\frac{1}{\gamma}$-contingent derivative $CG^{\frac{1}{\gamma}}(\bar{x},\bar{y})$ is directional Hölder metrically γ-subregular at $(0,0)$ in direction u_0 with modulus κ.*

Proof. Since G is directional Hölder metrically γ-subregular at $(\bar{x},\bar{y}) \in \operatorname{gph} G$ in direction $u_0 \in X$ with modulus κ, there exists $\delta > 0$ such that

$$d(x, G^{-1}(\bar{y}) \le \kappa[d(\bar{y}, G(x))]^\gamma, \quad \forall x \in B(\bar{x},\delta) \cap [\bar{x} + \operatorname{cone}(B(u_0,\delta))].$$

Let $u \in \operatorname{cone}(B(u_0,\delta))$ and $\epsilon > 0$ be arbitrary. Choose $v \in CG^{\frac{1}{\gamma}}(\bar{x},\bar{y})(u)$ such that $\|v\| < d(0, CG^{\frac{1}{\gamma}}(\bar{x},\bar{y})(u)) + \epsilon$. By the definition, there are sequences $t_n \downarrow 0$ and $(u_n, v_n) \to (u, v)$ such that $(\bar{x} + t_n u_n, \bar{y} + t_n{}^{\frac{1}{\gamma}} v_n) \in \operatorname{gph} G$. We have $\bar{x} + t_n u_n \in B(\bar{x},\delta) \cap [\bar{x} + \operatorname{cone}(B(u_0,\delta))]$ and

$$d(\bar{x} + t_n u_n, G^{-1}(\bar{y})) \le \kappa\,[d(\bar{y}, G(\bar{x} + t_n u_n))]^\gamma \le \kappa t_n \|v_n\|^\gamma \le \kappa t_n [d(0, CG^{\frac{1}{\gamma}}(\bar{x},\bar{y})(u)) + \epsilon]^\gamma$$

for k sufficiently large. Then choose $x_n \in G^{-1}(\bar{y})$ such that $\|x_n - \bar{x} - t_n u_n\| \leq \kappa t_n [d(0, CG^{\frac{1}{\gamma}}(\bar{x}, \bar{y})(u)) + 2\epsilon]^\gamma$. This implies the boundedness of the sequence $\{\frac{\|x_n - \bar{x}\|}{t_n}\}$. Hence, we may assume that $\{\frac{\|x_n - \bar{x}\|}{t_n}\}$ converges to some α. On the other hand, by Lemma 18, we also may assume that the sequence $\{\frac{x_n - \bar{x}}{\|x_n - \bar{x}\|}\}$ converges to some point a. Set $\bar{u}_n := \frac{x_n - \bar{x}}{t_n}$. Then $\bar{u}_n \in u_n + \kappa [d(0, CG^{\frac{1}{\gamma}}(\bar{x}, \bar{y})(u)) + 2\epsilon]^\gamma B_X$ and $\bar{u}_n \to \bar{u} := \alpha a$. Therefore, $\bar{u} \in u + \kappa [d(0, CG^{\frac{1}{\gamma}}(\bar{x}, \bar{y})(u)) + 2\epsilon]^\gamma B_X$. Since $\bar{y} \in G(\bar{x} + t_n \bar{u}_n)$ we have $0 \in CG^{\frac{1}{\gamma}}(\bar{x}, \bar{y})(\bar{u})$. Thus,

$$d(u, CG^{\frac{1}{\gamma}}(\bar{x}, \bar{y})^{-1}(0)) \leq \|u - \bar{u}\| \leq \kappa [d(0, CG^{\frac{1}{\gamma}}(\bar{x}, \bar{y})(u)) + 2\epsilon]^\gamma.$$

Take the limit on ϵ and the proof is complete. □

Now consider again the following mixed constraint system:

$$0 \in g(x) - F(x), \tag{27}$$

where, as the previous section, $F : X \rightrightarrows Y$ is a closed and convex set-valued map and $g : X \to Y$ is assumed to be continuously Fréchet differentiable in a neighbourhood of a point $\bar{x} \in (g - F)^{-1}(0)$. Set $G(x) := g(x) - F(x)$ and

$$\mathcal{C} := CG(\bar{x}, 0)^{-1}(0) = \{u \in X \ : \ Dg(\bar{x})(u) \in CF(\bar{x}, g(\bar{x}))(u)\}. \tag{28}$$

Proposition 20 *For the mixed smooth-convex constraint system (27), and for a given $\bar{x} \in G^{-1}(0) := (g - F)^{-1}(0)$, if G is directional metrically subregular at $(\bar{x}, 0) \in \operatorname{gph} G$ in direction $u_0 \in X$ with some modulus κ and if X is reflexive, then $CG(\bar{x}, 0)$ is also directionally metrically subregular at $(0, 0)$ in direction u_0 with modulus κ.*

Proof. By the hypothesis, there exists $\delta > 0$ such that

$$d(x, G^{-1}(0) \leq \kappa d(0, G(x)), \forall x \in B(\bar{x}, \delta) \cap [\bar{x} + \operatorname{cone}(B(u_0, \delta))].$$

Let $u \in \operatorname{cone}(B(u_0, \delta))$ and $\epsilon > 0$ be arbitrary. Choose $v \in CG(\bar{x}, 0)(u)$ such that

$$\|v\| < d(0, CG(\bar{x}, 0)(u)) + \epsilon.$$

By the definition, there are sequences $t_n \downarrow 0$ and $(u_n, v_n) \to (u, v)$ such that $(\bar{x} + t_n u_n, t_n v_n) \in \operatorname{gph} G$. For n sufficiently large, as in the proof of Proposition 19, there exists $x_n \in G^{-1}(0)$ such that

$$\|x_n - \bar{x} - t_n u_n\| \leq \kappa t_n [d(0, CG(\bar{x}, 0)(u)) + 2\epsilon].$$

By setting $\bar{u}_n := \frac{x_n - \bar{x}}{t_n}$, since $\{u_n\}$ is bounded and X is reflexive, then by passing to a subsequence if necessary, we may assume that $\{u_n\}$ weakly converges to some $\bar{u} \in X$. Therefore, $\left\{\frac{g(\bar{x} + t_n u_n) - g(\bar{x})}{t_n}\right\}$ also weakly converges to $Dg(\bar{x})(\bar{u})$. Since $\operatorname{gph} F$ is convex, then

$$\begin{aligned}(\bar{u}, Dg(\bar{x})(\bar{u})) \in \operatorname{cl}_w \operatorname{cone}(\operatorname{gph} F - (\bar{x}, g(\bar{x}))) &= \operatorname{cl} \operatorname{cone}(\operatorname{gph} F - (\bar{x}, g(\bar{x}))) \\ &= T(\operatorname{gph} F, (\bar{x}, g(\bar{x}))),\end{aligned}$$

where, $\mathrm{cl}_w A$, $\mathrm{cl}A$, and $\mathrm{cone}A$ stand respectively for the weak closure, the closure and the cone hull of some subset A. Consequently, $\bar{u} \in CG(\bar{x}, 0)^{-1}(0)$, and one has

$$d(u, \mathrm{C}G(\bar{x}, 0)^{-1}(0)) \leq \|u - \bar{u}\| \leq \kappa[d(0, CG(\bar{x}, 0)(u)) + 2\epsilon].$$

As $\varepsilon > 0$ is arbitrary, this completes the proof. □

Next, we derive a second order condition for the directional metric subregularity of the system (27). Let $u_0 \in X \setminus \{0\}$ be a direction under consideration. By meaning of Proposition 17, without loss of generality, in what follows, assume that $\|u_0\| = 1$ and $u_0 \in \mathcal{C}$. In the sequel, we make use of the following assumptions.

Assumption 1. *There exist $\eta, R > 0$ such that for every $x, x' \in B(\bar{x}, R) \cap [\bar{x} + \mathrm{cone}\,(B(u_0, R)]$, the following inequality holds*

$$\|g(x) - g(x') - Dg(\bar{x})(x - x')\| \leq \eta \max\{\|x - \bar{x}\|, \|x' - \bar{x}\|\}\|x - x'\|.$$

Assumption 2. *The strict second order directional derivative at $\bar{x}$ in direction u_0 :*

$$g''(\bar{x}; u_0) := \lim_{\substack{t \to 0^+ \\ u \to u_0}} \frac{g(\bar{x} + tu) - g(\bar{x}) - tDg(\bar{x})(u)}{t^2/2}$$

exists.

Assumptions 1 and 2 imply

$$\|g''(\bar{x}, u_0)\| \leq 2\eta. \tag{29}$$

Let us recall from ([32]) the notion of inner second order approximation mappings for convex sets.

Definition 21 *([32]) Let S be a closed convex subset of a Banach space Z, $A : X \to Z$ be a continuous linear map and $s \in S, u \in A^{-1}(T(S; s))$. Let ξ be a nonnegative real number. A nonemty set $\mathcal{I} \subset Z$ is called an inner second order approximation set for S at s with respect to A, u and ξ if*

$$\lim_{t \to 0^+} t^{-2} d(s + tAu + \frac{t^2}{2} w, S + t^2 \xi A B_X) = 0 \tag{30}$$

holds for all $w \in \mathcal{I}$.

The notion below is a uniform version of the inner second order approximation.

Definition 22 *Let S, A, ξ, u as in the definition above. A nonemty set $\mathcal{I} \subset Z$ is called a uniform inner second order approximation set for S at s with respect to A and ξ in the direction u if*

$$\lim_{\substack{v \to u, v \in A^{-1}(T(S;s)) \cap \|u\| \mathcal{S}_X \\ t \to 0^+}} t^{-2} d(s + tAv + \frac{t^2}{2} w, S + t^2 \xi A B_X) = 0 \tag{31}$$

holds uniformly for all $w \in \mathcal{I}$.

Denote by I_X the identify map of X. Then,

$$\mathcal{C} = (I_X, Dg(\bar{x}))^{-1}(T(\text{gph } F, (\bar{x}, g(\bar{x})))).$$

As usual, the support function of a set $C \subseteq X$ is denoted by $\sigma_C : X^* \to \mathbb{R} \cup \{+\infty\}$, and is defined by

$$\sigma_C(x^*) := \sup_{x \in C} \langle x^*, x \rangle, \quad x^* \in X^*.$$

The norm in the space $X \times Y$ is defined by $\|(x, y)\| := \|x\| + \|y\|$.

Let $u_0 \in \mathcal{C} \cap \mathcal{S}_X$ be a direction under consideration. In Theorem 23 and Lemma **??** below we assume that Assumptions 1 and 2 are fulfilled with respect to u_0.

Theorem 23 *Suppose that X, Y are Banach spaces.*

1. If the contingent derivative $CG(\bar{x}, 0)$ is directionally metrically subregular at $(0, 0)$ in the direction u_0 and there are real $\xi \geq 0$ and a uniform inner second order approximation $\mathcal{A}$ for gph F *at $(\bar{x}, g(\bar{x}))$ with respect to $(I_X, Dg(\bar{x}))$ and ξ in the direction u_0 such that for each sequence $\{({x_n}^*, {y_n}^*)\} \subset X^* \times \mathcal{S}_{Y^*}$ satisfying*

$$\lim_{n\to\infty} [\langle ({x_n}^*, {y_n}^*), (\bar{x}, g(\bar{x})) \rangle - \sigma_{\text{gph } F}({x_n}^*, y_n^*)] = \lim_{n\to\infty} \|Dg(\bar{x})^* {y_n}^* + {x_n}^*\| = 0$$

one has

$$\liminf_{n\to\infty} [\langle {y_n}^*, g''(\bar{x}, u_0) \rangle - \sigma_{\mathcal{A}}({x_n}^*, {y_n}^*)] < 0, \tag{32}$$

then G is directionally metrically subregular at $(\bar{x}, 0)$ in the direction u_0.

2. Conversely, if G is directionally metrically subregular at $(\bar{x}, 0)$ in the direction u_0 and

$$\limsup_{\substack{u \to u_0, u \in \mathcal{C} \cap \mathcal{S}_X \\ t \to 0^+}} \frac{d\left((\bar{x} + tu, g(\bar{x}) + tDg(\bar{x})(u)), \text{gph } F\right)}{t^2} \tag{33}$$

is finite, then there are real $\xi \geq 0$ and a uniform inner second order approximation set $\mathcal{A}$ for gph F *at $(\bar{x}, g(\bar{x}))$ with respect to $(I_X, Dg(\bar{x}))$ and ξ in the direction u_0 such that for each sequence $\{({x_n}^*, {y_n}^*)\} \subset X^* \times \mathcal{S}_{Y^*}$ satisfying*

$$\lim_{n\to\infty} [\langle ({x_n}^*, {y_n}^*), (\bar{x}, g(\bar{x})) \rangle - \sigma_{\text{gph } F}({x_n}^*, {y_n}^*)] = \lim_{n\to\infty} \|Dg(\bar{x})^* {y_n}^* + {x_n}^*\| = 0,$$

one has

$$\liminf_{n\to\infty} [\langle {y_n}^*, g''(\bar{x}, u_0) \rangle - \sigma_{\mathcal{A}}({x_n}^*, {y_n}^*)] \leq 0. \tag{34}$$

Moreover, if $G^{-1}(0)$ is tangentiable at $\bar{x}$ and the tangent cone $T(G^{-1}(0), \bar{x})$ is locally compact at the origin then the contingent derivative $CG(\bar{x}, 0)$ is directionally metrically subregular at $(0, 0)$ in the direction u_0.

Proof. 1. Suppose on the contrary that G is not directionally metrically subregular at $(\bar{x}, 0)$ in the direction u_0. Then by Theorem 11, there exist sequences $x_n \to \bar{x}$, $y_n \in F(x_n)$, $y_n^* \in \mathcal{S}_{Y^*}$, $x_n^* \in D^*F(x_n, y_n)(-y_n^*)$, $\epsilon_n \to 0^+$ such that

$$\frac{x_n - \bar{x}}{\|x_n - \bar{x}\|} \to u_0, \tag{35}$$

$$g(x_n) \notin F(x_n), \ \frac{\|g(x_n) - y_n\|}{\|x_n - \bar{x}\|} \to 0, \ \|Dg(\bar{x})^* y_n^* + x_n^*\| \to 0, \tag{36}$$

$$\|g(x_n) - y_n\| \le (1 + \epsilon_n) d(0, g(x_n) - F(x_n)), \tag{37}$$

$$|\langle y_n^*, g(x_n) - y_n\rangle - \|g(x_n) - y_n\|| \le \epsilon_n \|g(x_n) - y_n\|. \tag{38}$$

Immediately, from definitions of x_n^*, y_n^* and relation (36), we have

$$\lim_{n\to\infty} [\langle (x_n^*, y_n^*), (\bar{x}, g(\bar{x}))\rangle - \sigma_{\text{gph } F}(x_n^*, y_n^*)] = \lim_{n\to\infty} \langle (x_n^*, y_n^*), (\bar{x}, g(\bar{x})) - (x_n, y_n)\rangle = 0. \tag{39}$$

Since $CG(\bar{x}, 0)$ is directionally metrically subregular at $(0, 0)$ in the direction u_0, there exist $\kappa, R > 0$ such that for every $u \in B(0, R) \cap \text{cone}\,(B(u_0, R))$, one has $d(u, CG(\bar{x}, 0)^{-1}(0)) \le \kappa d(0, CG(\bar{x}, 0)(u))$. As $\text{C}G(\bar{x}, 0)(u) = Dg(\bar{x})(u) - \text{C}F(\bar{x}, g(\bar{x}))(u)$, then

$$d(u, \mathcal{C}) \le \kappa d(Dg(\bar{x})(u), CF(\bar{x}, g(\bar{x}))(u)) \quad \forall u \in B(0, R) \cap \text{cone}\,(B(u_0, R)).$$

Thus, for each n sufficiently large, there exist $u_n \in \mathcal{C} \cap \mathcal{S}_X, t_n \ge 0$ such that (note that $F(x_n) - g(\bar{x}) \subset CF(\bar{x}, g(\bar{x}))(x_n - \bar{x})$))

$$\begin{aligned}
\|x_n - \bar{x} - t_n u_n\| &\le \kappa d(Dg(\bar{x})(x_n - \bar{x}), CF(\bar{x}, g(\bar{x}))(x_n - \bar{x})) + \frac{\|x_n - \bar{x}\|^2}{n} \\
&\le \kappa d(Dg(\bar{x})(x_n - \bar{x}), F(x_n) - g(\bar{x})) + \frac{\|x_n - \bar{x}\|^2}{n} \\
&= \kappa d(g(\bar{x}) + Dg(\bar{x})(x_n - \bar{x}), F(x_n)) + \frac{\|x_n - \bar{x}\|^2}{n}.
\end{aligned}$$

Then Assumption 1 yields

$$\|x_n - \bar{x} - t_n u_n\| \le \kappa[d(g(x_n), F(x_n)) + \eta\|x_n - \bar{x}\|^2] + \frac{\|x_n - \bar{x}\|^2}{n} \tag{40}$$

which together with (36) gives

$$\left\| \frac{x_n - \bar{x}}{\|x_n - \bar{x}\|} - \frac{t_n}{\|x_n - \bar{x}\|} u_n \right\| \le \kappa \left[\frac{\|g(x_n) - y_n\|}{\|x_n - \bar{x}\|} + \eta\|x_n - \bar{x}\| \right] + \frac{\|x_n - \bar{x}\|}{n} \to 0 (n \to \infty)$$

Hence,

$$\frac{t_n}{\|x_n - \bar{x}\|} \to 1, \quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{x_n - \bar{x}}{\|x_n - \bar{x}\|} = u_0.$$

We have the following estimations:

$$\begin{aligned}
&\langle (x_n^*, y_n^*), (\bar{x} + t_n u_n, g(\bar{x} + t_n u_n))\rangle - \sigma_{gphF}(x_n^*, y_n^*) \\
&\quad = \langle x_n^*, \bar{x} + t_n u_n\rangle + \langle y_n^*, g(\bar{x} + t_n u_n)\rangle - \langle (x_n^*, y_n^*), (x_n, y_n)\rangle (\text{since } x_n^* \in D^*F(x_n, y_n)(-y_n^*)) \\
&\quad = \langle x_n^*, \bar{x} + t_n u_n - x_n\rangle + \langle y_n^*, g(\bar{x} + t_n u_n) - y_n\rangle \\
&\quad = \langle x_n^*, \bar{x} + t_n u_n - x_n\rangle + \langle y_n^*, g(\bar{x} + t_n u_n) - g(x_n)\rangle + \langle y_n^*, g(x_n) - y_n\rangle \\
&\quad \ge \langle x_n^*, \bar{x} + t_n u_n - x_n\rangle + \langle Dg(\bar{x})^* y_n^*, \bar{x} + t_n u_n - x_n\rangle - \eta\max\{\|t_n u_n\|, \|x_n - \bar{x}\|\}\|\bar{x} + t_n u_n - x_n\| + \\
&\quad + (1 - \epsilon_n)\|g(x_n) - y_n\| \text{ (by Assumption 1 and (38))} \\
&\quad \ge (1 - \epsilon_n) d(g(x_n), F(x_n)) - \|x_n^* + Dg(\bar{x})^* y_n^*\|.\|\bar{x} + t_n u_n - x_n\| - \eta_n \|\bar{x} + t_n u_n - x_n\| \\
&\quad (\eta_n := \eta\max\{\|t_n u_n\|, \|x_n - \bar{x}\|\} \to 0) \\
&\quad = (1 - \epsilon_n) d(g(x_n), F(x_n)) - \delta_n \|\bar{x} + t_n u_n - x_n\| (\delta_n := \|x_n^* + Dg(\bar{x})^* y_n^*\| + \eta_n \to 0 \text{ by } (36)) \\
&\quad \ge (1 - \epsilon_n - \kappa\delta_n) d(g(x_n), F(x_n)) - (\eta\kappa + \frac{1}{n})\delta_n \|\bar{x} - x_n\|^2 \text{ (by (40))}.
\end{aligned}$$

Therefore,

$$\frac{1}{t_n{}^2}[\langle (x_n^*, y_n^*), (\bar{x} + t_n u_n, g(\bar{x} + t_n u_n))\rangle - \sigma_{\text{gph } F}(x_n^*, y_n^*)]$$
$$\geq \frac{(1 - \epsilon_n - \kappa\delta_n)}{t_n{}^2} d(g(x_n), F(x_n)) - (\eta\kappa + \frac{1}{n})\delta_n \left(\frac{\|\bar{x} - x_n\|}{t_n}\right)^2.$$

Hence

$$\liminf_{n\to\infty} \frac{1}{t_n{}^2}[\langle (x_n^*, y_n^*), (\bar{x} + t_n u_n, g(\bar{x} + t_n u_n))\rangle - \sigma_{\text{gph } F}(x_n^*, y_n^*)] \geq 0. \tag{41}$$

On the other hand, since

$$\begin{aligned}
&\langle (x_n^*, y_n^*), (\bar{x} + t_n u_n, g(\bar{x}) + t_n Dg(\bar{x})(u_n))\rangle + \frac{t_n{}^2}{2}\sigma_{\mathcal{A}}(x_n^*, y_n^*) \\
&= \sup_{(w_1, w_2)\in\mathcal{A}} \langle (x_n^*, y_n^*), (\bar{x}, g(\bar{x})) + t_n(u_n, Dg(\bar{x})(u_n)) + \frac{t_n{}^2}{2}(w_1, w_2)\rangle \\
&\leq \sigma_{\text{gph } F}(x_n^*, y_n^*) + t_n{}^2\xi\|x_n^* + Dg(\bar{x})^* y_n^*\| + \circ(t_n{}^2) \\
&(\text{since } (\bar{x}, g(\bar{x})) + t_n(u_n, Dg(\bar{x})(u_n)) + \frac{t_n{}^2}{2}(w_1, w_2) \in \text{gph } F + t_n{}^2\xi(I_X, Dg(\bar{x}))B_X + \circ(t_n{}^2)\mathcal{B}_{X\times Y}),
\end{aligned}$$

one has

$$\begin{aligned}
&\langle (x_n^*, y_n^*), (\bar{x} + t_n u_n, g(\bar{x} + t_n u_n))\rangle - \sigma_{\text{gph } F}(x_n^*, y_n^*) \leq \\
&\leq \langle (x_n^*, y_n^*), (\bar{x} + t_n u_n, g(\bar{x} + t_n u_n))\rangle - \langle (x_n^*, y_n^*), (\bar{x} + t_n u_n, g(\bar{x}) + t_n Dg(\bar{x})(u_n))\rangle \\
&- \frac{t_n{}^2}{2}\sigma_{\mathcal{A}}(x_n^*, y_n^*) + t_n{}^2\xi\|x_n^* + Dg(\bar{x})^* y_n^*\| + \circ(t_n{}^2) \\
&= \langle y_n^*, g(\bar{x} + t_n u_n) - g(\bar{x}) - t_n Dg(\bar{x})(u_n))\rangle - \frac{t_n{}^2}{2}\sigma_{\mathcal{A}}(x_n^*, y_n^*) \\
&+ t_n{}^2\xi\|x_n^* + Dg(\bar{x})^* y_n^*\| + \circ(t_n{}^2).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{2}{t_n{}^2}[\langle (x_n^*, y_n^*), (\bar{x} + t_n u_n, g(\bar{x} + t_n u_n))\rangle - \sigma_{\text{gph } F}(x_n^*, y_n^*)] \leq \\
&\leq \left\langle y_n^*, \frac{2}{t_n{}^2}[g(\bar{x} + t_n u_n) - g(\bar{x}) - t_n Dg(\bar{x})(u_n)]\right\rangle - \sigma_{\mathcal{A}}(x_n^*, y_n^*) + \\
&+ 2\xi\|x_n^* + Dg(\bar{x})^* y_n^*\| + \frac{\circ(t_n{}^2)}{t_n{}^2} \\
&= \langle y_n^*, g''(\bar{x}, u_0)\rangle - \sigma_{\mathcal{A}}(x_n^*, y_n^*) + \langle y_n^*, \frac{2}{t_n{}^2}[g(\bar{x} + t_n u_n) - g(\bar{x}) - t_n Dg(\bar{x})(u_n)] - \\
&- g''(\bar{x}, u_0)\rangle + 2\xi\|x_n^* + Dg(\bar{x})^* y_n^*\| + \frac{\circ(t_n{}^2)}{t_n{}^2}
\end{aligned}$$

which together with (36), (54), (47) and Assumption 2 imply

$$\liminf_{n\to\infty} \frac{2}{t_n{}^2}[\langle (x_n^*, y_n^*), (\bar{x} + t_n u_n, g(\bar{x} + t_n u_n))\rangle - \sigma_{\text{gph } F}(x_n^*, y_n^*)] < 0$$

which contradicts to (41).

The second part of the theorem follows directly from the following lemma.

Lemma 24 *Suppose that G is directional metrically subregular at $(\bar{x}, 0)$ in the direction u_0. If (33) is finite, then the set $\mathcal{A} := \{(0, g''(\bar{x}, u_0))\}$ is a uniform inner second order approximation set for* gph F *at $(\bar{x}, g(\bar{x}))$ with respect to $(I_X, Dg(\bar{x}))$ and some $\xi > 0$ in the direction u_0.*

Proof. Since G is directional metrically subregular at $(\bar{x}, 0)$ with some modulus κ in the direction u_0, there exists $R \in (0, 1)$ such that

$$d(x, G^{-1}(0) \leq \kappa d(0, G(x)), \forall x \in B(\bar{x}, R) \cap [\bar{x} + \text{cone}\,(B(u_0, R))]. \tag{42}$$

By virtue of the finiteness of (33), there are $\delta \in (0, R)$ and $\gamma > 0$ with $\gamma\delta < R$ such that

$$d((\bar{x} + tu, g(\bar{x}) + tDg(\bar{x})(u)), \text{gph } F) < \gamma t^2 \quad \forall t \in (0, \delta), \;\; u \in B(u_0, \delta) \cap \mathcal{C} \cap \mathcal{S}_X. \tag{43}$$

Let $(t, u) \in (0, \delta/2) \times (B(u_0, \delta/2) \cap \mathcal{C} \cap \mathcal{S}_X)$ be given. Then we can find $(u', v) \in X \times Y$ with $(\bar{x} + tu', g(\bar{x}) + tv) \in \text{gph } F$ such that

$$\|(t(u' - u), t(v - Dg(\bar{x})(u)))\| \leq \gamma t^2. \tag{44}$$

It implies $\|u' - u\| < \gamma t < \gamma\delta/2 < R/2$. By (42), there is $\bar{u} \in X$ with $\bar{x} + t\bar{u} \in G^{-1}(0)$ such that

$$t\|u' - \bar{u}\| = \|\bar{x} + tu' - (\bar{x} + t\bar{u})\| \leq \kappa d(0, G(\bar{x} + tu')) + t^2.$$

Then taking (44) and Assumption 1 into account, one has

$$\begin{aligned}
t\|\bar{u} - u'\| \leq & \\
& \leq \kappa d(g(\bar{x} + tu'), F(\bar{x} + tu') + t^2 \\
& \leq \kappa d(g(\bar{x} + tu'), g(\bar{x}) + tv) + t^2 \\
& = \kappa\|g(\bar{x} + tu') - g(\bar{x}) - tv\| + t^2 \\
& \leq \kappa\|g(\bar{x} + tu') - g(\bar{x}) - tDg(\bar{x})(u')\| + \kappa\|t(Dg(\bar{x})(u') - v)\| + t^2 \\
& \leq \kappa\eta t^2\|u'\|^2 + \kappa\|tDg(\bar{x})(u' - u)\| + \kappa\|t(Dg(\bar{x})(u) - v)\| + t^2 \\
& \leq 2\kappa\eta t^2 + \kappa\gamma\|Dg(\bar{x})\|t^2 + \kappa\gamma t^2 + t^2.
\end{aligned}$$

By this inequality and (44), one has

$$\|\bar{u} - u\| \leq \xi t, \tag{45}$$

for some constant ξ. Next, we have the following estimation, by using Assumption 1:

$$\begin{aligned}
&\frac{2}{t^2}d((\bar{x},g(\bar{x}))+t(u,Dg(\bar{x})(u))+\frac{t^2}{2}(0,g''(\bar{x},u_0)),\text{gph } F+t^2\xi(I_X,Dg(\bar{x}))\mathcal{B}_X)\\
&\le \frac{2}{t^2}d((\bar{x}+tu,g(\bar{x})+tDg(\bar{x})(u)+\frac{t^2}{2}g''(\bar{x},u_0)),\text{gph } F-t(I_X,Dg(\bar{x}))(\bar{u}-u))\\
&= \frac{2}{t^2}d((\bar{x}+t\bar{u},g(\bar{x})+tDg(\bar{x})(\bar{u})+\frac{t^2}{2}g''(\bar{x},u_0)),\text{gph } F)\\
&\le \frac{2}{t^2}d((\bar{x}+t\bar{u},g(\bar{x})+tDg(\bar{x})(\bar{u})+\frac{{t_n}^2}{2}g''(\bar{x},u)),\{\bar{x}+t\bar{u}\}\times F(\bar{x}+t\bar{u}))\\
&= \frac{2}{t^2}d(g(\bar{x})+tDg(\bar{x})(\bar{u})+\frac{t^2}{2}g''(\bar{x},u_0)),F(\bar{x}+t\bar{u}))\\
&\le \frac{2}{t^2}\|g(\bar{x}+t\bar{u})-g(\bar{x})-tDg(\bar{x})(\bar{u})-\frac{t^2}{2}g''(\bar{x},u_0)\|\\
&\le \frac{2}{t^2}\|g(\bar{x}+t\bar{u})-g(\bar{x}+tu)-tDg(\bar{x})(\bar{u}-u)\|+\\
&+\frac{2}{t^2}\|g(\bar{x}+tu)-g(\bar{x})-tDg(\bar{x})(u)-\frac{t^2}{2}g''(\bar{x},u_0)\|\\
&\le 2\eta\max\{1,\|\bar{u}\|\}\|\bar{u}-u\|+\|\frac{2}{t^2}[g(\bar{x}+tu)-g(\bar{x})-tDg(\bar{x})(u)]-g''(\bar{x},u_0)\|.
\end{aligned}$$

By (45) and Assumption 2, the last right hand part of the above inequalities converges to 0 as $u\in\mathcal{C}\cap\mathcal{S}_X, (t,u)\to(0^+,u_0)$. Therefore, $\mathcal{A}:=\{(0,g''(\bar{x},u_0))\}$ is a uniform inner second order approximation for gph F at $(\bar{x},g(\bar{x}))$ with respect to $(I_X,Dg(\bar{x}))$ and ξ in the direction u_0. □

Let us return to the proof of the second part of theorem. By Lemma 24 there exists $\xi>0$ such that $\mathcal{A}:=\{(0,g''(\bar{x},u_0))\}$ is an inner second order approximation for gph F at $(\bar{x},g(\bar{x}))$ with respect to $(I_X,Dg(\bar{x})),\xi$ in the direction u_0. Then (34) holds immediately. The last assertion of Theorem 23 is obvious from Proposition 19. The proof is complete. □

A special case of the theorem above, consider the inclusion of the form:

$$0\in G(x):=g(x)-C,\ \ x\in X, \tag{46}$$

where a closed convex subset $C\subseteq Y$. One obtains the following corollary, which is a directional version of Theorem 5.4 in [32].

Corollary 25 *Suppose that X,Y are Banach spaces and consider the inclusion (46).*

1. If the contingent derivative $CG(\bar{x},0)$ is directionally metrically subregular at $(0,0)$ in the direction u_0 and there are real $\xi\ge0$ and a uniform inner second order approximation $\mathcal{A}$ for C at $g(\bar{x})$ with respect to $Dg(\bar{x})$ and ξ in the direction u_0 such that for each sequence $\{({x_n}^*,{y_n}^*)\}\subset X^*\times\mathcal{S}_{Y^*}$ satisfying

$$\lim_{n\to\infty}[\langle {y_n}^*,g(\bar{x})\rangle-\sigma_C(y_n^*)]=\lim_{n\to\infty}\|Dg(\bar{x})^*{y_n}^*\|=0,$$

one has

$$\liminf_{n\to\infty}[\langle {y_n}^*,g''(\bar{x},u_0)\rangle-\sigma_{\mathcal{A}}({y_n}^*)]<0, \tag{47}$$

then G is directionally metrically subregular at $(\bar{x},0)$ in the direction u_0.

2. Conversely, if G is directionally metrically subregular at $(\bar{x}, 0)$ in the direction u_0 and

$$\limsup_{\substack{u \to u_0, u \in \mathcal{C} \cap \mathcal{S}_X \\ t \to 0^+}} \frac{d\left(g(\bar{x}) + tDg(\bar{x})(u), C\right)}{t^2} \tag{48}$$

is finite, then there are real $\xi \geq 0$ and a uniform inner second order approximation set $\mathcal{A}$ for C at $g(\bar{x})$ with respect to $Dg(\bar{x})$ and ξ in the direction u_0 such that for each sequence $\{{y_n}^*\} \subset \mathcal{S}_{Y^*}$ satisfying

$$\lim_{n \to \infty} [\langle {y_n}^*, g(\bar{x}) \rangle - \sigma_C({y_n}^*)] = \lim_{n \to \infty} \|Dg(\bar{x})^* {y_n}^*\| = 0,$$

one has

$$\liminf_{n \to \infty} [\langle {y_n}^*, g''(\bar{x}, u_0) \rangle - \sigma_{\mathcal{A}}({y_n}^*)] \leq 0. \tag{49}$$

Moreover, if $G^{-1}(0)$ is tangentiable at $\bar{x}$ and the tangent cone $T(G^{-1}(0), \bar{x})$ is locally compact at the origin then the contingent derivative $CG(\bar{x}, 0)$ is directionally metrically subregular at $(0, 0)$ in the direction u_0.

We next established a second order sufficient condition for the metric $1/2-$subregularity of the system (27). In what follows, we asume that g is a mapping of C^2 class on a neighborhood of the given point $\bar{x} : 0 \in g(\bar{x}) - F(\bar{x})$. Remind that $u_0 \in \mathcal{C} \cap \mathcal{S}_X$ is a given direction under consideration.

Theorem 26 *Suppose that X, Y are Banach spaces and g is a continuously twice differentiable mapping on a neighborhood of $\bar{x}$. . If for each sequence $\{({x_n}^*, {y_n}^*)\} \subset X^* \times \mathcal{S}_{Y^*}$ satisfying*

$$\lim_{n \to \infty} [\langle ({x_n}^*, {y_n}^*), (\bar{x}, g(\bar{x})) \rangle - \sigma_{\text{gph } F}({x_n}^*, y_n^*)] = \lim_{n \to \infty} \|Dg(\bar{x})^* {y_n}^* + {x_n}^*\| = 0,$$

there are $u \in \mathcal{C}$; a real $\xi \geq 0$ and an inner second order approximation $\mathcal{A}(u)$ for gph F *at $(\bar{x}, g(\bar{x}))$ with respect to $(I_X, Dg(\bar{x}))$ and ξ in the direction u such that*

$$\liminf_{n \to \infty} [\langle {y_n}^*, D^2 g(\bar{x})(u, u) - D^2 g(\bar{x})(u - u_0, u - u_0) \rangle - \sigma_{\mathcal{A}(u)}({x_n}^*, {y_n}^*)] < 0, \tag{50}$$

then G is directionally metrically $1/2-$subregular at $(\bar{x}, 0)$ in the direction u_0.

Proof. Suppose on the contrary that G is not directionally metrically $1/2-$subregular at $(\bar{x}, 0)$ in the direction u_0. Then by Theorem 11, there exist sequences $\varepsilon_n \to 0^+$, $x_n := \bar{x} + t_n u_n$, $y_n := g(x_n) - t_n^2 v_n$, with $t_n \to 0^+$, $\|u_n\| = 1$, $v_n \to 0$, $y_n \in F(x_n)$, $y_n^* \in \mathcal{S}_{Y^*}$, $x_n^* \in D^* F(x_n, y_n)(-y_n^*)$ such that

$$\lim_{n \to \infty} u_n = u_0, \tag{51}$$

$$g(x_n) \notin F(x_n), \ t_n^{-1} \|v_n\|^{-1/2} \|Dg(x_n)^* y_n^* + x_n^*\| \to 0, \tag{52}$$

$$|\langle y_n^*, g(x_n) - y_n \rangle - \|g(x_n) - y_n\|| \leq \epsilon_n \|g(x_n) - y_n\|. \tag{53}$$

Therefore, we have

$$\lim_{n \to \infty} [\langle (x_n^*, y_n^*), (\bar{x}, g(\bar{x})) \rangle - \sigma_{\text{gph } F}(x_n^*, y_n^*)] = \lim_{n \to \infty} \langle (x_n^*, y_n^*), (\bar{x}, g(\bar{x})) - (x_n, y_n) \rangle = 0, \tag{54}$$

and

$$\lim_{n\to\infty}\frac{\|Dg(x_n)^*{y_n}^*+{x_n}^*\|}{t_n}=0. \tag{55}$$

We have

$$\begin{aligned}
&\langle(x_n^*,y_n^*),(\bar{x}+t_nu,g(\bar{x}+t_nu))\rangle-\sigma_{\text{gph }F}(x_n^*,y_n^*)\\
&=\langle x_n^*,\bar{x}+t_nu\rangle+\langle y_n^*,g(\bar{x}+t_nu)\rangle-\langle(x_n^*,y_n^*),(x_n,y_n)\rangle\\
&=\langle x_n^*,\bar{x}+t_nu-x_n\rangle+\langle y_n^*,g(\bar{x}+t_nu)-y_n\rangle\\
&=t_n\langle x_n^*,u-u_n\rangle+\langle y_n^*,g(\bar{x}+t_nu)-g(x_n)\rangle+\langle y_n^*,g(x_n)-y_n\rangle\\
&=t_n\langle x_n^*+Dg(x_n)^*y_n^*,u-u_n\rangle+\langle y_n^*,g(\bar{x}+t_nu)-g(x_n)-Dg(x_n)(u-u_n)\rangle++\langle y_n^*,g(x_n)-y_n\rangle\\
&=t_n\langle x_n^*+Dg(x_n)^*y_n^*,u-u_n\rangle+\frac{t_n^2}{2}\langle y_n^*,D^2g(x_n)(u-u_n,u-u_n)\rangle+t_n^2v_n+\circ({t_n}^2).
\end{aligned}$$

Therefore, by (55),

$$\begin{aligned}
&\frac{1}{{t_n}^2}[\langle(x_n^*,y_n^*),(\bar{x}+t_nu_n,g(\bar{x}+t_nu_n))\rangle-\sigma_{\text{gph }F}(x_n^*,y_n^*)]\\
&=\langle y_n^*,D^2g(x_n)(u-u_n,u-u_n)\rangle+\frac{\circ({t_n}^2)}{t_n^2}.
\end{aligned} \tag{56}$$

On the other hand, as in the proof of Theorem 23,

$$\begin{aligned}
&\frac{2}{{t_n}^2}[\langle(x_n^*,y_n^*),(\bar{x}+t_nu,g(\bar{x}+t_nu))\rangle-\sigma_{\text{gph }F}(x_n^*,y_n^*)]\leq\\
&\leq\left\langle y_n^*,\frac{2}{{t_n}^2}[g(\bar{x}+t_nu)-g(\bar{x})-t_nDg(\bar{x})(u)]\right\rangle-\sigma_{\mathcal{A}(u)}(x_n^*,y_n^*)+\\
&+2\xi\|x_n^*+Dg(\bar{x})^*y_n^*\|+\frac{\circ({t_n}^2)}{{t_n}^2}\\
&=\langle y_n^*,D^2g(\bar{x})(u,u)\rangle-\sigma_{\mathcal{A}}(x_n^*,y_n^*)+\langle y_n^*,\frac{2}{{t_n}^2}[g(\bar{x}+t_nu_n)-g(\bar{x})-t_nDg(\bar{x})(u_n)]-\\
&-D^2g(\bar{x})(u,u)\rangle+2\xi\|x_n^*+Dg(\bar{x})^*y_n^*\|+\frac{\circ({t_n}^2)}{{t_n}^2}.
\end{aligned}$$

This together with (56) imply

$$\liminf_{n\to\infty}[\langle{y_n}^*,D^2g(\bar{x})(u,u)-D^2g(\bar{x})(u-u_0,u-u_0)\rangle-\sigma_{\mathcal{A}(u)}({x_n}^*,{y_n}^*)]\geq 0,$$

a contradiction. The proof is completed. □

For the inclusion (46), one has the following corollary.

Corollary 27 *Suppose that X,Y are Banach spaces and g is continuously twice differentiable near $\bar{x}$. For the inclusion (46), if for each sequence $\{({x_n}^*,{y_n}^*)\}\subset X\times X^*\times\mathcal{S}_{Y^*}$ satisfying*

$$\lim_{n\to\infty}[\langle{y_n}^*,g(\bar{x})\rangle-\sigma_C(y_n^*)]=\lim_{n\to\infty}\|Dg(\bar{x})^*{y_n}^*\|=0,$$

there are $u\in\mathcal{C}$; a real number $\xi\geq 0$ and an inner second order approximation $\mathcal{A}(u)$ for C at $g(\bar{x})$ with respect to $Dg(\bar{x})$ and ξ in the direction u such that

$$\liminf_{n\to\infty}[\langle{y_n}^*,D^2g(\bar{x})(u,u)-D^2g(\bar{x})(u-u_0,u-u_0)\rangle-\sigma_{\mathcal{A}(u)}({y_n}^*)]<0,$$

then G is directionally metrically $1/2-$subregular at $(\bar{x},0)$ in the direction u_0.

Consider the special case of the system of inequalities/inequalities:

$$S = \{x \in X : \quad H(x) = 0, \;\; h_i(x) \le 0, \; i = 1, ..., m\}, \tag{57}$$

where, X, Y are Banach space; $H : X \to Y$ and $h_i : X \to \mathbb{R}$, $i = 1, ..., m$ are continuously twice differentiable functions (around the reference point)defined on the Banach space X. Set

$$h : X \to \mathbb{R}^m, \; h(x) := (h_1(x), ..., h_m(x))), \;\; g(x) := (H(x), h(x));$$

and

$$G(x) := g(x) + \{0_Y\} \times \mathbb{R}^m_+, \; x \in X. \tag{58}$$

Let $\bar{x} \in S = G^{-1}(0)$ be given. For the sake of simplicity, without loss of generality, assume that $g(\bar{x}) = 0$. Obviously, one has

$$CG(\bar{x}, 0)^{-1}(0) = \{u \in \text{Ker } H'(\bar{x}) : \;\; h_i'(\bar{x})u \le 0, \; \forall i = 1, ..., m\}. \tag{59}$$

For this particular case, we have the following corollary.

Corollary 28 *Suppose that X, Y are Banach spaces and let $F : X \rightrightarrows Y \times \mathbb{R}^m$ defined by (66). Let $\bar{x} \in X$ with $g(\bar{x}) = 0$ and $u_0 \in CG(\bar{x}, 0)^{-1}(0) \cap \mathcal{S}_X$ be given. Assume that g is continuously twice differentiable around $\bar{x}$. If for any sequence $\{z_n{}^*\} \subset (Y^* \times \mathbb{R}^m_+) \cap \mathcal{S}_{Y^* \times \mathbb{R}^m}$ satisfying*

$$\lim_{n \to \infty} \langle z_n^*, Dg(\bar{x}) \rangle = 0,$$

there exists $u \in CG(\bar{x}, 0)^{-1}(0)$ such that

$$\liminf_{n \to \infty} \langle z_n{}^*, D^2 g(u, u) - D^2 g(\bar{x})(u - u_0, u - u_0) \rangle < 0, \tag{60}$$

then G is directionally metrically $1/2-$subregular at $(\bar{x}, 0)$ in the direction u_0.

Proof. Set $C := \{0_Y\} \times \mathbb{R}^m_-$. Then, $G(x) = g(x) - C, \; x \in X$. It suffices to see that C is an inner second order approximation for C itself at $g(\bar{x})$ with respect to $Dg(\bar{x})$ in any direction $u \in CG(\bar{x}, 0)^{-1}(0)$. □

When Y is a finite dimensional space, the previous corollary yields immediately the following.

Corollary 29 *With the assumptions as in Corollary 28 and in addition, assume that Y is finite dimensional. If for any $z^* \in (Y^* \times \mathbb{R}^m_+) \times \mathcal{S}_{Y^* \times \mathbb{R}^m}$ satisfying*

$$\langle z^*, Dg(\bar{x}) \rangle = 0,$$

there exists $u \in CG(\bar{x}, 0)^{-1}(0)$ such that

$$\langle z^*, D^2 g(\bar{x})(u, u) - D^2 g(\bar{x})(u - u_0, u - u_0) \rangle < 0,$$

then G is directionally metrically $1/2-$subregular at $(\bar{x}, 0)$ in the direction u_0.

We recall the notion of $2-$*regularity* from [42, 43, 2]: Let $G : X \to \mathbb{R}^m$ (m is some positive integer) be a mapping of C^2-class near $\bar{x} : G(\bar{x}) = 0$. For given $u \in X \setminus \{0\}$, G is said to be $2-$regular at $\bar{x}$ with respect to u if

$$\text{Im } DG(\bar{x}) + D^2 G(u, \text{Ker } DG(\bar{x})) = \mathbb{R}^m.$$

Corollary 30 *Let $G : X \to \mathbb{R}^m$ be a mapping of C^2-class near $\bar{x} \in X$ with $G(\bar{x}) = 0$. If G is $2-$regular at $\bar{x}$ with respect to a given direction $u_0 \in \text{Ker } DG(\bar{x})\backslash\{0\}$, then G is directionally metrically $1/2-$subregular at $(\bar{x}, 0)$ in direction u_0.*

Proof. It suffices to show that the sufficient condition for the directional metric $1/2-$subregularity in Corollary 29 is satisfied . Indeed, let $z^* \in \mathbb{R}^m$ with $\|z^*\| = 1$ be such that $\langle z^*, DG(\bar{x})\rangle = 0$. If $D^2G(u_0, u_0) \neq 0$, then obviously,

$$\langle z^*, D^2g(\bar{x})(u, u) - D^2g(\bar{x})(u - u_0, u - u_0)\rangle < 0,$$

for $u = 0$ or $u = u_0$. Assume that $D^2G(u_0, u_0) = 0$. Since G is 2-regular at $\bar{x}$ with respect to u_0, there exist $u_1 \in X$ and $u_2 \in \text{Ker } DG(\bar{x})$ such that

$$DG(\bar{x})(u_1) + D^2G(u_0, u_2) = -z^*.$$

Consequently, $\langle z^*, D^2G(u_0, u_2)\rangle = -\|z^*\|^2 = 1$. Hence,

$$\langle z^*, D^2g(\bar{x})(u_2, u_2) - D^2g(\bar{x})(u_2 - u_0, u_2 - u_0)\rangle = 2\langle z^*, D^2G(u_0, u_2)\rangle = -2 < 0.$$

□

Example 31 Let $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x) = x_1^3 + x_1x_2 - x_2^2, \;\; x = (x_1, x_2) \in \mathbb{R}^2.$$

Then,

$$Dg(0) = 0 \quad \text{and} \quad D^2g(0) = \begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix}.$$

Consider the multifunction $G : \mathbb{R}^2 \rightrightarrows \mathbb{R}$ defined by $G(x) := g(x) + \mathbb{R}_+, \; x \in \mathbb{R}^2$. In view of the preceding corollary We shall show that G is directionally metrically$1/2-$subregular at $(0, 0)$ in any direction $u = (u_1, u_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, and therefore, it is metrically$1/2-$subregular at $(0, 0)$. Indeed, let a direction $u = (u_1, u_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ be given. Then, for some $a = (a_1, a_2) \in \mathbb{R}^2$, one has

$$D^2g(0)(u, u) = 2u_2(2u_1 - u_2); \;\; D^2g(0)(u, a) = 2[u_2a_1 + (u_1 - u_2)a_2],$$

and

$$D^2g(0)(a, a) - D^2g(0)(a - u, a - u) = 2D^2g(0)(u, a) - D^2g(0)(u, u).$$

So it is easy to check that there exists $a \in \mathbb{R}^2$ (depending on u) such that

$$D^2g(0)(a, a) - D^2g(0)(a - u, a - u) < 0.$$

In view of Corollary 29, G is directionally metrically$1/2-$subregular at $(0, 0)$ in direction u.

Example 32 Let now $g(x) = x_1^3 - x_2^2, \; x = (x_1, x_2) \in \mathbb{R}^2$ and define the multifunction $G := g + \mathbb{R}_+$ as in the preceding example. Then

$$Dg(0) = (0, 0); \quad D^2g(0) == \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}.$$

We see that for any direction $u = (u_1, u_2) \in \mathbb{R}^2$ with $u_2 \neq 0$, there is $a \in \mathbb{R}^2$ such that

$$D^2g(0)(a, a) - D^2g(0)(a - u, a - u) < 0,$$

and therefore G is directionally metrically $1/2-$subregular at $(0, 0)$ in this directions u, in view of Corollary 29. However, for $e = (1, 0)$,

$$D^2g(0)(a, a) - D^2g(0)(a - e, a - e) = 0 \text{ for all } a \in \mathbb{R}^2,$$

so Corollary 29 is not applicable for the direction $e = (1, 0)$.

For simple systems of one inequality, we have the following sufficient condition for the metric $1/2-$regularity.

Proposition 33 *Let X be a Banach space and let $h : X \to \mathbb{R}$ be continuously twice differentiable near $\bar{x} \in X$ with $h(\bar{x}) = 0$. If there exists $u \in X$ such that either*

(i) $Dh(\bar{x})(u) < 0$ *for all* $j = 1, ..., m$,

or

(ii) $Dh(\bar{x}) = 0$ *and* $D^2h(\bar{x})(u, u) = 0$,

then the multifunction $G : X \rightrightarrows \mathbb{R}$ defined by $G(x) := h(x) + \mathbb{R}_+$, $x \in X$, is metrically $1/2-$subregular around $(\bar{x}, 0)$.

Proof. Firstly, assume that (i) is satisfied for some $u \in X$ with $\|u\| = 1$. Then, there exist $\gamma, \delta > 0$ such that $Dh(x)(u) \leq -\gamma$ for all $x \in B(\bar{x}, \delta)$. By virtue of the mean value theorem, for all $x \in B(\bar{x}, \delta)$; all $t \in (0, \delta)$, there is $x_t \in (x + tu, x)$ such that

$$h(x + tu) = h(x) + tDh(x_t)(u) \leq h(x) - t\gamma.$$

Pick $\varepsilon \in (0, \delta)$ such that $h(x) < \min\{\delta, \delta\gamma\}$ for all $x \in B(\bar{x}, \varepsilon)$. For $x \in B(\bar{x}, \varepsilon)$ with $h(x) > 0$, by taking $t := h(x)/\gamma$ into account of the inequality above, one has $h(x + h(x)u/\gamma) < 0$. Thus $x + h(x)u/\gamma \in G^{-1}(0)$, and consequently, $d(x, G^{-1}(0)) \leq \|x - (x + h(x)u/\gamma)\| = h(x)/\gamma$. This shows that G is metrically subregular at $(\bar{x}, 0)$.

Suppose now that (ii) is satisfied for $u \in X$ with $\|u\| = 1$. Let $\gamma, \delta, \varepsilon \in (0, \delta)$ such that

$$D^2h(x)(u, u) \leq -\gamma \ \forall x \in B(\bar{x}, \delta); \ h(x) < \min\{\delta, \delta\gamma\} \ \forall x \in B(\bar{x}, \varepsilon).$$

Let $x \in B(\bar{x}, \varepsilon)$ with $h(x) > 0$. If $Dh(x)(u) \leq 0$, then for $t := 2^{1/2}h(x)^{1/2}/\gamma^{1/2}$, by the Taylor expansion, there is $z_t \in (x, x + tu)$ such that

$$h(x + tu) = h(x) + tDh(x)(u) + \frac{t^2}{2}D^2h(x_t)(u, u) \leq h(x) - \frac{t^2\gamma}{2} = 0.$$

Thus $x + tu \in G^{-1}(0)$, and therefore

$$d(x, G^{-1}(0)) \leq \|x - x - tu\| = t = 2^{1/2}h(x)^{1/2}/\gamma^{1/2}.$$

Otherwise, $Dh(x)(u) > 0$, by replacing u by $-u$, one has $x - tu \in G^{-1}(0)$ and the inequality above also holds. □

With a minor modification, we can show that (i) or (ii) is also sufficient for the metric $1/2-$regularrity of G at $(\bar{x}, 0)$. In fact, In [29], it was established that when X is finite dimensional, the condition above is a necessary and sufficient condition for the metric $1/2-$regularrity of G at $(\bar{x}, 0)$. Return to Example 32, the multifunction G in this example satisfies obviously the condition (ii) of the proposition above.

5 Applications: Tangent vectors to a zero set

Consider a closed multifunction $F : X \rightrightarrows Y$ between the two Banach spaces X, Y. Denote by

$$M = F^{-1}(0) = \{x \in X : \ \ 0 \in F(x)\}. \tag{61}$$

For given $\bar{x} \in M$, as above, $T(M, \bar{x})$ denotes the contingent cone to M at $\bar{x}$. The contingent cone to the zero set M plays the important role in some areas of mathematics, in particular, it is a key notion in the context of optimality conditions for constrained optimization problems.

When F is a single-valued mapping which is continuously differentiable at $\bar{x}$, due to the classical Lyusternik theorem, $T(M, \bar{x}) = \text{Ker } F'(\bar{x})$ provied $F'(x)$ is subjective. Without the subjectivity of $F'(x)$, some results on the tangent cones were established in [6, 42, 43, 66]. The following proposition gives the following general formula in terms of the higher order contingent derivative.

Proposition 34 *Let $F : X \rightrightarrows Y$ be a closed multifunction and let $\bar{x} \in M$.*

(i) For any $\gamma \in (0, 1]$, one has

$$T(M, \bar{x}) \subseteq CF^{1/\gamma}(\bar{x}, 0)^{-1}(0) := \{u \in X : \ \ CF^{1/\gamma}(\bar{x}, 0)(u) = 0\}. \tag{62}$$

(ii) Conversely, for $u \in CF^{1/\gamma}(\bar{x}, 0)^{-1}(0)$, if F is metrically $\gamma-$subregular at $(\bar{x}, 0)$ in direction u, then $u \in T(M, \bar{x})$.

As a result, if F is metrically $\gamma-$subregular at $(\bar{x}, 0)$, then

$$T(M, \bar{x}) = CF^{1/\gamma}(\bar{x}, 0)^{-1}(0).$$

Proof. The part (i) is obvious. For (ii), by the assumption, there are $\kappa, \delta > 0$ such that

$$d(x, M) \leq \kappa d(0, F(x))^{\gamma} \ \ \forall x \in B(\bar{x}, \delta) \cap [\bar{x} + \text{cone } B(u, \delta)].$$

Since $u \in CF^{1/\gamma}(\bar{x}, 0)^{-1}(0)$, there are sequences $(t_n) \to 0^+$, $(u_n, v_n) \to (u, 0)$ such that

$$t_n^{1/\gamma} v_n \in F(\bar{x} + t_n u_n) \quad n \in \mathbb{N}.$$

Then, when n is sufficiently large, $\bar{x} + t_n u_n \in B(\bar{x}, \delta) \cap [\bar{x} + \text{cone } B(u, \delta)]$, and therefore

$$d(\bar{x} + t_n u_n, M) \leq \kappa d(0, F(\bar{x} + t_n u_n))^{\gamma}.$$

By this inequality, for n sufficiently large, we can find $\bar{u}_n \in X$ with $\bar{x} + t_n \bar{u}_n \in M$ such that

$$t_n \|u_n - \bar{u}_n\| \leq (1 + 1/n) d(\bar{x} + t_n u_n, M) \leq \kappa(1 + 1/n) d(0, F(\bar{x} + t_n u_n))^{\gamma} \leq \kappa(1 + 1/n) t_n \|v_n\|^{1/\gamma}.$$

Thus

$$\|u_n - \bar{u}_n\| \leq \kappa(1 + 1/n)\|v_n\|^{1/\gamma} \to 0 \ \text{ as } n \to \infty.$$

This implies $(\bar{u}_n) \to u$ and therefore $u \in T(M, \bar{x})$. □

When $F : X \to Y$ is a single valued mapping which is continuous ly Fréchet differentiable at $\bar{x} \in M := \{x \in X : \ \ F(x) = 0\}$, then $CF(\bar{x})^{-1}(0) := CF^1(\bar{x}, 0)^{-1}(0) = \text{Ker } F'(\bar{x})$. Hence, and the converse inclusion holds provided F is metrically subregular at $\bar{x}$. Moreover, it is well known that the metric subregularity of F at $\bar{x}$ is equivalent to the sujectivity of $F'(\bar{x})$. Thus Proposition 34 covers the classical Lyusternik theorem.

When F is twice differentiable at $\bar{x}$, the second order contingent derivative is explicitely given as follows.

Proposition 35 *Let $F : X \to Y$ be a twice differentiable function at $\bar{x} \in M$ such that the image of $F'(\bar{x})$: Im $F'(\bar{x})$ is closed. Then, one has*

$$CF^2(\bar{x})(u) = \begin{cases} \frac{1}{2}F''(\bar{x})(u,u) + \text{Im } F'(\bar{x}) & \text{if} \quad u \in \text{Ker } F'(\bar{x}), \\ \emptyset & \text{otherwise.} \end{cases} \tag{63}$$

Proof. Let $u \in X$ and $v \in CF^2(\bar{x})(u)$. By the definition, there are sequence $(t_n) \to 0^+$, $(u_n) \to u$ such that

$$v = \lim_{n\to\infty} \frac{F(\bar{x} + t_n u_n) - F(\bar{x})}{t_n^2} = \lim_{n\to\infty} \frac{F(\bar{x} + t_n u_n)}{t_n^2}.$$

According to the Taylor formula,

$$\frac{F(\bar{x} + t_n u_n)}{t_n^2} = \frac{F'(\bar{x})(u_n)}{t_n} + \frac{1}{2}F''(\bar{x})(u_n, u_n) + o(t_n).$$

This imples that $v = \lim_{n\to\infty} \frac{F'(\bar{x})(u_n)}{t_n} + \frac{1}{2}F''(\bar{x})(u,u)$. Therefore,

$$F'(\bar{x})(u) = \lim_{n\to\infty} F'(\bar{x})(u_n) = 0 \quad \text{and} \quad v \in \frac{1}{2}F''(\bar{x})(u,u) + \text{Im } F'(\bar{x}).$$

Where, the latter relation is due to the closedness of Im $F'(\bar{x})$.

Conversely, suppose that $u \in \text{Ker } F'(\bar{x})$ and let

$$v \in \frac{1}{2}F''(\bar{x})(u,u) + \text{Im } F'(\bar{x}).$$

Then, there is $u' \in X$ such that $v = \frac{1}{2}F''(\bar{x})(u,u) + F'(\bar{x})(u')$. Pick a sequence $(t_n) \to 0^+$ and set $u_n = u + t_n u'$, $n \in \mathbb{N}$. One has

$$F(\bar{x} + t_n u_n) = t_n^2 F'(u') + \frac{t_n^2}{2}F''(\bar{x})(u_n, u_n) + o(t_n^2).$$

Thus

$$v = \lim_{n\to\infty} \frac{F(\bar{x} + t_n u_n)}{t_n^2},$$

which follows $v \in CF^2(\bar{x})(u)$. □

Combinning the above propositions, we obtain the following formula for the contingent cone to $M = F^{-1}(0)$.

Theorem 36 *Let $F : X \to Y$ be a twice differentiable at $\bar{x} \in M = \{x \in X : \; F(x) = 0\}$. If Im $F'(\bar{x})$ is closed and F is metrically $1/2-$subregularity at $\bar{x}$ in every direction $u \in \text{Ker } F'(\bar{x})$, then*

$$T(M, \bar{x}) = \left\{u \in \text{Ker } F'(\bar{x}) : \; F''(\bar{x})(u,u) \in \text{Im } F'(\bar{x})\right\}. \tag{64}$$

In [42, 43], it was established that the formula (64) holds under the so-called *2-regularity* condition. By virtue of Corollary 30, the theorem above covers the one in the mentioned papers.

More general, consider the system of inequalities/equalities:

$$S = \{x \in X : \quad H(x) = 0, \;\; h_i(x) \le 0, \; i = 1, ..., m\}, \tag{65}$$

where, X, Y are Banach space; $H : X \to Y$ and $h_i : X \to \mathbb{R}$, $i = 1, ..., m$ are continuously differentiable functions (around the reference point)defined on the Banach space X. By setting

$$h : X \to \mathbb{R}^m, \; g(x) := (h_1(x), ..., h_m(x))), \;\; g(x) := (H(x), h(x));$$

and

$$F(x) := g(x) + \{0_Y\} \times \mathbb{R}^m_+, \; x \in X, \tag{66}$$

then $S = F^{-1}(0)$. Let $\bar{x} \in S$ be given. For the sake of simplicity and without loss of generality, assume that all components of h are *active* at $\bar{x}$, i.e., $h(\bar{x}) = 0$. It is well-known that the Mangasarian-Fromovitz qualification condition:

$$(MF) \quad \begin{cases} H'(\bar{x}) \quad \text{is onto;} \\ \exists u \in X, \;\; h_i'(\bar{x})(u) < 0, \; \forall i = 1, ..., m, \end{cases}$$

is equivalent to the metric regularity of F at $(\bar{x}, 0)$. Thus, under (MF) qualification condition, Proposition 34 implies the classical formula for the contingent cone:

$$T(S, \bar{x}) = CF(\bar{x}, 0)^{-1}(0) = \{u \in \text{Ker } H'(\bar{x}) : \;\; h_i'(\bar{x})u \le 0, \; \forall i = 1, ..., m\}.$$

The following theorem gives a formula for the contingent cone under the directional metric 1/2-subregularity of F.

Theorem 37 *Suppose that g is twice differentiable at $\bar{x}$;* Im $g'(\bar{x})$ *is closed and the multifunction F is metrically 1/2-regular at $\bar{x}$ in every direction $u \in CF(\bar{x}, 0)^{-1}(0)$. Then one has*

$$T(S, x) = CF^2(\bar{x}, 0)^{-1}(0) = \left\{ u \in X : \quad \begin{array}{l} u \in \text{Ker } H'(\bar{x}), \; h_i'(\bar{x})u \le 0, \; i = 1, ..., m; \\ 0 \in \frac{1}{2} g''(\bar{x})(u, u) + \text{Im } g'(\bar{x}) + \{0_Y\} \times \mathbb{R}^m_+ \end{array} \right\}.$$

The theorem follows immediately from Proposition 34 and the following lemma.

Lemma 38 *Suppose that g is twice differentiable at $\bar{x}$. Then, one has*

$$CF^2(\bar{x})(u) = \begin{cases} \frac{1}{2} g''(\bar{x})(u, u) + \text{Im } g'(\bar{x}) + \{0_Y\} \times \mathbb{R}^m_+ & \text{if} \quad u \in CF(\bar{x}, 0)^{-1}(0), \\ \emptyset & \text{otherwise.} \end{cases} \tag{67}$$

Proof. Let $u \in X$ and $v \in CF^2(\bar{x}, 0)(u)$. There are sequences $(t_n) \to 0^+$, $(u_n) \to u$ and $(v_n) :== ((w_n, \lambda_n)) \to v := (w, \lambda)$ as $n \to \infty$ with $(w_n, \lambda_n); (w, \lambda)$ in $Y \times \mathbb{R}^m$, such that

$$(\bar{x} + t_n u_n, t_n^2 v_n) \in \text{gph } F, \; n \in \mathbb{N}.$$

Setting

$$\lambda := (\lambda^1, ..., \lambda^m); \quad \lambda_n = (\lambda_n^1, ..., \lambda_n^m) \in \mathbb{R}^m,\ n \in \mathbb{N},$$

then

$$\lim_{n\to\infty} \frac{F(\bar{x} + t_n u_n) - F(\bar{x})}{t_n^2} = \lim_{n\to\infty} w_n = w;$$

$$\frac{h_i(\bar{x} + t_n u_n)}{t_n^2} \leq \lambda_n^i \quad i = 1, ..., m;\ n \in \mathbb{N}.$$

According to the Taylor formula, one has, for $i = 1, ..., m$,

$$h_i(\bar{x} + t_n u_n) = t_n h_i'(\bar{x}) u_n + \frac{t_n^2}{2} h_i''(u_n, u_n) + o(t_n^2),$$

and

$$H(\bar{x} + t_n u_n) = t_n H'(\bar{x}) u_n + \frac{t_n^2}{2} H''(u_n, u_n) + o(t_n^2).$$

Thus, $u \in \text{Ker } H'(\bar{x})$, and

$$w = \lim_{n\to\infty} H'(\bar{x})(u_n/t_n) + \frac{1}{2} H''(u, u); \tag{68}$$

$$\limsup_{n\to\infty} \frac{h_i(\bar{x} + t_n u_n)}{t_n^2} = \limsup_{n\to\infty} \frac{h_i'(\bar{x}) u_n}{t_n} + \frac{1}{2} h_i''(u, u) \leq \lambda^i,\ i = 1, ..., m. \tag{69}$$

The latter relations imply

$$h_i'(\bar{x}) u = \lim_{n\to\infty} h_i'(\bar{x}) u_n \leq 0\ \forall i = 1, ..., m,$$

that is, $u \in CF(\bar{x}, 0)^{-1}(0)$, and that for each $k = 1, 2, ...$, there exists $n_k \in \mathbb{N}$ such that

$$\frac{h_i'(\bar{x}) u_{n_k}}{t_{n_k}} + \frac{1}{2} h_i''(u, u) \leq \lambda^i + 1/k,\ i = 1, ..., m.$$

By setting

$$z_k := H'(\bar{x})(u_{n_k}/t_{n_k}) + \frac{1}{2} H''(u, u);\ \alpha_k := (\lambda_1^1 + 1/k, ..., \lambda^m + 1/k),$$

then by virtue of (68) and (69), one has

$$(z_k, \alpha_k) \in \frac{1}{2} g''(\bar{x})(u, u) + \text{Im } g'(\bar{x}) + \{0_Y\} \times \mathbb{R}_+^m,\ \forall k = 1, 2, ...$$

As $\frac{1}{2} g''(\bar{x})(u, u) + \text{Im } g'(\bar{x}) + \{0_Y\} \times \mathbb{R}_+^m$ is a closed convex cone, by letting $k \to \infty$, one obtains

$$v = (w, \lambda) = \lim_{k\to\infty} (z_k, \alpha_k) \in \frac{1}{2} g''(\bar{x})(u, u) + \text{Im } g'(\bar{x}) + \{0_Y\} \times \mathbb{R}_+^m.$$

Conversely, let $u \in CF(\bar{x}, 0)^{-1}(0)$ and let

$$v := (w, \lambda^1, ..., \lambda^m) \in \frac{1}{2} g''(\bar{x})(u, u) + \text{Im } g'(\bar{x}) + \{0_Y\} \times \mathbb{R}_+^m.$$

Then, there is $u' \in X$ such that

$$w = \frac{1}{2} H''(\bar{x})(u, u) + H'(\bar{x})(u');$$

$$\lambda^i \geq \frac{1}{2}h_i''(\bar{x})(u,u) + h_i'(\bar{x})u',\ i = 1, ..., m.$$

Pick a sequence $(t_n) \to 0^+$ and set $u_n := u + t_n u'$, $n \in \mathbb{N}$. By using the Taylor formula, one has

$$\lim_{n\to\infty} \frac{H(\bar{x} + t_n u_n)}{t_n^2} = H'(u') + \frac{1}{2}H''(\bar{x})(u,u),$$

and

$$\lim_{n\to\infty} \frac{h_i(\bar{x} + t_n u_n)}{t_n^2} = h_i'(\bar{x})u' + \frac{1}{2}h_i''(\bar{x})(u,u),\ i = 1, ..., m.$$

Therefore, for each $k = 1, 2, ...$, we can find $n_k \in \mathbb{N}$ verifying

$$\lambda^i + 1/k \geq \frac{h_i(\bar{x} + t_{n_k} u_{n_k})}{t_{n_k}^2},\ \forall i = 1, ..., m.$$

Therefore, by setting

$$w_k = \frac{H(\bar{x} + t_{n_k} u_{n_k})}{t_{n_k}^2};\ \lambda_k := (\lambda^1 + 1/k, ..., \lambda^m + 1/k);\ v_k := (w_k, \lambda_k) \in \{0_Y\} \times \mathbb{R}^m$$

$$(\bar{x} + t_{n_k} u_{n_k}, t_{n_k}^2 v_k) \in \text{gph } F,\ k = 1, 2, ...$$

It follows that $v \in CF^2(\bar{x}, 0)(u)$. The proof is completed. □

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