Strong Inequalities for Chance–Constrained Programming

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Mar. 11, 2015

Abstract

As an essential substructure underlying a large class of chance-constrained programming problems with finite discrete distributions, the mixing set with $0-1$ knapsack has received considerable attentions in recent literature. In this study, we present a family of strong inequalities that subsume known inequalities for this set. We also find many other inequalities that can be explained by lifting on $0-1$ knapsack with continuous variable restrained in piecewise intervals. To develop stronger inequalities for the original chance-constrained model, earlier studies consider a relaxation of the intersection of multiple mixing sets by aggregating individual mixing sets into a single mixing set. Then, the inequalities can be generated from this resulting single mixing set. Nevertheless, it is unknown that if this procedure will derive facet-defining inequalities for the intersection of multiple mixing sets. Different from such traditional formulation aggregation strategy, we propose a new *blending* procedure that can produce strong inequalities by combining valid inequalities of all individual mixing sets. We further show that the inequalities are facet–defining under certain conditions. In the computational experiments, we illustrate the efficacy of the proposed inequalities with static probabilistic lot-sizing problems.

1 Introduction

Chance constraints appear in optimization formulations of many important planning and design applications that model service level or reliability requirements. When the randomness occurs only at the right–hand–side vector, the chance–constrained programming problem (CCP), with joint probabilistic constraints, can be formulated as follows

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & y = \mathbf{A}\mathbf{x} \\ & \mathbb{P}\{y \geq \mathbf{h}(\omega)\} \geq 1 - \tau \\ & \mathbf{x} \in \mathbb{R}^{m_1} \times \mathbb{Z}^{m_2} \end{aligned}$$

where we let ω be a random scenario in probability space Ω and $\mathbf{x}$ be a m–dimensional decision variable, and define scalars that $\mathbf{A}$ is $d \times m$ matrix, $\mathbf{h}(\omega)$ is d–dimensional column random vectors, $\mathbf{c}$ is m–dimensional cost vector, and τ is a threshold probability with $0 \leq \tau \leq 1$. The chance–constrained program with joint probabilistic constraints was first studied by Miller and Wagner [19]. A disjunctive programming reformulation for chance–constrained programs with discrete distributions is studied in [6, 8, 24] by using the concept of $1 - \epsilon$ efficient points [18]. The chance–constrained programs with discrete distributions has many applications, such as probabilistic lot-sizing [6, 32], health care [5, 25], probabilistic set covering [7, 23]. Recently, many studies extend CCP to multi-stage CCP by deriving valid inequalities for the deterministic equivalent formulation [32] or strong feasibility and optimality cuts for decomposition algorithms [15, 16, 27, 29].

In this paper, we consider a mixed–integer programming (MIP) reformulation of CCP with finite discrete distribution, which is proposed in [17]. Suppose Ω has finitely many realizations, i.e., $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ and π_i is the probability associated with ω_i $\forall i \in \{1, \dots, n\}$. Let h_{ri} be the r-th component of $\mathbf{h}(\omega_i)$. As described in [14, 17], we can assume $h_{ri} \geq 0$ without loss of generality. Throughout, we denote $[i, j] \equiv \{r \in \mathbb{Z} : i \leq r \leq j\}$. A deterministic equivalent of CCP is (see also [14, 17])

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{y} = \mathbf{A}\mathbf{x} \\ & y_r \geq h_{ri}(1 - z_i) \qquad \forall r \in [1, d], i \in [1, n] && (1) \\ & \sum_{i=1}^{n} \pi_i z_i \leq \tau && (2) \\ & \mathbf{x} \in \mathbb{R}^{m_1} \times \mathbb{Z}^{m_2}, \mathbf{z} \in \{0, 1\}^n \end{aligned}$$

where $z_i = 0$ indicates that demands in scenario i are required to be satisfied and $z_i = 1$ otherwise. The constraints (1) and (2) define a substructure of the MIP reformulation, which is the intersection of mixing sets with $0 - 1$ knapsack

$$\mathcal{Q} = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^d_+ \times \{0, 1\}^n : \sum_{i=1}^{n} \pi_i z_i \leq \tau,\ y_r \geq h_{ri}(1 - z_i),\ r \in [1, d], i \in [1, n] \right\}$$

and for each $r \in [1, d]$, we have

$$\mathcal{Q}_r = \left\{ (y_t, \mathbf{z}) \in \mathbb{R}_+ \times \{0,1\}^n : \sum_{i=1}^{n} \pi_i z_i \leq \tau, s_r \geq h_{ri}(1 - z_i) \ \forall i \in [1, n] \right\}.$$

By dropping the index r, we redefine the single mixing set with $0-1$ knapsack as

$$\mathcal{K} = \left\{ (y, z) \in \mathbb{R}_+ \times \{0,1\}^n : \sum_{i=1}^{n} \pi_i z_i \leq \tau, \ y \geq h_i(1 - z_i), \ i \in [1, n] \right\}.$$

Note that we can assume $h_1 \geq h_2 \geq \cdots \geq h_n \geq 0$ without loss of generality. If the $0-1$ knapsack is ignored, we have the following mixing set $\mathcal{M}$ [12]

$$\mathcal{M} = \{(y, z) \in \mathbb{R}_+ \times \{0,1\}^n : y \geq h_i(1 - z_i), \ i \in [1, n]\}.$$

The mixing set was first introduced by Günlük and Pochet [12] on general integer variables and then extensively studied in varying degrees of generality by [2, 11, 20, 30, 31] due to the fact that it arises in many applications. Atamtürk et al. [2] provided valid inequalities,

$$y + \sum_{j=1}^{a} (h_{t_j} - h_{t_{j+1}}) z_{t_j} \geq h_{t_1} \quad \forall T = \{t_1, \ldots, t_a\} \subseteq [1, n] \tag{3}$$

where $t_1 < \cdots < t_a$ and $h_{t_{a+1}} = 0$ and showed that (3) are sufficient to define the convex hull of feasible solutions of $\mathcal{M}$ [12, 11, 20].

To study the single mixing set with $0-1$ knapsack $\mathcal{K}$, [14, 17] have defined two important parameters ν and p. The parameter ν is defined such that

$$\sum_{i=1}^{\nu} \pi_i \leq \tau \text{ and } \sum_{i=1}^{\nu+1} \pi_i > \tau.$$

As noted by [17], we have $y \geq h_{\nu+1}$ and the mixing set can be strengthened as follows

$$\left\{ (y, z) \in \mathbb{R}_+ \times \{0,1\}^n : \sum_{i=1}^{n} \pi_i z_i \leq \tau, \ y + (h_i - h_{\nu+1}) z_i \geq h_i, \ i \in [1, \nu] \right\}. \tag{4}$$

Indeed, by using $y + (h_i - h_{\nu+1}) z_i \geq h_i$ in (4) to replace (1), we obtain a new formulation of CCP with a tighter LP relaxation and less constraints. Let $\{\langle 1 \rangle, \langle 2 \rangle, \ldots, \langle n \rangle\}$ be a permutation of set $[1, n]$ with

$$\pi_{\langle 1 \rangle} \leq \pi_{\langle 2 \rangle} \leq \cdots \leq \pi_{\langle n \rangle}.$$

The parameter p is defined such that

$$\sum_{i=1}^{p} \pi_{\langle i \rangle} \leq \tau \ \text{ and } \sum_{i=1}^{p+1} \pi_{\langle i \rangle} > \tau.$$

Note that in the case of equal probabilities, i.e., $\pi_i = 1/n \ \forall i \in [1, n]$, the knapsack constraint reduces to the following cardinality constraint

$$\sum_{i=1}^{n} z_i \leq p.$$

To the best of our knowledge, the single mixing set with $0-1$ knapsack was first studied by Luedtke et al. [17]. They extended the inequality (3) to the inequality

$$y + \sum_{j=1}^{a}(h_{t_j} - h_{t_{j+1}})z_{t_j} \geq h_{t_1} \quad \forall T = \{t_1, \ldots, t_a\} \subseteq [1, n] \tag{5}$$

where $t_1 < \cdots < t_a$ and $h_{t_{a+1}} = h_{v+1}$, and showed that it is facet–defining for $\mathcal{K}$ when $t_1 = 1$. Then, the result was generalized in [14] for general probabilities case, which is also included in this paper as Theorem 2.1. Recently, the result was investigated by Abdi et al. in [1]. In this paper, we further generalize results presented in [1, 14] and subsume known inequalities as special cases. In addition, we show that valid inequalities of $\mathcal{K}$ can be obtained by first starting from lifted cover inequalities of the $0-1$ knapsack, which is a well studied topic in MIP [3, 4, 28, 21, 9, 10], then involving the mixing set through lifting. The procedure includes lifting continuous variable y and some binary variables z_i, which are fixed at 0 or 1.

To derive strong inequalities for $\mathcal{Q}$, [14] developed a blending method on combining original formulation specifically for case $d = 2$. They try to find good scalars β_1, β_2 to obtained a weighted sum of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ such that

$$\mathcal{Q}' = \left\{ (s, \mathbf{z}) \in \mathbb{R}_+ \times \{0,1\}^n : \sum_{i=1}^{n} \pi_i z_i \leq \tau, s \geq (\beta_1 h_{1i} + \beta_2 h_{2i})(1 - z_i) \ \forall i \in [1, n] \right\}.$$

Then, the results on set $\mathcal{K}$ can be readily applied to $\mathcal{Q}'$. Nevertheless, it remains unknown that if this procedure will derive facet-defining inequalities of $\mathcal{Q}$. Instead of combining original constraints from $\mathcal{Q}_1$ and $\mathcal{Q}_2$, in this paper, we will define a different blending procedure to combine strong inequalities of $\mathcal{Q}_1$ and $\mathcal{Q}_2$. We show that, under certain conditions, this procedure leads to facet-defining inequalities of $\mathcal{Q}$.

The remainder of the paper is organized as follows. In Section 2, we present a family of strong inequalities for $\mathcal{K}$ that include earlier results as special cases. We also provide a sufficient condition under which the inequalities are facet–defining. In Section 3, we first assume a lifted cover inequality from $0-1$ knapsack by fixing binary variables. Then we perform the lifting procedure on continuous variable y and binary variables that are fixed at 0. At last, we derive a valid inequality of $\mathcal{K}$ by lifting binary variables that are fixed at 1. In Section 4, we give strong inequalities of $\mathcal{Q}$ by blending inequalities of $\mathcal{Q}_t \ \forall t \in [1, d]$ and show that they are facet–defining under certain conditions. In Section 5, computational results on a set static probabilistic lot-sizing instances are presented, which validates the efficacy of the proposed inequalities. Section 6 concludes this paper.

2 Strong inequalities derived from mixing set

In this section, we consider the set $\mathcal{K}$, which is a single mixing set with $0-1$ knapsack. As a generalization of inequalities (3) for the mixing set, earlier studies showed

Theorem 2.1 (Theorem 6 in [14]) *For $m \in \mathbb{Z}_+$ such that $m \leq \nu$, let $T = \{t_1, \dots, t_a\} \subseteq [1, m]$ with $t_1 < \dots < t_a$, $L \subseteq [m+2, n]$ and a permutation of elements in L, $\Pi_L = \{l_1, \dots, l_{p-m}\}$ such that $l_j \geq m+1+j$. The inequality*

$$y + \sum_{j=1}^{a} (h_{t_j} - h_{t_{j+1}}) z_{t_j} + \sum_{j=1}^{p-m} \delta_j (1 - z_{l_j}) \geq h_{t_1} \tag{6}$$

is valid for $\mathcal{K}$, where $t_{a+1} = m+1$, and

$$\delta_j = \begin{cases} h_{m+1} - h_{m+1+\min\{\nu-m,1\}} & j = 1 \\ \max\left\{ \delta_{j-1}, h_{m+1} - h_{m+1+\min\{\nu-m,j\}} - \sum_{i \in [1,j-1] \text{ and } l_i \geq m+1+j} \delta_i \right\} & j \in [2, p-m] \end{cases}$$

Next, we give a family of inequalities that contain inequality (6) as a special case.

Theorem 2.2 Given $m \in [1, \nu]$ and set $T = \{t_1, \dots, t_a\} \subseteq [1, m]$ with $t_1 < \dots < t_a$. For $q \in [0, p-m]$, let $L \subseteq [m + s_1 + 1, n]$ and $\Pi_L = \{l_1, \dots, l_q\}$ be a permutation of the elements in L with $l_j \geq m + 1 + s_j$ such that $0 \leq s_1 \leq \dots \leq s_q \leq s_{q+1} = \nu - m + 1$ satisfy

$$\sum_{i=1}^{m+s_j} \pi_i + \sum_{i=j}^{q} \pi_{k_i} > \tau \quad \forall j \in [1, q]$$

where $\{k_1, \dots, k_q\}$ is a permutation of L with $\pi_{k_1} \geq \dots \geq \pi_{k_q}$ and $t_a < t_{a+1} \leq m + s_1$. The $(T, \Pi_L, \mathbf{s})$ inequality

$$y + \sum_{j=1}^{a} (h_{t_j} - h_{t_{j+1}}) z_{t_j} + (h_{t_{a+1}} - f) z_{t_{a+1}} + \sum_{j=1}^{q} \delta_j (1 - z_{l_j}) \geq h_{t_1} \tag{7}$$

is valid for $\mathcal{K}$, where

$$\delta_j = \begin{cases} \max(f - h_{m+s_2}, 0) & j = 1 \\ \max\left\{ \delta_{j-1}, f - h_{m+s_{j+1}} - \sum_{i \in [1,j-1] \text{ and } l_i \geq m+1+s_j} \delta_i \right\} & j \in [2, q] \end{cases} \tag{8}$$

and $h_{m+s_1} \geq f$.

Before prove the theorem, we will show a few cases that the inequality (7) generalizes inequalities in earlier studies.

- Let $q = p - m$, $t_{a+1} = m + s_1$ and $f = h_{m+s_1}$. The definition of p in Section 1 implies that the summation of any $p+1$ many π_is for $i \in [1, n]$ is strictly greater than τ. Thus we have

$$\sum_{i=1}^{m+j} \pi_i + \sum_{i=j}^{p-m} \pi_{k_i} > \tau \quad \forall j \in [1, q].$$

Also, because

$$\sum_{i=1}^{\nu+1} \pi_i = \sum_{i=1}^{m+(\nu-m+1)} \pi_i > \tau,$$

we note that the conditions in Theorem 2.2 are satisfied when $s_j = \min\{j, \nu - m + 1\}$ $\forall j \in [1, q]$ and the inequality (7) becomes (6). It is easy to see that if we choose s_j such that

$$\sum_{i=1}^{m+s_j-1} \pi_i + \sum_{i=j}^{q} \pi_{k_i} \leq \tau \quad \forall j \in [1, q]$$

we have $s_j \leq \min\{j, \nu - m + 1\}$ $\forall j \in [1, q]$. So, the inequality (7) is at least as strong as (6).

Example 1 (Example 1 in [14]) *Let* $h = (40, 38, 34, 31, 26, 16, 8, 4, 2, 1)$ *for* $n = 10$, *and* $\pi_1 = \cdots = \pi_4 = \tau/4$ *and* $\pi_5 = \cdots = \pi_{10} = \tau/6$ *with* $\tau = 0.5$. *It is easy to check that* $\nu = 4$ *and* $p = 6$. *As in* [14], *inequality* (6) *with* $m = 1$, $t_1 = 1$ *and* $\Pi_L = \{4, 6, 7, 8, 9\}$ *gives*

$$\begin{aligned} y + (h_1 - h_2)z_1 &+ (h_2 - h_3)(1 - z_4) + (h_2 - h_3)(1 - z_6) + (h_2 - h_5 - \delta_2)(1 - z_7) \\ &+ (h_2 - h_5 - \delta_2)(1 - z_8) + (h_2 - h_5 - \delta_2)(1 - z_9) \geq h_1 \end{aligned}$$

or specifically,

$$y + 2z_1 + 4(1 - z_4) + 4(1 - z_6) + 8(1 - z_7) + 8(1 - z_8) + 8(1 - z_9) \geq 40$$

where $\delta_2 = h_2 - h_3$ *is the coefficient for term* $(1 - z_6)$. *This inequality is not facet–defining, since*

$$y + (h_1 - h_2)z_1 + (h_2 - h_3)(1 - z_4) + (h_2 - h_3)(1 - z_7) + (h_2 - h_3)(1 - z_8) \geq h_1 \quad (9)$$

or specifically,

$$y + 2z_1 + 4(1 - z_4) + 4(1 - z_7) + 4(1 - z_8) \geq 40$$

is valid and facet-defining. Inequality (9) *can be generated by letting* $m = 1$, $\Pi_L = \{4, 7, 8\}$, $t_{a+1} = m + s_1$ *and* $f = h_{m+s_1}$, *and we get* $s_1 = 1$, $s_2 = 2$ *and* $s_3 = 3$ *by simple calculations.*

- Let $a = 0$. It is easy to see that we cannot derive strong inequalities from Theorem 2.1 because an inequality with same parameters except $a = 1$ is stronger. However, the inequality (7) becomes

$$y + (h_{t_1} - f)z_{t_1} + \sum_{j=1}^{q} \delta_j(1 - z_{l_j}) \geq h_{t_1}$$

which could be a facet–defining inequality as shown in the next example,

Example 1 *1 (cont.) Let $m = 1$, $t_{a+1} = t_1 = 1$ and $\Pi_L = \{4, 5, 6\}$. It is easy to calculate that $s_1 = 1$, $s_2 = 2$, $s_3 = 3$ and $s_4 = 4$. If $f = 37 \leq h_{m+s_1} = h_2$, we get a facet–defining inequality*

$$y + (h_1 - f)z_1 + (f - h_3)(1 - z_4) + (f - h_4)(1 - z_5) + (f - h_5 - \delta_2)(1 - z_6) \geq h_{t_1}.$$

or specifically,

$$y + 3z_1 + 3(1 - z_4) + 3(1 - z_5) + 8(1 - z_6) \geq 40$$

- In [1], the authors scaled the probabilities by using an integer M such that $M\pi_i = a_i$ $\forall i \in [1, n]$ and $M\tau = \mu$, where a_i and μ are integers. They assumed that $a_i = 1$ $\forall i \in L$. We note that (7) implies the inequalities in [1] if we let $q = \mu - \sum_{i=1}^{m} a_i$ and $s_j = k(j) - m + 1$, where $k(j)$ is defined in the Theorem 7 of [1]. According to the definition in [1], we have

$$\sum_{i=1}^{k(j)+1} a_i - \sum_{i=1}^{m} a_i \geq j \;\forall j \in [1, q]$$

which implies that $\forall j \in [1, q]$

$$\begin{aligned} \sum_{i=1}^{m+s_j} \pi_i + \sum_{i=j}^{q} \pi_{k_i} &= \frac{1}{M}\left(\sum_{i=1}^{m+s_j} a_i + \sum_{i=j}^{q} a_{k_i}\right) \\ &= \frac{1}{M}\left(\sum_{i=1}^{k(j)+1} a_i + q - j + 1\right) \\ &= \frac{1}{M}\left(\sum_{i=1}^{k(j)+1} a_i - \sum_{i=1}^{m} a_i + \mu - j + 1\right) \\ &\geq \frac{1}{M}(M\tau + 1) > \tau \end{aligned}$$

Therefore, the choice of s_j $\forall j \in [1, q]$ satisfy the condition in Theorem 2.2.

Proof of Theorem 2.2 If $y \geq h_{t_1}$, then the inequality (7) is trivially satisfied. First, we suppose $a \geq 1$. If $y \geq h_{t_i}$ for some $i = 2, \ldots, a+1$ and $y < h_{t_j}$ for all $j \in [1, i-1]$, then we must have $z_{t_j} = 1$ for all $j \in [1, i-1]$. Thus,

$$
\begin{aligned}
y + \sum_{j=1}^{a} (h_{t_j} - h_{t_{j+1}}) z_{t_j} &\geq h_{t_i} + \sum_{j=1}^{i-1} (h_{t_j} - h_{t_{j+1}}) = h_{t_1} \\
&\geq h_{t_1} - (h_{t_{a+1}} - f) z_{t_{a+1}} - \sum_{j=1}^{q} \delta_j (1 - z_{l_j})
\end{aligned}
$$

and inequality (7) is satisfied when $y \geq h_{t_{a+1}}$. If $h_{t_{a+1}} > y \geq h_{m+s_1}$, then $z_{t_j} = 1 \ \forall j \in 1, \ldots, a+1$. Hence,

$$
\begin{aligned}
& y + \sum_{j=1}^{a} (h_{t_j} - h_{t_{j+1}}) z_{t_j} + (h_{t_{a+1}} - f) z_{t_{a+1}} \\
\geq \ & \max\{h_{\nu+1}, h_{m+s_1}\} + \sum_{j=1}^{a} (h_{t_j} - h_{t_{j+1}}) + (h_{t_{a+1}} - f) \\
\geq \ & \max\{h_{\nu+1}, h_{m+s_1}\} + h_{t_1} - f \geq h_{t_1} \qquad (10) \\
\geq \ & h_{t_1} - \sum_{j=1}^{q} \delta_j (1 - z_{l_j})
\end{aligned}
$$

where (10) holds because $h_{m+s_1} \geq f$. Now, we suppose $a = 0$, which implies that $T = \emptyset$ and $t_{a+1} = t_1$. If $h_{t1} > y \geq h_{m+s_1}$, we have

$$
\begin{aligned}
y + (h_{t_1} - f) z_{t_1} &\geq \max\{h_{\nu+1}, h_{m+s_1}\} + h_{t_1} - f \geq h_{t_1} \\
&\geq h_{t_1} - \sum_{j=1}^{q} \delta_j (1 - z_{l_j}).
\end{aligned}
$$

Therefore, (7) is valid when $y \geq h_{m+s_1}$. If $q = 0$, we have $m + s_1 = m + \nu - m + 1 = \nu + 1$. Since $y \geq h_{\nu+1}$. The proof is done. Hence, we assume $q \geq 1$, $y < h_{m+s_1}$ and $z_{t_j} = 1$ $\forall j = 1, \ldots, a+1$ in the rest of proof, which implies that

$$
\sum_{j=1}^{a} (h_{t_j} - h_{t_{j+1}}) z_{t_j} + (h_{t_{a+1}} - f) z_{t_{a+1}} = h_{t_1} - f.
$$

Then, we must have $h_{m+s_{i'}} > y \geq h_{m+s_{i'+1}}$ for some $i' = 1, \ldots, q$ because $y \geq h_{\nu+1}$ and $s_{q+1} = \nu - m + 1$. Thus, $z_j = 1$ for all $j = 1, \ldots, m + s_{i'}$, and we have

$$
\sum_{i=1}^{m+s_{i'}} \pi_i + \sum_{j=m+s_{i'}+1}^{n} \pi_i z_i \leq \tau \qquad (11)
$$

and

$$
\begin{aligned}
\sum_{j=1}^{q} \delta_j(1-z_{l_j}) &= \sum_{j\in[1,q] \text{ and } l_j\geq m+s_{i'}+1} \delta_i(1-z_{l_j}) \\
&= \sum_{j=i'+1}^{q} \delta_j(1-z_{l_j}) + \sum_{j\in[1,i'] \text{ and } l_j\geq m+s_{i'}+1} \delta_i(1-z_{l_j}) \\
&= \sum_{j=i'+1}^{q} \delta_j + \sum_{j\in[1,i'] \text{ and } l_j\geq m+s_{i'}+1} \delta_i - \sum_{j=i'+1}^{q} \delta_j z_{l_j} - \sum_{j\in[1,i'] \text{ and } l_j\geq m+s_{i'}+1} \delta_i z_{l_j}. \qquad (12)
\end{aligned}
$$

The first equality holds because $z_j = 1 \ \forall j \in [1, m+s_{i'}]$. The second equality holds because

$$l_j \geq m + s_j + 1 \geq m + s_{i'} + 1 \ \forall j \in [i'+1, q].$$

We claim that

$$\sum_{i=1}^{q} z_{l_j} \leq q - i' \qquad (13)$$

by introducing a contradiction. Suppose $\sum_{i=1}^{q} z_{l_j} \geq q - i' + 1$. Then we have

$$
\begin{aligned}
\tau &\geq \sum_{i=1}^{m+s_{i'}} \pi_i + \sum_{j=m+s_{i'}+1}^{n} \pi_i z_i \qquad (14) \\
&\geq \sum_{i=1}^{m+s_{i'}} \pi_i + \sum_{j\in[1,q] \text{ and } l_j\geq m+s_{i'}+1} \pi_{l_j} z_{l_j} \\
&\geq \sum_{i=1}^{m+s_{i'}} \pi_i + \sum_{i=i'}^{q} \pi_{l_j} z_{l_j} + \sum_{j\in[1,i'] \text{ and } l_j\geq m+s_{i'}+1} \pi_{l_j} z_{l_j} \qquad (15) \\
&\geq \sum_{i=1}^{m+s_{i'}} \pi_i + \sum_{i=i'}^{q} \pi_{k_j} > \tau \qquad (16)
\end{aligned}
$$

where the inequality (14) is just (11). Inequality (15) holds because $l_j \geq m + s_j + 1 \geq m+s_{i'}+1 \ \forall j \in [i', q]$. Inequality (16) holds because $\sum_{i=1}^{q} z_{l_j} \geq q-i'+1$ and $\pi_{k_1} \geq \cdots \geq \pi_{k_q}$. Thus, (13) holds. Note that $\delta_1 \leq \cdots \leq \delta_{q+1}$ is monotonic. With (13), we get

$$\sum_{j=i'+1}^{q} \delta_j z_{l_j} + \sum_{j\in[1,i'] \text{ and } l_j\geq m+s_{i'}+1} \delta_i z_{l_j} \leq \sum_{j=i'+1}^{q} \delta_j.$$

Then from (12), we have

$$\begin{aligned}\sum_{j=1}^{q}\delta_j(1-z_{l_j}) &= \sum_{j=i'+1}^{q}\delta_j + \sum_{j\in[1,i'] \text{ and } l_j\geq m+s_{i'}+1}\delta_i - \sum_{j=i'+1}^{q}\delta_j z_{l_j} - \sum_{j\in[1,i'] \text{ and } l_j\geq m+s_{i'}+1}\delta_i z_{l_j}\\ &\geq \sum_{j\in[1,i'] \text{ and } l_j\geq m+s_{i'}+1}\delta_i\\ &\geq \delta_{i'} + \sum_{j\in[1,i'-1] \text{ and } l_j\geq m+s_{i'}+1}\delta_i \geq f - h_{m+s_{i'+1}}.\end{aligned}$$

The last inequality holds because of the definition of $\delta_{i'}$. Therefore, we have

$$\begin{aligned}&y + \sum_{j=1}^{a}(h_{t_j}-h_{t_{j+1}})z_{t_j} + (h_{t_{a+1}}-f)z_{t_{a+1}} + \sum_{j=1}^{q}\delta_j(1-z_{l_j})\\ \geq\ & h_{m+s_{i'+1}} + h_{t_1} - f + f - h_{m+s_{i'+1}} \geq h_{t_1}.\end{aligned}$$

and (7) is valid for $\mathcal{K}$. □

The next proposition will give necessary conditions that (7) is facet–defining for $\mathcal{K}$.

Proposition 2.1 *If (7) is facet–defining for $\mathcal{K}$, then we have $t_1 = 1$ and*

$$\sum_{i=1}^{m+s_j-1}\pi_i + \sum_{i=j}^{q}\pi_{k_i} \leq \tau \quad \forall j\in[1,q]. \tag{17}$$

Proof First, we will prove the necessary conditions that $t_1 = 1$. Given a $(T,\Pi_L,\mathbf{s})$ inequality (7) with $t_1 > 1$. We can have a $(T',\Pi_L,\mathbf{s})$ inequality (7) with $T' = T\cup\{1\}$, i.e.,

$$y + (h_1-h_{t_1})z_1 + \sum_{j=1}^{a}(h_{t_j}-h_{t_{j+1}})z_{t_j} + (h_{t_{a+1}}-f)z_{t_{a+1}} + \sum_{j=1}^{q}\delta_j(1-z_{l_j}) \geq h_1$$

which implies

$$(h_1-h_{t_1})(z_1-1) + y + \sum_{j=1}^{a}(h_{t_j}-h_{t_{j+1}})z_{t_j} + (h_{t_{a+1}}-f)z_{t_{a+1}} + \sum_{j=1}^{q}\delta_j(1-z_{l_j}) \geq h_{t_1}.$$

As $(h_1-h_{t_1})(z_1-1)\leq 0$, $(T',\Pi_L,\mathbf{s})$ inequality is at least as strong as the $(T,\Pi_L,\mathbf{s})$ inequality.

Suppose (17) does not hold for a $(T,\Pi_L,\mathbf{s})$ inequality with coefficient $\delta_j\ \forall j\in[1,q]$. Thus, we have

$$\sum_{i=1}^{m+s_{i'}-1}\pi_i + \sum_{i=i'}^{q+1}\pi_{k_i} > \tau \quad \text{for some } i'\in[1,q]$$

and a $(T, \Pi_L, \mathbf{s}')$ inequality such that $s'_j = s_j \ \forall j \in [1,q] - \{i\}$ and $s'_{i'} = s_{i'} - 1$. It is clear that we have $\delta'_j \leq \delta_j \ \forall j \in [1,q]$ where δ'_j is the coefficient for $(T, \Pi_L, \mathbf{s}')$ inequality. So $(T, \Pi_L, \mathbf{s}')$ inequality is at least as strong as the $(T, \Pi_L, \mathbf{s})$ inequality. □

Because of the necessary conditions, we can assume $t_1 = 1$ and

$$s_j = \min \left\{ s'_j : \sum_{i=1}^{m+s'_j} \pi_i + \sum_{i=j}^{q} \pi_{k_i} > \tau \quad \forall j \in [1,q] \right\} \tag{18}$$

to obtain stronger inequalities without loss of generality.

Theorem 2.3 *The inequality* (7) *is facet-defining for* $\mathcal{K}$ *if* $t_1 = 1$, $\pi_{l_1} \geq \cdots \geq \pi_{l_{q+1}}$,

$$\sum_{i=1}^{q} \pi_{k_i} + \pi_j \leq \tau \quad \forall j \notin T \cup L \tag{19}$$

$$\sum_{i=1}^{m+s_1-1} \pi_i + \sum_{i=1}^{q} \pi_{l_i} \leq \tau \ \textit{and} \ \sum_{i=1}^{m+s_j} \pi_i + \sum_{i=j+1}^{q} \pi_{l_i} \leq \tau \quad \forall j \in [1,q] \tag{20}$$

and either one of the following cases holds

1. $t_{a+1} = m + s_1$ *and* $f = h_{m+s_1}$,
2. $f = 2h_{m+s_2} - h_{m+s_3}$, *if* $2h_{m+s_2} - h_{m+s_3} \leq h_{m+s_1}$ *and* $l_1 \geq m + s_2 + 1$.

Proof The proof is similar to the proof of Theorem 4 in [14]. However, since our inequality (7) is more general, we give a self–contained proof. First, let $y^0 = h_1$, $z^0_j = 1$ if $j \in L$ and $z^0_j = 0$ otherwise. Next, for each $j \notin (T \cup L)$, we have point $(y^j, z^j) = (y^0, \mathbf{z}^0 + e_j)$, where e_j is n dimensional unit vector with jth component equal to 1. The point is feasible because of (19).

For each $j \in [1,a]$, let $y^{t_j} = h_{t_{j+1}}$, $z^{t_j}_i = 1$ if $i \in [1, t_{j+1} - 1] \cup L$ and $z^{t_j}_i = 0$ otherwise. The point is feasible because the condition

$$\sum_{i=1}^{m+s_1-1} \pi_i + \sum_{i=1}^{q} \pi_{l_i} \leq \tau.$$

For $j = t_{a+1}$, if $t_{a+1} = m + s_1$ and $f = h_{m+s_1}$, we let $(y^{t_{a+1}}, z^{t_{a+1}}) = (y^0, \mathbf{z}^0 + e_{t_{a+1}})$. The point is feasible because of (19). If $f = 2h_{m+s_2} - h_{m+s_3}$, we have $\delta_1 = \delta_2 = f - h_{m+s_2}$ because $l_1 \geq m + s_2 + 1$. We let $y^{t_{a+1}} = h_{m+s_2+1}$ and

$$z^{l_j}_i = 1 \ \forall i \in [1, m+s_1] \cup \{l_3, \ldots, l_{q+1}\} \text{ and } z^{l_j}_i = 0 \text{ otherwise.}$$

The point is feasible because of (20). For $j \in [1, q+1]$, first we let $y^{l_1} = h_{m+1+s_1}$ and

$$z^{l_j}_i = 1 \ \forall i \in [1, m+s_1] \cup \{l_2, \ldots, l_{q+1}\} \text{ and } z^{l_j}_i = 0 \text{ otherwise.}$$

The point is feasible because of (20). Then, for $j \in [2, q+1]$, if

$$\delta_j = f - h_{m+1+\min\{\nu-m, s_j\}} - \sum_{i \in [1,j-1] \text{ and } l_i \geq m+1+s_j} \delta_i.$$

Let $y^{l_j} = h_{m+1+s_j}$ and

$$z_i^{l_j} = 1 \ \forall i \in [1, m+s_j] \cup \{l_{j+1}, \ldots, l_{q+1}\} \text{ and } z_i^{l_j} = 0 \text{ otherwise.}$$

If $\delta_j = \delta_{j-1}$, we let

$$(y^{l_j}, \mathbf{z}^{l_j}) = (y^{l_{j-1}}, \mathbf{z}^{l_{j-1}} + e_{l_{j-1}} - e_{l_j}).$$

Note that $\pi_{l_j} \geq \pi_{l_{j-1}}$. In either case, the point is feasible because of (20). These $n+1$ points on the face defined by inequality (7) are affinely independent. □

When all the scenarios have same probabilities, the $0-1$ knapsack constraint becomes a cardinality constraint. It is easy to check that we must have $q = p - m$ and $s_j = j$ $\forall j \in [1, p-m]$. Therefore, the inequality (7) can be simply presented as follows,

Corollary 2.3.1 *For $m \in [1,p]$, let $T = \{t_1, \ldots, t_a, t_{a+1}\} \subseteq [1, m+1]$ with $t_1 < \cdots < t_a < t_{a+1}$, $L \subseteq [m+2, n]$ and a permutation of the elements in L, $\Pi_L = \{l_1, \ldots, l_{p-m}\}$ such that $l_j \geq m+1+j$. The inequality*

$$y + \sum_{j=1}^{a} (h_{t_j} - h_{t_{j+1}}) z_{t_j} + (h_{t_{a+1}} - f) z_{t_{a+1}} + \sum_{j=1}^{p-m} \delta_j (1 - z_{l_j}) \geq h_{t_1} \tag{21}$$

is valid for $\mathcal{K}$, where

$$\delta_j = \begin{cases} \max(f - h_{m+2}, 0) & j = 1 \\ \max\left\{ \delta_{j-1}, f - h_{m+1+j} - \sum_{i \in [1,j-1] \text{ and } l_i \geq m+1+j} \delta_i \right\} & j \in [2, p-m] \end{cases}$$

and $h_{m+1} \geq f$. The inequality is facet-defining if $t_1 = 1$ and either one of the following cases holds

1. *$t_{a+1} = m+1$ and $f = h_{m+1}$,*
2. *$f = 2h_{m+2} - h_{m+3}$, if $2h_{m+2} - h_{m+3} \leq h_{m+1}$ and $l_1 \geq m+3$.*

The property of facet–defining simply follows Theorem 2.3.

Example 2 *1 (cont.) Suppose all scenarios have equal probabilities. Then we get $p = 6$. As stated in* [14], *the inequality* (6) *cannot explain following inequalities*

$$\begin{aligned} y &+ (h_1 - h_2)z_1 + (h_2 - h_3)z_2 + (h_3 - h_6 - \delta_1)z_3 + (h_6 - h_7)(1 - z_7) \\ &+ (h_6 - h_7)(1 - z_9) \geq h_1 \\ y &+ (h_1 - h_3)z_1 + (h_3 - h_6 - \delta_1)z_3 + (h_6 - h_7)(1 - z_7) + (h_6 - h_7)(1 - z_9) \geq h_1 \\ y &+ (h_1 - h_2)z_1 + (h_2 - h_6 - \delta_1)z_2 + (h_6 - h_7)(1 - z_7) + (h_6 - h_7)(1 - z_9) \geq h_1 \end{aligned} \tag{22}$$

which are explained by inequality (21). For example, inequality (22) has $m = 4$, $T = \{1, 2, 3\}$ where $t_{a+1} = 3$ and $f = 2h_6 - h_7$.

3 Strong inequalities derived from lifting

We realize that $\mathcal{K}$ contains a $0-1$ knapsack as a substructure which was well studied in literature [3, 4, 28, 21, 9, 10]. Strong inequalities of $0-1$ knapsack can be obtained by lifting cover inequalities. Thus, a possible way of deriving strong inequalities for $\mathcal{K}$ is considering cover inequalities and lifting procedures. For $m \in [1, \nu]$, let $N_0 = [1, m-1]$ and $N_1 = \{l_1, \ldots, l_q\} \subseteq [\nu + 1, n]$ with cardinality q such that

$$\sum_{i=1}^{m} \pi_i + \sum_{i \in N_1} \pi_i \leq \tau \text{ and } \sum_{i=1}^{m+1} \pi_i + \sum_{i \in N_1} \pi_i > \tau. \tag{23}$$

We note that the mixing inequalities $y \geq h_i(1 - z_i)$ $\forall i \in [1, n]$ become redundant if $y \geq h_1$. So the projection of the set $\mathcal{K} \cap \{y \geq h_1\} \cap \{\mathbf{z} \in \{0,1\}^n : z_i = 0 \ \forall i \in N_0 \text{ and } z_i = 1 \ \forall i \in N_1\}$ on $\mathbf{z}$ variables is simply a $0-1$ knapsack polytope $\mathcal{S}(N_0, N_1)$, where

$$\mathcal{S}(N_0, N_1) = \left\{ \mathbf{z} \in \{0,1\}^n : \sum_{i=1}^{n} \pi_i z_i \leq \tau, z_i = 0 \ \forall i \in N_0 \text{ and } z_i = 1 \ \forall i \in N_1 \right\}.$$

In this section we will show that a valid inequality for $\mathcal{K}$ can be lifted from a valid inequality of $\mathcal{S}(N_0, N_1)$. Because the lifting procedure involves continuous variable y, it can be done by allowing y on piecewise intervals when we lift with respect to variables z_i for $i \in N_1$.

Let C be a cover of the set $\mathcal{S}(N_0, N_1)$, and $C \subseteq E \subseteq [1, n] - N_0 - N_1$, where E defines a lifted cover inequality

$$\sum_{i \in E} \alpha_i z_i \leq \gamma \tag{24}$$

on $\mathcal{S}(N_0, N_1)$ with $\alpha_i > 0$ $\forall i \in E$. We define

$$\begin{aligned} G(\alpha) \ &= \ \max \sum_{i \in E} \alpha_i z_i \\ &\quad \text{s.t.} \ \sum_{i \in E} \pi_i z_i \leq \tau' - \alpha, \ z_i \in \{0,1\} \ \forall i \in E \end{aligned}$$

where $\tau' = \tau - \sum_{i = N_1} \pi_i$, and

$$\begin{aligned} \rho \leq G\left(\sum_{i=1}^{m-1} \pi_i\right) - \quad &\max \quad \alpha_m + \sum_{i \in E - \{m\}} \alpha_i z_i \\ &\text{s.t.} \quad \sum_{i \in E - \{m\}} \pi_i z_i \leq \tau' - \sum_{i=1}^{m} \pi_i, \ z_i \in \{0,1\} \ \forall i \in E - \{m\}. \end{aligned}$$

We denote $\mathcal{K}(N_1) = \mathcal{K} \cap \{\mathbf{z} \in \{0,1\}^n : z_i = 1 \ \forall i \in N_1\}$. Then in the next theorem, we will give valid inequality for the set $\mathcal{K}(N_1)$.

Theorem 3.1 *Given a superadditive function $\Phi(\alpha) \leq \gamma - G(\alpha)$. If $h_m > h_{m+1}$, then the inequality*

$$\sum_{i=1}^{m-1} \alpha_i z_i + \sum_{i \in E} \alpha_i z_i \leq \gamma + \frac{\rho}{h_m - h_{m+1}}(y - h_1) \tag{25}$$

is valid for the set $\mathcal{K}(N_1)$, where

$$\alpha_i = \Phi(\pi_i) + \frac{\rho}{h_m - h_{m+1}}(h_{i+1} - h_i) \tag{26}$$

Proof To prove (25) is valid for $\mathcal{K}(N_1)$, we can fix $z_i = 1 \ \forall i \in N_1$ for the rest of proof. First, we note that $y \geq h_{m+1}$ in the set $\mathcal{K}(N_1)$, because of condition (23). If $y \geq h_1$, we assume that $z_i = 1 \ \forall i \in Q \subseteq [1, m-1]$ and $z_i = 0 \ \forall i \in [1, m-1] - Q$ for a set Q. Then the knapsack constraint in $\mathcal{K}$ becomes

$$\sum_{i \in E} \pi_i z_i \leq \tau' - \sum_{i \in Q} \pi_i$$

and we have

$$\begin{aligned}
\sum_{i=1}^{m-1} \alpha_i z_i + \sum_{i \in E} \alpha_i z_i &\leq \sum_{i \in Q} \Phi(\pi_i) + \sum_{i \in E} \alpha_i z_i \qquad (27)\\
&\leq \Phi\left(\sum_{i \in Q} \pi_i\right) + \sum_{i \in E} \alpha_i z_i \\
&\leq \Phi\left(\sum_{i \in Q} \pi_i\right) + G\left(\sum_{i \in Q} \pi_i\right) \leq \gamma \\
&\leq \gamma + \frac{\rho}{h_m - h_{m+1}}(y - h_1)
\end{aligned}$$

where (27) holds because $\alpha_i \leq \Phi(\pi_i)$. Inequality (25) is valid.

If $h_{t-1} > y \geq h_t$ for some $t = 2, \ldots, m$. We must have $z_i = 1 \ \forall i \in [1, t-1]$, and also assume that $z_i = 1 \ \forall i \in Q \subseteq [t, m-1]$ and $z_i = 0 \ \forall i \in [1, m-1] - Q$ for a set Q. Then the knapsack constraint in $\mathcal{K}$ becomes

$$\sum_{i \in E} \pi_i z_i \leq \tau' - \sum_{i=1}^{t-1} \pi_i - \sum_{i \in Q} \pi_i$$

and we get

$$\begin{aligned}
&\sum_{i=1}^{m-1}\alpha_i z_i+\sum_{i\in E}\alpha_i z_i=\sum_{i=1}^{t-1}\alpha_i+\sum_{i=t}^{m-1}\alpha_i z_i+\sum_{i\in E}\alpha_i z_i\\
\leq\ & \sum_{i=1}^{t-1}\Phi(\pi_i)+\frac{\rho}{h_m-h_{m+1}}(h_t-h_1)+\sum_{i\in Q}\Phi(\pi_i)+\sum_{i\in E}\alpha_i z_i\\
\leq\ & \Phi\left(\sum_{i=1}^{t-1}\pi_i+\sum_{i\in Q}\pi_i\right)+\frac{\rho}{h_m-h_{m+1}}(h_t-h_1)+\sum_{i\in E}\alpha_i z_i\\
\leq\ & \Phi\left(\sum_{i=1}^{t-1}\pi_i+\sum_{i\in Q}\pi_i\right)+\frac{\rho}{h_m-h_{m+1}}(h_t-h_1)+G\left(\sum_{i=1}^{t-1}\pi_i+\sum_{i\in Q}\pi_i\right)\\
\leq\ & \gamma+\frac{\rho}{h_m-h_{m+1}}(y-h_1)
\end{aligned}$$

So the inequality (25) is valid.

Otherwise, we must have $h_m > y \geq h_{m+1}$ and $z_i = 1\ \forall i \in [1, m]$. So, we get

$$\begin{aligned}
&\sum_{i=1}^{m-1}\alpha_i z_i+\sum_{i\in E}\alpha_i z_i=\sum_{i=1}^{m-1}\alpha_i+\alpha_m+\sum_{i\in E-\{m\}}\alpha_i z_i\\
\leq\ & \sum_{i=1}^{m-1}\Phi(\pi_i)+\frac{\rho}{h_m-h_{m+1}}(h_m-h_1)+\alpha_m+\sum_{i\in E-\{m\}}\alpha_i z_i\\
=\ & \Phi\left(\sum_{i=1}^{m-1}\pi_i\right)+\frac{\rho}{h_m-h_{m+1}}(h_{m+1}-h_1)+\rho+\alpha_m+\sum_{i\in E-\{m\}}\alpha_i z_i\\
\leq\ & \Phi\left(\sum_{i=1}^{m-1}\pi_i\right)+\frac{\rho}{h_m-h_{m+1}}(h_{m+1}-h_1)+G\left(\sum_{i=1}^{m-1}\pi_i\right) \qquad (28)\\
\leq\ & \gamma+\frac{\rho}{h_m-h_{m+1}}(h_{m+1}-h_1)\\
\leq\ & \gamma+\frac{\rho}{h_m-h_{m+1}}(y-h_1)
\end{aligned}$$

where the inequality (28) holds because of the definition of ρ. Therefore, the inequality (25) is valid for the set $\mathcal{K}(N_1)$. □

Note that the definition of $G(\alpha)$ is very related to lifting procedure on $0-1$ knapsack. We have some typical choices of $\Phi(\alpha)$ as described in the next proposition.

Proposition 3.1 *Let $C = \{j_1, \ldots, j_r\}$ such that $\pi_{j_1} \geq \cdots \geq \pi_{j_r}$. We can have*

$$\Phi(\alpha) = h \ \textit{if} \ \sum_{k=1}^{h}\pi_{j_k} \leq \alpha < \sum_{k=1}^{h+1}\pi_{j_k};$$

furthermore, if C is a minimal cover and E is an extension of C, i.e.,

$$E - C \subseteq \{i \in [1, n] - C : \pi_i \geq \pi_{j_1}\}$$

then we can have

$$\Phi(\alpha) = \begin{cases} h & \text{if } \sum_{k=1}^{h} \pi_{j_k} \leq \alpha \leq \sum_{k=1}^{h+1} \pi_{j_k} - \lambda \\ h+1 & \text{if } \sum_{k=1}^{h+1} \pi_{j_k} - \lambda < \alpha < \sum_{k=1}^{h+1} \pi_{j_k} - 1 \end{cases}$$

where $\lambda = \sum_{i \in C} \pi_i - \tau' > 0$.

Proof First, we will show that $\Phi(\alpha)$ is superadditive. Let $\Phi(\alpha_1) = h_1$ and $\Phi(\alpha_2) = h_2$. If $\sum_{k=1}^{h_1} \pi_{j_k} \leq \alpha_1$ and $\sum_{k=1}^{h_2} \pi_{j_k} \leq \alpha_2$, then we get

$$\alpha_1 + \alpha_2 \geq \sum_{k=1}^{h_1} \pi_{j_k} + \sum_{k=1}^{h_2} \pi_{j_k} \geq \sum_{k=1}^{h_1+h_2} \pi_{j_k}$$

which implies that $\Phi(\alpha_1 + \alpha_2) \geq h_1 + h_2 = \Phi(\alpha_1) + \Phi(\alpha_2)$. Now, we assume that C is a minimal cover, which implies that $\lambda < \pi_i \ \forall i \in C$. If $\sum_{k=1}^{h_1} \pi_{j_k} \leq \alpha_1$ and $\sum_{k=1}^{h_2+1} \pi_{j_k} - \lambda \leq \alpha_2$, then we have

$$\alpha_1 + \alpha_2 \geq \sum_{k=1}^{h_1} \pi_{j_k} + \sum_{k=1}^{h_2+1} \pi_{j_k} - \lambda \geq \sum_{k=1}^{h_1} \pi_{j_k} + \sum_{k=1}^{h_2} \pi_{j_k} \geq \sum_{k=1}^{h_1+h_2} \pi_{j_k}$$

If $\sum_{k=1}^{h_1+1} \pi_{j_k} - \lambda \leq \alpha_1$ and $\sum_{k=1}^{h_2+1} \pi_{j_k} - \lambda \leq \alpha_2$, then we have

$$\alpha_1 + \alpha_2 \geq \sum_{k=1}^{h_1} \pi_{j_k} - \lambda + \sum_{k=1}^{h_2+1} \pi_{j_k} - \lambda \geq \sum_{k=1}^{h_1} \pi_{j_k} + \sum_{k=1}^{h_2} \pi_{j_k} \geq \sum_{k=1}^{h_1+h_2} \pi_{j_k}$$

Therefore, $\Phi(\alpha_1 + \alpha_2) \geq h_1 + h_2 = \Phi(\alpha_1) + \Phi(\alpha_2)$ and Φ is a superadditive function.

Next, we will show that $\Phi(\alpha) \leq \gamma - G(\alpha)$. The proof is very similar to the proof of Proposition 2.6 in Page 268 [21]. We provide a sketch here to make our proof self-contained. Note that we have

$$\sum_{k=h+1}^{r} \pi_{j_k} > \tau' - \sum_{k=1}^{h} \pi_{j_k} \geq \tau' - \alpha,$$

because $\sum_{k=1}^{h} \pi_{j_k} \leq \alpha$. Hence there is no feasible solution with $z_{j_k} = 1$ for $k = h+1, \ldots, r$. Since $\min\{\pi_{j_1}, \ldots, \pi_{j_h}\} \geq \max\{\pi_{j_{h+1}}, \ldots, \pi_{j_r}\}$, there exits an optimal solution $\hat{\mathbf{z}}$ for $G(\alpha)$ such that $\hat{z}_i = 0$ for $i = j_1, \ldots, j_h$. We define a new solution $\tilde{\mathbf{z}}$ such that $\tilde{z}_i = 1$ for $i = j_1, \ldots, j_h$ and $\tilde{z}_i = \hat{z}_i$ otherwise. It is easy to see that $\tilde{\mathbf{z}}$ is feasible for $G(0)$. Thus, we have

$$\gamma = G(0) \geq G(\alpha) + \sum_{k=1}^{h} \tilde{z}_{j_k} = G(\alpha) + h \geq G(\alpha) + \Phi(\alpha)$$

When E is an extension of the minimal cover C, the lifted cover inequality (24) has form

$$\sum_{i \in E} z_i \leq r - 1$$

We note that there exists an optimal solution for $G(\alpha)$ with $z_i = 0 \ \forall i \in E - C$, since $E - C \subseteq \{i \in [1, n] - C : \pi_i \geq \pi_{j_1}\}$. Thus, we have

$$\begin{aligned} G(\alpha) &= \max\left\{\sum_{j \in C} z_j : \sum_{j \in C} \pi_j z_j \leq \tau' - \alpha, z_j \in \{0, 1\} \ \forall j \in C\right\} \\ &= \max\left\{r + 1 - i : \sum_{k=i}^{r} \pi_{j_k} \leq \tau' - \alpha\right\} \end{aligned}$$

When $\sum_{k=1}^{h} \pi_{j_k} \leq \alpha \leq \sum_{k=1}^{h+1} \pi_{j_k} - \lambda$, we have

$$\sum_{k=h+2}^{r} \pi_{j_k} = \tau' - \left(\sum_{k=1}^{h+1} \pi_{j_k} - \lambda\right) \leq \tau' - \alpha \text{ and } \sum_{k=h+1}^{r} \pi_{j_k} = \tau' - \left(\sum_{k=1}^{h} \pi_{j_k} - \lambda\right) > \tau' - \alpha$$

It implies that $G(\alpha) = r + 1 - (h + 2)$ and $\Phi(\alpha) = h = r - 1 - G(\alpha) = \gamma - G(\alpha)$. When $\sum_{k=1}^{h+1} \pi_{j_k} - \lambda < \alpha$, we know that $\sum_{k=h+2}^{r} \pi_{j_k} > \tau' - \alpha$. Thus, $G(\alpha) < r + 1 - (h + 2)$, i.e., $G(\alpha) \leq r + 1 - (h + 2) - 1 = r - h - 2$. So $\Phi(\alpha) = h + 1 = r - 1 - (r - h - 2) \leq \gamma - G(\alpha)$. Therefore, in either case, we have $\Phi(\alpha) \leq \gamma - G(\alpha)$. □

It is clear that the inequality (25) is simply valid for the $0 - 1$ knapsack set $\mathcal{S}(\emptyset, N_1)$ when $\rho = 0$. Then performing lifting with respect to variables fixed at 1 can only provide us a valid inequality for the knapsack set $\mathcal{S}(\emptyset, \emptyset)$. So we limit ourselves to the case that $\rho > 0$ in Theorem 3.1 to study more interesting inequalities. Next, we will lift inequality (25) sequentially with respect to variables $z_i \ \forall i \in N_1$ in the order of $\{l_1, \ldots, l_q\}$. Suppose we already have lifting coefficients for $z_{l_1}, \ldots, z_{l_{r-1}}$ such that

$$\sum_{i=1}^{m-1} \alpha_i z_i + \sum_{i \in E} \alpha_i z_i \leq \gamma + \frac{\rho}{h_m - h_{m+1}}(y - h_1) + \sum_{j=1}^{r-1} \delta_{l_j}(1 - z_{l_j}) \tag{29}$$

is valid for the set $\mathcal{K} \cap \{\mathbf{z} \in \{0, 1\}^n : z_i = 1 \ \forall i \in \{l_r, \ldots, l_q\}\}$, and we are lifting inequality (29) with respect to variable z_{l_r}. By denoting the lifting coefficient as δ_{l_r}, we have

Proposition 3.2 *The lifting coefficient δ_{l_r} can be defined as*

$$\delta_{l_r} = \sum_{j=1}^{r-1} \delta_{l_j} + \frac{\rho}{h_m - h_{m+1}} h_1 - \gamma + \max_{t \in [1, \nu_r + 1]} \Delta_{t,r} - \frac{\rho}{h_m - h_{m+1}} h_t \tag{30}$$

where

$$\nu_r = \max\left\{ j : \sum_{i=1}^{j} \pi_i \le \tau' + \sum_{i=1}^{r} \pi_{l_i} \right\}$$

and for $t \in [1, \nu_r + 1]$ *we have*

$$\begin{aligned} \Delta_{t,r} \ge \sum_{i=1}^{t-1} \alpha_i \quad + \quad \max & \sum_{i \in E \cup [t,m-1]} \alpha_i z_i + \sum_{j=1}^{r-1} \delta_{l_j} z_{l_j} \\ \text{s.t.} & \sum_{i \in [t,n]-N_1} \pi_i z_i + \sum_{j=1}^{r-1} \pi_{l_j} z_{l_j} \le \tau' + \sum_{j=1}^{r} \pi_{l_j} - \sum_{i=1}^{t-1} \pi_i \\ & z_i \in \{0,1\} \quad \forall i \in [t,n] - \{l_r, \ldots, l_q\} \end{aligned} \tag{31}$$

Proof We need to show that

$$\sum_{i=1}^{m-1} \alpha_i z_i + \sum_{i \in E} \alpha_i z_i \le \gamma + \frac{\rho}{h_m - h_{m+1}}(y - h_1) + \sum_{j=1}^{r} \delta_{l_j}(1 - z_{l_j}) \tag{32}$$

is valid for the set $\mathcal{K} \cap \{\mathbf{z} \in \{0,1\}^n : z_i = 1 \ \forall i \in \{l_{r+1}, \ldots, l_q\} \text{ and } z_{l_r} = 0\}$.

Suppose $h_{t-1} > y \ge h_t$ for some $t \in [1, \nu+1]$, where we denote h_0 as $+\infty$. It implies that $z_j = 1 \ \forall j \in [1, t-1]$ ($[1, t-1] = \emptyset$ when $t = 1$). Then we note that the knapsack constraint in $\mathcal{K}$ becomes

$$\sum_{i \in [t,n]-N_1} \pi_i z_i + \sum_{j=1}^{r-1} \pi_{l_j} z_{l_j} \le \tau' + \sum_{j=1}^{r} \pi_{l_j} - \sum_{i=1}^{t-1} \pi_i$$

and we have

$$\sum_{i=1}^{m-1} \alpha_i z_i + \sum_{i \in E} \alpha_i z_i + \sum_{j=1}^{r} \delta_{l_j} z_{l_j} = \sum_{i=1}^{t-1} \alpha_i + \sum_{i \in E \cup [t,m-1]} \alpha_i z_i + \sum_{j=1}^{r-1} \delta_{l_j} z_{l_j} \le \Delta_{t,r}.$$

The definition of δ_{l_r} in (30) implies that

$$\begin{aligned} & \delta_{l_r} \ge \sum_{j=1}^{r-1} \delta_{l_j} + \frac{\rho}{h_m - h_{m+1}} h_1 - \gamma + \Delta_{t,r} - \frac{\rho}{h_m - h_{m+1}} h_t \\ \Rightarrow \quad & \Delta_{t,r} \le \gamma + \frac{\rho}{h_m - h_{m+1}}(h_t - h_1) + \sum_{j=1}^{r-1} \delta_{l_j} \\ \Rightarrow \quad & \Delta_{t,r} \le \gamma + \frac{\rho}{h_m - h_{m+1}}(y - h_1) + \sum_{j=1}^{r-1} \delta_{l_j} \end{aligned}$$

So the inequality (32) holds. □

Therefore, by summarizing Theorem 3.1 together with Proposition 3.2, we get

Theorem 3.2 *If $h_m > h_{m+1}$, then the inequality*

$$\sum_{i=1}^{m-1} \alpha_i z_i + \sum_{i \in E} \alpha_i z_i \leq \gamma + \frac{\rho}{h_m - h_{m+1}}(y - h_1) + \sum_{j=1}^{q} \delta_{l_j}(1 - z_{l_j}) \tag{33}$$

is valid for $\mathcal{K}$, where α_i $\forall i \in N_0$ are defined in (26) and δ_i $\forall i \in N_1$ are given by lifting procedure in (30).

Example 3 *1 (cont.) We suppose general probabilities as shown previously, where $\nu = 4$. Let $m = 3$, $N_0 = \{1, 2\}$ and $N_1 = \{10\}$. Note that the condition (23) holds. So, we have set $\mathcal{S}(N_0, N_1)$ that includes a knapsack as follows,*

$$\frac{\tau}{4}z_3 + \frac{\tau}{4}z_4 + \frac{\tau}{6}z_5 + \frac{\tau}{6}z_6 + \frac{\tau}{6}z_7 + \frac{\tau}{6}z_8 + \frac{\tau}{6}z_9 \leq \tau - \frac{\tau}{6}$$

The set $\{3, 4, 5, 6, 8\}$ gives a cover for this $0 - 1$ knapsack and we have the cover inequality

$$z_3 + z_4 + z_5 + z_6 + z_8 \leq 4$$

By lifting the cover inequality with respect to variable z_9, we have

$$z_3 + z_4 + z_5 + z_6 + z_8 + z_9 \leq 4$$

It is easy to check that we can have $\rho = 1$ and superadditive function $\Phi(\alpha) = 1$. Therefore, Theorem 3.1 gives the following inequality

$$\begin{aligned}\left(1 + \frac{1}{h_3 - h_4}(h_2 - h_1)\right) z_1 \quad &+ \quad \left(1 - \frac{1}{h_3 - h_4}(h_3 - h_2)\right) z_2 + z_3 \\ &+ \quad z_4 + z_5 + z_6 + z_8 + z_9 \leq 4 + \frac{1}{h_3 - h_4}(y - h_1),\end{aligned}$$

or specifically

$$\frac{1}{3}z_1 - \frac{1}{3}z_2 + z_3 + z_4 + z_5 + z_6 + z_8 + z_9 \leq 4 + \frac{1}{3}(y - h_1) \tag{34}$$

which is valid for the set $\mathcal{K} \cap \{\mathbf{z} \in \{0, 1\}^{10} : z_{10} = 1\}$. Then, by lifting inequality (34) with respect to z_{10}, we get

$$\Delta_{t1} = \begin{cases} 1 & \textit{when } t = 1 \\ 1 & \textit{when } t = 2 \\ 1 & \textit{when } t = 3 \\ 1 & \textit{when } t = 4 \\ \frac{8}{3} & \textit{when } t = 5 \end{cases}$$

So $\delta_1 = 8/3$ and we have valid inequality

$$\frac{1}{3}z_1 - \frac{1}{3}z_2 + z_3 + z_4 + z_5 + z_6 + z_8 + z_9 \leq 4 + \frac{1}{3}(y - 40) + \frac{8}{3}(1 - z_{10})$$

for $\mathcal{K}$, which is actually facet–defining. Similar procedures can produce many facet-defining inequalities as follows

$$
\begin{array}{rcl}
\frac{1}{3}z_1 - \frac{1}{3}z_2 + z_3 + z_4 + z_5 + z_7 + z_8 + z_9 & \leq & 4 + \frac{1}{3}(y-40) + \frac{8}{3}(1-z_{10}) \\
\frac{1}{3}z_1 - \frac{1}{3}z_2 + z_3 + z_4 + z_5 + z_6 + z_7 + z_9 & \leq & 4 + \frac{1}{3}(y-40) + \frac{8}{3}(1-z_{10}) \\
\frac{1}{3}z_1 - \frac{1}{3}z_2 + z_3 + z_4 + z_6 + z_7 + z_8 + z_9 & \leq & 4 + \frac{1}{3}(y-40) + \frac{8}{3}(1-z_5) \\
\frac{2}{4}z_1 + z_2 + 2z_3 + z_4 + z_5 + z_7 + z_8 + z_9 & \leq & 4 + \frac{1}{4}(y-40) + \frac{7}{4}(1-z_6) + \frac{9}{4}(1-z_{10}) \\
\frac{2}{4}z_1 + z_2 + 2z_3 + z_4 + z_5 + z_6 + z_7 + z_9 & \leq & 4 + \frac{1}{4}(y-40) + \frac{7}{4}(1-z_8) + \frac{9}{4}(1-z_{10}) \\
\frac{2}{4}z_1 + z_2 + z_3 + z_6 + z_7 + z_8 + z_9 + z_{10} & \leq & 3 + \frac{1}{4}(y-40) + \frac{8}{4}(1-z_4) + \frac{12}{4}(1-z_5) \\
\frac{2}{4}z_1 + z_2 + z_3 + z_5 + z_6 + z_7 + z_8 + z_9 & \leq & 3 + \frac{1}{4}(y-40) + \frac{8}{4}(1-z_4) + \frac{12}{4}(1-z_{10})
\end{array}
$$

It is easy to see that these inequalities are not included in the description of Theorem 2.2.

4 Intersection of multiple mixing sets with knapsack

Now we consider the set $\mathcal{Q}$ with $d > 1$. Instead of combining original constraints as in [14], in this section, we are going to combine proposed inequalities (7) for multiple mixing sets $\mathcal{Q}_r$, where $r \in [1, d]$, to derive valid inequalities for $\mathcal{Q}$ and give the conditions under which the inequalities are facet–defining.

For each $r \in [1, d]$, we define a mapping $\langle \cdot \rangle_r$ on $[1, n]$ such that

$$h_{t\langle 1 \rangle_r} \geq h_{t\langle 2 \rangle_r} \geq \cdots \geq h_{r\langle n \rangle_r}$$

Also, we use the notation that $\langle X \rangle_r = \{\langle i \rangle_r : \forall i \in X\}$ for set $X \subseteq [1, n]$. For each $\mathcal{Q}_r$, we define ν_r such that

$$\sum_{i=1}^{\nu_r} \pi_{\langle i \rangle_r} \leq \tau \text{ but } \sum_{i=1}^{\nu_r+1} \pi_{\langle i \rangle_r} > \tau$$

Note that the value p is independent of index r, since it is only based on the monotonic order of π_i.

Definition 1 *Given $\theta \in [1, n]$, for any $r \in [1, d]$, we let $m_r \in [1, \nu_r]$, and*

- $T_r = \{t_{r1}, t_{r2}, \ldots, t_{ra_r}\} \subseteq \{1, \ldots, m_r\}$ *with* $t_{r1} < t_{r2} < \cdots < t_{ra_r}$*;*

- $L_r \subseteq \{m_r+s_{r1}+1, \ldots, n\}$ *and a permutation of the elements in* L_r, $\Pi_{L_r} = \{l_{r1}, l_{r2}, \ldots, l_{r,q_r}\}$ *with* $l_{r1} = \theta$ *and* $l_{lj} \geq m_r + 1 + s_{rj}$ *such that* $s_{r1} \leq \cdots \leq s_{q_r+1} = \nu_r - m_r + 1$ *satisfy*

$$\sum_{i=1}^{m_r+s_{rj}} \pi_{\langle i \rangle_r} + \sum_{i=j}^{q_r} \pi_{\langle k_{ri} \rangle_r} > \tau \ \forall j \tag{35}$$

where $\{k_{r1}, \ldots, k_{rq_r}\}$ *is a permutation of set* L_r *with* $\pi_{\langle k_{r1} \rangle_r} \geq \cdots \geq \pi_{\langle k_{rq_r} \rangle_r}$.

- $\mathcal{I}_r$ *represents expression as follows*

$$y_r + \sum_{j=1}^{a_r} (h_{r\langle t_{rj} \rangle_r} - h_{r\langle t_{r,j+1} \rangle_r}) z_{\langle t_{rj} \rangle_r} + \sum_{j=1}^{q_r} \delta_{rj} (1 - z_{\langle l_{rj} \rangle_r})$$

where

$$\delta_{rj} = \begin{cases} h_{r,\langle m_r+s_{r1} \rangle_r} - h_{r,\langle m_r+s_{r1+1} \rangle_r} & j = 1 \\ \max\left\{ \delta_{r,j-1}, h_{r,\langle m_r+s_{r1} \rangle_r} - h_{r,\langle m_r+s_{rj+1} \rangle_r} - \sum_{i: i<j \text{ and } l_{ri} \geq m_r+1+s_{rj}} \delta_{ri} \right\} & j \in [2, q_r] \end{cases}$$

where the sets $\langle L_r \rangle_r - \{\theta\}$ $\forall r \in [1, d]$ *are mutually disjoint. We define blending inequality as*

$$\sum_{r \in [1,d]} \frac{1}{\delta_{r1}} \mathcal{I}_r - (1 - z_\theta) \geq \sum_{r \in [1,d]} \frac{1}{\delta_{r1}} h_{r\langle 1 \rangle_r} \tag{36}$$

Theorem 4.1 *The blending inequality is valid for* $\mathcal{Q}$ *if*

$$\sum_{i \in \bigcup_{r \in [1,d]} \langle [1, m_r+s_{r1}] \rangle_r} \pi_i > \tau. \tag{37}$$

Proof Note that if we only consider one mixing set $\mathcal{Q}_r$ for a given $r \in [1, d]$, then we can assume $\langle i \rangle_r = i$ $\forall i \in [1, n]$ without loss of generality. Thus the inequality

$$\mathcal{I}_r \geq h_{r\langle 1 \rangle_r} \tag{38}$$

reduces to the inequality (7) with $t_{r,a+1} = m_r + s_{r1}$ and $f = h_{r,\langle m_r+s_{r1} \rangle_r}$, i.e., it is valid for $\mathcal{Q}_r$. Since $\mathcal{Q}_r \supseteq \mathcal{Q}$, the inequality (38) is valid for $\mathcal{Q}$.

Because of the condition (37), we claim that $y_j \geq h_{j,m_j+s_{j1}}$ for at least one $j \in [1, d]$. Suppose the claim is not true. Then we have $y_r < h_{r,m_r+s_{r1}}$ for all $r \in [1, d]$. It implies that $z_i = 1$ $\forall i \in \langle [1, m_r + s_{r1}] \rangle_r$ and $\forall r \in [1, d]$, which contradicts to (37).

Then, we suppose $y_u \geq h_{u,m_u+s_{u1}}$ for $u \in [1,d]$. We will show that the inequality $\mathcal{I}_u - \delta_{u1}(1-z_\theta) \geq h_{u\langle 1\rangle_u}$ is valid for $\mathcal{Q}_u$. Without of loss generality, we can assume $\langle i\rangle_u = i$ $\forall i \in [1,n]$. So, we need to show that

$$y_u + \sum_{j=1}^{a_u}(h_{ut_{uj}} - h_{ut_{u,j+1}})z_{t_{uj}} + \sum_{j=2}^{q_u}\delta_{uj}(1-z_{l_{uj}}) \geq h_{u1}$$

is valid for $\mathcal{Q}_u$. To simplify the notation, we can drop subscript u and need to prove that

$$y + \sum_{j=1}^{a}(h_{t_j} - h_{t_{j+1}})z_{t_j} + \sum_{j=2}^{q}\delta_j(1-z_{l_j}) \geq h_1$$

is valid for $\mathcal{K}$ when $y \geq h_{m+s_1}$, which is already proved in the first paragraph of proof for Theorem 2.2. Now, we have $\mathcal{I}_u - \delta_{u1}(1-z_\theta) \geq h_{u\langle 1\rangle_u}$ is valid for $\mathcal{Q}_u$, i.e., valid for $\mathcal{Q}$.

Since (38) is valid for any $r \in [1,d]$, we get

$$\begin{aligned}\sum_{r\in[1,d]}\frac{1}{\delta_{r1}}\mathcal{I}_r - (1-z_\theta) &= \sum_{r\in[1,d]-\{u\}}\frac{1}{\delta_{r1}}\mathcal{I}_r + \frac{1}{\delta_{u1}}\mathcal{I}_u - (1-z_\theta)\\ &\geq \sum_{r\in[1,d]-\{u\}}\frac{1}{\delta_{r1}}h_{r\langle 1\rangle_r} + \frac{1}{\delta_{u1}}h_{u\langle 1\rangle_u} = \sum_{r\in[1,d]}\frac{1}{\delta_{r1}}h_{r\langle 1\rangle_r}.\end{aligned}$$

□

When all the scenarios have equal probabilities, the condition (37) becomes

$$\left|\bigcup_{r\in[1,d]}\langle[1,m_r+1]\rangle_r\right| > p. \tag{39}$$

So, we have

Corollary 4.1.1 *Suppose all the scenarios have equal probabilities, then the blending inequality is valid for $\mathcal{Q}$ if (39) holds.*

Next, we will show that the *blending inequality* could be facet–defining in some cases for $d = 2$ when all the scenarios have equal probabilities.

Theorem 4.2 *Consider $d = 2$. Let t_{11} denote $\langle 1\rangle_1$ and t_{21} denote $\langle 1\rangle_2$. Suppose all scenarios have equal probabilities. The blending inequality is facet-defining for $\mathcal{Q}$ if the sets $\langle T_r\rangle_r$, $\langle L_r\rangle_r - \{\theta\}$ $\forall r \in [1,2]$ are mutually disjoint, and we have*

1. $\langle 1\rangle_r \notin \langle[1,m_{\bar{r}}]\rangle_{\bar{r}} \bigcup \langle L_{\bar{r}}\rangle_{\bar{r}}$,
2. $\langle L_r\rangle_r - \{\theta\} \subsetneq \langle[1,m_{\bar{r}}]\rangle_{\bar{r}}$,

where $\{r, \bar{r}\} = \{1, 2\}$ *and* $r \neq \bar{r}$.

Proof For any $r \in \{1, 2\}$, note that when all scenarios have equal probabilities, the inequality $\mathcal{I}_r \geq h_{r\langle 1\rangle_r}$ is in the form of inequality (21) with $t_{r,a+1} = m_r + 1$ and $f = h_{r,\langle m_r+1\rangle_r}$. So we have $|L_r| = p - m_r$ for $r = 1, 2$.

To show that that the blending inequality is facet–defining for $conv(\mathcal{Q})$, we give $n + 2$ affinely independent points. The basic idea is that we can always set $y_r = h_{r\langle 1\rangle_r}$ and enumerate extreme points as in the proof of Theorem 2.3 for $y_{\bar{r}}$ because of condition 1. Because of the notation of multiple mixing sets, we give a self–contained proof.

First, we let $y_r^0 = h_{r\langle 1\rangle_r}$ for $r = 1, 2$, and $z_j^0 = 1$ if $j \in \langle L_1\rangle_1 \bigcup \langle L_2\rangle_2$. The point is feasible because condition 2 implies

$$\langle L_1\rangle_1 \cup \langle L_2\rangle_2 \subsetneq \langle L_1\rangle_1 \cup \langle [1, m_1]\rangle_1 \cup \{\theta\} = \langle L_1\rangle_1 \cup \langle [1, m_1]\rangle_1. \tag{40}$$

Thus $|\langle L_1\rangle_1 \bigcup \langle L_2\rangle_2| \leq p - 1$. Next, for each $j \notin \langle T_1 \cup L_1\rangle_1 \bigcup \langle T_2 \cup L_2\rangle_2$, we consider the point $(\mathbf{y}^j, \mathbf{z}^j) = (\mathbf{y}^0, \mathbf{z}^0 + e_j)$. For each $t_{1j} \in \{t_{11}, \ldots, t_{1a_1}\}$, we let $y_2^{t_{1j}} = h_{2\langle 1\rangle_2}$, $y^{t_{1j}} = h_{1t_{1,j+1}}$, $z_i^{t_{1j}} = 1$ if $i \in \langle [1, t_{1,j+1} - 1]\rangle_1 \cup \langle L_1\rangle_1 \cup \langle L_2\rangle_2$ and 0 otherwise. The point is feasible because (40) implies

$$|\langle [1, t_{1,j+1} - 1]\rangle_1 \cup \langle L_1\rangle_1 \cup \langle L_2\rangle_2| \leq |\langle L_1\rangle_1 \cup \langle [1, m_1]\rangle_1| = p.$$

Due to the symmetry, we have similar way to get points for each $t_{2j} \in \{t_{21}, \ldots, t_{2a_2}\}$ by letting $y_1^{t_{1j}} = h_{1\langle 1\rangle_1}$.

Let $y_1^{l_{11}} = h_{1\langle 1\rangle_1}$, $y_2^{l_{21}} = h_{m_2+2}$, $z_i^{l_{21}} = 1$ if $i \in \langle [1, m+1]\rangle_2$ and $z_{l_{2i}}^{l_{21}} = 1$ for $i > 1$, and 0 otherwise. The point is feasible because of condition 2. For each $j \in [2, p - m_2]$ if $\delta_{2j} = \delta_{2,j-1}$, we have

$$(\mathbf{y}^{l_{2j}}, \mathbf{z}^{l_{2j}}) = (\mathbf{y}^{l_{2,j-1}}, \mathbf{z}^{l_{2,j-1}} + e_{l_{2,j-1}} - e_{l_{2j}}),$$

otherwise we have that $y_1^{l_{1j}} = h_{1\langle 1\rangle_1}$, $y_2^{l_{2j}} = h_{m_2+1+j}$, $z_i^{l_{2j}} = 1$ if $i \in \langle [1, m + j]\rangle_2$ and $z_{l_{2i}}^{l_{2j}} = 1$ for $i > j$, and 0 otherwise. Due to the symmetry, we can also get points for each $l_{1j} \in \{l_{11}, \ldots, l_{1,p-m_1}\}$.

Note that $\langle T_1\rangle_1$, $\langle T_2\rangle_1$, $\langle L_1\rangle_1$, $\langle L_2\rangle_2$ are mutually disjoint except $\langle L_1\rangle_1 \cap \langle L_2\rangle_2 = \{\theta\}$. So, we get totally $n + 2$ points on the face defined by blending inequality and they are affinely independent.

□

Example 2 *Let* $n = 6$ *and* $p = 3$. *Suppose we have equal probabilities for all scenarios and*

$$(h_{11}, \ldots, h_{16}) = (28, 25, 15, 8, 5, 3) \text{ and } (h_{21}, \ldots, h_{26}) = (2, 5, 6, 8, 17, 10)$$

The inequality (21) gives that

$$y_1 + 3z_1 + 10(1 - z_3) + 17(1 - z_6) \geq 28 \quad \text{is valid for } \mathcal{Q}_1 \text{ with } m_1 = 1$$
$$y_2 + 9z_5 + 2(1 - z_3) \geq 17 \quad \text{is valid for } \mathcal{Q}_2 \text{ with } m_2 = 2$$

Let $\theta = 3$. *We have*

Table 1: Joint probability density function

Scenario	1	2	3	4	5	6	7	8	9
ω_1	0.75	0.5	0.5	0.25	0.25	0.25	0	0	0
ω_2	1.25	1.5	1.25	1.75	1.5	1.25	2	1.5	1.25
Probability	0.2	0.14	0.06	0.06	0.06	0.3	0.04	0.04	0.1

- $\langle 1\rangle_1 = 1 \notin \{5,6\} \bigcup \{3\} = \langle [1, m_2]\rangle_2 \bigcup \langle L_2\rangle_2$,
- $\langle 1\rangle_2 = 5 \notin \{1\} \bigcup \{3,6\} = \langle [1, m_1]\rangle_1 \bigcup \langle L_1\rangle_1$,
- $\langle L_1\rangle_1 - \{\theta\} = \{3,6\} - \{3\} \subseteq \{5,6\} = \langle [1, m_2]\rangle_2$,
- $\langle L_2\rangle_2 - \{\theta\} = \{3\} - \{3\} \subseteq \{1\} = \langle [1, m_1]\rangle_1$.

Thus, the conditions in Theorem 4.2 are satisfied and we have blending inequality

$$y_1 + 5y_2 + 3z_1 + 45z_5 + 10(1 - z_3) + 17(1 - z_6) \geq 113$$

which is facet–defining for $\mathcal{Q}$*.*

Example 3 *(Example 2 in [14]) We have the chance–constrained program*

$$\begin{array}{rl} \min & x_1 + x_2 \\ s.t. & P\left\{ \begin{array}{l} 2x_1 - x_2 \geq \omega_1 \\ x_1 + 2x_2 \geq \omega_2 \end{array} \right\} \geq 0.6 = 1 - \tau \\ & x_1, x_2 \geq 0 \end{array}$$

where ω_1 *and* ω_2 *are dependent random variables with joint probability density function given in Table 1. The optimal solution is* $(x, y) = (0.55, 0.35, 0.75, 1.25)$ *with objective value* 0.9*.*

For this example, we have $\tau = 0.4$*,* $p = 6$*,* $\nu_1 = 3$ *and* $\nu_2 = 5$*. Let* $y_1 = 2x_1 - x_2$ *and* $y_2 = x_1 + 2x_2$*. Then mixing set reformulation is*

$$\begin{array}{ll} y_1 + 0.75z_1 \geq 0.75 & y_2 + 2.00z_7 \geq 2.00 \\ y_1 + 0.50z_2 \geq 0.50 & y_2 + 1.75z_4 \geq 1.75 \\ y_1 + 0.50z_3 \geq 0.50 & y_2 + 1.50z_2 \geq 1.50 \\ y_1 + 0.25z_4 \geq 0.25 & y_2 + 1.50z_5 \geq 1.50 \\ y_1 + 0.25z_5 \geq 0.25 & y_2 + 1.50z_8 \geq 1.50 \\ y_1 + 0.25z_6 \geq 0.25 & y_2 + 1.25z_1 \geq 1.25 \\ \vdots & \vdots \\ \sum_{i=1}^{9} \pi_i z_i \leq & 0.4 = \tau \end{array}$$

and the tighter formulation (4) in [17] is

$$\begin{array}{rcl} y_1 + 0.50z_1 \geq 0.75 & \quad & y_2 + 0.75z_7 \geq 2.00 \\ y_1 + 0.25z_2 \geq 0.50 & & y_2 + 0.50z_4 \geq 1.75 \\ y_1 + 0.25z_3 \geq 0.50 & & y_2 + 0.25z_2 \geq 1.50 \\ y_1 \geq 0.25 & & y_2 + 0.25z_5 \geq 1.50 \\ & & y_2 + 0.25z_8 \geq 1.50 \\ \vdots & & y_2 \geq 1.25 \\ & & \vdots \end{array}$$

$$\sum_{i=1}^{9} \pi_i z_i \leq 0.4 = \tau$$

The initial linear programming (LP) relaxation solution by using tight formulation (4) is

$$(x, y) = (0.52, 0.365, 0.675, 1.25).$$

In [14], the author proposed 3 *inequalities in the form of (6) and claim that no more violated inequality (6) can be found. Then they added an inequality* $y_1 + y_2 \geq 2$ *to obtain the optimal solution. This new inequality is derived by combining the original formulation of two mixing sets.*

In this example, we present many facet-defining inequalities that are derived from previous theorems. First, we have facet–defining inequality

$$y_2 + 0.25z_2 + 0.25z_4 + 0.25z_7 \geq 2 \tag{41}$$

which is derived from (5) in [17] and also a special case of inequality (7). Then, we have another facet–defining inequality

$$z_2 + z_4 + z_6 \leq 2 + 4(y_1 - 0.75) + (1 - z_7) \tag{42}$$

which is derived from lifting. If we fix $z_7 = 1$ *and let* $m = 2$*, then we have cover inequality*

$$z_2 + z_4 + z_6 \leq 2.$$

From Theorem 3.1, we can easily calculate $\rho = 1$ *and* $\Phi(\pi_1) = \Phi(0.2) = 1$*. So we have*

$$z_2 + z_4 + z_6 \leq 2 + 4(y_1 - 0.75)$$

which is valid when $z_7 = 1$*. By performing lifting procedure on variable* z_7*, we can derive inequality (42). Note that inequalities (41) and (42) implies a valid inequality* $y_1 + y_2 - 0.25z_6 \geq 2$ *by simply summation two inequalities, which implies that* $y_1 + y_2 \geq 2$ *in [14] is not facet–defining. Instead, the new blending strategy proposed in this section gives a facet-defining inequality*

$$y_1 + y_2 + 0.25z_1 + 0.5z_7 + 0.25(1 - z_9) \geq 2.75 \tag{43}$$

by combining

$$y_1 + 0.25z_1 + 0.25(1 - z_9) \geq 0.75 \tag{44}$$
$$y_2 + 0.5z_7 + 0.25(1 - z_9) \geq 2 \tag{45}$$

where (44) and (45) are facet–defining inequalities for two individual mixing sets respectively, and can be derived by (7).

5 Computational study

We performed our computational tests in the Texas Tech High Performance Computing Center's node based system, where each node contains two Westmere 2.8 GHz 6-core processors with 24 GB main memory [13]. We used the callable libraries of CPLEX 12.2 where we run instances on a single thread with transitional branch-and-bound method in one hour time limit.

We tested branch-and-cut (B&C) algorithm with (7) and (33) separately on difficult and large instances of the static probabilistic lot-sizing (SPLS) model described in [32]. Let x_t be the decision variable of ordering quantity in period t, I_{it} be the inventory level at the end of period t under scenario i, and w_t be binary variables to indicate order setup. The deterministic equivalent of SPLS model is

$$\begin{aligned}
\max \quad & \sum_{t=1}^{d} \sum_{i=1}^{n} \pi_i (c_t x_t + h_{it} I_{it} + g_t w_t) & \\
\text{s.t.} \quad & y_t = \sum_{j=1}^{t} x_j & t \in \{1, \ldots, d\} \\
& y_t \geq D_{it}(1 - z_i) & t \in \{1, \ldots, d\}, i \in \{1, \ldots, n\} \\
& I_{it} \geq y_t - D_{it} & t \in \{1, \ldots, d\}, i \in \{1, \ldots, n\} \\
& 0 \leq x_t \leq M_t w_t & t \in \{1, \ldots, d\} \\
& \sum_{i=1}^{n} \pi_i z_i \leq \tau & \\
& I_{i,t} \geq 0, z_i, w_t \in \{0, 1\} & t \in \{1, \ldots, d\}, i \in \{1, \ldots, n\}
\end{aligned}$$

where D_{it} is the cumulative demand until period t, c_t and g_t are the variable and fixed costs of ordering, h_{it} is the variable holding cost in period t under scenario i, and M_t is the order capacity in period t.

Our instances were generated in a way as similar as in [14]. We assume that the demand in a time period is Uniform(1,50). Therefore, the cumulative demand in the first period, D_{i1} is generated from discrete uniform distribution between 1 and 50 for each scenario $i = 1, \ldots, n$, and cumulative demand for period $t > 1$, D_{it}, is generated by adding a Uniform(1,50) to the value of $D_{i,t-1}$ for each scenario $i = 1, \ldots, n$. The variable production costs and inventory

Algorithm 1 Find m, set Π_L and $s_1, \dots, s_q$

Let $U = \{j \in [1, n] : z_j^* = 1\} = \{u_1, \dots, u_{|U|}\}$ with

$$\pi_{u_1} \leq \cdots \leq \pi_{u_{|U|}}$$

if $\{j \in U : j \geq \nu + 2\} = \emptyset$ **then**
    break
$i \leftarrow 1$, $s'_0 = \nu + 1$
for $j = 1$ to $|U|$ **do**
    if $u_j \geq s'_{i-1} + 1$ **then**
        $l'_i \leftarrow u_j$
        $s'_i \leftarrow \min\left\{s : \sum_{k=1}^{s} \pi_k + \sum_{k=1}^{i} \pi_{u_k} > \tau\right\}$
        $i \leftarrow i + 1$
$q \leftarrow i - 1$, $s_1 = 1$ and $m = s'_q - 1$
for $j = 1$ to q **do**
    $l_j = l'_{q+1-j}$ and $s_j = s'_{q+1-j} - m$
Note that $l_i = l'_{q+1-i} \geq s'_{q-i} + 1 = m + s_{i+1} + 1 \geq m + s_i + 1$
$\Pi_L \leftarrow \{l_1, \dots, l_q\}$

holding costs are generated from a discrete uniform distribution between 1 and 10. The fixed costs follow a discrete uniform distribution between 500 and 600. We set the order capacity $M_t = 50$ and generate scenario probabilities from Uniform(0,1) distribution.

5.1 Separation

In our computational experiments, we generate inequalities (7) dynamically in the branch–and–bound procedure. Let $(\mathbf{y}^*, \mathbf{z}^*)$ be a fractional solution. We can find m, the set Π_L and corresponding $s_1, \dots, s_q$ by Algorithm 1. Note that, for a given m, the best set T in inequalities (7) can be found by solving a shortest path problem from the source 1 to the sink $m + s_1$ on a directed acyclic graph with vertices $\{1, \dots, m + s_1\}$ (see Section 3.1 in [14]). With all these parameters, we have an unique inequality (7). It will be added as a cutting plane if it cuts of $(\mathbf{y}^*, \mathbf{z}^*)$.

We only add inequalities (33) in the initial formulation, because it is difficult for us to show the efficacy of the inequalities (33) by adding them dynamically, considering the fact that the efficacy of the inequalities (33) heavily relies on the lifted cover inequalities (LCIs) and LCIs are well implemented in CPLEX for more than a decade. We note that minimal covers are strong inequalities for $0-1$ knapsack. So, we let $N_1 = \emptyset$, which implies that $m = \nu$, and find cover C by Algorithm 2. Then, we get inequality (24) with the extension E of the cover C. Since $\alpha_i = 1\ \forall i \in E$, the value of ρ can be easily obtained. At last, we derive inequalities (33) for each period t by applying Proposition 3.1. Overall, we will have d inequalities (33) added into initial formulation.

Algorithm 2 Find cover C

Sort π_i $\forall i \in 1, \dots, n$ such that $\pi_{\langle 1 \rangle} \leq \cdots \leq \pi_{\langle n \rangle}$
$C \leftarrow \emptyset$
for $j = 1$ to n **do**
  if $\langle j \rangle \geq \nu$ **then**
    $C \leftarrow C \cup \{j\}$
  if $\sum_{i \in C} \pi_i > \tau$ **then** break
for $j = 1$ to $|C|$ **do**
  if $\sum_{i \in C - \{j\}} \pi_i > \tau$ **then**
    $C \leftarrow C - \{j\}$
  else
    break

5.2 Computational results

A summary of percentage of reductions is reported in Table 2 and the results of all experiments are reported in Table 3. The column $d \times n \times \tau$ indicates that the instance has d periods and n scenario with service level $1 - \tau$. We solve the same instances in one hour time with the default setting of CPLEX, except using single thread and transitional branch–and–bound algorithm. In Table 3, The columns `CPX` include results without adding any user cuts. The experiments with inequality (33) added into formulation are summarized under the columns `Lift`, and the experiments with the branch–and–cut algorithm using inequality (7) are summarized under the columns `Mix`. The column `Cuts` indicates the number of inequality (7) were found during the branch–and–cut algorithm. When the instance is solved in one hour, we report the computational time in `Time (%Endgap)` column, otherwise we report, in the parenthesis, the percentage gap between the best lower bound and the best integer solution found in the search tree.

In Table 2, each row shows the average reduction over 5 instances. The `endgap` column is missing when all 5 instances are solved in one hour, and `time` column is missing when none of 5 instances is solved in one hour.

6 Conclusion and future research

In this paper, we study the mixing set with a $0 - 1$ knapsack constraint arising in chance–constrained programs. By using the mixing set inequality as a base inequality, we propose facet-defining inequalities that subsume the explicit inequalities described in [14, 17]. In addition, we have explained a large families of inequalities by performing lifting procedure on $0 - 1$ knapsack and continuous variable. Our computational tests illustrate the efficacy of the branch-and-cut algorithm by using those inequalities.

In the future research, it is interesting to find an effective approach to derive blending inequalities as cutting planes. We believe that an efficient way of maintaining a cut pool is

Table 2: Reduction summary of static probabilistic lot–sizing experiments

$d \times n \times \tau$	Mix Reductions (%)			Lift Reductions (%)		
	node	time	endgap	node	time	endgap
$30 \times 500 \times 0.05$	44	45	-	21	21	-
$30 \times 500 \times 0.10$	55	57	84	20	12	29
$30 \times 1000 \times 0.05$	-24	4	61	-1	-	13
$30 \times 1000 \times 0.10$	-6	-	19	7	-	6
$50 \times 500 \times 0.02$	41	36	-	16	20	-
$50 \times 500 \times 0.05$	25	16	49	5	6	6
$50 \times 1000 \times 0.02$	21	18	49	7	6	1
$50 \times 1000 \times 0.05$	-13	-	22	2	-	3

crucial for blending inequalities.

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Table 3: Static probabilistic lot–sizing experiments

$d \times n \times \tau$	%RootGap			Nodes			Time (%Endgap)			Cuts
	CPX	Lift	Mix	CPX	Lift	Mix	CPX	Lift	Mix	
$30 \times 500 \times 0.05$	6.74	5.68	6.43	16332	9250	9623	69	37	40	43
	8.68	8.54	6.7	54456	53272	31953	178	172	140	67
	6.39	7.64	8.13	95363	68903	56809	480	358	279	62
	6.90	6.54	6.77	34272	27342	23183	216	174	117	70
	6.99	7.01	6.56	30504	22225	7518	171	131	42	28
$30 \times 500 \times 0.1$	7.99	8.02	7.03	187490	136136	72870	2126	1795	691	26
	10.10	9.55	8.69	459972	373948	91988	(0.38)	3185	602	34
	7.72	8.64	8.15	231531	226441	131538	(0.71)	(0.45)	1569	102
	10.65	10.27	10.41	176200	151370	223419	(1.92)	(1.91)	(0.49)	94
	9.66	10.29	8.35	241004	153846	63379	(0.33)	2350	647	28
$30 \times 1000 \times 0.05$	8.37	8.98	8.38	161485	188247	205875	(2.27)	(1.94)	(1.73)	27
	9.23	9.14	8.2	246196	235185	393816	(1.55)	(1.32)	3455	26
	8.28	8.43	7.21	173096	214561	190750	(1.08)	(0.75)	3087	399
	8.61	9.53	9.01	236293	221994	243470	(2.38)	(1.99)	(1.17)	25
	9.09	8.45	8.41	217560	188995	250912	(1.22)	(1.34)	(0.41)	425
$30 \times 1000 \times 0.1$	10.11	10.54	10.21	88836	105125	124359	(4.83)	(4.4)	(3.69)	168
	9.78	10.07	10.8	127495	112276	161172	(4.52)	(4.31)	(4.18)	27
	12.9	12.83	10.35	119900	70569	82247	(4.62)	(4.37)	(3.51)	1576
	11.03	11	10.31	126734	121916	104060	(4.32)	(4.01)	(3.64)	1607
	10.35	10.8	10.74	68584	82884	94084	(4.46)	(4.11)	(3.4)	191
$50 \times 500 \times 0.02$	5.62	6.41	4.6	360303	307070	252199	760	695	575	55
	6.3	6.07	5.31	295575	232625	218234	719	469	534	50
	7.33	7.16	6.02	362209	305018	124174	536	382	184	95
	4.42	3.86	3.59	62601	46474	56271	57	47	52	35
	5.48	7.55	6.26	172977	163696	86269	331	333	188	82
$50 \times 500 \times 0.05$	6.27	6.45	5.6	700401	787756	564000	(1.1)	(1.12)	(0.69)	58
	6.4	6.58	5.76	526200	439113	437165	(1.16)	(1.07)	(0.9)	88
	5.31	6.3	5.33	705441	830800	565628	(0.53)	(0.28)	2515	63
	8.81	7.54	7.69	767737	526211	335577	3204	2164	1462	80
	8.33	7.5	8.19	695290	657489	641772	(0.65)	(0.77)	(0.15)	53
$50 \times 1000 \times 0.02$	5.95	5.55	4.71	658627	671186	606848	(0.65)	(0.62)	2880	49
	6.61	5.63	5.64	887240	824339	839817	(1.23)	(1.25)	(0.93)	42
	6.12	7.71	7.32	356404	318733	266792	1772	1466	1456	170
	4.2	6.1	6.33	330954	296770	162922	1299	1219	796	53
	5.55	4.99	5.06	226589	175028	76895	715	460	248	56
$50 \times 1000 \times 0.05$	8.56	8.13	9.68	180126	178146	226527	(3.5)	(3.21)	(2.67)	93
	8.16	8.28	7.21	195526	169969	183517	(3.21)	(3.1)	(2.45)	47
	7.94	8.43	7.67	145864	148838	169236	(3.35)	(3.33)	(2.46)	63
	9.08	9.23	7.94	143027	153431	116906	(3.64)	(3.56)	(3.06)	48
	8.07	9.65	8.02	161906	159287	243204	(3.7)	(3.64)	(2.85)	44