SECOND ORDER ANALYSIS OF STATE-CONSTRAINED CONTROL-AFFINE PROBLEMS

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Abstract. In this article we establish new second order necessary and sufficient optimality conditions for a class of control-affine problems with a scalar control and a scalar state constraint. These optimality conditions extend to the constrained state framework the Goh transform, which is the classical tool for obtaining an extension of the Legendre condition. We propose a shooting algorithm to solve numerically this class of problems and we provide a sufficient condition for its local convergence. We provide examples to illustrate the theory.

1. Introduction

Control-affine problems have been intensively studied since the 1960s and there is a wide literature on this subject. In what respect to second order conditions, the main feature of these systems that are affine in the control variable is that the second derivative of the pre-Hamiltonian function with respect to the control vanishes, and hence the classical *Legendre-Clebsch conditions* hold trivially and do not provide any useful information. Second and higher order necessary conditions for problems without control nor state constraints were first established in [31, 23, 22, 28, 3]. The case with control constraints and purely bang-bang solutions was investigated by [42, 4, 46, 40, 18] among many others, while the class of bang-singular solutions was analyzed in e.g. [47, 6, 20].

This article is devoted to the study of Mayer-type optimal control problems governed by the dynamics

$$\dot{x}_t = f_0(x_t) + u_t f_1(x_t), \quad \text{for a.a. } t \in [0,T],$$

subject to endpoint constraints

$$\Phi(x_0, x_T) \in K_\Phi,$$

control constraints

$$u_{\min} \leq u_t \leq u_{\max},$$

and a scalar state constraint of the form

$$g(x_t) \leq 0.$$

For this class of problems, we show necessary optimality conditions involving the regularity of the control and the state constraint multiplier at the junction points. Some of these necessary conditions which hold at the junction points were proved in [37]. Moreover, we provide second order necessary and sufficient optimality conditions in integral form obtained through the *Goh transformation* [23]. We also

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propose a shooting-like numerical scheme and we show a sufficient condition for its local quadratic convergence, that is also a second order sufficient condition for optimality (in some sense to be specified later on). Finally, we solve numerically an example of practical interest.

This investigation is strongly motivated by applications since it allows to deal with both control and state constraints, which appear naturally in realistic models. Many practical examples can be found in the existing literature, a non exhaustive list includes the prey-predator model [24], the Goddard problem in presence of a dynamic pressure limit [50, 25], an optimal production and maintenance system studied in [39], and a recent optimization problem on running strategies [2].

In what concerns second order analysis in the state constrained case, Malanowski and Maurer [34], Bonnans and Hermant [10] provided second order necessary and sufficient optimality conditions and related the sufficient conditions with the convergence of the shooting algorithm in the case where the strengthened Legendre-Clebsch condition holds. Second order necessary conditions for the general nonlinear case with phase constraints were also proved in [8, 29]. For control-affine problems with bounded scalar control variable, a scalar state constraint (as is the case in this article) and solutions (possibly) containing singular, bang-bang and constrained arcs, Maurer [37] proved necessary conditions (similar to those developed by McDanell and Powers [41]) that hold at the junction points of optimal solutions. In Maurer et al. [39] they extended to the state-constrained framework, a second order sufficient test for optimality given in Agrachev et al. [4] and Maurer-Osmolovskii [44] for optimal bang-bang solutions.

As it is commonly known nowadays, the application of the necessary conditions provided by Pontryagin's maximum principle leads to an associated two-point boundary-value problem (TPBVP) for the optimal trajectory and its corresponding multiplier. A natural way for solving numerically TPBVPs is the application of *shooting-like algorithms.* This type of algorithms has been used extensively to solve optimal control problems (see e.g. [13, 45] and references therein) and, in particular, it has been applied to control-affine problems both with and without state constraints. Maurer [36] proposed a shooting scheme for solving a problem with bang-singular solutions, which was generalized quite recently by Aronna, Bonnans and Martinon in [7], where they provided a sufficient condition for its local convergence. Both these articles [36] and [7] analyze the case when no state constraints are present. Practical control-affine problems with state constraints were solved numerically in several articles, a non extensive list includes Maurer and Gillessen [38], Oberle [43] and Fraser-Andrews [21].

Up to our knowledge, there is no result in the existing literature about: (i) second order necessary and sufficient conditions in integral (quadratic) form for control-affine problems with state constraints and solutions containing singular arcs; (ii) sufficient conditions for the convergence of shooting algorithms in this context.

The paper is organized as follows. In Section 2 we give the basic definitions and show necessary optimality conditions concerning the regularity of the optimal control and associated multipliers. Section 3 is devoted to second order necessary optimality conditions in integral form and to the Goh transformation, while second order sufficient conditions are provided in Section 4. A shooting-like method and a sufficient condition for its local quadratic convergence are given in Section 5. This algorithm is implemented in Section 6 to solve numerically a variation of the regulator problem. The Appendix is consecrated to the presentation of abstract results on second order necessary conditions.

Notations. Let $\mathbb{R}^k$ denote the $k-$dimensional real space, i.e. the space of column real vectors of dimension k, and by $\mathbb{R}^{k*}$ its corresponding dual space, which consists of $k-$dimensional row real vectors. With $\mathbb{R}^k_+$ and $\mathbb{R}^k_-$ we refer to the subsets of $\mathbb{R}^k$ consisting of vectors with nonnegative, respectively nonpositive, components. We write h_t for the value of function h at time t if h is a function that depends only on t, and by $h_{i,t}$ the ith component of h evaluated at t. Let $h(t+)$ and $h(t-)$ be, respectively, the right and left limits of h at t, if they exist. Partial derivatives of a function h of (t,x) are referred as $D_t h$ or $\dot{h}$ for the derivative in time, and $D_x h$, h_x or h' for the differentiations with respect to space variables. The same convention is extended to higher order derivatives. By $L^p(0,T)^k$ we mean the Lebesgue space with domain equal to the interval $[0,T] \subset \mathbb{R}$ and with values in $\mathbb{R}^k$. The notations $W^{q,s}(0,T)^k$ and $H^1(0,T)^k$ refer to the Sobolev spaces (see Adams [1] for further details on Sobolev spaces). We let $BV(0,T)$ be the set of functions with bounded total variation. In general, when there is no place for confusion, we omit the argument $(0,T)$ when referring to a space of functions. For instance, we write L^∞ for $L^\infty(0,T)$, or $(W^{1,\infty})^{k*}$ for the space of $W^{1,\infty}-$functions from $[0,T]$ to $\mathbb{R}^{k*}$. We say that a function $h : \mathbb{R}^k \to \mathbb{R}^d$ is of class C^ℓ if it is $\ell-$times continuously differentiable in its domain.

2. Framework

2.1. **The problem.** Consider the control and state spaces L^∞ and $(W^{1,\infty})^n$, respectively. We say that a control-state pair $(u,x) \in L^\infty \times (W^{1,\infty})^n$ is a *trajectory* if it satisfies both the *state equation*

$$\dot{x}_t = f_0(x_t) + u_t f_1(x_t), \quad \text{for a.a. } t \in [0,T], \tag{2.1}$$

and the finitely many *endpoint constraints* of equality and inequality type

$$\Phi(x_0, x_T) \in K_\Phi. \tag{2.2}$$

Here f_0 and f_1 are twice continuously differentiable and Lipschitz continuous vector fields over $\mathbb{R}^n$, Φ is of class C^2 from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^{n_1+n_2}$, and

$$K_\Phi := \{0\}_{\mathbb{R}^{n_1}} \times \mathbb{R}^{n_2}_-, \tag{2.3}$$

where $\{0\}_{\mathbb{R}^{n_1}}$ is the singleton consisting of the zero vector of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}_- := \{y \in \mathbb{R}^{n_2} : y_i \leq 0, \text{ for all } i = 1,\ldots,n_2\}$. Given $(u,x_0) \in L^\infty \times \mathbb{R}^n$, (2.1) has a unique solution. In addition, we consider the *cost functional* $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, then *bound control constraints*

$$u_{\min} \leq u_t \leq u_{\max}, \quad \text{for a.a. } t \in [0,T], \tag{2.4}$$

and a *scalar state constraint*

$$g(x_t) \leq 0, \quad \text{for all } t \in [0,T], \tag{2.5}$$

with ϕ and $g : \mathbb{R}^n \to \mathbb{R}$ of class C^2. Here we allow $u_{\min}$ and $u_{\max}$ to be either finite real numbers, or to assume the values $-\infty$ or $+\infty$, respectively, in the sense that problems with control constraints of the form $u_t \leq u_{\max}$ or $u_{\min} \leq u_t$ are also considered in our investigation, as well as problems in the absence of control constraints. We say that the trajectory (u,x) is *feasible* if it satisfies (2.4)-(2.5). Let us then consider the optimal control problem in the Mayer form

$$\min \phi(x_0,x_T); \qquad \text{subject to (2.1)-(2.5).} \tag{P}$$

2.2. **Regular extremals.** We set $f(u,x) := f_0(x) + uf_1(x)$, and define the *pre-Hamiltonian function* and the *endpoint Lagrangian,* respectively, by

$$\begin{cases} H(u,x,p) := pf(u,x) = p\big(f_0(x)+uf_1(x)\big), \\ \ell^{\beta,\Psi}(x_0,x_T) := \beta\phi(x_0,x_T) + \Psi\Phi(x_0,x_T), \end{cases} \tag{2.6}$$

where $p \in \mathbb{R}^{n*}$, $\beta \in \mathbb{R}_+$ and $\Psi \in \mathbb{R}^{(n_1+n_2)*}$.

Any function $\mu \in BV(0,T)$ (shortly, BV) has left limit on $(0,T]$ and right limits on $[0,T)$ and, therefore, the values μ_{0+} and μ_T are well-defined. Moreover, μ has a distributional derivative that belongs to the space $\mathcal{M}(0,T)$ (shortly, $\mathcal{M}$) of finite Radon measures. Conversely, any measure $\mathrm{d}\mu \in \mathcal{M}$ can be identified with the derivative of a function μ of bounded variation such that $\mu_T \in BV_0$, i.e., μ belongs to the space of bounded variation functions that vanish at time T.

Let (u,x) be a feasible trajectory. We say that $\Psi \in \mathbb{R}^{(n_1+n_2)*}$ is *complementary to the endpoint constraint* if

$$\Psi \in N_\Phi\big(\Phi(x_0,x_T)\big), \tag{2.7}$$

where $N_\Phi\big(\Phi(x_0,x_T)\big)$ denotes the *normal cone to* K_Φ at the point $\Phi(x_0,x_T)$, i.e.

$$N_\Phi\big(\Phi(x_0,x_T)\big) := \{\Psi \in \mathbb{R}^{(n_1+n_2)*} : \Psi_i \geq 0,\ \Psi_i\Phi_i(x_0,x_T)=0,\ i=n_1+1,\dots,n_2\}. \tag{2.8}$$

A bounded variation function μ is *complementary to the state constraint* if and only if

$$\mathrm{d}\mu \geq 0, \quad \text{and} \quad \int_0^T g(x_t)\mathrm{d}\mu_t = 0. \tag{2.9}$$

For $\Psi \in \mathbb{R}^{(n_1+n_2)*}$ and $\mu \in BV_0$, the *costate equation* associated with $(\beta,\Psi,\mathrm{d}\mu)$ is given by

$$-\,\mathrm{d}p_t = p_t f_x(u_t,x_t)\mathrm{d}t + g'(x_t)\,\mathrm{d}\mu_t, \quad \text{for a.a. } t\in[0,T], \tag{2.10}$$

with endpoint conditions

$$(-p_0,p_T) = D\ell^{\beta,\Psi}(x_0,x_T). \tag{2.11}$$

Given $(\beta,\Psi,\mathrm{d}\mu) \in \mathbb{R}\times\mathbb{R}^{(n_1+n_2)*}\times\mathcal{M}$, the boundary value problem (2.10)-(2.11) has at most one solution. In addition, the *condition of minimization* of the pre-Hamiltonian H implied by the Pontryagin Maximum Principle can be expressed as follows, for a.a. $t\in[0,T]$,

$$\begin{cases} u_t = u_{\min}, & \text{if } u_{\min} > -\infty \text{ and } p_tf_1(x_t) > 0, \\ u_t = u_{\max}, & \text{if } u_{\max} < +\infty \text{ and } p_tf_1(x_t) < 0, \\ p_tf_1(x_t) = 0, & \text{if } u_{\min} < u_t < u_{\max}. \end{cases} \tag{2.12}$$

Denote the quadruple of dual variables by $\lambda := (\beta,\Psi,p,\mathrm{d}\mu)$, element of the space

$$E^\Lambda := \mathbb{R}_+ \times \mathbb{R}^{(n_1+n_2)*} \times BV^{n*} \times \mathcal{M}. \tag{2.13}$$

The *Lagrangian* of the problem is

$$\mathcal{L}(u,x,\lambda) := \ell^{\beta,\Psi}(x_0,x_T) + \int_0^T p_t\big(f(u_t,x_t)-\dot{x}_t\big)\mathrm{d}t + \int_0^T g(x_t)\mathrm{d}\mu_t. \tag{2.14}$$

Note that the costate equation (2.10) expresses the stationarity of $\mathcal{L}$ with respect to the state. For a feasible trajectory $(u,x) \in L^\infty \times (W^{1,\infty})^n$, define the *set of Lagrange multipliers* as

(2.15)
$$\Lambda(u,x) := \{\lambda = (\beta, \Psi, p, \mathrm{d}\mu) \in E^\Lambda : \ (\beta, \Psi, \mathrm{d}\mu) \neq 0; \ (2.1)\text{-}(2.5) \text{ and } (2.7)\text{-}(2.12) \text{ hold}\}.$$

Remark 2.1. Let us recall that, since problem (P) is affine in the control variable, Lagrange multipliers and *Pontryagin multipliers* coincide.

In addition we set, for $\beta \geq 0$,

$$\Lambda_\beta(u,x) := \{(\beta, \Psi, p, \mathrm{d}\mu) \in \Lambda(u,x)\}. \tag{2.16}$$

Since $\Lambda(u,x)$ is a cone we have that

$$\Lambda(u,x) = \begin{cases} \mathbb{R}_+\Lambda_1(u,x), & \text{if } \Lambda_0(u,x) = \emptyset, \\ \Lambda_0(u,x) + \mathbb{R}_+\Lambda_1(u,x), & \text{otherwise.} \end{cases} \tag{2.17}$$

Consider a nominal feasible trajectory $(\hat{u}, \hat{x}) \in L^\infty \times (W^{1,\infty})^n$. Set

$$A_t := D_x f(\hat{u}_t, \hat{x}_t), \quad \text{for } t \in [0,T]. \tag{2.18}$$

¿From now on, when the argument of a function is omitted, we mean that it is evaluated in the nominal pair $(\hat{u}, \hat{x})$. In particular, we write Λ to refer to $\Lambda(\hat{u}, \hat{x})$.

For $(v, z^0) \in L^\infty \times \mathbb{R}^n$, let $z[v, z^0] \in (W^{1,\infty})^n$ denote the solution of the *linearized state equation*

$$\dot{z}_t = A_t z_t + v_t f_{1,t}, \quad \text{for a.a. } t \in [0,T], \tag{2.19}$$

with initial condition

$$z_0 = z^0. \tag{2.20}$$

Let $\bar{\Phi}_E$ denote the function from $L^\infty \times \mathbb{R}^n$ to $\mathbb{R}^{n_1}$ that, to each $(u, x_0) \in L^\infty \times \mathbb{R}^n$, assigns the value $\big(\Phi_1(x_0, x_T), \dots, \Phi_{n_1}(x_0, x_T)\big)$, where x is the solution of (2.1) associated with (u, x_0). For some results obtained in this article, we shall consider the following *qualification condition:*

(2.21)
$$\begin{cases} \text{(i)} & D\bar{\Phi}_E(\hat{u}, \hat{x}_0) \text{ is onto from } L^\infty \times \mathbb{R}^n \text{ to } \mathbb{R}^{n_1}, \\ \text{(ii)} & \text{there exists } (\bar{v}, \bar{z}_0) \in L^\infty \times \mathbb{R}^n \text{ in the kernel of } D\bar{\Phi}_E(\hat{u}, \hat{x}_0), \\ & \text{such that, for some } \varepsilon > 0, \text{ setting } \bar{z} = z[\bar{v}, \bar{z}^0], \text{ one has:} \\ & \hat{u}_t + \bar{v}_t \in [u_{\min} + \varepsilon, u_{\max} - \varepsilon], \text{ for a.a. } t \in [0,T], \\ & g(\hat{x}_t) + g'(\hat{x}_t)\bar{z}_t < 0, \text{ for all } t \in [0,T], \\ & \Phi_i'(\hat{x}_0, \hat{x}_T)(\bar{z}_0, \bar{z}_T) < 0, \text{ if } \Phi_i(\hat{x}_0, \hat{x}_T) = 0, \text{ for } n_1 + 1 \leq i \leq n_1 + n_2. \end{cases}$$

Here the notation $\hat{u}_t + \bar{v}_t \in [u_{\min} + \varepsilon, u_{\max} - \varepsilon]$ means that $\hat{u}_t + \bar{v}_t \in [u_{\min} + \varepsilon, +\infty)$ if $u_{\max} = +\infty$ and $u_{\min}$ is finite; $\hat{u}_t + \bar{v}_t \in (-\infty, u_{\max} - \varepsilon)$ if $u_{\min} = -\infty$ and $u_{\max}$ is finite; and $\hat{u}_t + \bar{v}_t \in \mathbb{R}$ if neither $u_{\min}$ nor $u_{\max}$ is finite.

Definition 2.2. A *weak minimum* for (P) is a feasible trajectory (u,x) such that $\phi(x_0, x_T) \leq \phi(\tilde{x}_0, \tilde{x}_T)$ for any feasible $(\tilde{u}, \tilde{x})$ for which $\|(\tilde{u}, \tilde{x}) - (u,x)\|_\infty$ is small enough.

Theorem 2.3. *Assume that* $(\hat{u}, \hat{x})$ *is a weak minimum for* (P)*. Then* (i) *the set* Λ *of Lagrange multipliers is nonempty,* (ii) *if the qualification condition* (2.21) *holds, then* Λ_0 *is empty, and* Λ_1 *is bounded and weakly* compact.*

Proof. It follows from [12, Theorem 3.9]. □

2.3. **Jump conditions.** Given a function of time $h:[0,T]\to\mathbb{R}^d$ for $d\in\mathbb{N}$, we define its *jump at time* $t\in[0,T]$ by

$$[h_t] := h(t_+) - h(t_-), \tag{2.22}$$

when the left and right limits, $h(t_-)$ and $h(t_+)$, respectively, exist and are finite. Here we adopt the convention $h(0_-):=h_0$ and $h(T_+):=h_T$. For any function of bounded variation the associated jump function is well-defined. For a function defined almost everywhere with respect to the Lebesgue measure, we will accord that its jump at time t is the jump at t of a representative of this function for which the left and right limit exist, provided that such a representative exists. By the costate equation (2.10), we have that, for any $(\beta,\Psi,p,\mathrm{d}\mu)\in\Lambda$,

$$\begin{cases} \text{(i)} & [p_t] = -[\mu_t]g'(\hat{x}_t), \\ \text{(ii)} & \big[H_u(t)\big] = [p_t]f_1(\hat{x}_t) = -[\mu_t]g'(\hat{x}_t)f_1(\hat{x}_t). \end{cases} \tag{2.23}$$

In addition, if $[\hat{u}_t]$ makes sense, then the jump in the derivative of the state constraint exists and satisfies

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}g(\hat{x}_t)\right] = [\hat{u}_t]g'(\hat{x}_t)f_1(\hat{x}_t). \tag{2.24}$$

Lemma 2.4. *Let $(\beta,\Psi,p,\mathrm{d}\mu)\in\Lambda$. Then, if $t\in[0,T]$ is such that $[H_u(t)]=0$ and $g'(\hat{x}_t)f_1(\hat{x}_t)\neq 0$, then μ is continuous at t.*

Proof. It follows from item (ii) in equation (2.23). □

Lemma 2.5. *Let $t\in(0,T)$ be such that $[\hat{u}_t]$ makes sense. Then the following conditions hold*

$$\begin{cases} \text{(i)} & [\hat{u}_t][H_u(t)] = 0, \\ \text{(ii)} & [\mu_t]\left[\frac{\mathrm{d}}{\mathrm{d}t}g(\hat{x}_t)\right] = 0. \end{cases} \tag{2.25}$$

Proof. Note that

$$[\mu_t]\left[\frac{\mathrm{d}}{\mathrm{d}t}g(\hat{x}_t)\right] = [\mu_t]g'(\hat{x}_t)f_1(\hat{x}_t)[\hat{u}_t] = -[H_u(t)][\hat{u}_t], \tag{2.26}$$

where the last equality follows from (2.23). This implies that (i) is equivalent to (ii), and so we only need to prove (i), which holds trivially when $[\mu_t]=0$. Hence, let us assume that $[\mu_t]\neq 0$. Then $g(\hat{x}_t)=0$ in view of (2.9) and, necessarily, $t\mapsto g(\hat{x}_t)$ attains its maximum at t, so that $\left[\frac{\mathrm{d}}{\mathrm{d}t}g(\hat{x}_t)\right]\leq 0$. Since $\mathrm{d}\mu\geq 0$, it follows that $[\mu_t]\geq 0$, and therefore by (2.26), $[H_u(t)][\hat{u}_t]\geq 0$. However, the converse inequality holds in view of (2.12). The conclusion follows. □

We say that the *state constraint is of first order* if

$$g'(\hat{x}_t)f_1(\hat{x}_t)\neq 0, \quad \text{when } g(\hat{x}_t)=0. \tag{2.27}$$

Corollary 2.6. *Assume that the state constraint is of first order. Then if the control has a jump at time $\tau\in(0,T)$ for which $g(\hat{x}_\tau)=0$, then μ is continuous at τ for any associated multiplier $(\beta,\Psi,p,\mathrm{d}\mu)\in\Lambda$.*

Proof. ¿From the identity (2.25) in Lemma 2.5, if $[\hat{u}_\tau]\neq 0$, then $[H_u(t)]=0$. The latter implies that $[\mu_\tau]=0$, in view of the second equality in (2.26) and since (2.27) holds. □

We refer to Remark 2.8 regarding the relation between latter Corollary 2.6 and Maurer [35].

Remark 2.7. Let us illustrate by means of this example that the associated μ might have jumps at the initial and or at final times. In fact, consider the problem

$$\min \int_0^T x_t \mathrm{d}t + x_T; \quad \dot{x}_T = u_t \in [-1,1]; \quad g(x_t) = -x_t \leq 0; \quad x_0 = 1, \tag{2.28}$$

with $T = 2$, and note that

$$\hat{u}_t := \begin{cases} -1 & \text{on } [0,1], \\ 0 & \text{on } (1,2] \end{cases}$$

is optimal for it. Over $[0,1)$ the state constraint is not active, and on $[1,2]$ it is. Since $H_u = 0$ on $[1,2]$, necessarily one has that $p = 0$ on $(1,2)$. The adjoint equation reads $\mathrm{d}p = -\mathrm{d}t + \mathrm{d}\mu$. Since $p_{T_+} = 1$, we have that $[p_T] = [\mu_T] = 1$.

2.4. **Arcs and junction points.** The *contact* set associated with the state constraint is defined as

$$C := \{t \in [0,T] : \ g(\hat{x}_t) = 0\}. \tag{2.29}$$

For $0 \leq a < b \leq T$, we say that (a,b) is a (maximal) *active arc* for the state constraint, or a C *arc,* if C contains (a,b), but no open interval in which (a,b) is strictly contained. Note that, since $t \mapsto g(\hat{x}_t)$ is continuous, the set C consists of a countable union of arcs, which can be ordered by size. We say that $\tau \in (0,T)$ is a *junction point of the state constraint* if it is the extreme point of a C arc.

We give similar definitions for the control constraint, paying attention to the fact that the control variable is not continuous, and is defined only almost everywhere. So, we define the contact and interior sets for the control bounds as

$$\begin{cases} B_- := \{t \in [0,T] : \ \hat{u}_t = u_{\min}\}, \\ B_+ := \{t \in [0,T] : \ \hat{u}_t = u_{\max}\}, \end{cases} \tag{2.30}$$

and set $B := B_- \cup B_+$. We shall make clear that if $u_{\min} = -\infty$ then $B_- = \emptyset$ and, analogously, if $u_{\max} = +\infty$ then $B_+ = \emptyset$. These sets are defined up to a null measure set and they can be identified with their characteristic functions. We define arcs in a similar way, using representatives of the characteristic functions. That is, we say that (a,b) is a *B_-, B_+ arc* (or simply B arc if we do not want to precise in which bound $\hat{u}$ lies) if (a,b) is included, up to a null measure set, in B_-, B_+, (in B_- or B_+) respectively, but no open interval strictly containing (a,b) is. We call *junctions times of the control constraint* the boundary of B_-, B_+ arcs in $(0,T)$.

Finally, let S denote the *singular set*

$$S := \{t \in [0,T] : u_{\min} < \hat{u}_t < u_{\max} \text{ and } g(\hat{x}_t) < 0\}. \tag{2.31}$$

Junction times are in general points $\tau \in (0,T)$ at which the trajectory $(\hat{x},\hat{u})$ switches from one type of arc $(B_-, B_+, C$ or $S)$ to another type. We may have, for instance, *CS junctions, BC junctions, etc.*

Remark 2.8. The result in Corollary 2.6 above was proved by Maurer [37] at times $\tau \in (0,T)$ being junction points of the state constraint, and for state constraints of any order.

Consider the following geometric hypotheses on the control structure:

(2.32)
$$\begin{cases} \text{(i)} & \text{the interval } [0,T] \text{ is (up to a zero measure set) the disjoint} \\ & \text{union of finitely many arcs of type } B, C \text{ and } S, \\ \text{(ii)} & \text{the control } \hat{u} \text{ is at uniformly positive distance of the bounds } u_{\min} \text{ and } u_{\max}, \\ & \text{over } C \text{ and } S \text{ arcs}, \\ \text{(iii)} & \text{the control } \hat{u} \text{ is discontinuous at CS and SC junctions.} \end{cases}$$

The example in Remark 2.7 above and the regulator problem in Section 6 verify these hypotheses.

Let us note that (2.27) can be written as

$$g'(\hat{x}_t) f_1(\hat{x}_t) \neq 0, \quad \text{on } C. \tag{2.33}$$

¿From

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} g(\hat{x}_t) = g'(\hat{x}_t)\big(f_0(\hat{x}_t) + \hat{u}_t f_1(\hat{x}_t)\big), \quad \text{on } C, \tag{2.34}$$

and whenever (2.33) holds, we get that

$$\hat{u}_t = -\frac{g'(\hat{x}_t) f_0(\hat{x}_t)}{g'(\hat{x}_t) f_1(\hat{x}_t)}, \quad \text{on } C. \tag{2.35}$$

In the remainder of the article we assume that $(\hat{x}, \hat{u})$ *satisfy hypotheses* (2.32) *and* (2.33) *above.*

2.5. **About** $\mathrm{d}\mu$**.** Observe that, along a constrained arc (a,b), over which $u_{\min} < \hat{u}_t < u_{\max}$ a.e., in view of (2.12), we have that $H_u = pf_1 = 0$, for any $(\beta, \Psi, p, \mathrm{d}\mu) \in \Lambda$. Differentiating in time this equation and using (2.10), we get

$$0 = \mathrm{d}H_u = p[f_1, f_0]\mathrm{d}t - g' f_1 \mathrm{d}\mu. \tag{2.36}$$

Thus, since the state constraint is of first order, i.e. (2.33) holds, $\mathrm{d}\mu$ has a density $\nu \geq 0$ over C given by the absolutely continuous function

$$\nu : [0,T] \to \mathbb{R}, \quad t \mapsto \nu_t := \frac{p_t [f_1, f_0](\hat{x}_t)}{g'(\hat{x}_t) f_1(\hat{x}_t)}, \tag{2.37}$$

where $[X,Y] := X'Y - Y'X$ denotes the *Lie bracket* associated to a pair of vector fields $X, Y : \mathbb{R}^n \to \mathbb{R}^n$.

3. Second order necessary conditions

In this section we state second order necessary conditions for weak optimality of problem (P). We start by defining the cones of critical directions and giving necessary conditions, obtained by applying the abstract result in Theorem A.9 in the Appendix. Afterwards, we present the Goh transformation [23] and we state second order conditions in the transformed variables.

3.1. **Critical directions.** Let us extend the use of $z[v, z^0]$ to denote the solution of (2.19)-(2.20) for $(v, z^0) \in L^2 \times \mathbb{R}^n$, and let us write T_Φ to refer to the *polar cone* of N_Φ (defined in (2.8)) at the point $(\hat{x}_0, \hat{x}_T)$, given by

$$T_\Phi := \{0\}_{\mathbb{R}^{n_1}} \times \{\eta \in \mathbb{R}^{n_2};\ \eta_i \leq 0 : \text{if } \Phi_{n_1+i}(\hat{x}_0, \hat{x}_T) = 0,\ i = 1, \dots, n_2\}. \tag{3.1}$$

For $v \in L^2$ and $z_0 \in \mathbb{R}^n$, consider the *linearization of the cost and endpoint constraints*

$$\begin{cases} \phi'(\hat{x}_0, \hat{x}_T)(z_0, z_T) \leq 0, \\ \Phi'(\hat{x}_0, \hat{x}_T)(z_0, z_T) \in T_\Phi, \end{cases} \tag{3.2}$$

where $z := z[v, z_0]$. Define the *critical cone* as the set

$$\mathcal{C} := \left\{ \begin{matrix} (v,z) \in L^\infty \times (W^{1,\infty})^n : (v,z) \text{ satisfies (2.19) and (3.2),} \\ v_t \geq 0 \text{ a.e. on } B_-,\ v_t \leq 0 \text{ a.e. on } B_+, \quad g'(\hat{x}_t) z_t \leq 0 \text{ on } C \end{matrix} \right\}. \tag{3.3}$$

Define the *strict critical cone,*

$$\mathcal{C}_S := \{(v,z) \in \mathcal{C} : v_t = 0 \text{ a.e. on } B_\pm,\ g'(\hat{x}_t) z_t = 0 \text{ on } C\}. \tag{3.4}$$

Note that the strict critical cone is a polyhedron of a closed subspace of $L^\infty \times (W^{1,\infty})^n$. Consider the following *weak complementarity condition*: there exists a Lagrange multiplier $(\beta, \Psi, p, \mathrm{d}\mu) \in \Lambda$ (defined in (2.15)) such that

$$\begin{cases} H_u(\hat{u}, \hat{x}, p) \neq 0, \quad \text{a.e. over } B, \\ \text{the support of } \mathrm{d}\mu \text{ is } C. \end{cases} \tag{3.5}$$

We have the following identity:

Proposition 3.1. *Assume that* (3.5) *holds, then* $\mathcal{C} = \mathcal{C}_S$.

3.2. **Radiality of critical directions.** In view of the definition of *radial critical directions* given in Section A.2 of the Appendix, an element (v, z) of $\mathcal{C}$ is *radial* if and only if, for some $\sigma > 0$, the following conditions are satisfied:

$$u_{\min} \leq \hat{u}_t + \sigma v_t \leq u_{\max}, \quad \text{a.e. on } [0,T], \tag{3.6}$$

$$g(\hat{x}_t) + \sigma g'(\hat{x}_t) z_t \leq 0, \quad \text{for all } t \in [0,T]. \tag{3.7}$$

Recall hypotheses (2.32) and (2.33).

Proposition 3.2. *Any critical direction (in $\mathcal{C}$) is radial.*

Proof. Let $(v, z) \in \mathcal{C}$. Relation (3.6) holds over B arcs, and on S and C arcs by (2.32)(ii). Relation (3.7) trivially holds over C arcs and, since the state constraint is not active over B and S arcs, it does also hold over these arcs, except perhaps in the vicinity of entry or exit points to C arcs. For $t \in [0,T]$, let δ_t denote the distance between t and C. By (2.32)(ii)-(iii) and (2.27), $\frac{\mathrm{d}}{\mathrm{d}t} g(\hat{x}_t)$ has a jump at the entry and exit points of any C arc. Let us check that, for $\varepsilon > 0$ and $\sigma > 0$ small enough, we have $g(\hat{x}_t) + \sigma g'(\hat{x}_t) z_t \leq 0$ for all $t \in [0,T]$ such that $\delta_t \in (0, \varepsilon)$. For such t, reducing $\varepsilon > 0$ if necessary, we have that $g(\hat{x}_t) \leq -c_1 \delta_t$. On the other hand, since $g'(\hat{x}_t) z_t$ is Lipschitz continuous and nonpositive over C arcs, we have that $g'(\hat{x}_t) z_t \leq c_2 \delta_t$, where $c_2 > 0$ depends on v. Therefore, $g(\hat{x}_t) + \sigma g'(\hat{x}_t) z_t \leq (\sigma c_2 - c_1)\delta_t < 0$ as soon as $\sigma < c_1/c_2$. The conclusion follows. ☐

We next give an example of a problem in which the optimal control is not discontinuous at the junction points of C arcs, but whose associated critical cone is nevertheless radial since it reduces to $\{0\}$.

Remark 3.3. Let us consider the problem

$$\begin{aligned} &\min \int_0^2 x_{1,t}\mathrm{d}t, \\ &\dot{x}_{1,t} = u_t \in [-1,1], \ \ \dot{x}_{2,t} = 1, \text{ a.e. on } [0,T], \\ &x_{1,0} = 1, \, x_{2,0} = 0; \ \ -(x_{2,t}-1)^2 - x_{1,t} \leq 0 \text{ over } [0,T]. \end{aligned} \tag{3.8}$$

Notice that $x_{2,t} = t$ and that the state constraint is of first order since $g(x) := -(x_2-1)^2 - x_1$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t}g(x_t) = -2(x_{2,t}-1) - u + t$. Thus, $u_t = -2(t-1)$ on a constraint arc. It is easy to see that the optimal control is of type B_-CB_-. More precisely,

$$\hat{u}_t = \begin{cases} -1, & t \in [0,1], \\ -2(t-1), & t \in (1,3/2], \\ -1, & t \in (3/2,2]. \end{cases} \tag{3.9}$$

Thus, $\hat{u}$ is continuous at the junction time $t = 3/2$. Yet, since no singular arc occurs, the critical cone reduces to $\{0\}$ and, therefore, any critical direction is radial.

3.3. **Statement of second order necessary conditions.** Next we state a second order necessary condition in terms of the Hessian of the Lagrangian $\mathcal{L}$ (which respect to (u,x)), which is given by the quadratic form

$$Q := Q^0 + Q^E + Q^g, \tag{3.10}$$

where

$$\begin{cases} Q^0(v,z,\lambda) := \displaystyle\int_0^T \big(z_t^\top H_{xx}\, z_t + 2v_t H_{ux}\, z_t\big)\mathrm{d}t, \\ Q^E(v,z,\lambda) := D^2\ell^{\beta,\Psi}(z_0,z_T)^2, \\ Q^g(v,z,\lambda) := \displaystyle\int_0^T z_t^\top\, g''\, z_t \mathrm{d}\mu_t, \end{cases} \tag{3.11}$$

We recall that the Lagrangian $\mathcal{L}$ was defined in (2.14).

Proposition 3.4. *For every multiplier* $\lambda \in \Lambda$, *we have*

$$D^2_{(u,x)^2}\mathcal{L}(\hat{u},\hat{x},\lambda)(v,z)^2 = Q(v,z,\lambda), \quad \textit{for all } (v,z) \in \mathcal{C}. \tag{3.12}$$

As a consequence of Proposition 3.2 and Theorem A.9 in the Appendix, the following result holds.

Theorem 3.5 (Second order necessary condition)**.** *Assume that* $(\hat{u},\hat{x})$ *is a weak minimum of problem* (P)*. Then*

$$\max_{\lambda\in\Lambda} Q(v,z,\lambda) \geq 0, \quad \textit{for all } (v,z) \in \mathcal{C}. \tag{3.13}$$

3.4. **Goh transformation and primitives of critical directions.** For $(v,z^0) \in L^\infty \times \mathbb{R}^n$, and $z := z[v,z^0]$ being the solution of the linearized Cauchy problem (2.19)-(2.20), let us set,

$$y_t := \int_0^t v_s \mathrm{d}s; \quad \xi_t := z_t - y_t f_1(\hat{x}_t), \quad \text{for } t \in [0,T]. \tag{3.14}$$

This change of variables is called *Goh transformation* [23]. Defining $E_t := A_t f_1(\hat{x}_t) - \frac{\mathrm{d}}{\mathrm{d}t} f_1(\hat{x}_t)$ (where A was defined in (2.18)), observe that ξ is solution of

$$\dot{\xi}_t = A_t\xi_t + y_t E_t, \tag{3.15}$$

on the interval $[0,T]$, with initial condition

$$\xi_0 = z^0.$$

Consider the set of *strict primitive critical directions*

$$\mathcal{P}_S := \left\{ (y,h,\xi) \in W^{1,\infty} \times \mathbb{R} \times (W^{1,\infty})^n : y_0 = 0,\ y_T = h,\ (\dot{y}, \xi + y f_1) \in \mathcal{C}_S \right\}.$$

The final value of y is involved in the definition since it becomes an independent variable when we consider the closure of $\mathcal{P}_S$ with respect to the $L^2 \times \mathbb{R} \times H^1-$topology. We provide a characterization of this closure in Theorem 3.7.

If $(v,z) \in \mathcal{C}_S$ is a strict critical direction, then (y,ξ) given by (3.14) satisfies the following conditions:

$$\begin{cases} \text{(i)} & g'(\hat{x}_t)(\xi_t + y_t f_1(\hat{x}_t)) = 0 \text{ on } C, \\ \text{(ii)} & y \text{ is constant on each } B \text{ arc}, \end{cases} \tag{3.16}$$

and if (v,z) satisfies the linearized endpoint relations (3.2), then

$$\begin{cases} \phi'(\hat{x}_0, \hat{x}_T)\big(\xi_0, \xi_T + h f_1(\hat{x}_T)\big) \leq 0, \\ \Phi'(\hat{x}_0, \hat{x}_T)\big(\xi_0, \xi_T + h f_1(\hat{x}_T)\big) \in T_\Phi, \end{cases} \tag{3.17}$$

where we set $h := y_T$. In the sequel we let $0 =: \hat{\tau}_0 < \hat{\tau}_1 < \cdots < \hat{\tau}_N := T$ denote the union of the set of junction times with $\{0,T\}$, and we write $\overline{\mathcal{P}_S}$ for the closure of $\mathcal{P}_S$ in $L^2 \times \mathbb{R} \times (H^1)^n$.

A B arc starting at time 0 (respectively ending at time T) is called a $B_{0\pm}$ (respectively $B_{T\pm}$) *arc.*

Proposition 3.6. *Any* $(y,h,\xi) \in \overline{\mathcal{P}_S}$ *verifies* (3.16)-(3.17) *and*

$$\begin{cases} \text{(i)} & y \textit{ is continuous at the } BC, CB \textit{ and } BB \textit{ junctions}, \\ \text{(ii)} & y_t = 0, \textit{ on } B_{0\pm} \textit{ if a } B_{0\pm} \textit{ arc exists}, \\ \text{(iii)} & y_t = h, \textit{ on } B_{T\pm} \textit{ if a } B_{T\pm} \textit{ arc exists}, \\ \text{(iv)} & \lim_{t\uparrow T} y_t = h, \textit{ if } T \in C. \end{cases} \tag{3.18}$$

Proof. Let $(y,h,\xi) \in \overline{\mathcal{P}_S}$ be the limit of a sequence $(y^k, y^k_T, \xi^k)_k \subset \mathcal{P}_S$ in the $L^2 \times \mathbb{R} \times (H^1)^n-$topology. By the Ascoli-Arzelà theorem, if y^k is equi-bounded and equi-Lipschitz continuous over an interval $[a,b]$, then $y^k \to y$ uniformly over $[a,b]$ and the limit function y is Lipschitz continuous on $[a,b]$. Applying this property, we obtain that y is constant over B arcs, and Lipschitz continuous over the union of adjacent BC, CB and BB arcs. We can prove the other statements by an analogous reasoning. □

Define the set

$$\mathcal{P}^2_S := \left\{ (y,h,\xi) \in L^2 \times \mathbb{R} \times (H^1)^n : \ \text{(3.15)-(3.18) hold} \right\}. \tag{3.19}$$

Then the following characterization holds.

Theorem 3.7. *We have that* $\mathcal{P}^2_S = \mathcal{P}_S$.

Proof. That $\overline{\mathcal{P}_S} \subset \mathcal{P}^2_S$ follows from Proposition 3.6. Let us prove the converse inclusion. Define the linear space

$$\mathcal{Z} := \left\{ \begin{array}{l} (y, y_T, \xi) \in W^{1,\infty} \times \mathbb{R} \times \left(\displaystyle\prod_{j=1}^{N} W^{1,\infty}(\hat{\tau}_{j-1}, \hat{\tau}_j)^n \right) \\ y_0 = 0,\ \text{(3.15) holds at each } (\hat{\tau}_{j-1}, \hat{\tau}_j), \text{ and (3.16) holds} \end{array} \right\}, \tag{3.20}$$

that is obtained from $\mathcal{P}_S$ by removing condition (3.17) and allowing ξ to be discontinuous at the junctions $\hat{\tau}_j$, $j=1,\dots N-1$.

Let $(y,h,\xi)\in\mathcal{P}_S^2$. For any $\varepsilon>0$, we now construct $(y_\varepsilon,y_{\varepsilon,T},\xi_\varepsilon)\in\mathcal{Z}$ such that $y_{\varepsilon,T}=h$ and

$$\|y-y_\varepsilon\|_2+\|\xi-\xi_\varepsilon\|_\infty=o(1). \tag{3.21}$$

First set

$$y_{\varepsilon,t}:=y_t,\quad \xi_{\varepsilon,t}:=\xi_t,\quad \text{on } B\cup C. \tag{3.22}$$

On S, let $y_\varepsilon\in W^{1,\infty}(0,T)$ be such that $\|y-y_\varepsilon\|_2<\varepsilon$, the values of y_ε at the junction times being fixed in the following way:

$$\begin{cases} y_\varepsilon(\hat{\tau}_j+)=y(\hat{\tau}_j-), & \text{if } \hat{\tau}_j>0 \text{ is an entry point of an } S \text{ arc},\\ y_\varepsilon(\hat{\tau}_j-)=y(\hat{\tau}_j+), & \text{if } \hat{\tau}_j<T \text{ is an exit point of an } S \text{ arc},\\ y_{\varepsilon,0}=0, \quad \text{if } 0\in S; \quad y_{\varepsilon,T}=h, \quad \text{if } T\in S. \end{cases} \tag{3.23}$$

Such a y_ε exists, see [6, Lemma 8.1]. Define ξ_ε over each S arc by integrating (3.15) over the respective arc with $y=y_\varepsilon$ and the initial condition $\xi_{\varepsilon,\tau}=\xi_\tau$, where τ denotes the entry point of the arc. Then $(y_\varepsilon,y_{\varepsilon,T},\xi_\varepsilon)\in\mathcal{Z}$ satisfies (3.21). In particular, we have

$$|\xi_\varepsilon(\hat{\tau}_j-)-\xi_\varepsilon(\hat{\tau}_j+)|=|\xi_\varepsilon(\hat{\tau}_j-)-\xi(\hat{\tau}_j)|=o(1),\quad \text{for all } j=1,\dots,N-1, \tag{3.24}$$

$$|\varphi'(\hat{x}_0,\hat{x}_T)(\xi_{\varepsilon,0},\xi_{\varepsilon,T}+f_1(\hat{x}_T)y_{\varepsilon,T})-\varphi'(\hat{x}_0,\hat{x}_T)(\xi_0,\xi_T+f_1(\hat{x}_T)h)|=o(1), \tag{3.25}$$

$$|\Phi'(\hat{x}_0,\hat{x}_T)(\xi_{\varepsilon,0},\xi_{\varepsilon,T}+f_1(\hat{x}_T)y_{\varepsilon,T})-\Phi'(\hat{x}_0,\hat{x}_T)(\xi_0,\xi_T+f_1(\hat{x}_T)h)|=o(1). \tag{3.26}$$

Notice that, the cone $\mathcal{P}_S$ is obtained from $\mathcal{Z}$ by adding the constraints

$$\xi(\hat{\tau}_j-)-\xi(\hat{\tau}_j+)=0,\quad \text{for all } j=1,\dots,N-1, \tag{3.27}$$

$$\Phi'(\hat{x}_0,\hat{x}_T)(\xi_0,\xi_T+f_1(\hat{x}_T)h)\in T_\Phi. \tag{3.28}$$

In view of Hoffman's lemma [27] and estimates (3.24)-(3.26), we get that there exists $(\tilde{y}_\varepsilon,\tilde{y}_{\varepsilon,T},\tilde{\xi}_\varepsilon)\in\mathcal{P}_S$ such that

$$\|\tilde{y}_\varepsilon-y_\varepsilon\|_2+\|\tilde{\xi}_\varepsilon-\xi_\varepsilon\|_\infty=o(1). \tag{3.29}$$

Finally, from (3.21) and (3.29) we have that

$$\|\tilde{y}_\varepsilon-y\|_2+\|\tilde{\xi}_\varepsilon-\xi\|_\infty=o(1), \tag{3.30}$$

and hence, the density of $\mathcal{P}_S$ in $\mathcal{P}_S^2$ follows. □

3.5. **Goh transformation on the Hessian of Lagrangian.** Next, we want to express each of the quadratic functions in (3.10)-(3.11) as functions of (y,h,ξ). For the terms that are quadratic in z, it suffices to replace z by $\xi+f_1(\hat{x})y$. With this aim, set for $(y,h,\xi)\in L^2\times\mathbb{R}\times(H^1)^n$ and $\lambda:=(\beta,\Psi,p,\mathrm{d}\mu)\in\Lambda$,

$$\begin{aligned} \Omega_T(y,h,\xi,\lambda) &:= 2h\,H_{ux,T}\xi_T+h\,H_{ux,T}f_1(\hat{x}_T)h,\\ \Omega^0(y,h,\xi,\lambda) &:= \int_0^T \big(\xi_t^\top H_{xx}\xi_t+2y_tM\xi_t+y_tRy_t\big)\mathrm{d}t,\\ \Omega^E(y,h,\xi,\lambda) &:= D^2\ell^{\beta,\Psi}(\xi_0,\xi_T+f_1(\hat{x}_T)h)^2,\\ \Omega^g(y,h,\xi,\lambda) &:= \int_0^T (\xi_t+f_1(\hat{x}_t)y)^\top g''(\hat{x}_t)(\xi_t+f_1(\hat{x}_t)y)\mathrm{d}\mu_t,\\ \Omega &:= \Omega_T+\Omega^0+\Omega^E+\Omega^g, \end{aligned} \tag{3.31}$$

with

$$M := f_1^\top H_{xx} - \dot{H}_{ux} - H_{ux}A, \tag{3.32}$$

$$R := f_1^\top H_{xx} f_1 - 2H_{ux}E - \frac{\mathrm{d}}{\mathrm{d}t}(H_{ux}f_1). \tag{3.33}$$

Here, when we omit the argument of M, R, H or its derivatives, we mean that they are evaluated at $(\hat{x}, \hat{u}, \lambda)$.

Remark 3.8. Easy computations show that R does not depend on u. More precisely, R is given by

$$R(\hat{x}_t, \lambda_t) = p_t\big[[f_0, f_1], f_1\big](\hat{x}_t) + g'(\hat{x}_t) f_1'(\hat{x}_t) f_1(\hat{x}_t)\nu_t, \tag{3.34}$$

where ν is the density of $\mathrm{d}\mu$ given in (2.37).

Proposition 3.9. *Let $(v, z) \in L^2 \times (H^1)^n$ be a solution of* (2.19) *and let (y, ξ) be defined by Goh transformation* (3.14). *Then, for any $\lambda \in \Lambda$,*

$$Q(v, z, \lambda) = \Omega(y, y_T, \xi, \lambda). \tag{3.35}$$

Proof. Take (v, z) and (y, ξ) as in the statement. It is straightforward to prove that

$$Q^E(v, z, \lambda) = \Omega^E(y, y_T, \xi, \lambda), \quad \text{and} \quad Q^g(v, z, \lambda) = \Omega^g(y, y_T, \xi, \lambda). \tag{3.36}$$

In order to prove the equality between Q^0 and Ω^0, let us replace each occurrence of z in Q^0, by its expression in the Goh transformation (3.14), i.e. change z by $\xi + f_1(\hat{x})y$. The first term in Q^0 can be written as

$$\int_0^T z^\top H_{xx} z \mathrm{d}t = \int_0^T (\xi + f_1 y)^\top H_{xx}(\xi + f_1 y)\mathrm{d}t. \tag{3.37}$$

Let us consider the second term in Q^0 :

$$\int_0^T v H_{ux}(\xi + f_1 y)\mathrm{d}t = \int_0^T \big(v H_{ux}\xi + v H_{ux} f_1 y\big)\mathrm{d}t. \tag{3.38}$$

Integrating by parts the first term in previous equation we get

$$\begin{aligned} \int_0^T v H_{ux}\xi \mathrm{d}t &= [y H_{ux}\xi]_0^T - \int_0^T y\big(\dot{H}_{ux}\xi + H_{ux}\dot{\xi}\big)\mathrm{d}t \\ &= y_T H_{ux,T}\xi_T - \int_0^T y\big(\dot{H}_{ux}\xi + H_{ux}A\xi + H_{ux}Ey\big)\mathrm{d}t. \end{aligned} \tag{3.39}$$

For the second term in the right hand-side of (3.38) we obtain

$$\int_0^T v H_{ux} f_1 y \mathrm{d}t = [y H_{ux} f_1 y]_0^T - \int_0^T y\left(\frac{\mathrm{d}}{\mathrm{d}t}(H_{ux}f_1)y + H_{ux}f_1 v\right)\mathrm{d}t. \tag{3.40}$$

This identity yields the following

$$\int_0^T v H_{ux} f_1 y \mathrm{d}t = \tfrac{1}{2} y_T H_{ux,T} f_1 y_T - \tfrac{1}{2}\int_0^T y \frac{\mathrm{d}}{\mathrm{d}t}(H_{ux}f_1) y \,\mathrm{d}t. \tag{3.41}$$

¿From (3.37), (3.39) and (3.41) we get the desired result. □

3.6. **Second order necessary condition in the new variables.** We can obtain the following new necessary condition in the variables after Goh's transformation.

Theorem 3.10. *If* $(\hat{u}, \hat{x})$ *is a weak minimum, then*

$$\max_{\lambda \in \Lambda} \Omega(y, h, \xi, \lambda) \geq 0, \quad \text{for all } (y, h, \xi) \in \mathcal{P}_S^2. \tag{3.42}$$

Proof. Let us assume first that the qualification condition (2.21)(i) does not hold. Therefore, there exists a nonzero element Ψ_E in $[\operatorname{Im} D\bar{\Phi}_E(\hat{u}, \hat{x}_0)]^\perp$. Hence, the multiplier λ composed by such Ψ_E, and having $(\beta, \Psi_I, \mathrm{d}\mu) := 0$ and the associated costate p, is a Lagrange multiplier, as well as its opposite $-\lambda$. Therefore, either $\Omega(y, h, \xi, \lambda)$ or $\Omega(y, h, \xi, -\lambda)$ is greater or equal zero, and the conclusion follows.

Let us now consider th case when (2.21)(i) holds true. Then, the corresponding abstract problem (P_A) (defined in the Appendix A.1) verifies the qualification condition (A.5)(i). Recall the definition of $\hat{\Lambda}$ in (A.10), and the corresponding set $\hat{\Lambda}_1$ (see (2.16)). In view of Theorem A.5, $\hat{\Lambda}_1$ is non empty and bounded. Furthermore, due to Banach-Alaoglu Theorem and since the space of continuous functions in $[0, T]$ is separable, we get that $\hat{\Lambda}_1$ is weakly* sequentially compact.

Consider now $(y, h, \xi) \in \mathcal{P}_S^2$. By Theorem 3.7, there exists a sequence $(y^k, y_T^k, \xi^k) \subset \mathcal{P}_S$ such that

$$(y^k, y_T^k, \xi^k) \to (y, h, \xi), \quad \text{in the } L^2 \times \mathbb{R} \times (H^1)^n\text{-topology.} \tag{3.43}$$

By Proposition 3.9, for all $\lambda \in \Lambda$,

$$\Omega(y^k, y_T^k, \xi^k, \lambda) = Q(v^k, z^k, \lambda), \tag{3.44}$$

where $v^k := \frac{\mathrm{d}}{\mathrm{d}t} y^k$ and $z^k := \xi^k + f_1 y^k$. For each (v^k, z^k), due to Theorem 3.5, there exists $\lambda^k \in \Lambda$ for which

$$Q(v^k, z^k, \lambda^k) \geq 0. \tag{3.45}$$

Let $\hat{\lambda}_k$ be the corresponding element of $\hat{\Lambda}_1$ given by the bijection (A.11), and consider $\bar{\lambda}_k \in \Lambda$ such that $\hat{\lambda}_k = (1, \bar{\lambda}_k)$. Then

$$Q(v^k, z^k, \bar{\lambda}^k) \geq 0, \tag{3.46}$$

in view of (3.45) and since $\bar{\lambda}_k$ is obtained from λ_k by multiplying by a positive scalar. Since $\hat{\Lambda}_1$ is weakly* sequentially compact, then there exists a subsequence $(\hat{\lambda}^{k_j})_j$ weakly* convergent to $\hat{\lambda} = (1, \bar{\lambda}) \in \hat{\Lambda}_1$, where $\bar{\lambda} \in \Lambda$. Thus, $\bar{\lambda}_k \stackrel{*}{\rightharpoonup} \bar{\lambda}$ in Λ and we get

$$\Omega(y, h, \xi, \bar{\lambda}) = \lim_{k \to \infty} \Omega(y^k, y_T^k, \xi^k, \bar{\lambda}^k) = \lim_{k \to \infty} Q(v^k, z^k, \bar{\lambda}^k) \geq 0. \tag{3.47}$$

This concludes the proof.

□

4. Second order sufficient conditions

In this section we show a second order sufficient condition in terms of the uniform positivity of Ω and guaranteeing that the nominal solution $(\hat{u}, \hat{x})$ is a strict Pontryagin minimum whenever this condition is satisfied.

To state the main result of this section (see Theorem 4.5) we need to introduce the following concepts.

Definition 4.1. We say that $(\hat{u}, \hat{x})$ is a *Pontryagin minimum* of problem (P) if for any $M > 0$, there exists $\varepsilon_M > 0$ such that $(\hat{u}, \hat{x})$ is a minimum in the set of feasible trajectories (u, x) satisfying

$$\|x - \hat{x}\|_\infty + \|u - \hat{u}\|_1 < \varepsilon_M, \quad \|u - \hat{u}\|_\infty < M. \tag{4.1}$$

A sequence $(v_k) \subset L^\infty$ is said to *converge to 0 in the Pontryagin sense* if $\|v_k\|_1 \to 0$ and there exists $M > 0$ such that $\|v_k\|_\infty \leq M$, for all $k \in \mathbb{N}$.

Definition 4.2. Let us define the function $\gamma : L^2 \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, given by

$$\gamma(y, h, x_0) := \int_0^T y_t^2 \mathrm{d}t + h^2 + |x_0|^2. \tag{4.2}$$

We say that $(\hat{u}, \hat{x})$ satisfies the $\gamma-$*growth condition in the Pontryagin sense* if there exists $\rho > 0$ such that, for every sequence of feasible variations $(v_k, \delta x_k)$ having (v_k) convergent to 0 in the Pontryagin sense and $\delta x_{k,0} \to 0$, one has

$$J(\hat{u} + v_k) - J(\hat{u}) \geq \rho\gamma(y_k, y_{k,T}, \delta x_{k,0}), \tag{4.3}$$

for k large enough, where $y_{k,t} := \int_0^t v_{k,s}\mathrm{d}s$, for $t \in [0, T]$.

Definition 4.3. We say that $(\hat{u}, \hat{x})$ satisfies *strict complementarity condition for the control constraints* if the following conditions hold:

(i) $\max_{\lambda \in \Lambda} H_u(\hat{u}_t, \hat{x}_t, p_t) > 0$, for all t in the interior of B_-,
(ii) $\min_{\lambda \in \Lambda} H_u(\hat{u}_t, \hat{x}_t, p_t) < 0$, for all t in the interior of B_+,
(iii) $\max_{\lambda \in \Lambda} H_u(\hat{u}_0, \hat{x}_0, p_0) > 0$, if $0 \in B_-$, and $\max_{\lambda \in \Lambda} H_u(\hat{u}_T, \hat{x}_T, p_T) > 0$, if $T \in B_-$, $\min_{\lambda \in \Lambda} H_u(\hat{u}_0, \hat{x}_0, p_0) < 0$, if $0 \in B_+$, and $\min_{\lambda \in \Lambda} H_u(\hat{u}_T, \hat{x}_T, p_T) > 0$, if $T \in B_+$.

Consider the following *extended cones of critical directions* (compare to (3.19)):

$$\hat{\mathcal{P}}^2 := \{(y, h, \xi) \in L^2 \times \mathbb{R} \times (H^1)^n : \text{(3.15)-(3.17), (3.18)(ii)-(iii) hold}\} \tag{4.4}$$

$$\hat{\hat{\mathcal{P}}}^2 := \{(y, h, \xi) \in \hat{\mathcal{P}}^2 : \text{(3.18)(iv) holds}\}, \tag{4.5}$$

and

$$\mathcal{P}_*^2 := \begin{cases} \hat{\hat{\mathcal{P}}}^2, & \text{if } T \in C \text{ and } [\mu(T)] > 0, \text{ for some } (\beta, \Psi, p, \mathrm{d}\mu) \in \Lambda, \\ \hat{\mathcal{P}}^2, & \text{otherwise.} \end{cases} \tag{4.6}$$

Let us recal the definition of Legendre form (see e.g. [26]):

Definition 4.4. Let W be a Hilbert space. We say that a quadratic mapping $\mathcal{Q} : W \to \mathbb{R}$ is a *Legendre form* if it is sequentially weakly lower semi continuous over W, such that, if $w_k \to w$ weakly in W and $\mathcal{Q}(w_k) \to \mathcal{Q}(w)$, then $w_k \to w$ strongly.

Theorem 4.5. *Suppose that the following conditions hold true:*

(i) $(\hat{u}, \hat{x})$ *satisfies strict complementarity for the control constraint,*
(ii) $\beta > 0$, *for each* $(\beta, \Psi, p, \mathrm{d}\mu) \in \Lambda$,
(iii) *for each* $\lambda \in \Lambda$, $\Omega(\cdot, \lambda)$ *is a Legendre form in* $\{(y, h, \xi) \in L^2 \times \mathbb{R} \times (H^1)^n :$ (3.15) *holds*$\}$ *and there exists* $\rho > 0$ *such that*

$$\max_{\lambda \in \Lambda} \Omega(y, h, \xi, \lambda) \geq \rho\gamma(y, h, \xi_0), \quad \text{for all } (y, h, \xi) \in \mathcal{P}_*^2. \tag{4.7}$$

Then $(\hat{u}, \hat{x})$ *is a Pontryagin minimum satisfying* $\gamma-$*growth.*

Remark 4.6. A necessary and sufficient condition for first assertion in item (iii) of Theorem 4.5 to hold is that there exists $\alpha>0$, such that, for every $\lambda\in\Lambda$,

$$R(\hat{x}_t,\lambda_t)+f_1(\hat{x}_t)^\top g''(\hat{x}_t)f_1(\hat{x}_t)\nu_t>\alpha,\quad \text{on } [0,T], \tag{4.8}$$

where ν is the density of $\mathrm{d}\mu$, which vanishes on $[0,T]\backslash C$, and on the set C is given by the expression (2.37). See [26] for more details.

For the lemma below recall the definition of $\mathcal{L}$ which was given in (2.14).

Lemma 4.7. *Let $(u,x)\in L^2\times(H^1)^n$ be a solution of* (2.1)*, and set $(\delta x,\delta u):=(x,u)-(\hat{x},\hat{u})$. Then, the following expression for the Lagrangian function holds for every multiplier $\lambda\in\Lambda$,*

$$\mathcal{L}(u,x,\lambda)=\mathcal{L}(\hat{u},\hat{x},\lambda)+\tilde{Q}(\delta u,\delta x,\lambda)+r(\delta u,\delta x,\lambda), \tag{4.9}$$

where

$$\tilde{Q}(\delta u,\delta x,\lambda):=$$

$$\tfrac12 D^2\ell^{\beta,\Psi}(\delta x_0,\delta x_T)^2+\int_0^T\left\{H_u\delta u+\tfrac12 H_{xx}\delta x^2+H_{ux}(\delta u,\delta x)\right\}\mathrm{d}t+\tfrac12\int_{[0,T]}g''\delta x^2\mathrm{d}\mu,$$

and

$$\begin{aligned} r(\delta u,\delta x)&:=\mathcal{O}\big(|(\delta x_0,\delta x_T)|^3\big)\\ &+\int_0^T\left\{\tfrac12 H_{uxx}(\delta u,\delta x,\delta x)+p\sum(\hat{u}_i+\delta u_i)\mathcal{O}\big(|\delta x|^3\big)\right\}\mathrm{d}t+\int_{[0,T]}\mathcal{O}\big(|\delta x|^3\big)\mathrm{d}\mu.\end{aligned}$$

Proof. Let us consider the following second order Taylor expansions, written in compact form,

$$\begin{aligned}\ell^{\beta,\Psi}(x_0,x_T)&=\ell^{\beta,\Psi}+D\ell^{\beta,\Psi}(\delta x_0,\delta x_T)+\tfrac12 D^2\ell^{\beta,\Psi}(\delta x_0,\delta x_T)^2+\mathcal{O}\big(|(\delta x_0,\delta x_T)|^3,\\ f_i(x_t)&=f_{i,t}+Df_i\delta x_T+\tfrac12 D^2f_i\delta x_t^2+\mathcal{O}\big(|\delta x|^3\big),\\ g(x_t)&=\tfrac12 g+Dg\delta x_t+\tfrac12 g''\delta x_t^2+\mathcal{O}\big(|\delta x|^3\big).\end{aligned} \tag{4.10}$$

Observe that

$$D\ell^{\beta,\Psi}(\delta x_0,\delta x_T)=[p\,\delta x]_0^T=\int_0^T p(-f_x\delta x+\dot{\delta x})\mathrm{d}t-\int_0^T g'(\hat{x})\delta x\mathrm{d}\mu. \tag{4.11}$$

Using the identities (4.10) and (4.11) in the following equation

$$\mathcal{L}(u,x,\lambda)=\ell^{\beta,\Psi}(x_0,x_T)+\int_0^T p\big(\sum u_if_i-\dot{x}\big)\mathrm{d}t+\int_0^T g(x)\mathrm{d}\mu, \tag{4.12}$$

we obtain the desired expression for $\mathcal{L}(u,x,\lambda)$. The result follows. □

In view of previous Lemma 4.7 and [6, Lemma 8.4] we can prove the following:

Proposition 4.8. *Let (v_k) be a sequence converging to 0 in the Pontryagin sense and $(x_{k,0})$ a sequence in $\mathbb{R}^n$ converging to $\hat{x}_0$. Set $u_k:=\hat{u}+v_k$, and let x_k be the corresponding solution of equation* (2.1) *with initial value equal to $x_{k,0}$. Then, for every $\lambda\in\Lambda$, one has*

$$\mathcal{L}(u_k,x_k,\lambda)=\mathcal{L}(\hat{u},\hat{x},\lambda)+\int_0^T H_u(t)v_{k,t}dt+Q(v_k,z_k,\lambda)+o(\gamma_k), \tag{4.13}$$

where $z_k := z[v_k, x_{k,0} - \hat{x}_0]$, $y_{k,t} := \int_0^t v_{k,s}\mathrm{d}s$ *and* $\gamma_k := \gamma(y_k, y_{k,T}, x_{k,0} - \hat{x}_0)$.

Remark 4.9. Notice that Lemma 8.4 in [6] was proved for γ depending only on (y,h) since the initial value x_0 was fixed throughout the article, while here γ depend also on the initial variation of the state. The extension of Lemmas 8.3 and 8.4 in [6] for the case with variable initial state are immediate, and the proofs are given in detail in [5].

Proposition 4.10. *Let* $(p, \mathrm{d}\mu) \in (BV)^{n*} \times \mathcal{M}$ *verify* (2.10)-(2.11), *and let* $(v,z) \in L^2 \times (H^1)^n$ *satisfy* (2.19). *Then,*

$$\int_{[0,T]} g'(\hat{x}_t) z_t \mathrm{d}\mu_t + D\ell^{\beta,\Psi}(z_0, z_T) = \int_0^T H_u(t) v_t \mathrm{d}t. \tag{4.14}$$

Proof. Note that

$$\begin{aligned} \int_{[0,T]} g'(\hat{x}_t) z_t \mathrm{d}\mu_t &= -\int_0^T \mathrm{d}p_t\, z_t \mathrm{d}t - \int_0^T p_t f_x(\hat{u}_t, \hat{x}_t) z_t \mathrm{d}t \\ &= -\int_0^T \mathrm{d}p_t\, z_t \mathrm{d}t - \int_0^T p_t(\dot{z}_t - f_1(\hat{x}_t) v_t) \mathrm{d}t \\ &= -[pz]_{0-}^{T+} + \int_0^T H_u(t) v_t \mathrm{d}t. \end{aligned} \tag{4.15}$$

The wanted result follows from latter equation and (2.11). □

Proof of Theorem 4.5. If the conclusion does not hold, there should exist a sequence $(v_k, x_{k,0}) \subset L^\infty \times \mathbb{R}^n$ of non identically zero functions having (v_k) converging to 0 in the Pontryagin sense and $x_{k,0} \to \hat{x}_0$, and such that, setting $u_k := \hat{u} + v_k$, the corresponding solutions x_k of (2.1) are feasible and

$$J(u_k) \leq J(\hat{u}) + o(\gamma_k), \tag{4.16}$$

where $y_{k,t} := \int_0^t v_{k,s}\mathrm{d}s$ and $\gamma_k := \gamma(y_k, y_{k,T}, x_{k,0} - \hat{x}_0)$. Set $z_k := z[v_k, z_{k,0}]$ with $z_{k,0} := x_{k,0} - \hat{x}_0$. Take any $\lambda \in \Lambda$, and then multiply inequality (4.16) by β (which is positive in view of assumption (ii) of the present theorem), afterwards add the non-positive term $\Psi \cdot \Phi(x_{k,0}, x_{k,T}) + \int_{[0,T]} g(x_k)\mathrm{d}\mu$ to the left hand-side of the resulting inequality, and obtain the following:

$$\mathcal{L}(u_k, x_k, \lambda) \leq \mathcal{L}(\hat{u}, \hat{x}, \lambda) + o(\gamma_k). \tag{4.17}$$

Set $(\bar{y}_k, \bar{h}_k, \bar{\xi}_{k,0}) := (y_k, y_{k,T}, z_{k,0})/\sqrt{\gamma_k}$. Note that the elements of this sequence have unit norm in $L^2 \times \mathbb{R} \times \mathbb{R}^n$. By the Banach-Alaoglu Theorem, extracting if necessary a sequence, we may assume that there exists $(\bar{y}, \bar{h}, \bar{\xi}_0) \in L^2 \times \mathbb{R} \times \mathbb{R}^n$ such that

$$\bar{y}_k \rightharpoonup \bar{y}, \text{ and } (\bar{h}_k, \bar{\xi}_{k,0}) \to (\bar{h}, \bar{\xi}_0), \tag{4.18}$$

where the first limit is given by the weak topology of L^2.

We shall next prove that $(\bar{y}, \bar{h}, \bar{\xi})$ is in $\mathcal{P}_*^2$, where $\bar{\xi}$ is the solution of (3.15) associated to $\bar{y}$ and the initial condition $\bar{\xi}_0$.

Recall the definition of $\mathcal{P}_*^2$ given in equation (4.6). In the sequel we show that $(\bar{y}, \bar{h}, \bar{\xi})$ verifies (3.16)(ii). ¿From (4.13), (4.17) and the equivalence between Q and Ω stated in Proposition 3.9, it follows that

$$-\Omega(\xi_k, y_k, h_k, \lambda) + o(\gamma_k) \geq \int_0^T H_u(t) v_{k,t} \mathrm{d}t \geq 0, \tag{4.19}$$

where ξ_k is solution of (3.15) corresponding to y_k. The last inequality in (4.19) holds in view of (2.12) and since $\hat{u} + v_k$ satisfies the control constraint (2.4). By the continuity of the mapping Ω over the space $L^2 \times \mathbb{R} \times (H^1)^n$ and from (4.19) we deduce that

$$0 \leq \int_0^T H_u(t) v_{k,t} \mathrm{d}t \leq O(\gamma_k).$$

Hence, since the integrand in previous inequality is nonnegative for all $k \in \mathbb{N}$, we have that

$$\lim_{k\to\infty} \int_0^T H_u(t)\varphi_t \frac{v_{k,t}}{\sqrt{\gamma_k}} dt = 0, \tag{4.20}$$

for any nonnegative Lipschitz continuous function $\varphi : [0,T] \to \mathbb{R}$. Let us consider, in particular, such a function φ having its support included in a maximal interval (c,d) where the control constraint is active. Integrating by parts in (4.20) and in view of (4.18), we obtain

$$0 = \lim_{k\to\infty} \int_c^d \frac{\mathrm{d}}{\mathrm{d}t}\left(H_u(t)\varphi_t\right) \bar{y}_{k,t} \mathrm{d}t = \int_c^d \frac{\mathrm{d}}{\mathrm{d}t}\left(H_u(t)\varphi_t\right) \bar{y}_t \mathrm{d}t.$$

Over a B arc, v_k has constant sign and therefore, $\bar{y}$ is either nondecreasing or nonincreasing on (c,d). Thus, we can integrate by parts in the previous equation to get

$$\int_c^d H_u(t)\varphi_t \mathrm{d}\bar{y}_t = 0. \tag{4.21}$$

Take $t_0 \in (c,d)$. By the strict complementary condition for the control constraint assumed here (see Definition 4.3), there exists $\hat{\lambda} = (\hat{\beta}, \hat{\Psi}, \hat{p}, \mathrm{d}\hat{\mu}) \in \Lambda$ such that $H_u(\hat{u}_{t_0}, \hat{x}_{t_0}, \hat{p}_{t_0}) > 0$. Hence, in view of the continuity of H_u on B,[1] there exists $\varepsilon > 0$ such that $H_u(\hat{u}_t, \hat{x}_t, \hat{p}_t) > 0$ on $(t_0 - 2\varepsilon, t_0 + 2\varepsilon) \subset (c,d)$. Choose φ such that $\operatorname{supp}\varphi \subset (t_0 - 2\varepsilon, t_0 + 2\varepsilon)$, and $H_u(\hat{u}_t, \hat{x}_t, \hat{p}_t)\varphi_t = 1$ on $(t_0 - \varepsilon, t_0 + \varepsilon)$. Since $\mathrm{d}\bar{y} \geq 0$, equation (4.21) yields

$$\begin{aligned} 0 &= \int_c^d H_u(\hat{u}_t, \hat{x}_t, \hat{p}_t)\varphi_t \mathrm{d}\bar{y}_t \geq \int_{t_0-\varepsilon}^{t_0+\varepsilon} H_u(\hat{u}_t, \hat{x}_t, \hat{p}_t)\varphi_t \mathrm{d}\bar{y}_t \\ &= \int_{t_0-\varepsilon}^{t_0+\varepsilon} \mathrm{d}\bar{y}_t = \bar{y}_{t_0+\varepsilon} - \bar{y}_{t_0-\varepsilon}. \end{aligned} \tag{4.22}$$

As both ε and $t_0 \in (c,d)$ are arbitrary we deduce that

$$\mathrm{d}\bar{y}_t = 0, \quad \text{on } B. \tag{4.23}$$

Hence, $(\bar{y}, \bar{h}, \bar{\xi})$ satisfies (3.16)(ii).

Assume now that a $B_{0\pm}$ arc exists, and let us prove that $\bar{y} = 0$ on $B_{0\pm}$ (see the definition of $B_{0\pm}$ in the paragraph preceding Proposition 3.6). Let $B_{0\pm}$ be equal to $[0,t_1]$ for some $t_1 > 0$. Assume without loss of generality that $\hat{u} = u_{\min}$ on $[0,t_1]$. Notice that by the strict complementarity condition for the control constraint (i.e. condition (i) of the present theorem) there exists $\lambda' = (\beta', \Psi', p', \mathrm{d}\mu') \in \Lambda$ and $\varepsilon, \delta > 0$ such that $H_u(\hat{u}_t, \hat{x}_t, p'_t) > \delta$ for all $t \in [0,\varepsilon]$, and thus, by considering in (4.20) a nonnegative Lipschitz continuous function $\varphi : [0,T] \to \mathbb{R}$ being equal to

[1] Actually H_u continuous on B since p does not jump on B.

$1/\delta$ on $[0,t]$, we obtain $\bar{y}_{k,t} = \displaystyle\int_0^t \frac{v_{k,s}}{\sqrt{\gamma_k}} \mathrm{d}s \to 0$, since $v_k \geq 0$ on $[0,t_1]$. Hence $\bar{y} = 0$ on $[0,\varepsilon]$. This last assertion, together with (4.23), imply that $\bar{y} = 0$ on $B_{0\pm}$.

Suppose that T is in a boundary arc $B_{T\pm}$. Let $B_{T\pm} = [t_N, T]$. Then, we can derive that for some $\varepsilon > 0$, $\bar{y}_{k,T} - \bar{y}_{k,t} = \displaystyle\int_t^T \frac{v_{k,s}}{\sqrt{\gamma_k}} \mathrm{d}s \to 0$ for all $t \in [T-\varepsilon, T]$, by an argument analogous to the one above. Thus, $\bar{y}_t = \bar{h}$ on $[T-\varepsilon, T]$, and hence, by (4.23) we get that

$$\bar{y} = \bar{h}, \text{ on } (t_N, T], \text{ if } T \in B. \tag{4.24}$$

Therefore, $(\bar{y}, \bar{h}, \bar{\xi})$ verifies (3.18)(ii)-(iii)

Let us prove that $(\bar{y}, \bar{h}, \bar{\xi})$ satisfies (3.16)(i). For all $t \in [0,T]$, a first order Taylor expansion gives

$$0 \geq g(x_{k,t}) = g(\hat{x}_t) + g'(\hat{x}_t)\delta x_{k,t} + O(|\delta x_{k,t}|^2). \tag{4.25}$$

¿From latter estimate and [6, Lemma 8.12], we deduce that

$$g'(\hat{x}_t) z_{k,t} \leq o(\sqrt{\gamma}_k) + O(|\delta x_{k,t}|^2), \quad \text{for all } t \in C. \tag{4.26}$$

Let $\varphi \geq 0$ be some continuous function with support in C. ¿From (4.26), we get that

$$\begin{aligned} \int_0^T \varphi_t g'(\hat{x}_t)(\xi_{k,t} + f_1(\hat{x}_t) y_{k,t}) \mathrm{d}t &= \int_0^T \varphi_t g'(\hat{x}_t) z_{k,t} \mathrm{d}t \\ &\leq \|\varphi\|_\infty \int_0^T \big(o(\sqrt{\gamma}_k) + O(|\delta x_{k,t}|^2)\big) \mathrm{d}t \leq o(\sqrt{\gamma}_k), \end{aligned} \tag{4.27}$$

where the last inequality follows from [6, Lemma 8.4]. Therefore, dividing by $\sqrt{\gamma_k}$ and passing to the limit, we obtain

$$\int_0^T \varphi_t g'(\hat{x}_t)(\bar{\xi}_t + f_1(\hat{x}_t)\bar{y}_t) \mathrm{d}t \leq 0. \tag{4.28}$$

Since φ is an arbitrary nonnegative continuous function with support in C (and C is a finite union of intervals), we deduce that

$$g'(\hat{x}_t)(\bar{\xi}_t + f_1(\hat{x}_t)\bar{y}_t) \leq 0, \quad \text{for a.a. } t \in C. \tag{4.29}$$

In view of (4.26), we have that

$$g'(\hat{x}_T)(\bar{\xi}_{k,T} + f_1(\hat{x}_T)\bar{y}_{k,T}) \leq o(1). \tag{4.30}$$

Thus,

$$g'(\hat{x}_T)(\bar{\xi}_T + f_1(\hat{x}_T)\bar{h}) \leq 0. \tag{4.31}$$

Take $(\beta, \Psi, p, \mathrm{d}\mu) \in \Lambda$. In view of Proposition 4.10 and since $u_{\min} \leq \hat{u} + v_k \leq u_{\max}$, we have

$$D\ell^{\beta,\Psi}(\hat{x}_0, \hat{x}_T)(z_{k,0}, z_{k,T}) + \int_{[0,T]} g'(\hat{x}_t) z_{k,t} \mathrm{d}\mu_t = \int_0^T H_u(t) v_{k,t} \mathrm{d}t \geq 0. \tag{4.32}$$

On the other hand, a first order Taylor expansion of $\ell^{\beta,\Psi}$ and [6, Lemma 8.12] lead to

$$D\ell^{\beta,\Psi}(\hat{x}_0, \hat{x}_T)(z_{k,0}, z_{k,T}) = \ell^{\beta,\Psi}(x_{k,0}, x_{k,T}) - \ell^{\beta,\Psi}(\hat{x}_0, \hat{x}_T) + o(\sqrt{\gamma}_k). \tag{4.33}$$

Hence, by (4.32),

$$
\begin{aligned}
0 &\leq \ell^{\beta,\Psi}(x_{k,0},x_{k,T}) - \ell^{\beta,\Psi}(\hat{x}_0,\hat{x}_T) + o(\sqrt{\gamma_k}) + \int_{[0,T]} g'(\hat{x}_t) z_{k,t}\mathrm{d}\mu_t \\
&\leq \beta\phi(x_{k,0},x_{k,T}) - \beta\phi(\hat{x}_0,\hat{x}_T) + o(\sqrt{\gamma_k}) + \int_{[0,T]} g'(\hat{x}_t) z_{k,t}\mathrm{d}\mu_t,
\end{aligned}
\tag{4.34}
$$

where the last inequality holds since $\sum_{i=n_1+1}^{n_1+n_2} \Psi_i \Phi_i(x_{k,0},x_{k,T}) \leq 0$. Observe now that, due to (4.16), $\beta\phi(x_{k,0},x_{k,T}) - \beta\phi(\hat{x}_0,\hat{x}_T) \leq o(\gamma_k)$. Hence, by latter estimate and from (4.34), we deduce that

$$
\frac{1}{\sqrt{\gamma_k}} \int_{[0,T]} g'(\hat{x}_t) z_{k,t}\mathrm{d}\mu_t = \int_{[0,T]} g'(\hat{x}_t)(\bar{\xi}_{k,t} + f_1(\hat{x}_t)\bar{y}_{k,t})\mathrm{d}\mu_t \geq o(1).
\tag{4.35}
$$

Since $\mathrm{d}\mu_t$ has an essentially bounded density over $[0,T)$, we have that

$$
\begin{aligned}
0 &\leq \liminf \int_{[0,T]} g'(\hat{x}_t)(\bar{\xi}_{k,t} + f_1(\hat{x}_t)\bar{y}_{k,t})\mathrm{d}\mu_t \\
&= \lim \Big(\int_{[0,T)} g'(\hat{x}_t)(\bar{\xi}_{k,t} + f_1(\hat{x}_t)\bar{y}_{k,t})\mathrm{d}\mu_t + g'(\hat{x}_T)(\bar{\xi}_{k,T} + f_1(\hat{x}_T)\bar{y}_{k,T})[\mu(T)]\Big) \\
&= \int_{[0,T)} g'(\hat{x}_t)(\bar{\xi}_t + f_1(\hat{x}_t)\bar{y}_t)\mathrm{d}\mu_t + g'(\hat{x}_T)(\bar{\xi}_T + f_1(\hat{x}_T)\bar{h})[\mu(T)].
\end{aligned}
\tag{4.36}
$$

In view of (4.29), (4.31) and the complementary condition (3.5), we get from (4.36) that

$$
g'(\hat{x}_t)(\bar{\xi}_t + f_1(\hat{x}_t)\bar{y}_t) = 0, \quad \text{for a.a. } t \in C,
\tag{4.37}
$$

and

$$
g'(\hat{x}_T)(\bar{\xi}_T + f_1(\hat{x}_T)\bar{h}) = 0, \quad \text{whenever } [\mu(T)] > 0.
\tag{4.38}
$$

Then, $(\bar{y},\bar{h},\bar{\xi})$ satisfies (3.16)(i) and

$$
\bar{h} = -\frac{g'(\hat{x}_T)\bar{\xi}_T}{g'(\hat{x}_T)f_1(\hat{x}_T)} = \lim_{t\uparrow T} -\frac{g'(\hat{x}_t)\bar{\xi}_t}{g'(\hat{x}_t)f_1(\hat{x}_t)} = \lim_{t\uparrow T} \bar{y}(t),
\tag{4.39}
$$

if $[\mu(T)] > 0$ for some $(\beta,\Psi,p,\mathrm{d}\mu) \in \Lambda$, so that, in this case, (3.18)(iv) holds.

We shall now prove (3.17). Let $i = 1,\dots,n_1+n_2$, then

$$
\begin{aligned}
\Phi_i'(\hat{x}_0,\hat{x}_T)(\bar{\xi}_0,\bar{\xi}_T + f_1(\hat{x}_T)\bar{h}) &= \lim \Phi_i'(\hat{x}_0,\hat{x}_T)\frac{(\xi_{k,0},\xi_{k,T} + f_1(\hat{x}_T)y_{k,T})}{\sqrt{\gamma_k}} \\
&= \lim \Phi_i'(\hat{x}_0,\hat{x}_T)\frac{(z_{k,0},z_{k,T})}{\sqrt{\gamma_k}}.
\end{aligned}
\tag{4.40}
$$

A first order Taylor expansion of Φ_i at $(\hat{x}_0,\hat{x}_T)$ and [6, Lemma 8.12] yield

$$
\Phi_i'(\hat{x}_0,\hat{x}_T)\frac{(z_{k,0},z_{k,T})}{\sqrt{\gamma_k}} = \frac{\Phi_i(x_{k,0},x_{k,T}) - \Phi_i(\hat{x}_0,\hat{x}_T)}{\sqrt{\gamma_k}} + o(1).
\tag{4.41}
$$

Thus, from (4.40)-(4.41) we get

$$
\Phi_i'(\hat{x}_0,\hat{x}_T)(\bar{\xi}_0,\bar{\xi}_T + f_1(\hat{x}_T)\bar{h}) = 0, \quad \text{for } i = 1,\dots,n_1,
$$

$$
\Phi_i'(\hat{x}_0,\hat{x}_T)(\bar{\xi}_0,\bar{\xi}_T + f_1(\hat{x}_T)\bar{h}) \leq 0, \text{ for } i = n_1+1,\dots,n_1+n_2, \text{ with } \Phi_i(\hat{x}_0,\hat{x}_T) = 0.
$$

For the endpoint cost, we can obtain expressions analogous to (4.40)-(4.41), and then, from (4.16) we get

$$\phi'(\hat{x}_0,\hat{x}_T)(\bar{\xi}_0,\bar{\xi}_T+f_1(\hat{x}_T)\bar{h})\leq 0. \tag{4.42}$$

Hence, (3.17) is verified.

We conclude that $(\bar{y},\bar{h},\bar{\xi})\in\mathcal{P}^2_*$.

¿From equation (4.13) in Proposition 4.8 we obtain that

$$\Omega(y_k,h_k,\xi_k,\lambda)=\mathcal{L}(u_k,x_k,\lambda)-\mathcal{L}(\hat{u},\hat{x},\lambda)-\int_0^T H_u(t)v_{k,t}\mathrm{d}t-o(\gamma_k)\leq o(\gamma_k), \tag{4.43}$$

where the last inequality follows from (4.17) and since $H_u(t)v_{k,t}\geq 0$, a.e. on $[0,T]$. Hence,

$$\liminf_{k\to\infty}\Omega(y_k,h_k,\xi_k,\lambda)\leq\limsup_{k\to\infty}\Omega(y_k,h_k,\xi_k,\lambda)\leq 0. \tag{4.44}$$

Let us recall that, for each $\lambda\in\Lambda$, the mapping $\Omega(\cdot,\lambda)$ is a Legendre form in the Hilbert space $\{(y,h,\xi)\in L^2\times\mathbb{R}\times(H^1)^n : (3.15)\text{ holds}\}$ (in view of hypothesis (iii) of the current theorem). In particular, for $\bar{\lambda}\in\Lambda$ reaching the maximum in (4.7) for the critical direction $(\bar{y},\bar{h},\bar{\xi})$, one has

$$\rho\gamma(\bar{y},\bar{h},\bar{\xi}_0)\leq\Omega(\bar{y},\bar{h},\bar{\xi},\bar{\lambda})=\liminf\Omega(\bar{y}_k,\bar{h}_k,\bar{\xi}_k,\bar{\lambda})\leq 0, \tag{4.45}$$

where the equality holds since Ω is a Legendre form and the last inequality follows from (4.44). Hence, in view of (4.45) we get that $(\bar{y},\bar{h},\bar{\xi}_0)=0$ and $\lim\Omega(\bar{y}_k,\bar{h}_k,\bar{\xi}_k,\bar{\lambda})=0$. Consequently, $(\bar{y}_k,\bar{h}_k,\bar{\xi}_{k,0})$ converges strongly to $(\bar{y},\bar{h},\bar{\xi}_0)=0$, which is a contradiction since $(\bar{y}_k,\bar{h}_k,\bar{\xi}_{k,0})$ has unit norm in $L^2\times\mathbb{R}\times\mathbb{R}^n$. We conclude that $(\hat{u},\hat{x})$ is a Pontryagin minimum satisfying $\gamma-$growth in the Pontryagin sense. □

5. Shooting formulation

Here we aim to use the necessary conditions provided by the PMP to write a numerical scheme that allows us to calculate the candidates for optimal solution. The idea is to use optimality conditions to write the control as a function of the state and the costate, and then to integrate a differential system written only in terms of the latter variables. The main feature of this shooting formulation is that the value of the control is computed differently depending on the type of arc. The technique used to compute the optimal candidates is as explained next. We first estimate the structure of the control, i.e. the concatenation of different type of arcs that form the control and the approximate values of the junction times. This is done in practice by some *direct method* such as solving the NLP associated to the corresponding discretized problem. Afterwards, we write the state and costate equations in which we eliminate the control by using a different formula on each sub arc. The latter equations are completely determined by the values of the junction times, the initial values of x and p and the magnitude of the jump of the multiplier $\mathrm{d}\mu$ whenever the final time T is in a constrained arc.

We consider the problem

(P') $\quad \min\phi(x_0,x_1);\ \dot{x}=f_0(x)+uf_1(x),\ 0\leq u\leq 1,\ \Phi(x_0,x_1)=0,\ g(x)\leq 0,$

which is obtained from (P) by removing the inequality endpoint constraints. We set $u_{\min}:=0$ and $u_{\max}:=1$ for the sake of simplicity.

Recall the concept of Pontryagin minimum given in Definition 4.1. We want to provide a numerical scheme for approximating Pontryagin minima of (P').

Assume for the remainder of the section that $(\hat{u}, \hat{x})$ *is a Pontryagin minimum for* (P') *that verifies* (2.32) *and* (2.33).

5.1. **Reformulation of the problem.** We make a transformation of the optimal control problem (P') in the spirit of [7], by taking as decision variables (state variables) the entry and exit times of boundary arcs. We aim to prove that any Pontryagin minimum for the original problem can be transformed into a weak minimum of an unconstrained transformed problem, in which the switching times are optimization parameters.

Consider a Pontryagin minimum $(\hat{u}, \hat{x})$ of (P'), and let

$$0 =: \hat{\tau}_0 < \hat{\tau}_1 < \cdots < \hat{\tau}_N := T \tag{5.1}$$

denote its associated switching times. Recall the definition of the sets C, B_-, B_+ and S given in Section 2.4 above. Set $\hat{I}_k := [\hat{\tau}_{k-1}, \hat{\tau}_k]$, for $k = 1, \dots, N$, and

$$\mathcal{I}(S) := \{k = 1, \dots, N : \hat{I}_k \text{ is a singular arc}\}. \tag{5.2}$$

Analogously, define $\mathcal{I}(C), \mathcal{I}(B_-), \mathcal{I}(B_+)$. For each $k = 1, \dots, N$, consider a state variable $x^k \in (W^{1,\infty})^n$, and for each $k \in \mathcal{I}(S)$, consider the control variable $u^k \in L^\infty$.

In the following reformulation of problem (P') we consider a control variable for each singular arc of $\hat{u}$. On the set B, we fix the control to the corresponding bound. On the other hand, from (2.35) we get that, on C, we have $\hat{u}_t = \Gamma(\hat{x}_t)$ where Γ is a function from $\mathbb{R}^n$ to $\mathbb{R}$ given by

$$\Gamma(x) := -\frac{g'(x) f_0(x)}{g'(x) f_1(x)}.$$

Consider the optimal control problem (TP') on the interval $[0, 1]$ given by

$$\min \phi(x_0^1, x_1^N), \tag{5.3}$$

$$\dot{x}^k = (\tau_k - \tau_{k-1})(f_0(x^k) + u^k f_1(x^k)), \quad \text{for } k \in \mathcal{I}(S), \tag{5.4}$$

$$\dot{x}^k = (\tau_k - \tau_{k-1}) f_0(x^k), \quad \text{for } k \in \mathcal{I}(B_-), \tag{5.5}$$

$$\dot{x}^k = (\tau_k - \tau_{k-1})(f_0(x^k) + f_1(x^k)), \quad \text{for } k \in \mathcal{I}(B_+), \tag{5.6}$$

$$\dot{x}^k = (\tau_k - \tau_{k-1}) \left(f_0(x^k) + \Gamma(x^k) f_1(x^k)\right), \quad \text{for } k \in \mathcal{I}(C), \tag{5.7}$$

$$\dot{\tau}_k = 0, \quad \text{for } k = 1, \dots, N-1, \tag{5.8}$$

$$\Phi(x_0^1, x_1^N) = 0, \tag{5.9}$$

$$g(x_0^k) = 0, \quad \text{for } k \in \mathcal{I}(C), \tag{5.10}$$

$$x_1^k = x_0^{k+1}, \quad \text{for } k = 1, \dots, N-1. \tag{5.11}$$

Remark 5.1. Since we use the expression (2.35) to eliminate the control from equation (2.34), we impose $g(x_0^k) = 0$ in the formulation of (TP') in order to guarantee that the state constraint is active along x^k with $k \in \mathcal{I}(C)$.

Set for each $k = 1, \dots, N$:

$$\begin{aligned} \hat{x}_s^k &:= \hat{x}\big(\hat{\tau}_{k-1} + (\hat{\tau}_k - \hat{\tau}_{k-1})s\big), \quad \text{for } s \in [0,1],\ k = 1, \dots, N, \\ \hat{u}_s^k &:= \hat{u}\big(\hat{\tau}_{k-1} + (\hat{\tau}_k - \hat{\tau}_{k-1})s\big), \quad \text{for } s \in [0,1],\ k \in \mathcal{I}(S). \end{aligned} \tag{5.12}$$

Lemma 5.2. *Let* $(\hat{u}, \hat{x})$ *be a Pontryagin minimum of problem* (P'). *Then* $\big((\hat{u}^k)_{k \in \mathcal{I}(S)}, (\hat{x}^k)_{k=1}^N, (\hat{\tau}_k)_{k=1}^{N-1}\big)$ *is a weak solution of (TP').*

Proof. Let $\varepsilon > 0$ be as in Definition 4.1 and $\big((u^k),(x^k),(\tau_k)\big)$ be feasible for (TP') with

$$\|u^k - \hat{u}^k\|_\infty < \bar{\varepsilon} \quad \text{and} \quad |\tau_k - \hat{\tau}_k| \le \bar{\delta}, \quad \text{for all } k = 1,\dots,N, \tag{5.13}$$

for some $\bar{\varepsilon}, \bar{\delta} > 0$ to be determined later.

Set $I_k := [\tau_{k-1}, \tau_k]$, and consider the functions $s_k : I_k \to [0,1]$ given by $s_{k,t} := \dfrac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}}$. Define $u : [0,T] \to \mathbb{R}$ as

$$u_t := \begin{cases} 0, & \text{if } t \in I_k,\ k \in \mathcal{I}(B_-), \\ 1, & \text{if } t \in I_k,\ k \in \mathcal{I}(B_+), \\ \Gamma\big(x^k(s_{k,t})\big), & \text{if } t \in I_k,\ k \in \mathcal{I}(C), \\ u^k(s_{k,t}), & \text{if } t \in I_k,\ k \in \mathcal{I}(S). \end{cases} \tag{5.14}$$

Let x be the state corresponding to the control u and the same initial condition $x(0) = x_0^1$. We aim to show that if $\bar{\varepsilon}, \bar{\delta} > 0$ are small enough, then (u,x) is feasible for (P') and (4.1) holds. Notice that, due to Gronwall's Lemma, any trajectory (u,x) defined as above verifies the estimate

$$\|u - \hat{u}\|_1 + \|x - \hat{x}\|_\infty < \mathcal{O}(\bar{\varepsilon} + \bar{\delta}). \tag{5.15}$$

Moreover, for $k = 1,\dots,N$, we have that $x(t) = x^k(s_{k,t})$, for all $t \in I_k$.

We analyze first the control constraints. Take $k = 1,\dots,N$. If $k \in \mathcal{I}(B_-) \cup \mathcal{I}(B_+)$, then $u_t \in \{0,1\}$ for a.a. $t \in I_k$. On the other hand, observe that, by (2.32), there exists $\rho_1 > 0$ such that

$$\rho_1 < \hat{u}_t < 1 - \rho_1, \quad \text{over } C \text{ and } S \text{ arcs}. \tag{5.16}$$

Suppose now that $k \in \mathcal{I}(S)$. Then, in view of (5.13) and (5.16) we can see that the control constraints hold on I_k, provided that $\bar{\varepsilon} \le \rho_1$. Finally, consider the case where $k \in \mathcal{I}(C)$. Notice that in this case (5.16) is equivalent to

$$\rho_1 < \Gamma(\hat{x}_t) < 1 - \rho_1, \quad \text{on } \hat{I}_k. \tag{5.17}$$

Hence, by standard continuity arguments and for $\bar{\varepsilon}, \bar{\delta}$ sufficiently small, we get

$$0 < \Gamma(x_t) < 1, \quad \text{on } I_k, \tag{5.18}$$

and hence, (u,x) verifies the control constraints.

Let us now consider the state constraint. Take first $k \in \mathcal{I}(C)$. Then $g(x_{\tau_k}) = g(x_0^k) = 0$ and, by definition of (u,x), we have that $\frac{\mathrm{d}}{\mathrm{d}t} g(x_t) = 0$ for all $t \in I_k$. Therefore, x satisfies the state constraint on I_k for $k \in \mathcal{I}(C)$. Next, observe that, due to (2.32), for any $t \in [0,T]$ sufficiently away from a C arc, one has $g(\hat{x}_t) \le -\rho$ for some small $\rho > 0$. Thus, by (5.15) we get that $g(x_t) < 0$ for appropriate $\bar{\varepsilon}, \bar{\delta}$. On the other hand, for $t \in [0,T]$ close to a C arc, we proceed as follows. Assume, without loss of generality, that t is near an entry point τ_k (of a C arc). In view of hypothesis (2.32) and of the relation (2.34), we have that $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=\hat{\tau}_k-} g(\hat{x}_s) > 0$, therefore, $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=\tau_k-} g(x_s) > 0$ as well, if $\bar{\varepsilon}, \bar{\delta}$ are sufficiently small. Consequently, $g(x_t) < 0$. Hence, x verifies the state constraint on $[0,T]$.

Finally note that x satisfies the endpoint constraints. Hence (u,x) is feasible for the original problem (P').

We can easily see that $\bar{\delta}, \bar{\varepsilon}$ can be taken in such a way that (u,x) satisfies (4.1). Consequently, $\phi(x_0,x_1) \geq \phi(\hat{x}_0,\hat{x}_1)$, or equivalently,

$$\phi(x_0^1, x_1^N) \geq \phi(\hat{x}_0^1, \hat{x}_1^N), \tag{5.19}$$

which proves that $\big((\hat{u}^k)_{k\in\mathcal{I}(S)}, (\hat{x}^k)_{k=1}^N, (\hat{\tau}_k)_{k=1}^{N-1}\big)$ is a weak solution of (TP') as it was desired. □

5.2. **The shooting function.** We shall start by rewriting the problem (TP') as follows

$$\min\ \tilde{\phi}(X_0,X_1), \tag{5.20}$$

$$\dot{X} = \tilde{f}_0(X) + \sum_{k\in\mathcal{I}(S)} U^k \tilde{f}_k(X), \tag{5.21}$$

$$\tilde{\Phi}(X_0,X_1) = 0, \tag{5.22}$$

where $X := \big((x^k)_{k=1}^N, (\tau_k)_{k=1}^{N-1}\big)$, $U := (u^k)_{k\in\mathcal{I}(S)}$, the vector field $\tilde{f}_0 : \mathbb{R}^{Nn+N-1} \to \mathbb{R}^{Nn+N-1}$ is defined as follows: for $k = 1,\dots,N$,

$$\big(\tilde{f}_0(X)\big)_{i=(k-1)n+1}^{kn} := \begin{cases} (\tau_k-\tau_{k-1})f_0(x^k), & \text{for } k\in\mathcal{I}(S)\cup\mathcal{I}(B_-),\\ (\tau_k-\tau_{k-1})\big(f_0(x^k)+f_1(x^k)\big), & \text{for } k\in\mathcal{I}(B_+),\\ (\tau_k-\tau_{k-1})\left(f_0(x^k)+\Gamma(x^k)f_1(x^k)\right), & \text{for } k\in\mathcal{I}(C),\end{cases}$$

and $\big(\tilde{f}_0(X)\big)_{i=nN+1}^{Nn+N-1} := 0$; for $k\in\mathcal{I}(S)$ the vector field $\tilde{f}_k : \mathbb{R}^{Nn+N-1} \to \mathbb{R}^{Nn+N-1}$ is given by

$$\big(\tilde{f}_k(X)\big)_{i=(k-1)n+1}^{kn} := (\tau_k-\tau_{k-1})f_1(x^k),$$

and $\big(\tilde{f}_k(X)\big)_i := 0$ for the remaining index i, the new cost $\tilde{\phi} : \mathbb{R}^{2(Nn+N-1)} \to \mathbb{R}$ is

$$\tilde{\phi}(X_0,X_1) := \phi(x_0^1, x_1^N),$$

and the function $\tilde{\Phi} : \mathbb{R}^{2(Nn+N-1)} \to \mathbb{R}^{\mathrm{d}_{\tilde{\Phi}}}$ with $\mathrm{d}_{\tilde{\Phi}} := n_1 + |\mathcal{I}(C)| + n(N-1)$ is defined as

$$\tilde{\Phi}(X_0,X_1) := \begin{pmatrix} \Phi(x_0^1,x_1^N)\\ \big(g(x_0^k)\big)_{k\in\mathcal{I}(C)}\\ \big(x_1^k - x_0^{k+1}\big)_{k=1}^{N-1}\end{pmatrix}.$$

Hence the Hamiltonian for problem (TP') is given by

$$\tilde{H} = P\Big(\tilde{f}_0(X) + \sum_{k=1}^N U^k\tilde{f}_k(X)\Big) = \sum_{k=1}^N (\tau_k-\tau_{k-1})H^k, \tag{5.23}$$

where P denotes the costate associated to (TP'), $H^k := p^k\big(f_0(x^k)+w^kf_1(x^k)\big)$, with

$$w^k := \begin{cases} u^k, & \text{if } k\in\mathcal{I}(S),\\ 0, & \text{if } k\in\mathcal{I}(B_-),\\ 1, & \text{if } k\in\mathcal{I}(B_+),\\ \Gamma(x^k), & \text{if } k\in\mathcal{I}(C),\end{cases} \tag{5.24}$$

and p^k denotes the $n-$dimensional vector with components $P_{(k-1)n+1},\dots,P_{kn}$.

In order to formulate a shooting algorithm, we need to be able to write the control as a function of the state and costate. It is a standard technique to eliminate the control on S arcs from the stationarity condition $\tilde{H}_U = 0$. In this control-affine case, the control does not appear neither in $\tilde{H}_U$ nor in its time derivative $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{H}_U$. Hence, what it is usually done (see [36, 7]) is to impose the *strengthened Legendre-Clebsch condition:*

$$-\frac{\partial}{\partial \tilde{u}}\ddot{\tilde{H}}_{\tilde{u}} \succ 0, \text{ where } \tilde{u} := (u^k)_{k\in\mathcal{I}(S)}, \tag{5.25}$$

and the symbol $\succ$ mean *positive definiteness.* Simple calculations show that $-\frac{\partial}{\partial \tilde{u}}\ddot{\tilde{H}}_{\tilde{u}}$ is a $|\mathcal{I}(S)| \times |\mathcal{I}(S)|-$diagonal matrix with positive entries equal to

$$-(\tau_k - \tau_{k-1})\frac{\partial}{\partial u^k}\ddot{H}^k_{u^k}, \quad \text{for } k \in \mathcal{I}(S). \tag{5.26}$$

Hence, thanks to the hypothesis (5.25), for each $k \in \mathcal{I}(S)$, one can compute the control u^k from the identity

$$\ddot{H}^k_{u^k} = 0. \tag{5.27}$$

Apart from previous equation (5.27), in order to ensure the stationarity $H^k_{u^k} = 0$, we add the following endpoint conditions:

$$0 = H^k_{u^k}(0) = p^k_0 f_1(x^k_0), \quad 0 = \dot{H}^k_{u^k} = p^k_0[f_1, f_0](x^k_0), \quad \text{for } k \in \mathcal{I}(S). \tag{5.28}$$

The endpoint Lagrangian associated to (TP') is given by

$$\tilde{\ell}^\Psi := \phi(x^1_0, x^N_1) + \sum_{j=1}^{n_1}\Psi_j\Phi_j(x^1_0, x^N_1) + \sum_{k\in\mathcal{I}(C)}\gamma_k g(x^k_0) + \sum_{k=1}^{N-1}\theta_k(x^k_1 - x^{k+1}_0). \tag{5.29}$$

The costate equation for p^k is

$$\dot{p}^k = -(\tau_k - \tau_{k+1})D_{x^k}H^k, \tag{5.30}$$

with endpoint conditions

$$p^1_0 = -D_{x^1_0}\tilde{\ell}^\Psi = -D_{x^1_0}\phi - \sum_{j=1}^{n_1}\Psi_j D_{x^1_0}\Phi_j - \chi_{\mathcal{I}(C)}(1)\gamma_1 g'(x^1_0), \tag{5.31}$$

$$p^k_1 = \theta^k, \quad \text{for } k = 1, \ldots, N-1, \tag{5.32}$$

$$p^k_0 = \theta^{k-1} - \chi_{\mathcal{I}(C)}(k)\gamma_k g'(x^k_0), \quad \text{for } k = 2, \ldots, N, \tag{5.33}$$

$$p^N_1 = D_{x^N_1}\phi + \sum_{j=1}^{n_1}\Psi_j D_{x^N_1}\Phi_j. \tag{5.34}$$

For the costate p^{τ_k} we have the dynamics

$$\dot{p}^{\tau_k} = -H^k + H^{k+1}, \ p^{\tau_k}_0 = 0, \ p^{\tau_k}_1 = 0, \quad \text{for } k = 1, \ldots, N-1. \tag{5.35}$$

It is known that we can reformulate optimization parameters as states with zero dynamics, and that then the application of the Pontryagin's maximum principle gives that the Lagrangian of the problem has a zero derivative with respect to optimization parameters. In the case of the parameter τ_k this means that

$$\int_0^1 (H^k - H^{k+1})\mathrm{d}t = 0, \tag{5.36}$$

and hence, p^{τ_k} vanishes identically. Recall that, since the problem is autonomous, the pre-Hamiltonian $t \mapsto \tilde{H}(U_t, X_t, P_t)$ is constant on $[0,1]$ along any optimal solution (a proof can be found in [51, 11]). Actually, it can be easily seen that each H^k is constant on $[0,1]$ and, consequently, from (5.36) we obtain that

$$H_1^k = H_0^{k+1}, \quad \text{for } k = 1, \dots, N-1. \tag{5.37}$$

Remark 5.3. We shall notice that the reparameterization used in (5.12), which transforms a trajectory of (P') into one of (TP'), cannot be, in general, applied to obtain the costate P as a reparameterization of p. This is due to the fact that the costate equation for (TP') is not a reparameterization of the costate equation for (P'). In fact, on one hand we have that p^k is solution of (5.30), or equivalently, that, for $k \in \mathcal{I}(C)$,

$$-\dot{p}^k = (\tau_k - \tau_{k+1}) p^k \frac{\partial}{\partial x^k}\big(f_0(x^k) + \Gamma(x^k) f_1(x^k)\big), \tag{5.38}$$

while on the other hand, the costate equation for (P) is given in (2.10). On the C arcs, we can differentiate the stationarity condition $H_u = 0$ as done in (2.36) and obtain the expression (2.37) for the measure $\mathrm{d}\mu$.

Simple examples show that the costate equation (5.38) is not in general a reparameterization of (2.10) (in which $\mathrm{d}\mu$ is replaced by ν given in (2.37)), and hence the multipliers P and p are not linked by a change of time variable. This fact actually makes sense, since it is expected that the sensitivity changes after the transformation of the problem is performed.

The shooting function associated with (TP') that we propose here is, eliminating the variable θ,

(5.39)

$$\mathcal{S} : \mathbb{R}^{Nn+N-1} \times \mathbb{R}^{Nn+n_1+|\mathcal{I}(C)|,*} \to \mathbb{R}^{(N-1)n+N-1+n_1+|\mathcal{I}(C)|+2|\mathcal{I}(S)|} \times \mathbb{R}^{(N+1)n,*},$$

$$\big((x_0^k),(\tau_k),(p_0^k),\Psi,\gamma\big) \mapsto \begin{pmatrix} \Phi(x_0^1, x_1^N) \\ \big(g(x_0^k)\big)_{k\in\mathcal{I}(C)} \\ (x_1^k - x_0^{k+1})_{k=1,\dots,N-1} \\ p_0^1 + D_{x_0^1}\tilde{\ell}^{\Psi} \\ p_1^k - p_0^{k+1} - \chi_{\mathcal{I}(C)}(k)\gamma_k g'(x_0^k) \\ p_1^N - D_{x_1^N}\tilde{\ell}^{\Psi} \\ (H_1^k - H_0^{k+1})_{k=1,\dots,N-1} \\ \big(p_0^k f_1(x_0^k)\big)_{k\in\mathcal{I}(S)} \\ \big(p_0^k [f_1,f_0](x_0^k)\big)_{k\in\mathcal{I}(S)} \end{pmatrix},$$

where $\big((x^k),(p^k)\big)$ is the solution of (5.4)-(5.7), (5.30) with initial values $(x_0^k),(p_0^k)$, and with u^k, for $k \in \mathcal{I}(S)$, given by the stationarity condition (5.27).

Notice that if $\omega := \big((x_0^k),(\tau_k),(p_0^k),\Psi,\gamma\big)$ is such that

$$\mathcal{S}(\omega) = 0, \tag{5.40}$$

then, the associated solution $\big((x^k),(p^k),(u^k)\big)$ verifies the Pontryagin Maximum Principle for (TP'). Briefly speaking, in order to find the extremal points of (TP'), we shall solve (5.40).

We shall notice that the system (5.40) has $2Nn+N-1+\mathrm{d}_\Phi+|\mathcal{I}(C)|$ unknowns and $2Nn+N-1+\mathrm{d}_\Phi+|\mathcal{I}(C)|+2|\mathcal{I}(S)|$ equations. Hence, as soon as a singular arc occurs, the system has more equations than unknowns, i.e. it is overdetermined.

Therefore, we employ the *Gauss-Newton method* to solve it. For a description of this method, the reader is referred to Fletcher [19].

The *shooting method* we propose here consists of solving (5.40) by the Gauss-Newton method. Given that the right-hand side of (5.40) is zero, the Gauss-Newton method converges locally quadratically if $\mathcal{S}$ has a one-to-one Lipschitz continuous derivative at the solution. In view of the regularity hypotheses done in Section 2, we know that $\mathcal{S}'$ is Lipschitz continuous. The convergence result can then be stated as follows:

Proposition 5.4. *The shooting algorithm converges locally quadratically around* $\hat{\omega}$ *if* $\mathcal{S}'(\hat{\omega})$ *is one-to-one.*

5.3. **Sufficient condition for the convergence of the shooting algorithm.** The main result of this section is Theorem 5.11 which gives a sufficient condition for the local convergence of the shooting algorithm, that is also a sufficient condition for weak optimality of problem (TP') (see Theorem 5.10 below).

We shall start by presenting the following hypothesis that is required to state the sufficient condition in Theorem 5.10. We say that the problem (TP') satisfies the following *qualification condition for the endpoint constraints* at the nominal trajectory $(\hat{U},\hat{X})$ if the mapping

$$\bar{\Phi}:\mathbb{R}^{Nn+N-1}\times(L^\infty)^{|\mathcal{I}(S)|}\to\mathbb{R}^{\mathrm{d}_{\tilde{\Phi}}},\quad (X_0,U)\mapsto\tilde{\Phi}(X_0,X_1), \tag{5.41}$$

where X_t is the solution of (5.21) associated to (X_0,U) is such that

$$D\bar{\Phi}(\hat{X}_0,\hat{U}) \text{ is surjective.} \tag{5.42}$$

Under the hypothesis just presented, the Pontryagin Maximum Principle holds in its normal form.

Theorem 5.5. *Let* $(\hat{U},\hat{X})$ *be a weak solution for (TP') satisfying the qualification condition* (5.42). *Then there exists a unique* $\tilde{\lambda}:=(\tilde{\Psi},P)\in\mathbb{R}^{\mathrm{d}_{\tilde{\Phi}}*}\times(W^{1,\infty})^{Nn+N-1*}$ *such that* P *is solution of*

$$-\dot{P}_t=D_X\tilde{H}(\hat{U}_t,\hat{X}_t,P_t),\quad a.e.\ on\ [0,T], \tag{5.43}$$

with transversality conditions

$$P_0=-D_{X_0}\tilde{\ell}^{\tilde{\Psi}}(\hat{X}_0,\hat{X}_1),\quad P_1=D_{X_T}\tilde{\ell}^{\tilde{\Psi}}(\hat{X}_0,\hat{X}_1), \tag{5.44}$$

and with

$$\tilde{H}_U(\hat{U}_t,\hat{X}_t,P_t)=0. \tag{5.45}$$

Since there is a unique associated multiplier, we omit from now on the dependence on $\tilde{\lambda}$ for the sake of simplicity of the presentation. Moreover, we omit the dependence on the nominal solution $(\hat{U},\hat{X})$.

Consider the quadratic mapping on the space $(L^\infty)^{|\mathcal{I}(S)|}\times(W^{1,\infty})^{Nn+N-1}$, defined as

$$\tilde{Q}(V,Z):=\tfrac{1}{2}D^2\tilde{\ell}(Z_0,Z_1)^2+\tfrac{1}{2}\int_0^1\big[Z^\top\tilde{H}_{XX}Z+2V\tilde{H}_{UX}Z\big]\mathrm{d}t. \tag{5.46}$$

In what follows we recall a sufficient condition for optimality for problem (TP') proved in Dmitruk [15, 16]. Let us define the critical cone associated to (TP'). Consider first the linearized state equation:

$$\dot{Z} = \tilde{A}Z + \tilde{B}V, \quad \text{a.e. on } [0,1], \tag{5.47}$$

where $F(U,X) := \tilde{f}_0(X) + \sum_{k\in\mathcal{I}(S)} U^k \tilde{f}_k(X)$, $\tilde{A} := F_X$, $\tilde{B} := F_U$; and let the linearized endpoint constraints be given by

$$D\tilde{\Phi}(Z_0, Z_T) = 0. \tag{5.48}$$

Consider the *critical cone*

$$\tilde{\mathcal{C}} := \{(V,Z) \in (L^\infty)^{|\mathcal{I}(S)|} \times (W^{1,\infty})^{Nn+N-1} : \text{(5.47)-(5.48) hold}\}. \tag{5.49}$$

The following result follows (see [32] for a proof).

Theorem 5.6 (Second order necessary condition)**.** *If* $(\hat{U},\hat{X})$ *is a weak minimum for (TP') that verifies* (5.42)*, then*

$$\tilde{Q}(V,Z) \geq 0, \quad \text{for all } (V,Z) \in \tilde{\mathcal{C}}. \tag{5.50}$$

The Goh transformation on the system (5.47) gives

$$Y_t := \int_0^t V_s \mathrm{d}s, \quad \Xi_t := Z_t - \tilde{E}_t Y_t, \tag{5.51}$$

where $\tilde{E} := \tilde{A}\tilde{B} - \frac{\mathrm{d}}{\mathrm{d}t}\tilde{B}$. Notice that, if $(V,Z) \in \tilde{\mathcal{C}}$, then (Y,Ξ) defined by the above transformation (5.51) is solution of the *transformed linearized equation*

$$\dot{\Xi} = \tilde{A}\Xi + \tilde{E}Y, \tag{5.52}$$

and satisfies the *transformed linearized endpoint constraints*

$$D\tilde{\Phi}(\Xi_0, \Xi_T + \tilde{B}_T h) = 0, \tag{5.53}$$

where we set $h := Y_T$.

Consider the function

$$\rho(\zeta_0,\zeta_1,h) := D^2\tilde{\ell}(\zeta_0, \zeta_1 + \tilde{B}_1 h)^2 + h\tilde{H}_{UX,1}(2\zeta_1 + \tilde{B}_1 h), \tag{5.54}$$

and the quadratic mapping

$$\tilde{\Omega}(Y,\tilde{h},\Xi) := \tfrac{1}{2}\rho(\Xi_0,\Xi_1,\tilde{h}) + \tfrac{1}{2}\int_0^1 \{\Xi^\top \tilde{H}_{XX}\Xi + 2Y\tilde{M}\Xi + Y\tilde{R}Y\}\mathrm{d}t, \tag{5.55}$$

for $(Y,\tilde{h},\Xi) \in (L^2)^{|\mathcal{I}(S)|} \times \mathbb{R} \times (H^1)^{Nn+N-1}$ and

$$\tilde{M} := \tilde{f}_1^\top \tilde{H}_{XX} - \frac{\mathrm{d}}{\mathrm{d}t}\tilde{H}_{UX} - \tilde{H}_{UX}\tilde{A}, \quad \tilde{R} := \tilde{f}_1^\top \tilde{H}_{XX}\tilde{f}_1 - 2\tilde{H}_{UX}\tilde{E} - \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{H}_{UX}\tilde{f}_1). \tag{5.56}$$

Remark 5.7. Let us recall that the second order necessary condition for optimality stated by Goh [23] (and nowadays known as *Goh condition*) implies that: if $(\hat{U},\hat{X})$ is a weak minimum for (TP') verifying (5.42), then

$$\tilde{H}_{UX}\tilde{B} \text{ is symmetric}, \tag{5.57}$$

or, equivalently, $P\cdot D\tilde{f}_i\tilde{f}_j = P\cdot D\tilde{f}_j\tilde{f}_i$ for all $i,j = 1,\dots,N$. In the recent literature, this condition can be encountered as $P\cdot[\tilde{f}_i,\tilde{f}_j] = 0$. We shall mention that this necessary condition was first stated by Goh in [23] for the case with neither control nor state constraints, and extended in [6, 20] for problems containing control constraints.

Notice that, when the control is scalar (i.e. $|\mathcal{I}(S)| = 1$), then (5.57) is verified since $\tilde{H}_{UX}\tilde{B}$ is also a scalar. Furthermore, given the special structure of the dynamics of (TP'), we can see that $D\tilde{f}_i\,\tilde{f}_j = 0$ for all $1 \leq i \neq j \leq N$, and hence the matrix in (5.57) is diagonal and, therefore, the Goh condition holds trivially even in the general case (i.e. when $|\mathcal{I}(S)| > 1$).

¿From previous Remark 5.7 and [6, Theorem 4.4] we get the following result:

Proposition 5.8. *For all* $(V, Z) \in (L^\infty)^{|\mathcal{I}(S)|} \times (W^{1,\infty})^{Nn+N-1}$ *and* (Y, Ξ) *verifying* (5.51)*, it holds:*

$$\tilde{Q}(V, Z) = \tilde{\Omega}(Y, Y_T, \Xi),$$

Define, for $(\Xi_0, Y, \tilde{h}) \in \mathbb{R}^n \times (L^2)^{|\mathcal{I}(S)|} \times \mathbb{R}$, the *order function*

$$\gamma(\Xi_0, Y, \tilde{h}) := |\Xi_0|^2 + \int_0^1 |Y_t|^2 \mathrm{d}t + |\tilde{h}|^2. \tag{5.58}$$

Definition 5.9 ($\gamma-$growth). A feasible trajectory $(\hat{U}, \hat{X})$ of (TP') satisfies the $\gamma-$*growth condition in the weak sense* if there exists a positive constant c such that, for every sequence of feasible variations $\big((\delta X_0^k, V^k)\big)_k$ converging to 0 in $(L^\infty)^{|\mathcal{I}(S)|} \times (W^{1,\infty})^{Nn+N-1}$, one has that

$$\tilde{\phi}(X_0^k, X_1^k) - \tilde{\phi}(\hat{X}_0, \hat{X}_1) \geq c\gamma(\Xi_0^k, Y^k, Y_T^k), \tag{5.59}$$

for k large enough, where (Y^k, Ξ^k) are given by Goh transformation (5.51) and X^k is the solution of (5.21) associated to $(\hat{X}_0 + \delta X_0^k, \hat{U} + V^k)$.

Consider the *transformed critical cone*

$$\tilde{\mathcal{P}}_S^2 := \{(Y, \tilde{h}, \Xi) \in (L^2)^{|\mathcal{I}(S)|} \times \mathbb{R} \times (H^1)^{Nn+N-1} : \text{(5.52)-(5.53) hold}\}. \tag{5.60}$$

The following characterization of $\gamma-$growth holds (see [15] or [16, Theorem 3.1] for a proof).

Theorem 5.10. *Let* $(\hat{U}, \hat{X})$ *be such that* (5.42) *is verified. Then* $(\hat{U}, \hat{X})$ *is a weak minimum of (TP') that satisfies* $\gamma-$*growth in the weak sense if and only if* (5.57) *holds and there exists* $c > 0$ *such that*

$$\tilde{\Omega}(Y, \tilde{h}, \Xi) \geq c\gamma(\Xi_0, Y, \tilde{h}), \quad \text{on } \tilde{\mathcal{P}}_S^2. \tag{5.61}$$

The main theorem of the section is the following:

Theorem 5.11. *If* $(\hat{U}, \hat{X})$ *is a weak minimum of problem (TP') satisfying the constraint qualification* (5.42) *and the uniform positivity condition* (5.61)*, then the shooting algorithm is locally quadratically convergent.*

Proof. It follows from [7, Theorem 5.4]. □

6. Application of the shooting algorithm: a regulator problem

Consider the following variation of the regulator problem:

$$\begin{gathered}\min \tfrac{1}{2}\int_0^5 (x_{1,t}^2 + x_{2,t}^2)\mathrm{d}t + \tfrac{1}{2}x_{2,5}^2,\\ \dot{x}_{1,t} = x_{2,t}, \quad \dot{x}_{2,t} = u_t \in [-1, 1],\end{gathered} \tag{6.1}$$

with a state constraint and initial conditions

$$x_{2,t} \geq -0.2, \quad x_{1,0} = 0, \quad x_{2,0} = 1. \tag{6.2}$$

To write the problem in the Mayer form, we define an extra state variable given by the dynamics

$$\dot{x}_{3,t} = \tfrac{1}{2}(x_{1,t}^2 + x_{2,t}^2), \quad x_{3,0} = 0. \tag{6.3}$$

The resulting problem is then

$$\begin{aligned} &\min\, x_{3,5} + \tfrac{1}{2}x_{2,5}^2,\\ &\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad \dot{x}_3 = \tfrac{1}{2}(x_1^2 + x_2^2),\\ &x_{1,0} = 0, \quad x_{2,0} = 1, \quad x_{3,0} = 0,\\ &-1 \leq u \leq 1, \;\; x_2 \geq -0.2. \end{aligned} \tag{6.4}$$

To this problem we can associate the functions

$$g(x) := -x_2 - 0.2, \quad f_0(x) := \begin{pmatrix} x_2 \\ 0 \\ \frac{1}{2}(x_1^2 + x_2^2) \end{pmatrix}, \quad f_1(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

where $g : \mathbb{R}^3 \to \mathbb{R}$, $f_0, f_1 : \mathbb{R}^3 \to \mathbb{R}^3$. The pre-Hamiltonian of (6.4) is given by

$$H(u, x, p) := p_1 x_2 + p_2 u + \tfrac{1}{2} p_3 (x_1^2 + x_2^2),$$

where $p := (p_1, p_2, p_3)$ is the costate.

We solve numerically the problem by BOCOP software [9] through which we find that the optimal control $\hat{u}$ is a concatenation of a bang arc in the lower bound, followed by a constrained arc and ended with a singular arc. Briefly, we can write that the optimal control has a B_-CS structure. The control is equal to -1 on the B_- arc, and to $-\dfrac{g'(\hat{x})f_0(\hat{x})}{g'(\hat{x})f_1(\hat{x})} = 0$ on C according to (2.35), where $\hat{x}$ is the associated state. On the singular arc S we have

$$\hat{u} = -\frac{p\big[[f_1, f_0], f_0\big](\hat{x})}{p\big[[f_1, f_0], f_1\big](\hat{x})}, \tag{6.5}$$

where the involved Lie brackets are

$$[f_1, f_0] = \begin{pmatrix} -1 \\ 0 \\ -x_2 \end{pmatrix}, \quad \big[[f_1, f_0], f_0\big] = \begin{pmatrix} 0 \\ 0 \\ x_1 \end{pmatrix}, \quad \big[[f_1, f_0], f_1\big] = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}. \tag{6.6}$$

On the other hand, the costate equation on the singular arc gives

$$\begin{aligned} &\dot{p}_1 = -p_3 \hat{x}_1, \quad p_{1,5} = 0,\\ &\dot{p}_2 = -p_1 - p_3 \hat{x}_2, \quad p_{2,5} = \hat{x}_{2,5},\\ &\dot{p}_3 = 0, \quad p_{3,5} = 1. \end{aligned} \tag{6.7}$$

Thus,

$$p_3 \equiv 1, \tag{6.8}$$

and from (6.5)-(6.6) we get

$$\hat{u} = \hat{x}_1, \quad \text{on } S. \tag{6.9}$$

Moreover, the first order optimality conditions imply that

$$0 = H_u = p_2, \quad \text{on } C \cup S. \tag{6.10}$$

6.1. **Checking the sufficient conditions.** Let us show that condition (i) in Theorem 4.5 is verified. Since $\hat{u}$ starts with a B_- arc, then

$$\hat{x}_{2,t} = 1 - t, \tag{6.11}$$

until it saturates the constraint, i.e. for $t \in [0, 1.2]$. On this interval, we have that

$$\hat{x}_{1,t} = -\frac{(1-t)^2}{2} + \frac{1}{2}. \tag{6.12}$$

Note that, from (6.10) and (6.7), we get that $p_1 = -\hat{x}_2$ on $C \cup S$. Thus, $p_1(1.2) = 0.2$, and from (6.7), (6.8) and (6.12), we obtain

$$p_{1,t} = 0.2 + \int_{1.2}^{t} \left(\frac{(1-s)^2}{2} - \frac{1}{2} \right) \mathrm{d}s, \quad \text{for } t \in [0, 1.2). \tag{6.13}$$

Therefore, from latter equation together with (6.7), (6.8) and (6.11), we get

$$\dot{p}_{2,t} = \frac{(1-t)^3}{6} + \frac{3t}{2} - 1.8 + \frac{0.008}{6} < 0, \quad \text{for } t \in [0, 1.2), \tag{6.14}$$

and, since $p_2(1.2) = H_u(1.2) = 0$,

$$H_u = p_2 > 0, \quad \text{on } [0, 1.2). \tag{6.15}$$

Consequently, the strict complementarity condition for the control constraint holds. Furthermore, condition (ii) of Theorem 4.5 is trivially verified since no endpoint constraint is considered.

Let us now verify (4.7). The dynamics for the linearized state is

$$\begin{aligned} &\dot{z}_1 = z_2, \quad \dot{z}_2 = v, \quad \dot{z}_3 = \hat{x}_1 z_1 + \hat{x}_2 z_2, \\ &z_{1,0} = z_{2,0} = z_{3,0} = 0. \end{aligned} \tag{6.16}$$

Let $\mathcal{C}_S$ and $\mathcal{P}_*^2$ denote the strict critical cone and the extended cone (defined in (4.4)) at the optimal trajectory $(\hat{u}, \hat{x})$, respectively. Since $\hat{u}$ is $B_- CS$, then for any $(v, z) \in \mathcal{C}_S$, $v = 0$ on the initial interval B_-. Consequently,

$$y = 0, \quad \text{on } B_-, \text{ for all } (y, \xi, h) \in \mathcal{P}_*^2. \tag{6.17}$$

On the other hand, the dynamics for the transformed linearized state is

$$\begin{aligned} &\dot{\xi}_1 = \xi_2 + y, \quad \dot{\xi}_2 = 0, \quad \dot{\xi}_3 = \hat{x}_1 \xi_1 + \hat{x}_2 \xi_2 + \hat{x}_2 y, \\ &\xi_{1,0} = \xi_{2,0} = \xi_{3,0} = 0. \end{aligned} \tag{6.18}$$

Take $(y, \xi, h) \in \mathcal{P}_*^2$. Then,

$$\xi_2 \equiv 0, \quad \text{on } [0, 5]. \tag{6.19}$$

In view of condition (3.16)(i), we have that $-\xi_2 - y = 0$ on C. Thus, due to (6.19), we get

$$y = 0, \quad \text{on } B_- \cup C. \tag{6.20}$$

Thus, from (6.20) and (6.19), we get $\xi_1 = \xi_3 = 0$ on $B_- \cup C$. As for the last component h, we get from (3.17) that

$$\hat{x}_{2,5}(\xi_{1,5} + h) + \xi_{3,5} = 0. \tag{6.21}$$

Since $H_u = 0$ on $C \cup S$, then $p_2 = 0$ on $C \cup S$, and thus $0 = p_{2,5} = \hat{x}_{2,5}$. Then, from (6.21) we deduce that there is no restriction on h. We obtain

$$\mathcal{P}_*^2 = \left\{ \begin{array}{l} (y, \xi, h) \in L^2 \times (H^1)^n \times \mathbb{R} : y = \xi_1 = \xi_2 = 0 \text{ on } B_- \cup C, \\ \xi_2 = 0 \text{ on } [0,5],\ \dot{\xi}_1 = y \text{ and } \dot{\xi}_3 = \hat{x}_1 \xi_1 + \hat{x}_2 y \ \text{ on } S \end{array} \right\}. \tag{6.22}$$

The quadratic forms Q and Ω are given by

$$Q(v,z) := \int_0^T (z_1^2 + z_2^2)\mathrm{d}t + z_{2,5}^2; \quad \Omega(y,h,\xi) := \int_0^T (\xi_1^2 + y^2)\mathrm{d}t + h^2, \tag{6.23}$$

Thus, Ω is a Legendre form on $\{(y,h,\xi) \in L^2 \times \mathbb{R} \times (H^1)^3 : (3.15) \text{ holds}\}$ and is coercive on $\mathcal{P}_*^2$. Hence the hypotheses and the sufficient condition in Theorem 4.5 are verified and, consequently, $(\hat{u}, \hat{x})$ is a Pontryagin minimum satisfying $\gamma-$growth.

6.2. **Transformed problem.** Here we transform (6.4) to obtain a problem with neither control nor state constraints, as done in section 5.1.

The optimal control associated to (6.4) has a B_-CS structure, as already said above. Then we triplicate the number of state variables obtaining the new variables $X_1, \dots, X_9$ and we consider two switching times that we write X_{10}, X_{11}. The new problem has only one control variable that corresponds to the singular arc and which we call V. The reformulation of (6.4) is then as follows

$$\begin{aligned}
&\min X_{9,1} + \tfrac{1}{2} X_{8,1}^2,\\
&\quad \dot{X}_1 = X_{10} X_2,\\
&\quad \dot{X}_2 = -X_{10},\\
&\quad \dot{X}_3 = X_{10} \tfrac{1}{2}(X_1^2 + X_2^2),\\
&\quad \dot{X}_4 = (X_{11} - X_{10}) X_5,\\
&\quad \dot{X}_5 = 0,\\
&\quad \dot{X}_6 = (X_{11} - X_{10}) \tfrac{1}{2}(X_4^2 + X_5^2),\\
&\quad \dot{X}_7 = (5 - X_{11}) X_8,\\
&\quad \dot{X}_8 = (5 - X_{11}) V,\\
&\quad \dot{X}_9 = (5 - X_{11}) \tfrac{1}{2}(X_7^2 + X_8^2),\\
&\quad \dot{X}_{10} = 0,\\
&\quad \dot{X}_{11} = 0,\\
&\quad X_{1,0} = 0, \quad X_{2,0} = 1, \quad X_{3,0} = 0,\\
&\quad X_{1,1} = X_{4,0}, \quad X_{2,1} = X_{5,0}, \quad X_{3,1} = X_{6,0},\\
&\quad X_{4,1} = X_{7,0}, \quad X_{5,1} = X_{8,0}, \quad X_{6,1} = X_{9,0},\\
&\quad X_{2,1} = 0.2, \text{ (or } g(x_2(\tau_2)) = 0).
\end{aligned} \tag{6.24}$$

¿From (6.9) we deduce that $V = X_7$.

We solved problem (6.24) by the shooting algorithm proposed in Section 5.The results are shown in Figure 1 in the original variables u and x.

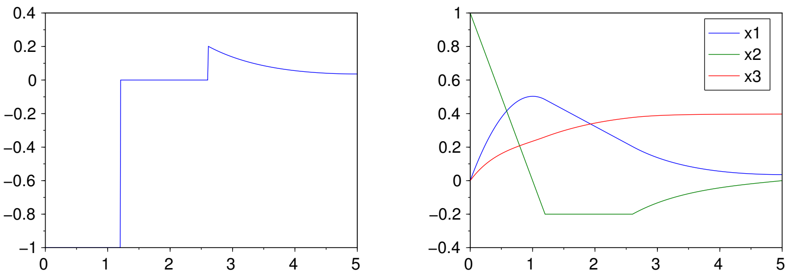

FIGURE 1. Optimal control and states

APPENDIX A. ON SECOND-ORDER NECESSARY CONDITIONS

A.1. **General constraints.** We will study an abstract optimization problem of the form

$$\min f(x); \quad G_E(x) = 0, \ G_I(x) \in K_I, \tag{A.1}$$

where X, Y_E, Y_I are Banach spaces, $f : X \to \mathbb{R}$, $G_E : X \to Y_E$, and $G_I : X \to Y_I$ are functions of class C^2, and K_I is a closed convex subset of Y_I with nonempty interior. The subindex E is used to refer to 'equalities' and I to 'inequalities'.

Setting

$$Y := Y_E \times Y_I, \quad G(x) := (G_E(x), G_I(x)), \quad K := \{0\}_{K_E} \times K_I, \tag{A.2}$$

we can rewrite (A.1) in a more compact form:

$$\min f(x); \quad G(x) \in K. \tag{P_A}$$

We use $F(P_A)$ to denote the set of feasible solutions of (P_A).

Remark A.1. We refer to [12] for a systematic study of problem (P_A). Here we will take advantage of the product structure that one can find in essentially all practical applications, to introduce a non qualified version of second order necessary conditions specialized to the case of quasi radial directions, that extends in some sense [12, Theorem 3.50]. See Kawasaki [30] for non radial directions.

The *tangent cone* (in the sense of convex analysis) to K_I at $y \in K_I$ is defined as

$$T_{K_I}(y) := \{z \in Y_I : \operatorname{dist}(y + tz, K_I) = o(t), \ \text{ with } t \geq 0\}, \tag{A.3}$$

and the *normal cone* to K_I at $y \in K_I$ is

$$N_{K_I}(y) := \{z^* \in Y_I^* : \langle z^*, y' - y \rangle \leq 0, \ \text{for all } y' \in K_I\}. \tag{A.4}$$

In what follows, we shall study a nominal feasible solution $\hat{x} \in F(P_A)$ that may satisfy or not the *qualification condition*

(A.5)

$$\begin{cases} \text{(i)} & DG_E(\hat{x}) \text{ is onto,} \\ \text{(ii)} & \text{there exists } z \in \operatorname{Ker} DG_E(\hat{x}) \text{ such that } \ G_I(\hat{x}) + DG_I(\hat{x})z \in \operatorname{int}(K_I). \end{cases}$$

The latter condition coincides with the qualification condition in (2.21) which was introduced for the optimal control problem (P).

Remark A.2. Condition (A.5) is equivalent to the Robinson qualification condition in [48]. See the discussion in [12, Section 2.3.4].

The *Lagrangian function* of problem (P_A) is defined as

$$L(x,\lambda) := \beta f(x) + \langle \lambda_E, G_E(x)\rangle + \langle \lambda_I, G_I(x)\rangle, \tag{A.6}$$

where we set $\lambda := (\beta, \lambda_E, \lambda_I) \in \mathbb{R}_+ \times Y_E^* \times Y_I^*$. Define the *set of Lagrange multipliers* associated with $x \in F(P_A)$ as

$$\Lambda(x) := \{\lambda \in \mathbb{R}_+ \times Y_E^* \times N_{K_I}(G_I(x)) : \lambda \neq 0,\ D_x L(x,\lambda) = 0\}. \tag{A.7}$$

Let $y_I \in \operatorname{int}(K_I)$, $y_I \neq G_I(\hat{x})$. We consider the following auxiliary problem, where $(x,\gamma) \in X \times \mathbb{R}$:

$$\begin{aligned} \min_{x,\gamma} \gamma;\quad & f(x) - f(\hat{x}) \le \gamma, \quad G_E(x) = 0, \quad \gamma \ge -1/2, \\ & y_I + (1+\gamma)^{-1}\big(G_I(x) - y_I\big) \in K_I. \end{aligned} \tag{AP_A}$$

Note that we recognize the idea of a *gauge function* (see e.g. [49]) in the last constraint.

Lemma A.3. *Assume that $\hat{x}$ is a local solution of (P_A). Then $(\hat{x},0)$ is a local solution of (AP_A).*

Proof. We easily check that $(\hat{x},0) \in F(AP_A)$. Now take $(x,\gamma) \in F(AP_A)$. Let us prove that if $-1/2 \le \gamma < 0$, then x can not be closed to $\hat{x}$ (in the norm of the Banach space X). Assuming that $-1/2 \le \gamma < 0$, we get $G_E(x) = 0$, $G_I(x) \in K_I + (-\gamma)y_I \subseteq K_I$, and $f(x) < f(\hat{x})$. Since $\hat{x}$ is a local solution of (P_A), the x can not be too closed to $\hat{x}$. The conclusion follows. □

The Lagrangian function of (AP_A) is

$$\beta_0\gamma + \beta(f(x) - f(\hat{x}) - \gamma) + \langle \lambda_E, G_E(x)\rangle + \langle \lambda_I, y_I + (1+\gamma)^{-1}(G_I(x) - y_I)\rangle. \tag{A.8}$$

or equivalently

$$L(x,\lambda) + (\beta_0 - \beta)\gamma + \big((1+\gamma)^{-1} - 1\big)\langle \lambda_I, G_I(x) - y_I\rangle. \tag{A.9}$$

Setting $\hat{\lambda} = (\beta_0, \beta, \lambda_E, \lambda_I)$, we see that the set of Lagrange multipliers of the auxiliary problem (AP_A) at $(\hat{x},0)$ is

$$\hat{\Lambda} := \left\{ \begin{array}{l} \hat{\lambda} \in \mathbb{R}_+ \times \mathbb{R}_+ \times Y_E^* \times N_{K_I}(G_I(\hat{x})) : \lambda \neq 0, \\ D_x L(\hat{x},\lambda) = 0;\ \beta + \langle \lambda_I, G_I(\hat{x}) - y_I\rangle = \beta_0 \end{array} \right\}. \tag{A.10}$$

Proposition A.4. *Suppose that* (A.5)*(i) holds. Then, the mapping*

$$(\beta, \lambda_E, \lambda_I) \mapsto \frac{(\beta + \langle \lambda_I, G_I(x) - y_I\rangle, \beta, \lambda_E, \lambda_I)}{\beta + \langle \lambda_I, G_I(x) - y_I\rangle} \tag{A.11}$$

is a bijection between $\Lambda(\hat{x})$ and $\hat{\Lambda}_1$ (recall the definition in (2.16)*).*

Proof. Since (A.5)(i) holds, then we necessarily have that $(\beta, \lambda_I) \neq 0$ for all $\lambda = (\beta, \lambda_E, \lambda_I) \in \Lambda(\hat{x})$. Therefore, if $\lambda_I = 0$ then $\beta > 0$ and $\beta + \langle \lambda_I, G_I(x) - y_I\rangle > 0$. If by the contrary, $\lambda_I \neq 0$, then $\langle \lambda_I, G_I(x) - y_I\rangle > 0$ and again, $\beta + \langle \lambda_I, G_I(x) - y_I\rangle > 0$. Hence, the mapping in (A.11) is well-defined and is a bijection from $\Lambda(\hat{x})$ to $\hat{\Lambda}_1$, as we wanted to whow. □

Theorem A.5. *Let $\hat{x}$ be a local solution of (P_A), such that $DG_E(\hat{x})$ is surjective. Then $\hat{\Lambda}_1$ is non empty and bounded.*

Proof. By lemma A.3, $(\hat{x},0)$ is a local solution of (AP_A). In addition the qualification condition for the latter problem at the point $(\hat{x},0)$ states as follows: there exists $(z,\delta)\in \operatorname{Ker} DG_E(\hat{x})\times\mathbb{R}$ such that

$$\begin{gathered} Df(\hat{x})z<\delta, \quad \delta>0,\\ G_I(\hat{x})+DG_I(\hat{x})z-\delta(G_I(\hat{x})-y_I)\in \operatorname{int}(K_I). \end{gathered}$$

These conditions trivially hold for $(z,\delta)=(0,1)$. Hence, in view of classical results by e.g. Robinson [48], the conclusion follows. □

A.2. **Second order necessary optimality conditions.** Let us introduce the notation $[a,b]$ to refer to the segment $\{\rho a+(1-\rho)b;\ \text{for } \rho\in[0,1]\}$ for any pair of points a,b in an arbitrary vector space Z.

Definition A.6. Let $y\in K$. We say that $z\in Y$ is a *radial direction* to K at y if $[y,y+\varepsilon z]\subset K$ for some $\varepsilon>0$, and a *quasi-radial direction* if $\operatorname{dist}(y+\sigma z,K)=o(\sigma^2)$ for $\sigma>0$.

Note that any radial direction is also quasi-radial, and both radial and quasi radial directions are tangent. With $\hat{x}\in F(P_A)$, we associate the *critical cone*

$$C(\hat{x}):=\{z\in X: Df(\hat{x})z\le 0,\ DG_E(\hat{x})z=0,\ DG_I(\hat{x})z\in T_K(G_I(\hat{x}))\}. \tag{A.12}$$

Definition A.7. We say that $z\in C(\hat{x})$ is a *radial (quasi radial) critical direction* for problem (P_A) if $DG_I(\hat{x})z$ is a radial (quasi radial) direction to K_I at $G_I(\hat{x})$. We write $C_{QR}(\hat{x})$ for the *set of quasi radial critical directions.* The critical cone $C(\hat{x})$ is *quasi radial* if $C_{QR}(\hat{x})$ is a dense subset of $C(\hat{x})$.

It is immediate to check that $C_{QR}(\hat{x})$ is a convex cone.

We next state *primal second order necessary conditions* for the problem (P_A). Consider the following optimization problem, where $z\in X$, $w\in X$ and $\theta\in\mathbb{R}$:

$$(Q_z)\qquad \begin{cases} \min\limits_{w,\theta}\ \theta,\\ Df(\hat{x})w+D^2f(\hat{x})(z,z)\le\theta,\\ DG_E(\hat{x})w+D^2G_E(\hat{x})(z,z)=0,\\ DG_I(\hat{x})w+D^2G_I(\hat{x})(z,z)-\theta(G(\hat{x})-y_I)\in T_K(G_I(\hat{x})). \end{cases}$$

Theorem A.8. *Let $(\hat{x},0)$ be a local solution of (AP_A), such that $DG_E(\hat{x})$ is surjective, and let $h\in C_{QR}(\hat{x})$. Then problem (Q_z) is feasible, and has a nonnegative value.*

Proof. We shall first show that (Q_z) is feasible. Since $DG_E(\hat{x})$ is surjective, there exists $w\in X$ such that $DG_E(\hat{x})w+D^2G_E(\hat{x})(z,z)=0$. Since $T_K(G_I(\hat{x}))$ is a cone, the last equation divided by $\theta>0$ is equivalent to

$$\theta^{-1}\big(DG_I(\hat{x})w+D^2G_I(\hat{x})(z,z)\big)+y_I-G(\hat{x})\in T_K(G_I(\hat{x})). \tag{A.13}$$

Since $y_I\in\operatorname{int}(K_I)$, we have that $y_I-G(\hat{x})\in\operatorname{int}T_K(G_I(\hat{x}))$, and therefore the last constraint of (Q_z) holds when θ is large enough. So it does the first constraint, and hence, (Q_z) is feasible.

We next have to show that we cannot have $(w,\theta_0)\in F(AP_A)$ with $\theta_0<0$. Let us suppose, on the contrary, that there is such a feasible solution (w,θ_0). Let $\theta:=\frac{1}{2}\theta_0$.

Then $Df(\hat{x})w + D^2 f(\hat{x})(z,z) < \theta$. Using (A.13) and $y_I \in \operatorname{int}(K_i)$, we can easily show that, for some $\varepsilon > 0$:

$$DG_I(\hat{x})w + D^2 G_I(\hat{x})(z,z) - \theta(G(\hat{x}) - y_I) + \varepsilon B \in T_K(G_I(\hat{x})). \tag{A.14}$$

Consider, for $\sigma > 0$, the path

$$x_\sigma := \hat{x} + \sigma z + \tfrac{1}{2}\sigma^2 w; \qquad \gamma_\sigma := \tfrac{1}{2}\sigma^2\theta. \tag{A.15}$$

By a second order Taylor expansion we obtain that $G_E(x_\sigma) = o(\sigma^2)$. Since $DG_E(\hat{x})$ is onto, by Lyusternik's theorem [33], there exists a path $x'_\sigma = x_\sigma + o(\sigma^2)$, such that $G_E(x'_\sigma) = 0$. Assuming, without loss of generality, that $G_I(\hat{x}) = 0$, we get

$$G_I(x'_\sigma) = \sigma DG_I(\hat{x})z + \tfrac{1}{2}\sigma^2\left[DG_I(\hat{x})w + D^2 G_I(\hat{x})(z,z)\right] + o(\sigma^2). \tag{A.16}$$

Setting

$$\begin{cases} k_1(\sigma) := (1 - \tfrac{1}{2}\sigma)^{-1}\sigma DG_I(\hat{x})z, \\ k_2(\sigma) := \sigma\big(DG_I(\hat{x})w + D^2 G_I(\hat{x})(z,z)\big), \end{cases} \tag{A.17}$$

we can rewrite (A.16) as

$$G_I(x'_\sigma) = (1 - \tfrac{1}{2}\sigma)k_1(\sigma) + \tfrac{1}{2}\sigma k_2(\sigma) + o(\sigma^2). \tag{A.18}$$

Since z is a quasi radial critical direction, there exists $k'_1(\sigma) \in K_I$ such that

$$\sigma DG_I(\hat{x})z = k'_1(\sigma) + o(\sigma^2), \tag{A.19}$$

and so,

$$G_I(x'_\sigma) \in (1 - \tfrac{1}{2}\sigma)K_I + \tfrac{1}{2}\sigma k_2(\sigma) + o(\sigma^2). \tag{A.20}$$

Using (A.14) and $G_I(\hat{x}) = 0$ we obtain

$$k_2(\sigma) - \sigma\theta(G(\hat{x}) - y_I) + \sigma\varepsilon B \in K_I, \tag{A.21}$$

and so for $\sigma > 0$ small enough

$$G_I(x'_\sigma) \in (1 - \tfrac{1}{2}\sigma)K_I + \tfrac{1}{2}\sigma K_I - \tfrac{1}{2}\sigma^2\theta(y_I - G(\hat{x})) \tag{A.22}$$

and so

$$(1 + \gamma_\sigma)^{-1} G_I(x'_\sigma) \in (1 + \gamma_\sigma)^{-1} K_I + \gamma_\sigma y_I \subset K_I. \tag{A.23}$$

We also easily check that $f(x'_\sigma) < \gamma_\sigma$, and so, we have constructed a feasible path for (AP_A) that contradicts the local optimality of $(\hat{x}, 0)$. □

We now present dual second order necessary conditions. Consider the problem

$$\max_{\lambda \in \Lambda(\hat{x})} D^2_{xx} L(\hat{x}, \lambda)(z,z). \tag{A.24}$$

Theorem A.9. *Let $\hat{x}$ be a local minimum of (P_A), that satisfies the qualification condition* (A.5)*. Then, for every $z \in C_{QR}(\hat{x})$,*

$$\max_{\lambda \in \Lambda(\hat{x})} D^2_{xx} L(\hat{x}, \lambda)(z,z) \geq 0. \tag{A.25}$$

Proof. Since problem (Q_z) is qualified with a finite nonnegative value, by the convex duality theory [17], its dual has a nonnegative value and a nonempty set of solutions. The Lagrangian of problem (Q_z) in qualified form ($\beta_0 = 1$) can be written as

$$D_x L(\hat{x}, \lambda)w + D^2_{xx} L(\beta, \hat{x}, \lambda)(z,z) + \big(1 - \beta + \langle \lambda_I, G(\hat{x}) - y_I \rangle\big)\theta \tag{A.26}$$

where $\lambda = (\beta, \lambda_E, \lambda_I)$ as before, and so, the dual problem of (Q_z) can be written as

$$\operatorname*{Max}_{\lambda \in \Lambda(\hat{x})} D^2_{xx} L(\beta, \hat{x}, \lambda)(z, z); \quad \beta + \langle \lambda_I, G(\hat{x}) - y_I \rangle)\theta = 1.$$

The conclusion follows. □

Remark A.10. Whereas the above theorem follows from Cominetti [14] or Kawasaki [30], our proof avoids the concepts of second order tangent set and its associated calculus, used in these references. This considerably simplifies the proof.

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