Solution Analysis for the Pseudomonotone Second-order Cone Linear Complementarity Problem

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Abstract. In this paper, we perform a thorough analysis related to the solutions for the *pseudomonotone* second-order cone linear complementarity problems (SOCLCP). Based upon Tao's recent work [Tao, *J. Optim. Theory Appl.*, 159(2013), pp. 41–56] on pseudomonotone LCP on Euclidean Jordan algebras, we made two noticeable contributions: First, we introduce the concept of J_n-*eigenvalue* of a matrix, and prove that the associated matrix of the pseudomonotone SOCLCP always admits a J_n-eigenvalue; the notion of the J_n-eigenvalue turns out be a key for the pseudomonotone SOCLCP as it not only generalizes the results on the SOCLCP with the globally uniquely solvable (GUS) property, but also offers insight into the corresponding LCP and lays a foundation for computing the solution of SOCLCP. Second, we characterize the range of the pseudomonotone SOCLCP, and establish an explicit and complete description of the range of the SOCLCP.

Key words. Pseudomonotone; SOCLCP; J_n-eigenvalue; Range of the SOCLCP; GUS property

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1 Introduction

Given a matrix $M \in \mathbb{R}^{n\times n}$ and a vector $\boldsymbol{q} \in \mathbb{R}^n$, the so-called second-order cone linear complementarity problem (SOCLCP), denoted by LCP$(M, \mathbb{K}^n, \boldsymbol{q})$, is to find a vector $\boldsymbol{x} \in \mathbb{R}^n$ satisfying the following conditions:

$$\mathbb{K}^n \ni \boldsymbol{x} \perp M\boldsymbol{x} + \boldsymbol{q} \in \mathbb{K}^n, \tag{1.1}$$

where $\mathbb{K}^n$ is the second-order cone

$$\mathbb{K}^n = \{[x_1, \boldsymbol{x}_2] \in \mathbb{R} \times \mathbb{R}^{n-1} : \|\boldsymbol{x}_2\|_2 \le x_1\}.$$

The solution set of (1.1) is denoted by SOL$(M, \mathbb{K}^n, \boldsymbol{q})$.

Using the notion of the Jordan product of two vectors (for definition see [4, (2.1)] or [10]), the SOCLCP can be viewed as one of the classes of the linear complementarity problems over *symmetric cone*, which includes the nonnegative orthant $\mathbb{R}^n_+$, the second-order cone $\mathbb{K}^n$ and the semi-definite cone. For the linear complementarity problem over the general symmetric cones, many useful and interesting properties have been established *via* the notion of Euclidean Jordan algebra; see e.g., [6, 7, 10, 12, 18]. However, an LCP over a particular symmetric cone definitely contains special properties which on the one hand offers insight into the corresponding LCP, and lays a foundation for computing the solution on the other hand. A prime example of this statement is the classical LCP over $\mathbb{R}^n_+$ [2, 5]. This motivates a lot of studies (e.g., [4, 5, 12, 13, 14, 23]) for a particular LCP in the literature including the present paper.

For the SOCLCP particularly, Yang and Yuan [20] investigated the so-called globally uniquely solvable (GUS) property and derived a set of linear algebra properties for the sufficient and necessary conditions for the GUS property. In particular, it is well known [6, 7, 10] that any positive definite matrix ensures the GUS property and therefore the set of solutions SOL$(M, \mathbb{K}^n, \boldsymbol{q})$ is a singleton; however, when M is only monotone, difficulties arise and SOL$(M, \mathbb{K}^n, \boldsymbol{q})$ becomes complicated. In [13], Kong et. al. characterized SOL$(M, \mathbb{K}^n, \boldsymbol{q})$ which is a convex set. To guarantee the convexity of SOL$(M, \mathbb{K}^n, \boldsymbol{q})$, it is known, to the authors' best knowledge, that the weakest condition on M is the so-called *pesudomonotonicity* (see [5, Def. 2.3.1] or [16, 17] and also (2.2)), which was firstly introduced by Karamardian [11], and studied e.g., by Crouzeix et. al. [3], Tao [16, 17] and Gowda [8, 9], as a natural generalization of monotonicity. Since then, several important related properties of linear transformations on Euclidean Jordan algebras have been established in e.g., [16, 17], and this paper attempts to make further developments on the SOCLCP with a pesudomonotone M.

The contributions of this paper are as follows:

(1) We introduce the concept of *J_n-eigenvalue* of a matrix and prove that the matrix M of the pseudomonotone SOCLCP always admites a J_n-eigenvalue. The notion of the J_n-eigenvalue is the key as it not only generalizes the results on the SOCLCP with the GUS property established in [20], but also injects new idea and lays a foundation for numerical algorithms for the related SOCLCP;

(2) By delineating a geometric picture (Section 3) of the SOCLCP, we establish an interesting property, namely, the $\mathbb{K}_s$-*nested-cone property*, and develop a thorough characterization (Section 4) for the range of the pseudomonotone SOCLCP. The $\mathbb{K}_s$-nested-cone property and the notion of J_n-eigenvalue together provide the key for designing efficient algorithms for the related SOCLCP; the work of this idea has been well demonstrated in [21, 22, 23] for the SOCLCP with the GUS property;

(3) We characterize the solution set of a pseudomonotone SOCLCP, which generalizes the results of [13] for the monotone SOCLCP.

The rest of the paper is organized in the following way. In Section 2, we introduce some notations, definitions and some preliminary results that will be frequently used in the subsequent discussions. In Section 3, we first introduce the concept of J_n-eigenvalue and prove that the matrix M of a pseudomonotone SOCLCP always has a J_n-eigenvalue; we next develop some interesting geometric properties and prove the $\mathbb{K}_s$-nested-cone property. A thorough characterization of the range of the pseudomonotone SOCLCP is established in Section 4, and concluding remarks together with possible future works are drawn in Section 5.

2 Notations and preliminary results

In this section, we introduce the notations, definitions, and preliminary results which will be used throughout the paper. We separate this section into two parts.

2.1 Notations and preliminary results of convex cones

Throughout the paper, all vectors are column vectors and are typeset in bold lower case letters, and x_i stands for the ith element of $\boldsymbol{x}$. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, the inner product is denoted by $\boldsymbol{x}^\top \boldsymbol{y}$ or $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$, and $\boldsymbol{y}^\perp \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^\top \boldsymbol{y} = 0\}$. Moreover, we denote

$$(\boldsymbol{x}, \boldsymbol{y}) \triangleq \{\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} : 0 < \lambda < 1\} \quad \text{and} \quad [\boldsymbol{x}, \boldsymbol{y}] \triangleq \{\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} : 0 \le \lambda \le 1\}.$$

For a matrix $A \in \mathbb{R}^{n\times n}$, $\mathrm{rank}(A)$ stands for the rank of A, $\mathrm{Ker}(A)$ denotes the kernel of A and $\mathcal{R}(A) = \{A\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n\}$ is the range of A; related with the matrix A, we introduce the set $\mathcal{A}_{\boldsymbol{y}}^{-1} = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{y}\}$ regardless of whether A is nonsingular or not.

For a set $\mathcal{C} \subset \mathbb{R}^n$, the boundary and interior of $\mathcal{C}$ is denoted by $\mathrm{bd}(\mathcal{C})$ and $\mathrm{int}(\mathcal{C})$, respectively. The *relative interior* of a convex set $\mathcal{C}$, denoted by $\mathrm{ri}(\mathcal{C})$ [15, p.44], is defined as the interior which results when $\mathcal{C}$ is regarded as a subset of its *affine hull*. Precisely, $\mathrm{ri}(\mathcal{C})$ can be denoted as [15, Theorem 6.4]

$$\mathrm{ri}(\mathcal{C}) \triangleq \{\boldsymbol{z} \in \mathcal{C} : \forall \boldsymbol{x} \in \mathcal{C}\ \exists \mu > 1 \text{ such that } \mu \boldsymbol{z} + (1-\mu)\boldsymbol{x} \in \mathcal{C}\}.$$

If $\mathcal{C}$ has nonempty interior, then $\mathrm{ri}(\mathcal{C}) = \mathrm{int}(\mathcal{C})$.

If $\mathcal{C}$ is a closed convex set, the *normal cone* of $\mathcal{C}$ at $\boldsymbol{x} \in \mathcal{C}$ is defined by

$$\mathbb{N}_{\mathcal{C}}(\boldsymbol{x}) \triangleq \{\boldsymbol{z} \in \mathbb{R}^n : \boldsymbol{z}^\top (\boldsymbol{y} - \boldsymbol{x}) \le 0, \quad \forall \boldsymbol{y} \in \mathcal{C}\}.$$

When $\mathcal{C}$ is a closed convex cone, if $\mathrm{int}(\mathcal{C}) \neq \emptyset$ and $\mathcal{C} \cap (-\mathcal{C}) = \{\mathbf{0}\}$, $\mathcal{C}$ is called a *proper cone* [1, p. 43]. An immediate observation is that $\mathbb{K}^n$ is a proper cone. Now we list some properties that will be used in next sections.

Lemma 2.1. *Let $A \in \mathbb{R}^{n\times n}$ and $\mathcal{C}$ be a convex cone. Then $\mathrm{ri}(A\mathcal{C}) = A(\mathrm{ri}(\mathcal{C}))$. If $\mathcal{C}$ is further a proper convex cone, then the following statements hold:*

(1) *$\mathrm{int}(A\mathcal{C}) \neq \emptyset$ if and only if A is nonsingular. If A is nonsingular, then $\mathrm{bd}(A\mathcal{C}) = A(\mathrm{bd}(\mathcal{C}))$ and $A\mathcal{C}$ is a proper cone;*

(2) *Assume that $\mathbf{0} \neq \boldsymbol{b} \in \mathcal{C}$ and $\mathbb{N}_{\mathcal{C}}(\boldsymbol{b}) = \{t\boldsymbol{a} : t \geq 0\}$ for some $\boldsymbol{a} \in \mathbb{R}^n$. If $\boldsymbol{a}^\top \boldsymbol{z} < 0$, then for any $t > 0$ sufficiently small, we have*

$$\boldsymbol{b} + t\boldsymbol{z} \in \mathrm{int}(\mathcal{C}). \tag{2.1}$$

Proof. The assertions $\mathrm{ri}(A\mathcal{C}) = A(\mathrm{ri}(\mathcal{C}))$ and (1) follow from [15, Theorem 6.6] and [15, Theorem 9.1], respectively. For (2), if $\{\boldsymbol{b} + t\boldsymbol{z} : t < 0\} \cap \mathrm{int}(\mathcal{C}) = \emptyset$, then by the convex separation theorem [15, Theorem 11.3], there exists a nonzero $\boldsymbol{r} \in \mathbb{R}^n$ such that $\boldsymbol{r}^\top(\boldsymbol{b} + t\boldsymbol{z}) \geq \boldsymbol{r}^\top \boldsymbol{y}$ for all $t > 0$ and all $\boldsymbol{y} \in \mathcal{C}$. Letting $t \to 0$ gives $\boldsymbol{r}^\top \boldsymbol{b} \geq \boldsymbol{r}^\top \boldsymbol{y}$, yielding $\boldsymbol{r} \in \mathbb{N}_{\mathcal{C}}(\boldsymbol{b})$ and so $\boldsymbol{r} = \mu \boldsymbol{a}$ for some $\mu > 0$. Furthermore, $\boldsymbol{r}^\top(\boldsymbol{b} + t\boldsymbol{z}) \geq \boldsymbol{r}^\top \boldsymbol{y}$ for *all* $t > 0$ implies also $\boldsymbol{a}^\top \boldsymbol{z} \geq 0$, a contradiction. Thus, there exists $t_0 > 0$ such that $\boldsymbol{b} + t_0 \boldsymbol{z} \in \mathcal{C}$ and by the convexity of $\mathcal{C}$, it follows that $\boldsymbol{b} + t\boldsymbol{z} \in \mathcal{C}$ for all $0 < t < t_0$. □

Lemma 2.2. *Let $\mathcal{C}_1, \mathcal{C}_2$ be two convex cones and $\mathcal{C}_1$ be closed. If $\mathrm{int}(\mathcal{C}_1) \cap \mathrm{ri}(\mathcal{C}_2) \neq \emptyset$ and $\mathrm{bd}(\mathcal{C}_1) \cap \mathrm{ri}(\mathcal{C}_2) = \emptyset$, then we have $\mathcal{C}_2 \subset \mathcal{C}_1$.*

Proof. Pick $\boldsymbol{x} \in \mathrm{int}(\mathcal{C}_1) \cap \mathrm{ri}(\mathcal{C}_2)$. If there exists $\boldsymbol{y} \in \mathcal{C}_2$ but $\boldsymbol{y} \notin \mathcal{C}_1$, there exists $\boldsymbol{z} \in (\boldsymbol{x}, \boldsymbol{y})$ such that $\boldsymbol{z} \in \mathrm{bd}(\mathcal{C}_1)$. Since $\boldsymbol{x} \in \mathrm{ri}(\mathcal{C}_2)$ and $\boldsymbol{y} \in \mathcal{C}_2$, by [15, Theorem 6.1], we have $\boldsymbol{z} \in \mathrm{ri}(\mathcal{C}_2)$, which contradicts $\mathrm{bd}(\mathcal{C}_1) \cap \mathrm{ri}(\mathcal{C}_2) = \emptyset$. □

Now we list some frequently used properties of $\mathbb{K}^n$.

Lemma 2.3. *Let*

$$J_n \triangleq \mathrm{diag}(1, -1, -1, \cdots, -1) \in \mathbb{R}^{n\times n}.$$

The following statements hold:

(1) *For nonzero vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{K}^n$, $\boldsymbol{x}^\top \boldsymbol{y} = 0$ if and only if $\boldsymbol{x} \in \mathrm{bd}(\mathbb{K}^n)$, $\boldsymbol{y} \in \mathrm{bd}(\mathbb{K}^n)$ and $\boldsymbol{y} = \mu J_n \boldsymbol{x}$ for some $\mu > 0$;*

(2) *Let $\boldsymbol{x} \in \mathbb{K}^n$. Then $\boldsymbol{x}^\top \boldsymbol{y} > 0$ for all nonzero vector $\boldsymbol{y} \in \mathbb{K}^n$ if and only if $\boldsymbol{x} \in \mathrm{int}(\mathbb{K}^n)$;*

(3) *$\boldsymbol{x} \in \mathbb{K}^n$ if and only if $\boldsymbol{x}^\top \boldsymbol{y} \geq 0$ for all $\boldsymbol{y} \in \mathbb{K}^n$;*

(4) *For all $\boldsymbol{q} \notin \mathbb{K}^n \cup (-\mathbb{K}^n)$, there exist $\boldsymbol{x} \in \mathrm{bd}(\mathbb{K}^n)$ and $\boldsymbol{y} \in \mathrm{bd}(\mathbb{K}^n)$ such that $\boldsymbol{q} = \boldsymbol{x} - \boldsymbol{y}$;*

(5) *For a given nonzero $\boldsymbol{x} \in \mathrm{bd}(\mathbb{K}^n)$, $\{\boldsymbol{z} : \boldsymbol{x}^\top \boldsymbol{z} > 0\} = \{tJ_n \boldsymbol{x} : t < 0\} + \mathrm{int}(\mathbb{K}^n)$;*

(6) *For* $\mathbf{0} \neq \boldsymbol{x} \in \mathrm{bd}(\mathbb{K}^n)$,

$$\mathbb{N}_{\mathbb{K}^n}(\boldsymbol{x}) = \{-tJ_n\boldsymbol{x} : t \geq 0\}.$$

Proof. Except for (5), all the properties can be found in [20]. For (5), it is easy to see that $\{\boldsymbol{z} : \boldsymbol{x}^\top \boldsymbol{z} > 0\} \supseteq \{tJ_n\boldsymbol{x} : t < 0\} + \mathrm{int}(\mathbb{K}^n)$. Now, pick any $\boldsymbol{z}$ such that $\boldsymbol{x}^\top\boldsymbol{z} > 0$, i.e., $z_1x_1 + \boldsymbol{z}_2^\top\boldsymbol{x}_2 > 0$. Let $\boldsymbol{y}(t) = \boldsymbol{z} - tJ_n\boldsymbol{x}$ and we need to prove $\boldsymbol{y}(t) \in \mathrm{int}(\mathbb{K}^n)$ for some $t < 0$. To this end, we note that $x_1 = \|\boldsymbol{x}_2\|_2$ and

$$y_1^2(t) = z_1^2 + t^2x_1^2 - 2tz_1x_1, \quad \|\boldsymbol{y}_2(t)\|_2^2 = \|\boldsymbol{z}_2\|_2^2 + t^2\|\boldsymbol{x}_2\|_2^2 + 2t\boldsymbol{z}_2^\top\boldsymbol{x}_2.$$

Therefore, we know $y_1^2(t) > \|\boldsymbol{y}_2(t)\|_2^2$ for sufficiently large $|t|$ with $t < 0$ which also ensures $y_1(t) > 0$, and consequently, $\boldsymbol{y}(t) \in \mathrm{int}(\mathbb{K}^n)$ for such t. □

2.2 Pseudomonocity and the range of $\mathrm{LCP}(M, \mathbb{K}^n, \boldsymbol{q})$

We say that M is *pseudomonotone* on $\mathbb{K}^n$ (see [5, Def. 2.3.1] or [16, 17]) if

$$\boxed{\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{K}^n, \quad \langle M\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{x}\rangle \geq 0 \Rightarrow \langle M\boldsymbol{y}, \boldsymbol{y} - \boldsymbol{x}\rangle \geq 0.} \tag{2.2}$$

According to (2.2), it is evident that a *monotone* matrix M, or equivalently $\boldsymbol{x}^\top M\boldsymbol{x} \geq 0$ for all $\boldsymbol{x}$, is automatically pseudomonotone. More discussions on the pseudomonotone LCP can be found in, for example, [3, 9, 16, 17] and the references therein. The following assertions which are stated using $\mathbb{K}^n$ also hold for a general symmetric cone of a Euclidean Jordan algebra, and the proof can be found in [8] and [17].

Lemma 2.4. *If M is pseudomonotone on $\mathbb{K}^n$, then we have:*

(1) *If* $\mathrm{rank}(M) \geq 2$, *then* $M^\top$ *is also pseudomonotone;*

(2) $\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$ *is convex for any* $\boldsymbol{q}$*;*

(3) $\boldsymbol{x}^\top M\boldsymbol{x} \geq 0$ *for all* $\boldsymbol{x} \in \mathbb{K}^n$ *and*

$$\boldsymbol{x} \in \mathbb{K}^n, M\boldsymbol{x} \in \mathbb{K}^n \text{ and } \langle \boldsymbol{x}, M\boldsymbol{x}\rangle = 0 \Rightarrow -M^\top\boldsymbol{x} \in \mathbb{K}^n.$$

If M is pseudomonotone on $\mathbb{K}^n$ and $\mathrm{rank}(M) = 1$, we can give an explicit formula for M as follows.

Lemma 2.5. *If M is pseudomonotone on $\mathbb{K}^n$ and $\mathrm{rank}(M) = 1$, then either $M = \boldsymbol{a}\boldsymbol{b}^\top$ for some nonzero $\boldsymbol{a} \in \mathbb{K}^n$ and $\boldsymbol{b} \in \mathrm{int}(\mathbb{K}^n)$, or $M = \boldsymbol{u}\boldsymbol{u}^\top$ for some $\boldsymbol{u} \neq \mathbf{0}$.*

Proof. Assume that $M = \boldsymbol{a}\boldsymbol{b}^\top$ for some nonzero vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$. By Lemma 2.4 (3),

$$\boldsymbol{a}^\top\boldsymbol{x} \cdot \boldsymbol{b}^\top\boldsymbol{x} \geq 0 \ \text{ for all } \ \boldsymbol{x} \in \mathbb{K}^n. \tag{2.3}$$

We first consider the case $\boldsymbol{b} \notin \mathbb{K}^n \cup (-\mathbb{K}^n)$, which implies that $\mathrm{int}(\mathbb{K}^n) \cap \mathrm{Ker}(M) = \mathrm{int}(\mathbb{K}^n) \cap \boldsymbol{b}^\perp \neq \emptyset$. Pick a nonzero $\boldsymbol{r} \in \mathrm{int}(\mathbb{K}^n) \cap \mathrm{Ker}(M)$. Then there exists $\delta > 0$

such that $\boldsymbol{x} = \boldsymbol{r} + t\boldsymbol{b} \in \mathrm{int}(\mathbb{K}^n)$ if $|t| < \delta$. From $\boldsymbol{b}^\top \boldsymbol{r} = 0$ and (2.3), we know that $\boldsymbol{a}^\top(\boldsymbol{r} + t\boldsymbol{b}) > 0$ if $0 < t < \delta$ and $\boldsymbol{a}^\top(\boldsymbol{r} + t\boldsymbol{b}) < 0$ if $-\delta < t < 0$, which implies that $\boldsymbol{a}^\top \boldsymbol{r} = 0$. Since $\boldsymbol{r}$ is chosen arbitrarily in the set $\mathrm{int}(\mathbb{K}^n) \cap \mathrm{Ker}(M)$, we conclude that $\boldsymbol{a}^\top \boldsymbol{z} = 0$ for any $\boldsymbol{z} \in \mathrm{int}(\mathbb{K}^n) \cap \mathrm{Ker}(M)$. On the other hand, for any $\boldsymbol{s} \in \boldsymbol{b}^\perp = \mathrm{Ker}(M)$, $\boldsymbol{r} + t\boldsymbol{s} \in \mathrm{int}(\mathbb{K}^n) \cap \mathrm{Ker}(M)$ if t is sufficiently small, and so $\boldsymbol{a}^\top(\boldsymbol{r} + t\boldsymbol{s}) = 0$, which implies $\boldsymbol{a}^\top \boldsymbol{s} = 0$. Thus $\boldsymbol{b}^\perp \subset \boldsymbol{a}^\perp$, and so $\boldsymbol{a} = \lambda \boldsymbol{b}$ for some nonzero scalar λ. From (2.3), it follows that $\lambda > 0$, and so $M = (\sqrt{\lambda}\boldsymbol{b})(\sqrt{\lambda}\boldsymbol{b})^\top$.

Now, for the case $\boldsymbol{b} \in \mathbb{K}^n$, we will first show $\boldsymbol{a} \in \mathbb{K}^n$. Otherwise, there exists $\boldsymbol{x}_0 \in \mathrm{int}(\mathbb{K}^n)$ such that $\boldsymbol{a}^\top \boldsymbol{x}_0 < 0$. If $\delta > 0$ is sufficiently small, then $\boldsymbol{x}_0 + t\boldsymbol{b} \in \mathrm{int}(\mathbb{K}^n)$ and $\boldsymbol{a}^\top(\boldsymbol{x}_0 + t\boldsymbol{b}) < 0$ for all $|t| < \delta$. Thus $\boldsymbol{a}^\top(\boldsymbol{x}_0 + t\boldsymbol{b}) \cdot \boldsymbol{b}^\top(\boldsymbol{x}_0 + t\boldsymbol{b}) < 0$, which contradicts (2.3), and thus $\boldsymbol{a} \in \mathbb{K}^n$. If $\boldsymbol{b} \in \mathrm{int}(\mathbb{K}^n)$, the assertion then follows; otherwise, assume $\boldsymbol{b} \in \mathrm{bd}(\mathbb{K}^n)$. Let $\boldsymbol{x} = J_n \boldsymbol{b}$ and $\boldsymbol{y} = t\boldsymbol{b}$, where $t > 0$. Then $\langle \boldsymbol{a}\boldsymbol{b}^\top \boldsymbol{x}, \boldsymbol{y} - \boldsymbol{x} \rangle = 0$, which by (2.2) yields $\langle \boldsymbol{a}\boldsymbol{b}^\top \boldsymbol{y}, \boldsymbol{y} - \boldsymbol{x} \rangle \geq 0$, i.e., $t\boldsymbol{a}^\top \boldsymbol{b} \geq \boldsymbol{a}^\top J_n \boldsymbol{b}$. Letting $t \to 0$, we have $\boldsymbol{a}^\top J_n \boldsymbol{b} = 0$, and so by Lemma 2.3 (1) it follows that $\boldsymbol{a} = \lambda \boldsymbol{b}$ for some $\lambda > 0$. We obtain that $M = (\sqrt{\lambda}\boldsymbol{b})(\sqrt{\lambda}\boldsymbol{b})^\top$ again.

Last, for the case $\boldsymbol{b} \in (-\mathbb{K}^n)$, we can show similarly that $-\boldsymbol{a} \in \mathbb{K}^n$ and the rest of the proof follows analogously. □

To deal with $\mathrm{LCP}(M, \mathbb{K}^n, \boldsymbol{q})$ with the fixed pair $(\mathbb{K}^n, M)$, we now define the range of the pair $(\mathbb{K}^n, M)$ by [5, p.187]

$$\mathcal{R}(\mathbb{K}^n, M) \triangleq \{\boldsymbol{q} \in \mathbb{R}^n \ : \ \mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q}) \neq \emptyset\}.$$

It should be pointed out that, in general, neither convexity nor closeness of $\mathcal{R}(\mathbb{K}^n, M)$ can be guaranteed. From the definition (1.1), it is easy to derive the following relation.

Lemma 2.6. *The range* $\mathcal{R}(\mathbb{K}^n, M)$ *satisfies* $(-M\mathbb{K}^n) \cup \mathbb{K}^n \subset \mathcal{R}(\mathbb{K}^n, M) \subset \mathbb{K}^n - M\mathbb{K}^n$.

If $\mathcal{R}(\mathbb{K}^n, M) = \mathbb{R}^n$, we say that M has the Q-property. Among the class of matrices with the Q-property, the subclass for which $\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$ is a singleton is the set of matrices with GUS property, and we refer to [20, 21, 23] for more discussions on the GUS property and the corresponding algorithms.

3 J_n-eigenpair and the $\mathbb{K}_s$-nested-cone property

In the remaining of this paper, we assume that M is pseudomonotone on $\mathbb{K}^n$.

3.1 The characterization of $\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$

We first characterize the set of the solutions of the SOCLCP as in Theorem 3.1. This generalizes the results in [13] established for the monotone case.

Theorem 3.1. *Suppose that* $\boldsymbol{q} \in \mathcal{R}(\mathbb{K}^n, M)$.

(1) *If* $\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q}) \cap \mathrm{int}(\mathbb{K}^n) \neq \emptyset$, *then* $\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q}) = (-\mathcal{M}_{\boldsymbol{q}}^{-1}) \cap \mathbb{K}^n$.

(2) *If* $\boldsymbol{q} \notin (-M\mathbb{K}^n) \cup \mathbb{K}^n$, *then* $\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$ *has a unique element* $\boldsymbol{x}$, *and moreover,* $\boldsymbol{0} \neq \boldsymbol{x} \in \mathrm{bd}(\mathbb{K}^n)$.

(3) *If* $\boldsymbol{q} \in \mathrm{int}(\mathbb{K}^n)$, *then* $\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q}) = \{\boldsymbol{0}\}$.

Proof. (1). First, note that for any $\boldsymbol{x}_0 \in \mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q}) \cap \mathrm{int}(\mathbb{K}^n)$, since $\boldsymbol{x}_0^\top(M\boldsymbol{x}_0 + \boldsymbol{q}) = 0$, by Lemma 2.3 (2), we must have $M\boldsymbol{x}_0 + \boldsymbol{q} = \boldsymbol{0}$. Pick any $\boldsymbol{x} \in \mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$ and define $\boldsymbol{x}(\lambda) \triangleq \lambda\boldsymbol{x}_0 + (1-\lambda)\boldsymbol{x}$ with $0 \leq \lambda \leq 1$. From Lemma 2.4 (2), it follows that $\boldsymbol{x}(\lambda) \in \mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$. Since $\boldsymbol{x}_0 \in \mathrm{int}(\mathbb{K}^n)$, by [15, Theorem 6.1], $\boldsymbol{x}(\lambda) \in \mathrm{int}(\mathbb{K}^n)$ for all $0 < \lambda \leq 1$. From $\boldsymbol{x}(\lambda)^\top(M\boldsymbol{x}(\lambda) + \boldsymbol{q}) = \boldsymbol{0}$, by Lemma 2.3 (2) again, it follows that $M\boldsymbol{x}(\lambda) + \boldsymbol{q} = \boldsymbol{0}$. Letting $\lambda \to 0$ yields $M\boldsymbol{x} + \boldsymbol{q} = \boldsymbol{0}$ and hence $\boldsymbol{x} \in (-\mathcal{M}_{\boldsymbol{q}}^{-1}) \cap \mathbb{K}^n$.

(2). Let $\boldsymbol{x} \in \mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$ be arbitrary. If $\boldsymbol{x} \in \mathrm{int}(\mathbb{K}^n)$, by Lemma 2.3 (2) we have $M\boldsymbol{x} + \boldsymbol{q} = \boldsymbol{0}$, which contradicts the fact $\boldsymbol{q} \notin (-M\mathbb{K}^n)$. Thus $\boldsymbol{x} \in \mathrm{bd}(\mathbb{K}^n)$ and so $\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q}) \cap \mathrm{int}(\mathbb{K}^n) = \emptyset$. From the assumption $\boldsymbol{q} \notin \mathbb{K}^n$, it follows that $\boldsymbol{x} \neq \boldsymbol{0}$. By Lemma 2.3 (1), we have

$$M\boldsymbol{x} + \boldsymbol{q} = sJ_n\boldsymbol{x} \quad \text{for some} \quad s > 0. \tag{3.1}$$

Pick any $\boldsymbol{y} \in \mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$ and we know $\boldsymbol{y} \neq \boldsymbol{0}$ as $\boldsymbol{q} \notin \mathbb{K}^n$. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly independent, it is true that $(\boldsymbol{x}+\boldsymbol{y})/2 \in \mathrm{int}(\mathbb{K}^n)$ and hence $(\boldsymbol{x}+\boldsymbol{y})/2 \in \mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q}) \cap \mathrm{int}(\mathbb{K}^n)$, a contradiction. Thus $\boldsymbol{y} = t_0\boldsymbol{x}$ for some $t_0 > 0$ and by Lemma 2.3 (1) again, $M\boldsymbol{y} + \boldsymbol{q} = tJ_n\boldsymbol{x}$ for some $t > 0$, which together with (3.1) gives $M(t_0 - 1)\boldsymbol{x} = (t-s)J_n\boldsymbol{x}$. If $t_0 \neq 1$, then $M\boldsymbol{x} = \frac{t-s}{t_0-1}J_n\boldsymbol{x}$, and substituting it into (3.1), we obtain $\boldsymbol{q} = \delta J_n\boldsymbol{x}$ for some δ. The condition $\boldsymbol{q} \notin \mathbb{K}^n$ leads to $\delta < 0$, which together with (3.1) implies that $M\boldsymbol{x} = (s-\delta)J_n\boldsymbol{x}$, and therefore, $\boldsymbol{q} = \frac{\delta}{s-\delta}M\boldsymbol{x} \in (-M\mathbb{K}^n)$, a contradiction again. Thus, $t_0 = 1$, i.e., $\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q}) = \{\boldsymbol{x}\}$.

(3). For this case, it is clear that $\boldsymbol{0} \in \mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$. Assume that there is a nonzero $\boldsymbol{y} \in \mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$, then $\boldsymbol{y}^\top(M\boldsymbol{y} + \boldsymbol{q}) = 0$. Since $\boldsymbol{y} \in \mathbb{K}^n$ and $\boldsymbol{q} \in \mathrm{int}(\mathbb{K}^n)$, it is true by Lemma 2.3 (2) that $\boldsymbol{y}^\top\boldsymbol{q} > 0$, which then implies $\boldsymbol{y}^\top M\boldsymbol{y} < 0$, a contradiction with Lemma 2.4 (3). This completes the proof. □

Based on Theorem 3.1, we have further the following corollary.

Corollary 3.1. *The following assertions hold.*

(1) *If* $\boldsymbol{q} \in (-M(\mathrm{int}(\mathbb{K}^n)))$, *then* $\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q}) = (-\mathcal{M}_{\boldsymbol{q}}^{-1}) \cap \mathbb{K}^n$.

(2) *If* $\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$ *has two linearly independent solutions, then*

$$\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q}) = (-\mathcal{M}_{\boldsymbol{q}}^{-1}) \cap \mathbb{K}^n.$$

Proof. The statement of (1) is straightforward from Theorem 3.1 (1), and only (2) needs a proof. Assume that $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$, then by Lemma 2.4 (2), $(\boldsymbol{x}+\boldsymbol{y})/2 \in \mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$. Since $\boldsymbol{y} \neq \mu\boldsymbol{x}$ for any $\mu \geq 0$, $(\boldsymbol{x}+\boldsymbol{y})/2 \in \mathrm{int}(\mathbb{K}^n)$, which together with Theorem 3.1 (1) implies that $\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q}) = (-\mathcal{M}_{\boldsymbol{q}}^{-1}) \cap \mathbb{K}^n$. □

According to Theorem 3.1 and Corollary 3.1, there are only two possible types for the set $\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$: one is $(-\mathcal{M}_{\boldsymbol{q}}^{-1}) \cap \mathbb{K}^n$ and the other is $[\alpha_1 \boldsymbol{x}, \alpha_2 \boldsymbol{x}]$, for some $\alpha_1, \alpha_2 \geq 0$ with $\alpha_2 = +\infty$ included. In fact, by Theorem 3.1 and Corollary 3.1 we can obtain (the detailed proof is omitted here for simplicity) all the results of [13, Theorem 3.2] for the monotone case; in other words, we have extended the condition from the monotonicity to the pesudomonotonicity case for $\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$.

3.2 J_n-eigenpair and the $\mathbb{K}_s$-nested-cone property

we next introduce the important notion of J_n-eigenpair of a matrix, which plays a vital role in this paper.

$$\boxed{\begin{array}{c}\text{We say } \gamma \in \mathbb{R} \text{ is a } J_n\text{-eigenvalue and } \mathbf{0} \neq \boldsymbol{x} \in \mathbb{K}^n \text{ is the} \\ \text{corresponding } J_n\text{-eigenvector of } M \text{ if } M\boldsymbol{x} = \gamma J_n \boldsymbol{x}.\end{array}} \tag{3.2}$$

For simplicity, we also call the pair $(\gamma, \boldsymbol{x})$ satisfying (3.2) as the J_n-eigenpair of M. Equivalently, we can say that (3.2) holds if and only if $\boldsymbol{x} \in \mathbb{K}^n$ is an eigenvector of $J_n M$ associated with the eigenvalue γ. It should be noticed that not every matrix M admits a J_n-eigenpair since the eigenvector of $J_n M$ is not necessarily in $\mathbb{K}^n$.

As in [20], we define $M_s \triangleq M - sJ_n$ for $s \in \mathbb{R}$. The following fact will be used many times:

$$\text{If } \boldsymbol{q} = -M_s \boldsymbol{x} \text{ for some } s \geq 0 \text{ and } \boldsymbol{x} \in \mathrm{bd}(\mathbb{K}^n), \text{ then } \boldsymbol{x} \in \mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q}). \tag{3.3}$$

Let

$$\mathbb{K}_s \triangleq M_s \mathbb{K}^n = \{M_s \boldsymbol{x} : \boldsymbol{x} \in \mathbb{K}^n\}.$$

With this notation, we have obviously $\mathbb{K}_0 = M\mathbb{K}^n$.

Now we need a measure to estimate the distance between $\mathbb{K}_s$ and $\mathbb{K}_t$. For this purpose, we first introduce the distance of a point $\boldsymbol{x}$ to a set $\mathcal{C}$. Define

$$\mathrm{dist}(\boldsymbol{x}, \mathcal{C}) \triangleq \min_{\boldsymbol{y} \in \mathcal{C}} \|\boldsymbol{x} - \boldsymbol{y}\|_2.$$

For two convex cones $\mathcal{C}_1$ and $\mathcal{C}_2$, the Pompeiu-Hausdorff distance between $\mathcal{C}_1$ and $\mathcal{C}_2$ is given by (see [19, 20])

$$d(\mathcal{C}_1, \mathcal{C}_2) \triangleq \max \left\{ \max_{\boldsymbol{u} \in \mathcal{C}_1, \|\boldsymbol{u}\|_2 = 1} \mathrm{dist}(\boldsymbol{u}, \mathcal{C}_2), \max_{\boldsymbol{v} \in \mathcal{C}_2, \|\boldsymbol{v}\|_2 = 1} \mathrm{dist}(\boldsymbol{v}, \mathcal{C}_1) \right\}. \tag{3.4}$$

In [20, Lemma 9], Yang and Yuan have proved that

$$\lim_{s \to 0} d(\mathbb{K}_s, M\mathbb{K}^n) = 0, \quad \text{and} \quad \lim_{s \to +\infty} d(\mathbb{K}_s, -\mathbb{K}^n) = 0, \tag{3.5}$$

and for this reason, we define $K_\infty \triangleq -\mathbb{K}^n$.

The rest of this section is devoted to delineating a geometric picture of $\mathbb{K}_s$.

First, it is interesting to note from the following results that the boundary of $\mathbb{K}_s$ can not intersect with the interior of $\mathbb{K}_\infty$; in fact, it reveals interestingly that the intersection can only be a ray which is generated by a J_n-eigenvector of M.

Lemma 3.1. *For any $s \in \mathbb{R}$, the following statements hold:*

(1) *For any $\boldsymbol{x} \in \operatorname{bd}(\mathbb{K}^n)$, $M_s\boldsymbol{x} \notin (-\operatorname{int}(\mathbb{K}^n))$;*

(2) *If $M_s\boldsymbol{x} = -\boldsymbol{a}$ for some nonzero vector $\boldsymbol{x} \in \operatorname{bd}(\mathbb{K}^n)$ and $\boldsymbol{a} \in \mathbb{K}^n$ (possibly $\mathbf{0}$), then $\boldsymbol{x}$ is a J_n-eigenvector of M and $\boldsymbol{a} = tJ_n\boldsymbol{x}$ for some $t \geq 0$;*

(3) *If $s > 0$ and $M_s\boldsymbol{x} = M\boldsymbol{a}$ for some nonzero vector $\boldsymbol{x} \in \operatorname{bd}(\mathbb{K}^n)$ and $\boldsymbol{a} \in \mathbb{K}^n$ (possibly $\mathbf{0}$), then $\boldsymbol{a} = \lambda\boldsymbol{x}$ for some $0 \leq \lambda \neq 1$, which implies that $\boldsymbol{x}$ is a J_n-eigenvector of M.*

Proof. (1). Assume $\boldsymbol{x} \neq \mathbf{0}$. If $M_s\boldsymbol{x} \in (-\operatorname{int}(\mathbb{K}^n))$, by Lemma 2.3 (2) and $\boldsymbol{x}^\top J_n\boldsymbol{x} = 0$,

$$0 > \boldsymbol{x}^\top M_s\boldsymbol{x} = \boldsymbol{x}^\top(M\boldsymbol{x} - sJ_n\boldsymbol{x}) = \boldsymbol{x}^\top M\boldsymbol{x},$$

which contradicts Lemma 2.4 (3).

(2). Assume $M_s\boldsymbol{x} = -\boldsymbol{a}$. By Lemma 2.3 (2) again, we have

$$0 \geq -\boldsymbol{x}^\top\boldsymbol{a} = \boldsymbol{x}^\top M_s\boldsymbol{x} = \boldsymbol{x}^\top(M\boldsymbol{x} - sJ_n\boldsymbol{x}) = \boldsymbol{x}^\top M\boldsymbol{x} \geq \mathbf{0},$$

the last inequality is due to Lemma 2.4 (3). Thus $\boldsymbol{x}^\top\boldsymbol{a} = 0$, which together with Lemma 2.3 (1) implies that $\boldsymbol{a} = tJ_n\boldsymbol{x}$ for some $t \geq 0$, and $M\boldsymbol{x} = (s-t)J_n\boldsymbol{x}$ and so $\boldsymbol{x}$ is a J_n-eigenvector.

(3). Let $\boldsymbol{q} = -M_s\boldsymbol{x} = -M\boldsymbol{a}$. From (3.3), it follows that $\boldsymbol{x}, \boldsymbol{a} \in \operatorname{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$. If $\boldsymbol{x}$ and $\boldsymbol{a}$ are linearly independent, by Corollary 3.1, $M\boldsymbol{x} = -\boldsymbol{q}$ and so $s = 0$, a contradiction. Thus $\boldsymbol{a} = \lambda\boldsymbol{x}$ for some $\lambda \geq 0$. Obviously, $\lambda \neq 1$ and so $M\boldsymbol{x} = sJ_n\boldsymbol{x}/(1-\lambda)$. □

For $s = 0$, much more can be said as the next lemma shows.

Lemma 3.2. *The following statements hold:*

(1) $M\mathbb{K}^n \cap (-\operatorname{int}(\mathbb{K}^n)) = \emptyset$;

(2) *If $M\mathbb{K}^n \cap (-\mathbb{K}^n) \neq \{\mathbf{0}\}$, there exists a J_n-eigenvector $\boldsymbol{x} \in \operatorname{bd}(\mathbb{K}^n)$ associated with a negative J_n-eigenvalue such that*

$$M\mathbb{K}^n \cap (-\mathbb{K}^n) = \{tJ_n\boldsymbol{x} : t \leq 0\}. \tag{3.6}$$

In particular, if $\boldsymbol{y} \in \mathbb{K}^n$ and $\mathbf{0} \neq M\boldsymbol{y} \in (-\mathbb{K}^n)$, then $\boldsymbol{y} = \lambda\boldsymbol{x}$ for some $\lambda > 0$; as a result, M has a unique negative J_n-eigenvalue and the subspace spanned by the associated J_n-eigenvectors is one-dimensional.

Proof. (1). If $M\boldsymbol{x} = -\boldsymbol{z}$ for some $\boldsymbol{x} \in \mathbb{K}^n$ and $\boldsymbol{z} \in \operatorname{int}(\mathbb{K}^n)$, then $\boldsymbol{x}^\top M\boldsymbol{x} = -\boldsymbol{x}^\top\boldsymbol{z}$. It is obvious that $\boldsymbol{x} \neq \mathbf{0}$, which together with Lemma 2.3 (2) implies that $\boldsymbol{x}^\top\boldsymbol{z} > 0$. Thus $\boldsymbol{x}^\top M\boldsymbol{x} < 0$, contradicting Lemma 2.4 (3).

(2). If $M\boldsymbol{x} = -\boldsymbol{z}$ for some nonzero vectors $\boldsymbol{x}, \boldsymbol{z} \in \mathbb{K}^n$, then $\boldsymbol{x}^\top M\boldsymbol{x} = -\boldsymbol{x}^\top\boldsymbol{z} \leq 0$, where the inequality is due to Lemma 2.3 (1). By Lemma 2.4, $\boldsymbol{x}^\top M\boldsymbol{x} \geq 0$ and so $\boldsymbol{x}^\top\boldsymbol{z} = 0$. Use Lemma 2.3 (1) again to get $\boldsymbol{x} \in \operatorname{bd}(\mathbb{K}^n)$ and $\boldsymbol{z} = sJ_n\boldsymbol{x}$ for some $s > 0$, that is $\boldsymbol{x}$ is a J_n-eigenvector associated with a negative J_n-eigenvalue $-s$. If there exists another nonzero

$\boldsymbol{y} \in \mathbb{K}^n$ such that $M\boldsymbol{y} = -\boldsymbol{r}$ for some $\boldsymbol{0} \neq \boldsymbol{r} \in \mathbb{K}^n$, then similarly, we have $\boldsymbol{r} = tJ_n\boldsymbol{y}$ for some $t > 0$. In fact furthermore, $\boldsymbol{r}$ and $\boldsymbol{z}$ are linearly dependent, because otherwise, it is easy to verify that $\boldsymbol{r}+\boldsymbol{z} \in \mathrm{int}(\mathbb{K}^n)$, which together with $M(\boldsymbol{y}+\boldsymbol{x}) = -(\boldsymbol{r}+\boldsymbol{z})$ implies that $M\mathbb{K}^n \cap (-\mathrm{int}(\mathbb{K}^n)) \neq \emptyset$, contradicting the assertion (1). Therefore, there is $t_0 > 0$ such that $\boldsymbol{r} = t_0\boldsymbol{z}$ leading to $\boldsymbol{y} = \lambda\boldsymbol{x}$ for some $\lambda > 0$. The rest part is obvious. □

It deserves to point out that even for a monotone matrix M, the case in (3.6) can occur as the following example shows, where M has a negative J_n-eigenvalue -1.

Example 3.1. *Let*

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then M is monotone, $M\mathbb{K}^2 \cap (-\mathbb{K}^2) = \{t[-1,1]^\top : t \geq 0\}$ and $M\boldsymbol{x} = -J_2\boldsymbol{x}$.

In [20], Yang and Yuan have proved that when M has the GUS property, there exists $\tau > 0$ such that

- if $s > t > \tau$ then $\mathrm{int}(\mathbb{K}_t) \supset \mathbb{K}_s \backslash \{\boldsymbol{0}\}$ and $\mathrm{int}(\mathbb{K}_t) \supset (-\mathbb{K}^n)\backslash\{\boldsymbol{0}\} = \mathbb{K}_\infty\backslash\{\boldsymbol{0}\}$, while
- if $0 < t < s < \tau$ then $\mathrm{int}(\mathbb{K}_s) \supset \mathbb{K}_t\backslash\{\boldsymbol{0}\}$ and $\mathrm{int}(\mathbb{K}_t) \supset M\mathbb{K}^n\backslash\{\boldsymbol{0}\} = \mathbb{K}_0\backslash\{\boldsymbol{0}\}$.

We call this property the *interior $\mathbb{K}_s$-nested-cone property*, based on which Zhang and Yang proposed the bisection-Newton method for solving (1.1) in [21] (see also [23]).

If M is pseudomonotone, the above interior $\mathbb{K}_s$-nested-cone property does not hold any more. But the $\mathbb{K}_s$-nested-cone property is still true as the following proposition and Proposition 3.3 together show.

Proposition 3.1. (1) *For all sufficiently large $s > 0$, we have $\mathbb{K}_s \supset (-\mathbb{K}^n) = \mathbb{K}_\infty$.*

(2) *If $s > t$ and $\mathbb{K}_t \supset (-\mathbb{K}^n)$, then $\mathbb{K}_t \supset \mathbb{K}_s \supset (-\mathbb{K}^n)$.*

Proof. (1). Let $\boldsymbol{x} \in \mathrm{int}(\mathbb{K}^n)$. Then $-J_n\boldsymbol{x} \in \mathrm{int}(-\mathbb{K}^n)$. Note that there exists $s_0 > 0$ such that if $s > s_0$, $M_s\boldsymbol{x} = s(M\boldsymbol{x}/s - J_n\boldsymbol{x}) \in \mathrm{int}(-\mathbb{K}^n)$. Moreover, for sufficiently large s_0, we can also assume that M_s is nonsingular for all $s > s_0$. By Lemma 2.1 (1), $M_s\boldsymbol{x} \in \mathrm{int}(\mathbb{K}_s)$ and so $\mathrm{int}(\mathbb{K}_s) \cap \mathrm{int}(-\mathbb{K}^n) \neq \emptyset$, and also we have $M_s(\mathrm{bd}(\mathbb{K}^n)) = \mathrm{bd}(M_s(\mathbb{K}^n))$. From Lemma 3.1 (1), it follows that $\mathrm{bd}(M_s(\mathbb{K}^n)) \cap \mathrm{int}(-\mathbb{K}^n) = \emptyset$. Thus the assumptions of Lemma 2.2 are satisfied and the assertion follows from Lemma 2.2.

(2). Pick any $\boldsymbol{a} \in -\mathrm{int}(\mathbb{K}^n)$. Since $\mathbb{K}_t \supset (-\mathbb{K}^n)$, by Lemma 2.1 (1), M_t is nonsingular. Then there exists $\boldsymbol{x} \in \mathrm{int}(\mathbb{K}^n)$ such that $M_t\boldsymbol{x} = \boldsymbol{a}$, which together with

$$M_s\boldsymbol{x} = M_t\boldsymbol{x} - (s-t)\boldsymbol{x} \tag{3.7}$$

implies that $M_s\boldsymbol{x} \in (-\mathrm{int}(\mathbb{K}^n))$. Let $\mathcal{B}(\boldsymbol{x};\delta)$ be the open ball centering at $\boldsymbol{x}$ with radius δ. If δ is sufficiently small, then $M_t\mathcal{B}(\boldsymbol{x};\delta) \subset (-\mathrm{int}(\mathbb{K}^n))$ and $\mathcal{B}(\boldsymbol{x};\delta) \subset \mathrm{int}(\mathbb{K}^n)$. From (3.7), it follows that $M_s\mathcal{B}(\boldsymbol{x};\delta)$ is an open set in $-\mathrm{int}(\mathbb{K}^n)$. Thus M_s is nonsingular and

$\mathrm{int}(\mathbb{K}_s)\cap(-\mathrm{int}(\mathbb{K}^n)) \neq \emptyset$. By Lemma 3.1 (1), we have $\mathrm{bd}(M_s(\mathbb{K}^n))\cap\mathrm{int}(-\mathbb{K}^n) = \emptyset$, which together with Lemma 2.2 implies $\mathbb{K}_s \supset (-\mathbb{K}^n)$.

Lastly, it is obvious that $\mathrm{int}(\mathbb{K}_s) \cap \mathrm{int}(\mathbb{K}_t) \neq \emptyset$ because $\mathbb{K}_s \supset (-\mathbb{K}^n)$ and $\mathbb{K}_t \supset (-\mathbb{K}^n)$. Now we will prove that $M_t\boldsymbol{x} \notin \mathrm{int}(\mathbb{K}_s)$ for any $\boldsymbol{x} \in \mathrm{bd}(\mathbb{K}^n)$, which implies that $\mathrm{bd}(M_t(\mathbb{K}^n)) \cap \mathrm{int}(\mathbb{K}_s) = \emptyset$, and hence the assertion follows from Lemma 2.2. To this end, assume that there exists $\boldsymbol{x} \in \mathrm{bd}(\mathbb{K}^n)$ such that $M_t\boldsymbol{x} \in \mathrm{int}(\mathbb{K}_s)$, then from $M_s\boldsymbol{x} = M_t\boldsymbol{x} - (s-t)\boldsymbol{x}$ and $-\boldsymbol{x} \in \mathbb{K}_s$, we have $M_s\boldsymbol{x} \in \mathrm{int}(\mathbb{K}_s)$. But from the nonsingularity of M_s and $\boldsymbol{x} \in \mathrm{bd}(\mathbb{K}^n)$, it follows that $M_s\boldsymbol{x} \in \mathrm{bd}(\mathbb{K}_s)$, a contradiction. □

Similar to the GUS property case, we can define the threshold τ and prove that the matrix M_τ is singular. To this end, let

$$\tau(M) \triangleq \inf\{s : -\mathbb{K}^n \subset \mathbb{K}_s, s \geq 0\}, \tag{3.8}$$

and if no confusion arises, we simply write $\tau(M)$ as τ. By Proposition 3.1, τ is well-defined, and obviously $\tau \geq 0$. In general, the value of τ is possibly 0. If $s > \tau$, from the proof of Proposition 3.1, we know that M_s is nonsingular and so $\mathbb{K}_s$ is a proper cone.

Proposition 3.1 is the half part of the $\mathbb{K}_s$-nested-cone property which is concerned with the case $s > \tau$; and the other part of the $\mathbb{K}_s$-nested-cone property related with $0 < s < \tau$ will be established later in Proposition 3.3.

Lemma 3.3. *Let τ be defined in* (3.8)*.*

(1) *M_τ is singular and $\mathbb{K}_\tau \cap (-\mathrm{int}(\mathbb{K}^n)) = \emptyset$.*

(2) $\mathrm{Ker}(M_\tau) \cap \mathbb{K}^n \neq \{\mathbf{0}\}$.

(3) *If $\tau > 0$, then $\mathrm{rank}(M_\tau) = n - 1$, and there exists $\boldsymbol{w} \in \mathbb{K}^n$ such that*

$$\mathrm{Ker}(M_\tau) = \{t\boldsymbol{w} : t \in \mathbb{R}\}. \tag{3.9}$$

(4) *If $\mathrm{rank}(M) \geq 2$, then $\tau(M) = \tau(M^\top)$.*

Proof. (1). By the definition (3.8), $M_s(\mathbb{K}^n) \supset (-\mathbb{K}^n)$ for any $s > \tau$. For any $\boldsymbol{x} \in \mathbb{K}^n$ and any $s > \tau$, there exists $\boldsymbol{a}_s \in \mathbb{K}^n$ such that $M_s\boldsymbol{a}_s = -\boldsymbol{x}$. We prove the singularity of M_τ by contradiction. Assume M_τ is nonsingular. By continuity, if s is sufficiently close to τ then M_s is nonsingular too; moreover, $\lim_{s\to\tau} M_s^{-1} = M_\tau^{-1}$. Thus

$$-M_\tau^{-1}\boldsymbol{x} = -\lim_{s\to\tau} M_s^{-1}\boldsymbol{x} = \lim_{s\to\tau} \boldsymbol{a}_s \in \mathbb{K}^n.$$

Since $\boldsymbol{x}$ is arbitrary, we have $M_\tau(\mathbb{K}^n) \supset (-\mathbb{K}^n)$. If $\tau = 0$, we obtain a contradiction to Lemma 3.2 (1) immediately. Assume then $\tau > 0$. For $\boldsymbol{x} \in \mathrm{int}(\mathbb{K}^n)$, by $M_\tau(\mathbb{K}^n) \supset (-\mathbb{K}^n)$, there exists $\boldsymbol{y} \in \mathbb{K}^n$ such that $M_\tau\boldsymbol{y} = -\boldsymbol{x}$. Since M_τ is nonsingular, we must have $\boldsymbol{y} \in \mathrm{int}(\mathbb{K}^n)$. If s is sufficiently close to τ, then M_s is nonsingular and so $M_s\boldsymbol{y} \in (-\mathrm{int}(\mathbb{K}^n))$. Then $\mathrm{int}(\mathbb{K}_s) \cap (-\mathrm{int}(\mathbb{K}^n)) \neq \emptyset$, which together with Lemmas 2.2 and 3.1 (1) leads to $\mathbb{K}_s \supset (-\mathbb{K}^n)$. Since s can be less than τ, we obtain a contradiction to (3.8) and M_τ is singular.

If $\mathbb{K}_\tau \cap (-\text{int}(\mathbb{K}^n)) \neq \emptyset$, there exists $\boldsymbol{z} \in \mathbb{K}^n$ such that $M_\tau \boldsymbol{z} \in (-\text{int}(\mathbb{K}^n))$. Pick a nonzero $\boldsymbol{w} \in \text{Ker}(M_\tau)$, and we can choose a suitable $t \in \mathbb{R}$ such that $\boldsymbol{z} + t\boldsymbol{w} \in \text{bd}(\mathbb{K}^n)$. Note that $M_\tau(\boldsymbol{z} + t\boldsymbol{w}) = M_\tau \boldsymbol{z} \in (-\text{int}(\mathbb{K}^n))$, which is a contradiction to Lemma 3.1 (1).

(2). From (3.8), for all $s > \tau$, we know that $\mathbb{K}_s \supset (-\mathbb{K}^n)$. Pick any $\boldsymbol{a} \in (-\text{int}(\mathbb{K}^n))$. Then there exists $\boldsymbol{x}_s \in \mathbb{K}^n$ such that $M_s \boldsymbol{x}_s = \boldsymbol{a}$. If $\boldsymbol{x}_s$ is bounded as s approaches τ, then $\boldsymbol{x} \triangleq \lim_{s \downarrow \tau} \boldsymbol{x}_s$ exists and satisfies $M_\tau \boldsymbol{x} = \boldsymbol{a}$, a contradiction to the assertion (1). Thus, we must have $\|\boldsymbol{x}_s\|_2 \to \infty$ as s tends to τ. Note $\boldsymbol{x}_s / \|\boldsymbol{x}_s\|_2$ has a cluster, say $\boldsymbol{w} \in \mathbb{K}^n$, and from $M_s \frac{\boldsymbol{x}_s}{\|\boldsymbol{x}_s\|_2} \to \boldsymbol{0}$ as $s \to \tau$, we know $M_\tau \boldsymbol{w} = \boldsymbol{0}$.

(3). Pick any nonzero $\boldsymbol{w} \in \text{Ker}(M_\tau)$ and we will show that $\boldsymbol{w} \in \mathbb{K}^n \cup (-\mathbb{K}^n)$. If not, by Lemma 2.3 (4), there exist $\boldsymbol{x}, \boldsymbol{y} \in \text{bd}(\mathbb{K}^n)$ such that $\boldsymbol{w} = \boldsymbol{x} - \boldsymbol{y}$. From $\boldsymbol{w} \notin \mathbb{K}^n \cup (-\mathbb{K}^n)$, it follows that $\boldsymbol{y} \neq t\boldsymbol{x}$ for any $t \in \mathbb{R}$. Since $\boldsymbol{w} \in \text{Ker}(M_\tau)$, we have $M_\tau \boldsymbol{x} = M_\tau \boldsymbol{y}$. Let $\boldsymbol{p} = -M_\tau \boldsymbol{x}$. Then $\text{SOL}(M, \mathbb{K}^n, \boldsymbol{p})$ has two linearly independent solutions $\boldsymbol{x}$ and $\boldsymbol{y}$. By Corollary 3.1, we have $M\boldsymbol{x} + \boldsymbol{p} = M\boldsymbol{y} + \boldsymbol{p} = \boldsymbol{0}$, which together with $\boldsymbol{p} = -M_\tau \boldsymbol{x} = -M_\tau \boldsymbol{y}$ implies that $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{0}$. Thus, we have $\boldsymbol{w} = \boldsymbol{0}$, a contradiction. Consequently, $\boldsymbol{w} \in \mathbb{K}^n \cup (-\mathbb{K}^n)$, implying that $\text{Ker}(M_\tau) \subset \mathbb{K}^n \cup (-\mathbb{K}^n)$. Thus the dimension of $\text{Ker}(M_\tau)$ is one and so (3.9) holds.

(4). By the definition in (3.8), for all $s > \tau(M)$, $M_s \mathbb{K}^n \supset (-\mathbb{K}^n)$ and so M_s is nonsingular. Since $M_s^\top = (M_s)^\top$, $M_s^\top$ is nonsingular too, which together with the statement (1) implies that $\tau(M^\top) \geq \tau(M)$. Since $M^\top$ is also pseudomonotone, by the previous arguments, we have $\tau((M^\top)^\top) \geq \tau(M^\top)$, and so $\tau(M) = \tau(M^\top)$.

□

Because the J_n-eigenvalue τ plays such an important role, the assignment to $\tau > 0$ which refers to the value defined by (3.8) is reserved in the rest of the paper; moreover, Lemma 3.3 (3) implies that if $\tau > 0$, $\boldsymbol{w}$ is the unique J_n-eigenvector of M associated with τ. Because of this reason, the assignment to $\boldsymbol{w}$ will also be reserved.

As an interesting case for $\text{rank}(M) = 1$, we are able to derive an explicit formula for τ as follows.

Proposition 3.2. *Suppose that* $\text{rank}(M) = 1$. *The following statements hold:*

(1) *If* $M = \boldsymbol{a}\boldsymbol{b}^\top$ *for some* $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{K}^n$ *and* $\boldsymbol{b} \in \text{int}(\mathbb{K}^n)$, *then* $\tau = \boldsymbol{b}^\top J_n \boldsymbol{a} > 0$ *and* $\text{Ker}(M_\tau) = \{t J_n \boldsymbol{a} : t \in \mathbb{R}\}$;

(2) *If* $M = \boldsymbol{u}\boldsymbol{u}^\top$, *where* $\boldsymbol{u} \notin \text{int}(\mathbb{K}^n) \cup (-\text{int}(\mathbb{K}^n))$, *then* $\tau = 0$ *and there exists* $\boldsymbol{w} \in \mathbb{K}^n$ *such that* $M\boldsymbol{w} = \boldsymbol{0}$.

Proof. (1). If $M\boldsymbol{x} = \boldsymbol{a}\boldsymbol{b}^\top \boldsymbol{x} = \tau J_n \boldsymbol{x}$ for some nonzero $\boldsymbol{x} \in \mathbb{K}^n$, then $\boldsymbol{x} = \lambda J_n \boldsymbol{a}$ for some $\lambda \geq 0$. Substitute it into $\boldsymbol{a}\boldsymbol{b}^\top \boldsymbol{x} = \tau J_n \boldsymbol{x}$ to get $\tau = \boldsymbol{b}^\top J_n \boldsymbol{a}$. Obviously $\tau > 0$.

(2). Assume that $\boldsymbol{u}\boldsymbol{u}^\top \boldsymbol{x} = \tau J_n \boldsymbol{x}$, where $\boldsymbol{0} \neq \boldsymbol{x} \in \mathbb{K}^n$. If $\tau > 0$, then $\boldsymbol{x} = \frac{\boldsymbol{u}^\top \boldsymbol{x}}{\tau} J_n \boldsymbol{u}$, which implies $\boldsymbol{u} \in \mathbb{K}^n \cup (-\mathbb{K}^n)$, and by the assumption $\boldsymbol{u} \notin \text{int}(\mathbb{K}^n) \cup (-\text{int}(\mathbb{K}^n))$, it leads to $\boldsymbol{u} \in \text{bd}(\mathbb{K}^n) \cup (-\text{bd}(\mathbb{K}^n))$. Thus we have $\boldsymbol{u}^\top \boldsymbol{x} = 0$, which yields $\tau = 0$, a contradiction. The existence of $\boldsymbol{w}$ follows from Lemma 3.3 (2). □

Remark 3.1. *We point out that if* $\operatorname{rank}(M) = 1$*, it is also true by Proposition 3.2 that* $\tau(M) = \tau(M^\top)$*. But on the other hand,* $M^\top$ *may not be pseudomonotone as we can consider the case* $M = \boldsymbol{a}\boldsymbol{b}^\top$ *for some* $\mathbf{0} \neq \boldsymbol{a} \in \operatorname{bd}(\mathbb{K}^n)$ *and* $\boldsymbol{b} \in \operatorname{int}(\mathbb{K}^n)$*, for which according to Lemma 2.5,* $M^\top$ *is not pseudomonotone.*

Based upon our previously established results, we know that if $\tau > 0$ then $\operatorname{rank}(M_\tau) = n-1$, or equivalently $\operatorname{rank}(M_\tau^\top) = n-1$. Hence, there exists $\mathbf{0} \neq \boldsymbol{v} \in \mathbb{R}^n$ such that

$$M_\tau^\top \boldsymbol{v} = (M^\top - \tau J_n)\boldsymbol{v} = \mathbf{0}, \tag{3.10}$$

which implies that $\mathcal{R}(M_\tau) = \boldsymbol{v}^\perp$.

Lemma 3.4. *Assume that* $\tau > 0$*. Let* $\boldsymbol{v}$ *be defined by* (3.10)*. Then either* $\boldsymbol{v} \in \mathbb{K}^n$ *or* $-\boldsymbol{v} \in \mathbb{K}^n$*.*

Proof. If $M = \boldsymbol{a}\boldsymbol{b}^\top$ for some nonzero $\boldsymbol{a} \in \mathbb{K}^n$ and $\boldsymbol{b} \in \operatorname{int}(\mathbb{K}^n)$, then $\boldsymbol{v} = J_n\boldsymbol{b}$ satisfies (3.10); if $\operatorname{rank}(M) \geq 2$, by Lemma 2.4 (1), $M^\top$ is pseudomonotone too, and by Lemma 3.3 (3), there exists $\hat{\boldsymbol{v}} \in \mathbb{K}^n$ such that $M_\tau^\top \hat{\boldsymbol{v}} = \mathbf{0}$; furthermore, according to (3.10) and $\operatorname{rank}(M_\tau^\top) = n-1$, $\hat{\boldsymbol{v}} = t\boldsymbol{v}$ for some $0 \neq t \in \mathbb{R}$, which establishes the assertion. ☐

Lemma 3.4 shows that $\boldsymbol{v} \in \mathbb{K}^n$ is a J_n-eigenvector of $M^\top$ associated with the positive J_n-eigenvalue $\tau > 0$. For this reason, in our following discussions, the assignment to $\boldsymbol{v}$ is also reserved.

The following theorem is one of our main results of this section.

Theorem 3.2. *Let* τ *be defined by* (3.8)*.*

(1) *If* $\tau = 0$*, then* M *has no other* J_n*-eigenvalue.*

(2) *If* $\tau > 0$*, then*

$$M\boldsymbol{w} = \tau J_n \boldsymbol{w}, \ \ M^\top \boldsymbol{v} = \tau J_n \boldsymbol{v}, \tag{3.11}$$

and moreover, M *has no other non-negative* J_n*-eigenvalue.*

(3) *The following four statements are equivalent:*

  (a) M *has a unique negative* J_n*-eigenvalue;*

  (b) M *has a negative* J_n*-eigenvalue;*

  (c) $\operatorname{rank}(M) > 1$ *and* $\boldsymbol{v} \in \operatorname{bd}(\mathbb{K}^n)$*;*

  (d) $\boldsymbol{v}$ *is a* J_n*-eigenvector of* M *associated with a negative* J_n*-eigenvalue.*

(4) *The equivalence between (a)* $\sim$ *(d) in (3) holds if* M *and* $\boldsymbol{v}$ *are replaced by* $M^\top$ *and* $\boldsymbol{w}$*, respectively. Moreover, under either condition (a)* $\sim$ *(d),* $\boldsymbol{w}$ *and* $\boldsymbol{v}$ *are linearly independent.*

Proof. (1) By the definition of τ, we know for all $s > \tau$, $\mathbb{K}_s \supset (-\mathbb{K}^n)$, and by Lemma 2.1 (1), M_s is nonsingular, which implies that M has no positive J_n-eigenvalue. Suppose t is a negative J_n-eigenvalue, and $\boldsymbol{y}$ is the corresponding J_n-eigenvector, then $\boldsymbol{y}$ and $\boldsymbol{w}$ are linearly independent, and $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{w} \in \mathrm{int}(\mathbb{K}^n)$. Moreover, $M\boldsymbol{x} = tJ_n\boldsymbol{y}$, and from Lemma 3.2 (2), we know that $\boldsymbol{x} \in \mathrm{bd}(\mathbb{K}^n)$, a contradiction.

(2) The equations in (3.11) have been established in Lemmas 3.4 and 3.3 (3), respectively. For the other statement, suppose $t \in [0, \tau)$ is a J_n-eigenvalue of M such that $M\boldsymbol{x} = tJ_n\boldsymbol{x}$ and $\boldsymbol{x} \in \mathbb{K}^n$. By (3.11), we have

$$0 = \boldsymbol{v}^\top M_\tau \boldsymbol{x} = (t-\tau)\boldsymbol{v}^\top J_n \boldsymbol{x} \Rightarrow \boldsymbol{v}^\top J_n \boldsymbol{x} = 0$$

which implies that $\boldsymbol{v} \in \mathrm{bd}(\mathbb{K}^n)$ and $\boldsymbol{v} = \lambda \boldsymbol{x}$ for some $\lambda > 0$ (by Lemma 2.3 (1)). Therefore, $\boldsymbol{v}^\top M \boldsymbol{v} = 0$, and by Lemma 2.4 (3), $M^\top \boldsymbol{v} \in (-\mathbb{K}^n)$, a contradiction to (3.11).

(3). First $(a) \Rightarrow (b)$ is evident. For $(b) \Rightarrow (c)$, assume that there is $\hat{\tau} > 0$ such that $M\boldsymbol{y} = -\hat{\tau} J_n \boldsymbol{y}$ for some $\boldsymbol{y} \in \mathbb{K}^n$, then by Proposition 3.2, we have $\mathrm{rank}(M) > 1$. Moreover, from (3.11),

$$0 \le \tau \boldsymbol{v}^\top J_n \boldsymbol{y} = \boldsymbol{v}^\top M \boldsymbol{y} = -\hat{\tau}\boldsymbol{v}^\top J_n \boldsymbol{y} \le 0 \Rightarrow \boldsymbol{v}^\top J_n \boldsymbol{y} = 0 \Rightarrow \boldsymbol{v} \in \mathrm{bd}(\mathbb{K}^n).$$

For $(c) \Rightarrow (d)$, from $M^\top \boldsymbol{v} = \tau J_n \boldsymbol{v}$ and $\boldsymbol{v} \in \mathrm{bd}(\mathbb{K}^n)$, it follows that $\boldsymbol{v}^\top M^\top \boldsymbol{v} = 0$. By Lemma 2.4 (3), we have $M\boldsymbol{v} \in (-\mathbb{K}^n)$, which together with Lemma 3.2 (2) implies (d). Last, if $\boldsymbol{v}$ is a J_n-eigenvector of M associated with a negative J_n-eigenvalue, then $M\mathbb{K}^n \cap (-\mathbb{K}^n) \neq \{\mathbf{0}\}$, and thus, $(d) \Rightarrow (a)$ follows directly from Lemma 3.2 (2). The proof is completed.

(4). The first assertion can be established similarly as (3). Now, assume $\tilde{\tau} > 0$ such that

$$M^\top \boldsymbol{w} = -\tilde{\tau} J_n \boldsymbol{w}; \tag{3.12}$$

then the linear independence of $\boldsymbol{w}$ and $\boldsymbol{v}$ follows from (3.11) and (3.12). □

Lastly, we establish the other main result of this section, that is, the remaining part (i.e., the part for $0 < s < \tau$) of $\mathbb{K}_s$-nested-cone property. Propositions 3.3 and 3.1 (i.e., the part for $s > \tau$) together are called the $\mathbb{K}_s$-nested-cone property in this paper.

Proposition 3.3. *Assume $\tau > 0$.*

(1) *If $0 < s < \tau$, we have $\mathbb{K}_s \supset \mathbb{K}_0 = M\mathbb{K}^n$.*

(2) *If $0 < t < s < \tau$, then $\mathbb{K}_t \subset \mathbb{K}_s$ and M_s is nonsingular for all $s \in (0, \tau)$.*

Proof. (1). For the case $\mathrm{rank}(M) = 1$, we know that $M = \boldsymbol{a}\boldsymbol{b}^\top$ for some $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{K}^n$ and $\boldsymbol{b} \in \mathrm{int}(\mathbb{K}^n)$, Thus $M\mathbb{K}^n = \{t\boldsymbol{a} : t \ge 0\}$. For $0 < s < \tau = \boldsymbol{b}^\top J_n \boldsymbol{a}$, we have $M_s J_n \boldsymbol{a} = (\boldsymbol{b}^\top J_n \boldsymbol{a} - s)\boldsymbol{a}$, which implies $\mathbb{K}_s \supset M\mathbb{K}^n$.

Assume now $\mathrm{rank}(M) \ge 2$. Note from Lemma 2.1 (2) that $\mathrm{ri}(M\mathbb{K}^n) = M(\mathrm{int}(\mathbb{K}^n))$. If $\mathbf{0} \in \mathrm{ri}(M\mathbb{K}^n)$, then $M\boldsymbol{x} = \mathbf{0}$ for some $\boldsymbol{x} \in \mathrm{int}(\mathbb{K}^n)$, which implies that $\mathbf{0}$ is a J_n-eigenvalue of M, a contradiction to Theorem 3.2 (2), and so $\mathbf{0} \notin \mathrm{ri}(M\mathbb{K}^n)$.

Fix $s \in (0, \tau)$. Since τ is the unique positive J_n-eigenvalue of M, we have $\text{Ker}(M_s) \cap \mathbb{K}^n = \{\mathbf{0}\}$, which by [15, Theorem 9.1] implies that $\mathbb{K}_s$ is closed. Since $\text{ri}(M\mathbb{K}^n) = M(\text{int}(\mathbb{K}^n))$ and $\mathbf{0} \notin \text{ri}(M\mathbb{K}^n)$, by Lemma 3.1 (3), we have

$$M_s(\text{bd}(\mathbb{K}^n)) \cap \text{ri}(M\mathbb{K}^n) = \emptyset.$$

Using this equality and the closeness of $\mathbb{K}_s$, if $\text{int}(\mathbb{K}_s) \cap \text{ri}(M\mathbb{K}^n) \neq \emptyset$, by Lemma 2.2, then the assertion follows.

Thus we only need to prove that $\text{int}(\mathbb{K}_s) \cap \text{ri}(M\mathbb{K}^n) \neq \emptyset$. If not, by the convex separation theorem [15, Theorem 11.3], there exists $\boldsymbol{d} \neq 0$ such that

$$\boldsymbol{d}^\top M\boldsymbol{x} \geq \boldsymbol{d}^\top (M\boldsymbol{y} - sJ_n\boldsymbol{y}), \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{K}^n. \tag{3.13}$$

Letting $\boldsymbol{y} = \mathbf{0}$ in (3.13), we have $\boldsymbol{d}^\top M\boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \in \mathbb{K}^n$. Thus $M^\top \boldsymbol{d} \in \mathbb{K}^n$. Letting $\boldsymbol{x} = \boldsymbol{y}$ in (3.13) yields $\boldsymbol{d}^\top (J_n\boldsymbol{y}) \geq 0$ for all $\boldsymbol{y} \in \mathbb{K}^n$, which implies $\boldsymbol{d} \in \mathbb{K}^n$. By (3.13), we have $\boldsymbol{d}^\top M_s\boldsymbol{y} \leq 0$ for all $\boldsymbol{y} \in \mathbb{K}^n$, and thereby $M_s^\top \boldsymbol{d} \in (-\mathbb{K}^n)$. Recall $\text{rank}(M) \geq 2$. Then $M^\top$ is pseudomonotone also and $\tau(M^\top) = \tau(M) = \tau$ by Lemma 3.3 (4).

We consider separately the two cases: $M_s^\top \boldsymbol{d} = \mathbf{0}$ and $M_s^\top \boldsymbol{d} \neq \mathbf{0}$. If $M_s^\top \boldsymbol{d} = \mathbf{0}$, then $M^\top$ has another positive J_n-eigenvalue s, a contradiction. Assume $M_s^\top \boldsymbol{d} \neq \mathbf{0}$. If $\boldsymbol{d} \in \text{int}(\mathbb{K}^n)$, pick any $\hat{s} > \tau$, and thus $M_{\hat{s}}^\top \boldsymbol{d} = M_s^\top \boldsymbol{d} - (\hat{s} - s)J_n\boldsymbol{d} \in (-\text{int}(\mathbb{K}^n))$. Note that $M_\tau^\top \boldsymbol{d} = \rho M_s^\top \boldsymbol{d} + (1 - \rho)M_{\hat{s}}^\top \boldsymbol{d}$ for some $0 < \rho < 1$. From $M_s^\top \boldsymbol{d} \in (-\mathbb{K}^n)$ and $M_{\hat{s}}^\top \boldsymbol{d} \in (-\text{int}(\mathbb{K}^n))$, it follows that $M_\tau^\top \boldsymbol{d} \in (-\text{int}(\mathbb{K}^n))$, which contradicts Lemma 3.3 (1), and thus we must have $\boldsymbol{d} \in \text{bd}(\mathbb{K}^n)$. By $M_s^\top \boldsymbol{d} \in (-\mathbb{K}^n)$ and Lemma 3.1 (2), there exists $\kappa \in \mathbb{R}$ such that $M^\top \boldsymbol{d} = \kappa J_n\boldsymbol{d}$. Since $\tau > 0$ is a J_n eigenvalue, Theorem 3.2 (2) implies $\kappa < 0$ and $M^\top \boldsymbol{d} = \kappa J_n\boldsymbol{d} \in (-\mathbb{K}^n)$, which contradicts the fact $M^\top \boldsymbol{d} \in \mathbb{K}^n$. This proves (1).

(2). For any $\boldsymbol{x} \in \mathbb{K}^n$, from $M_t\boldsymbol{x} = (1 - \frac{t}{s})M\boldsymbol{x} + \frac{t}{s}M_s\boldsymbol{x}$ and $M\boldsymbol{x} \in \mathbb{K}_0 \subset \mathbb{K}_s$, we know $M_t\boldsymbol{x} \in \mathbb{K}_s$, and therefore, $\mathbb{K}_t \subset \mathbb{K}_s$. Since one can find a $t \in (0, s)$ such that M_t is nonsingular, it follows $\text{int}(\mathbb{K}_t) \neq \emptyset$, which together with $\mathbb{K}_t \subset \mathbb{K}_s$ implies $\text{int}(\mathbb{K}_s) \neq \emptyset$ and hence M_s is nonsingular. □

We remark that if $\tau = 0$ and $s > 0$, $M\mathbb{K}^n \subset \mathbb{K}_s$ does not hold any longer. However, we will not give an example to illustrate that situation here because it will become clear in the next section.

4 The range $\mathcal{R}(\mathbb{K}^n, M)$

The analysis of the range of a particular LCP is very useful and important in both theory and computations. For example, for the classical LCP$(\boldsymbol{q}, M)$ over $\mathbb{R}^n_+$, the range of LCP$(\boldsymbol{q}, M)$ is the union of finitely many polyhedra in $\mathbb{R}^n$ (see [5, Theorem 2.5.15]), and so is closed; furthermore, in [2, Proposition 3.2.1], it is proved that the matrix M belongs to the class $\mathbf{Q}_0$ (refer to [2, Section 3.2]) if and only if the range of LCP$(\boldsymbol{q}, M)$ is convex.

In this section, we will perform a thorough analysis for the range $\mathcal{R}(\mathbb{K}^n, M)$. In particular, we shall answer the following three questions: under what conditions, does M have the Q property (i.e., $\mathcal{R}(\mathbb{K}^n, M) = \mathbb{R}^n$), is the range $\mathcal{R}(\mathbb{K}^n, M)$ convex, and is the range $\mathcal{R}(\mathbb{K}^n, M)$ closed, respectively? First we prove a preliminary result.

Lemma 4.1. *For any $s \geq 0$, $(-\mathbb{K}_s) \subset \mathcal{R}(\mathbb{K}^n, M)$.*

Proof. For any $\boldsymbol{q} \in (-\mathbb{K}_s)$, if $\boldsymbol{q} \in (-M\mathbb{K}^n) \cup \mathbb{K}^n$, by Lemma 2.6, we have $\boldsymbol{q} \in \mathcal{R}(\mathbb{K}^n, M)$. Thus we only need to consider the case $\boldsymbol{q} \notin (-M\mathbb{K}^n) \cup \mathbb{K}^n$.

Case I: $0 < s < \tau$ (This case vanishes if $\tau = 0$).

Define $s(\boldsymbol{q}) = \inf\{0 < \delta \leq s : \boldsymbol{q} \in (-\mathbb{K}_\delta)\}$. Since τ is the positive J_n-eigenvalue of M, by Theorem 3.2 (2), 0 is not a J_n-eigenvalue. Then $\mathrm{Ker}(M) \cap \mathbb{K}^n = \{\mathbf{0}\}$. By [15, Theorem 9.1], $M\mathbb{K}^n$ is closed, which together with $\boldsymbol{q} \notin (-M\mathbb{K}^n)$ implies that $d(\mathcal{C}, -M\mathbb{K}^n) > 0$, where $\mathcal{C} = \{t\boldsymbol{q} : t \geq 0\}$. By (3.4) and (3.5), it is easy to prove that $s(\boldsymbol{q}) > 0$, and therefore, for all $t \in [s(\boldsymbol{q}), s]$, M_t is nonsingular by Proposition 3.3.

Let $\boldsymbol{x}_t = -M_t^{-1}\boldsymbol{q}$, where $t \in [s(\boldsymbol{q}), s]$. It is trivial to see $-M_{s(\boldsymbol{q})}^{-1}\boldsymbol{q} \in \mathbb{K}^n$ if $s(\boldsymbol{q}) = s$. Otherwise, by the definition of $s(\boldsymbol{q})$, we must have $\boldsymbol{x}_t \in \mathbb{K}^n$ for all $t \in (s(\boldsymbol{q}), s]$. From

$$\mathbb{K}^n \ni \lim_{t \downarrow s(\boldsymbol{q})} \boldsymbol{x}_t = -\lim_{t \downarrow s(\boldsymbol{q})} M_t^{-1}\boldsymbol{q} = -M_{s(\boldsymbol{q})}^{-1}\boldsymbol{q},$$

it follows that $-M_{s(\boldsymbol{q})}^{-1}\boldsymbol{q} \in \mathbb{K}^n$. Now we prove $-M_{s(\boldsymbol{q})}^{-1}\boldsymbol{q} \in \mathrm{bd}(\mathbb{K}^n)$. If not, one has $-\boldsymbol{q} \in \mathrm{int}(\mathbb{K}_{s(\boldsymbol{q})})$. By [20, Lemma 3], there exists $t < s(\boldsymbol{q})$ such that $-\boldsymbol{q} \in \mathrm{int}(\mathbb{K}_t)$, which contradicts the definition of $s(\boldsymbol{q})$. Let $\boldsymbol{x} = -M_{s(\boldsymbol{q})}^{-1}\boldsymbol{q}$. Then $\boldsymbol{x} \in \mathrm{bd}(\mathbb{K}^n)$ and $M_{s(\boldsymbol{q})}\boldsymbol{x} = -\boldsymbol{q}$, which together with (3.3) yields $\boldsymbol{x} \in \mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$. Thus $\boldsymbol{q} \in \mathcal{R}(\mathbb{K}^n, M)$.

Case II: $s > \tau$.

Define $s(\boldsymbol{q}) = \sup\{\delta \geq s : \boldsymbol{q} \in (-\mathbb{K}_\delta)\}$. Recall that $\mathcal{C} = \{t\boldsymbol{q} : t \geq 0\}$. Since $\boldsymbol{q} \notin (-\mathbb{K}^n)$, we have $d(\mathcal{C}, -\mathbb{K}^n) > 0$, which together with (3.5) implies that $s(\boldsymbol{q}) < \infty$. The rest of the proof is similar as the **Case I** and is omitted.

Case III: $s = \tau$.

If $s = \tau = 0$, then by Lemma 2.6, $-\mathbb{K}_0 = -M\mathbb{K}^n \subset \mathcal{R}(\mathbb{K}^n, M)$. Otherwise, we assume $s = \tau > 0$ and according to Theorem 3.2, there exists $\mathbf{0} \neq \boldsymbol{w} \in \mathbb{K}^n$ such that $\boldsymbol{w} \in \mathrm{Ker}(M_\tau)$. Let $\boldsymbol{x} \in \mathbb{K}^n$ such that $M_\tau \boldsymbol{x} = -\boldsymbol{q}$. Since $\mathbb{K}^n$ is a proper cone, there exists t such that $\boldsymbol{x} - t\boldsymbol{w} \in \mathrm{bd}(\mathbb{K}^n)$. Thus $M_\tau(\boldsymbol{x} - t\boldsymbol{w}) = -\boldsymbol{q}$, which together with (3.3) implies that $\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q}) \neq \emptyset$. Then $\boldsymbol{q} \in \mathcal{R}(\mathbb{K}^n, M)$ and the proof is completed. □

To characterize the range $\mathcal{R}(\mathbb{K}^n, M)$, we separate our following analysis into two parts: $\tau > 0$ and $\tau = 0$.

4.1 The case $\tau > 0$

We first treat the case $\tau > 0$. Recall that Theorem 3.2 says that $\boldsymbol{w} \in \mathbb{K}^n$ is a J_n-eigenvector, and we shall prove next that

(i) if $\boldsymbol{w} \in \mathrm{int}(\mathbb{K}^n)$, then $\mathcal{R}(\mathbb{K}^n, M) = \mathbb{R}^n$, or equivalently M has the Q property, and

(ii) if $\boldsymbol{w} \in \mathrm{bd}(\mathbb{K}^n)$, then $\mathcal{R}(\mathbb{K}^n, M)$ is the union of an open half space and a line.

The following theorem is for the term (i).

Theorem 4.1. *Assume that $\tau > 0$ and $\boldsymbol{w} \in \mathbb{K}^n$ and $\boldsymbol{v} \in \mathbb{K}^n$ are the associated J_n-eigenvectors of M and $M^\top$, respectively. If $\boldsymbol{w} \in \text{int}(\mathbb{K}^n)$ then*

(1) *if $\boldsymbol{v} \in \text{int}(\mathbb{K}^n)$, then for all $0 < s < \tau$, M_s has the GUS property;*

(2) *M has the Q property.*

Proof. (1). If $\boldsymbol{w} \in \text{int}(\mathbb{K}^n)$, similar to the proofs of Propositions 3.1 and 3.3, we can show by Lemmas 3.1, 3.2, Theorem 3.2 and [20, Lemma 5], that if $s > t > \tau$,

$$-\mathbb{K}^n\backslash\{\mathbf{0}\} \subset \text{int}(\mathbb{K}_s) \ \text{ and } \ \mathbb{K}_s\backslash\{\mathbf{0}\} \subset \text{int}(\mathbb{K}_t), \tag{4.1}$$

and if $0 < t < s < \tau$,

$$M\mathbb{K}^n\backslash\{\mathbf{0}\} \subset \text{int}(\mathbb{K}_t) \ \text{ and } \ \mathbb{K}_t\backslash\{\mathbf{0}\} \subset \text{int}(\mathbb{K}_s). \tag{4.2}$$

Fix $s \in (0, \tau)$. For all $\mathbf{0} \neq \boldsymbol{x} \in \text{bd}(\mathbb{K}^n)$, by (4.2), similar to the proof of [20, Lemma 20], we can show that $\boldsymbol{x}^\top M_s^{-1}\boldsymbol{x} > 0$; furthermore, by Lemma 2.4 (3) we also have $\boldsymbol{x}^\top M_s \boldsymbol{x} = \boldsymbol{x}^\top M\boldsymbol{x} \geq 0$. Thus all conditions of [20, Theorem 2] holds for M_s, and therefore, M_s has the GUS property. This proves the assertion (1).

(2). Since $\boldsymbol{w} \in \text{int}(\mathbb{K}^n)$, for any $\boldsymbol{x} \in \mathbb{R}^n$, there exists $t \in \mathbb{R}$ such that $\boldsymbol{x} + t\boldsymbol{w} \in \text{bd}(\mathbb{K}^n)$. Thus $\mathcal{R}(M_\tau) = \cup_{\boldsymbol{x}\in\text{bd}(\mathbb{K}^n)} M_\tau \boldsymbol{x}$. On the other hand, by Lemma 3.4 and (3.10), $\mathcal{R}(M_\tau) = \boldsymbol{v}^\perp$, and hence

$$\boldsymbol{v}^\perp = \cup_{\boldsymbol{x}\in\text{bd}(\mathbb{K}^n)} M_\tau \boldsymbol{x}. \tag{4.3}$$

Let $\widetilde{\mathcal{C}} = \cup_{0<s<\tau}\mathbb{K}_s$ and $\widehat{\mathcal{C}} = \cup_{s>\tau}\mathbb{K}_s$. By (4.1) and (4.2), both $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$ are convex sets. Similar to the proof of [20, Theorem 16], we can show that $\widetilde{\mathcal{C}} \supset \{\boldsymbol{y} \in \mathbb{R}^n : \boldsymbol{v}^\top\boldsymbol{y} > 0\}$ and $\widehat{\mathcal{C}} \supset \{\boldsymbol{y} \in \mathbb{R}^n : \boldsymbol{v}^\top\boldsymbol{y} < 0\}$.

Let $\boldsymbol{p} \in \mathbb{R}^n$ be arbitrary. (a) If $\boldsymbol{v}^\top\boldsymbol{p} < 0$, then $-\boldsymbol{p} \in \widetilde{\mathcal{C}}$ and so there exists $s \in (0, \tau)$ such that $\boldsymbol{p} \in (-\mathbb{K}_s)$. By Lemma 4.1, we have $\boldsymbol{p} \in \mathcal{R}(\mathbb{K}^n, M)$. (b) If $\boldsymbol{v}^\top\boldsymbol{p} > 0$, then $-\boldsymbol{p} \in \widehat{\mathcal{C}}$ and so there exists $s > \tau$ such that $\boldsymbol{p} \in (-\mathbb{K}_s)$. Using Lemma 4.1 again we obtain $\text{SOL}(M, \mathbb{K}^n, \boldsymbol{p}) \neq \emptyset$. (c) If $\boldsymbol{v}^\top\boldsymbol{p} = 0$, from (4.3), it follows that $-\boldsymbol{q} = M_\tau\boldsymbol{x}$ for some $\boldsymbol{x} \in \text{bd}(\mathbb{K}^n)$, which yields $\boldsymbol{x} \in \text{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$. The proof is completed. □

The following two lemmas are technical preparations for the case $\boldsymbol{w} \in \text{bd}(\mathbb{K}^n)$.

Lemma 4.2. *Assume that $A \in \mathbb{R}^{n\times n}$ and $\mathbf{0} \neq \boldsymbol{u} \in \text{Ker}(A) \cap \text{bd}(\mathbb{K}^n)$. Let $\mathcal{H} = \{\boldsymbol{d} : (J_n\boldsymbol{u})^\top\boldsymbol{d} > 0\}$. Then*

$$A\mathbb{K}^n = A\mathcal{H} \cup \{\mathbf{0}\}. \tag{4.4}$$

Proof. Let $\boldsymbol{d} \in \mathcal{H}$ be arbitrary. By Lemma 2.3 (6), $\mathbb{N}_{\mathbb{K}^n}(\boldsymbol{u}) = \{-tJ_n\boldsymbol{u} : t \geq 0\}$, which together with Lemma 2.1 (2) implies that $\boldsymbol{u} + t\boldsymbol{d} \in \mathbb{K}^n$ for some $t > 0$. Thus $A\boldsymbol{d} = A(\boldsymbol{u}/t + \boldsymbol{d}) \in A\mathbb{K}^n$. Since $\boldsymbol{d} \in \mathcal{H}$ is arbitrary, $A\mathcal{H} \subset A\mathbb{K}^n$. On the other hand, for any $\boldsymbol{x} \in \mathbb{K}^n$, if $\boldsymbol{x} \neq t\boldsymbol{u}$ for any $t \geq 0$, from Lemma 2.3 (1), it follows that $(J_n\boldsymbol{u})^\top\boldsymbol{x} > 0$ and so $\boldsymbol{x} \in \mathcal{H}$; if $\boldsymbol{x} = t\boldsymbol{u}$, then $A\boldsymbol{x} = \mathbf{0}$. Thus we have $A\mathbb{K}^n \subseteq A\mathcal{H} \cup \{\mathbf{0}\}$, and so (4.4) holds. □

Lemma 4.3. *Assume that $\tau > 0$ and $\boldsymbol{w} \in \mathbb{K}^n$ and $\boldsymbol{v} \in \mathbb{K}^n$ are the associated J_n-eigenvectors of M and $M^\top$, respectively. If $\boldsymbol{w} \in \mathrm{bd}(\mathbb{K}^n)$, then*

$$M_\tau \mathbb{K}^n = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{w}^\top \boldsymbol{x} < 0, \boldsymbol{v}^\top \boldsymbol{x} = 0\} \cup \{\boldsymbol{0}\}.$$

Proof. Denote $\mathcal{E} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{w}^\top \boldsymbol{x} < 0, \boldsymbol{v}^\top \boldsymbol{x} = 0\}$, and $\mathcal{H} = \{\boldsymbol{d} : (J_n \boldsymbol{w})^\top \boldsymbol{d} > 0\}$. Then (4.4) holds with $A = M_\tau$ and $\boldsymbol{u} = \boldsymbol{w}$.

We first show $M_\tau \mathbb{K}^n \subseteq \mathcal{E} \cup \{\boldsymbol{0}\}$.

If $\mathrm{rank}(M) > 1$, from (3.12), it follows that

$$\boldsymbol{d} \in \mathcal{H} \Leftrightarrow (J_n \boldsymbol{w})^\top \boldsymbol{d} > 0 \Leftrightarrow \left(\frac{M_\tau^\top \boldsymbol{w}}{-(\tau + \tilde{\tau})} \right)^\top \boldsymbol{d} > 0 \Leftrightarrow \boldsymbol{w}^\top (M_\tau \boldsymbol{d}) < 0. \tag{4.5}$$

Note that by $\boldsymbol{v} \in \mathrm{Ker}(M_\tau^\top)$, we have $M_\tau \mathbb{K}^n \subset \boldsymbol{v}^\perp$, which together with (4.4) and (4.5) implies that $M_\tau \mathbb{K}^n \subseteq \mathcal{E} \cup \{\boldsymbol{0}\}$.

If $\mathrm{rank}(M) = 1$, by Proposition 3.2, $M = \boldsymbol{a}\boldsymbol{b}^\top$ for some $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{K}^n$ and $\boldsymbol{b} \in \mathrm{int}(\mathbb{K}^n)$, and $\boldsymbol{w} = J_n \boldsymbol{a}$, $\boldsymbol{v} = J_n \boldsymbol{b}$. It is obvious that $M_\tau \mathbb{K}^n \subset \boldsymbol{v}^\perp$. From $\boldsymbol{w} \in \mathrm{bd}(\mathbb{K}^n)$, it follows that $\boldsymbol{a} \in \mathrm{bd}(\mathbb{K}^n)$ and so $\boldsymbol{a}^\top J_n \boldsymbol{a} = 0$. For any $\boldsymbol{0} \neq \boldsymbol{q} \in M_\tau \mathbb{K}^n$, we have $\boldsymbol{q} = M_\tau \boldsymbol{x}$ for some $\boldsymbol{x} \in \mathbb{K}^n$. Note that $\boldsymbol{x} \neq t J_n \boldsymbol{a}$ for any t. Thus $\boldsymbol{w}^\top \boldsymbol{q} = \boldsymbol{w}^\top M_\tau \boldsymbol{x} = -\tau \boldsymbol{a}^\top \boldsymbol{x} < 0$, and so $M_\tau(\mathbb{K}^n) \subseteq \mathcal{E} \cup \{\boldsymbol{0}\}$.

We next show $M_\tau \mathbb{K}^n \supset \mathcal{E}$.

Pick any $\boldsymbol{x} \in \mathcal{E}$. Then $\boldsymbol{v}^\top \boldsymbol{x} = 0$, which together with $\mathcal{R}(M_\tau) = \boldsymbol{v}^\perp$ implies that $\boldsymbol{x} = M_\tau \boldsymbol{z}$ for some $\boldsymbol{z}$. From $\boldsymbol{w}^\top \boldsymbol{x} < 0$, it follows that $\boldsymbol{w}^\top (M_\tau \boldsymbol{z}) < 0$. Our last task is to prove $\boldsymbol{z} \in \mathcal{H}$. Once this holds, letting $A = M_\tau$ and $\boldsymbol{u} = \boldsymbol{w}$ in (4.4) yields $\boldsymbol{x} \in M_\tau \mathbb{K}^n$. Thus $\mathcal{E} \subset M_\tau(\mathbb{K}^n)$ and so $M_\tau(\mathbb{K}^n) = \mathcal{E} \cup \{\boldsymbol{0}\}$.

To prove $\boldsymbol{z} \in \mathcal{H}$, we first consider the case $\mathrm{rank}(M) = 1$, i.e., $M = \boldsymbol{a}\boldsymbol{b}^\top$ for some $\boldsymbol{0} \neq \boldsymbol{a} \in \mathrm{bd}(\mathbb{K}^n)$, $\boldsymbol{b} \in \mathrm{int}(\mathbb{K}^n)$, and $\boldsymbol{w} = J_n \boldsymbol{a}$, $\tau = \boldsymbol{a}^\top J_n \boldsymbol{b}$, from which we have $0 > \boldsymbol{w}^\top (M_\tau \boldsymbol{z}) = -\tau \boldsymbol{a}^\top \boldsymbol{z}$, yielding $\boldsymbol{a}^\top \boldsymbol{z} > 0$. Thus $\boldsymbol{z} \in \mathcal{H}$. If $\mathrm{rank}(M) > 1$, $\boldsymbol{z} \in \mathcal{H}$ can be deduced from (4.5) and $\boldsymbol{w}^\top (M_\tau \boldsymbol{z}) < 0$ directly with $\boldsymbol{z}$ instead of $\boldsymbol{d}$. The proof is completed. □

Now we treat the case $\boldsymbol{w} \in \mathrm{bd}(\mathbb{K}^n)$.

Theorem 4.2. *Assume that $\tau > 0$ and $\boldsymbol{w} \in \mathbb{K}^n$ and $\boldsymbol{v} \in \mathbb{K}^n$ are the associated J_n-eigenvectors of M and $M^\top$, respectively. If $\boldsymbol{w} \in \mathrm{bd}(\mathbb{K}^n)$, then*

$$\mathcal{R}(\mathbb{K}^n, M) = \{\boldsymbol{q} : \boldsymbol{w}^\top \boldsymbol{q} > 0\} \cup \{t J_n \boldsymbol{w} : t \in \mathbb{R}\}. \tag{4.6}$$

Proof. We use $\mathcal{F}$ to denote the set on the righthand of (4.6). We shall first show that $\mathbb{K}^n - M\mathbb{K}^n \subseteq \mathcal{F}$, which together with Lemma 2.6 implies $\mathcal{R}(\mathbb{K}^n, M) \subseteq \mathcal{F}$. Pick any nonzero vector $\boldsymbol{p} \in \mathbb{K}^n - M\mathbb{K}^n$. If $\mathrm{rank}(M) = 1$, by Proposition 3.2, $M = \boldsymbol{a}\boldsymbol{b}^\top$ for some $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{K}^n$ and $\boldsymbol{b} \in \mathrm{int}(\mathbb{K}^n)$, and $\boldsymbol{w} = J_n \boldsymbol{a}$, then $M\mathbb{K}^n = \{\lambda \boldsymbol{a} : \lambda \geq 0\}$ and so $\boldsymbol{p} = \boldsymbol{x} - \mu \boldsymbol{a}$ with $\boldsymbol{x} \in \mathbb{K}^n$ and $\mu \geq 0$. It is easy to see that $\boldsymbol{w}^\top \boldsymbol{p} = \boldsymbol{w}^\top \boldsymbol{x} \geq 0$ and thereby $\boldsymbol{p} \in \mathcal{F}$. If $\mathrm{rank}(M) > 1$, then $\boldsymbol{p} = \boldsymbol{x} - M\boldsymbol{y}$ for some vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{K}^n$. By (3.12), $\boldsymbol{w}^\top \boldsymbol{p} = \boldsymbol{w}^\top \boldsymbol{x} + \tilde{\tau}(J_n \boldsymbol{w})^\top \boldsymbol{y} \geq 0$. If $\boldsymbol{w}^\top \boldsymbol{p} = 0$, then $\boldsymbol{x} = t_1 J_n \boldsymbol{w}$ and $\boldsymbol{y} = t_2 \boldsymbol{w}$, where $t_1, t_2 \geq 0$, which implies that $\boldsymbol{p} = tJ_n\boldsymbol{w}$ for some t. Thus $\boldsymbol{p} \in \mathcal{F}$. Since $\boldsymbol{p} \in \mathbb{K}^n - M\mathbb{K}^n$ is arbitrary, we have $\mathbb{K}^n - M\mathbb{K}^n \subseteq \mathcal{F}$.

Now we prove that $\mathcal{F} \subseteq \mathcal{R}(\mathbb{K}^n, M)$. Assume that $\boldsymbol{q} = tJ_n\boldsymbol{w}$. If $t \geq 0$, it is obvious that $\boldsymbol{q} \in \mathcal{R}(\mathbb{K}^n, M)$; if $t < 0$, it is easy to see that $M(-\frac{t}{\tau}\boldsymbol{w}) = -\boldsymbol{q}$ and therefore $-\frac{t}{\tau}\boldsymbol{w} \in \mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$, which implies $\boldsymbol{q} \in \mathcal{R}(\mathbb{K}^n, M)$.

Assume that $\boldsymbol{q} \in \{\boldsymbol{q} : \boldsymbol{w}^\top \boldsymbol{q} > 0\}$. For the case $\mathrm{rank}(M) = 1$, $\boldsymbol{w} = J_n\boldsymbol{a} \in \mathrm{bd}(\mathbb{K}^n)$ and $\boldsymbol{v} = J_n\boldsymbol{b} \in \mathrm{int}(\mathbb{K}^n)$ and $\boldsymbol{v}^\top J_n\boldsymbol{w} > 0$; while for $\mathrm{rank}(M) > 1$, by Theorem 3.2 (4), $\boldsymbol{w}$ and $\boldsymbol{v}$ are linearly independent. Thus $\boldsymbol{v}^\top J_n\boldsymbol{w} > 0$ and so $\mathbb{R}^n = \{tJ_n\boldsymbol{w} : t \in \mathbb{R}\} + \boldsymbol{v}^\perp$, which implies that

$$\boldsymbol{q} = \bar{t}J_n\boldsymbol{w} + \boldsymbol{z}, \tag{4.7}$$

where $\bar{t} \in \mathbb{R}$ and $\boldsymbol{z} \in \boldsymbol{v}^\perp$. By $\boldsymbol{w}^\top\boldsymbol{q} > 0$ and (4.7), we have

$$\boldsymbol{w}^\top\boldsymbol{z} > 0. \tag{4.8}$$

From Lemma 4.3, it follows that $-\boldsymbol{z} \in M_\tau\mathbb{K}^n \subset \boldsymbol{v}^\perp$. Assume that $-\boldsymbol{z} = M_\tau\boldsymbol{x}$ for some $\boldsymbol{x} \in \mathbb{K}^n$. Note that $\boldsymbol{x} + \delta\boldsymbol{w} \in \mathrm{bd}(\mathbb{K}^n)$ for some $\delta \in \mathbb{R}$, we can assume without loss of generality that $\boldsymbol{x} \in \mathrm{bd}(\mathbb{K}^n)$ and so $\boldsymbol{q} = \bar{t}J_n\boldsymbol{w} - M_\tau\boldsymbol{x}$.

Let $\widetilde{\mathcal{C}} = \cup_{0<s<\tau}\mathbb{K}_s$ and $\widehat{\mathcal{C}} = \cup_{s>\tau}\mathbb{K}_s$, both of which, according to Propositions 3.1 and 3.3, are convex. From

$$\mathbb{K}_s \ni M_s\boldsymbol{x} \to M_\tau\boldsymbol{x} \text{ as } s \text{ tends to } \tau,$$

we know that $M_\tau\boldsymbol{x} \in \mathrm{bd}(\widetilde{\mathcal{C}})$ and $M_\tau\boldsymbol{x} \in \mathrm{bd}(\widehat{\mathcal{C}})$. We next consider the three cases of $\bar{t}$.

Case I: $\bar{t} > 0$.

If $\bar{t} > 0$, then $-\bar{t}J_n\boldsymbol{w} \in (-\mathbb{K}^n)$. Since $-\mathbb{K}^n \subset \mathbb{K}_s$ for $s > \tau$, we have $-\bar{t}J_n\boldsymbol{w} \in \mathbb{K}_s \subset \widehat{\mathcal{C}}$. For $\lambda \in [0, 1]$, define

$$\boldsymbol{q}(\lambda) \triangleq (1-\lambda)(-2\bar{t}J_n\boldsymbol{w}) + \lambda(2M_\tau\boldsymbol{x}) = -2\bar{t}J_n\boldsymbol{w} + 2\lambda(M_\tau\boldsymbol{x} + \bar{t}J_n\boldsymbol{w}). \tag{4.9}$$

From $-\boldsymbol{z} = M_\tau\boldsymbol{x}$ and (4.8), it follows that $\boldsymbol{w}^\top(M_\tau\boldsymbol{x} + \bar{t}J_n\boldsymbol{w}) < 0$, which based on Lemma 2.1 (2) (by setting $\mathcal{C} = -\mathbb{K}^n$ and $\boldsymbol{b} = -2\bar{t}J_n\boldsymbol{w}$) leads to $\boldsymbol{q}(\lambda) \in \mathrm{int}(-\mathbb{K}^n)$ for sufficiently small λ. For such λ, since $-\mathbb{K}^n \subset \mathbb{K}_s$ for $s > \tau$, we have $\boldsymbol{q}(\lambda) \in \mathrm{int}(\mathbb{K}_s)$. Thus

$$\boldsymbol{q}(\lambda) \in \mathrm{int}(\widehat{\mathcal{C}}) \quad \text{if} \quad \lambda \quad \text{is sufficiently small.} \tag{4.10}$$

Note that

$$-\boldsymbol{q} = \boldsymbol{q}(\lambda)\eta + 2(1-\eta)M_\tau\boldsymbol{x}, \quad \text{with} \quad \eta = \frac{1}{2(1-\lambda)}$$

which together with (4.10) and $2M_\tau\boldsymbol{x} \in \mathrm{bd}(\widehat{\mathcal{C}})$ implies that $-\boldsymbol{q} \in \widehat{\mathcal{C}}$. By the definition of $\widehat{\mathcal{C}}$, there exists $\delta > \tau$ such that $\boldsymbol{q} \in -\mathbb{K}_\delta$, which together with Lemma 4.1 yields $\boldsymbol{q} \in \mathcal{R}(\mathbb{K}^n, M)$.

Case II: $\bar{t} < 0$.

If $\bar{t} < 0$, then $-\bar{t}J_n\boldsymbol{w} = -\bar{t}M\boldsymbol{w}/\tau \in M\mathbb{K}^n$. Fixing $s \in (0, \tau)$, by Proposition 3.3, we have $M\mathbb{K}^n \subset \mathbb{K}_s$, which together with $\mathbb{K}_s \subset \widetilde{\mathcal{C}}$ implies that $-\bar{t}J_n\boldsymbol{w} \in \widetilde{\mathcal{C}}$. For $\lambda \in [0, 1]$, define $\boldsymbol{q}(\lambda)$ as in (4.9). Now we shall show that

$$\boldsymbol{w}^\top M_s\boldsymbol{r} \leq 0, \quad \forall \boldsymbol{r} \in \mathbb{K}^n. \tag{4.11}$$

If $\text{rank}(M) = 1$, i.e., (by Proposition 3.2) $M = \boldsymbol{a}\boldsymbol{b}^\top$ and $\boldsymbol{w} = J_n\boldsymbol{a}$, for some $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{K}^n$ and $\boldsymbol{b} \in \text{int}(\mathbb{K}^n)$, (4.11) holds obviously with the aid of the facts that $\boldsymbol{a}^\top J_n\boldsymbol{a} = 0$ and $\boldsymbol{r}^\top\boldsymbol{r} \geq 0$ for any $\boldsymbol{r} \in \mathbb{K}^n$. If $\text{rank}(M) > 1$, by (3.12), $M^\top\boldsymbol{w} = -\tilde{\tau}J_n\boldsymbol{w}$, and with $\boldsymbol{w}^\top J_n\boldsymbol{r} \geq 0$ for any $\boldsymbol{r} \in \mathbb{K}^n$, it yields (4.11).

On the other hand, note that $J_n\boldsymbol{w} \in \text{bd}(\mathbb{K}^n)$, and by [20, Lemma 18], it follows that $\mathbb{N}_{\mathbb{K}_s}(-2\bar{t}J_n\boldsymbol{w})$ is a ray; since $-2\bar{t}J_n\boldsymbol{w} \in M\mathbb{K}^n \subset \mathbb{K}_s$ and by (4.11), we have $\mathbb{N}_{\mathbb{K}_s}(-2\bar{t}J_n\boldsymbol{w}) = \{t\boldsymbol{w} : t \geq 0\}$. Furthermore, from $-\boldsymbol{z} = M_\tau\boldsymbol{x}$ and (4.8), it holds that $\boldsymbol{w}^\top(M_\tau\boldsymbol{x} + \bar{t}J_n\boldsymbol{w}) < 0$; based on Lemma 2.1 (2) with $\mathcal{C} = \mathbb{K}_s$ and $\boldsymbol{b} = -2\bar{t}J_n\boldsymbol{w}$, we have $\boldsymbol{q}(\lambda) \in \text{int}(\mathbb{K}_s)$ for sufficiently small λ, and thereby $\boldsymbol{q}(\lambda) \in \text{int}(\widetilde{\mathcal{C}})$. Similar to **Case I**,

$$-\boldsymbol{q} = \boldsymbol{q}(\lambda)\eta + 2(1-\eta)M_\tau\boldsymbol{x}, \quad \text{with} \ \ \eta = \frac{1}{2(1-\lambda)}$$

which together with $2M_\tau\boldsymbol{x} \in \text{bd}(\widetilde{\mathcal{C}})$ implies that $-\boldsymbol{q} \in \widetilde{\mathcal{C}}$. Then $-\boldsymbol{q} \in \mathbb{K}_\delta$ for some $\delta \in (0, \tau)$, and by Lemma 4.1 again, $\boldsymbol{q} \in \mathcal{R}(\mathbb{K}^n, M)$.

Case III: $\bar{t} = 0$.

If $\bar{t} = 0$, then $-\boldsymbol{q} = M_\tau\boldsymbol{x}$ (recall $\boldsymbol{x} \in \text{bd}(\mathbb{K}^n)$), which together with (3.3) implies that $\boldsymbol{q} \in \mathcal{R}(\mathbb{K}^n, M)$. The proof is completed. □

As an illustration of the range $\mathcal{R}(\mathbb{K}^n, M)$ for Theorem 4.2, Figure 1 demonstrates $\mathcal{R}(\mathbb{K}^n, M)$ in $\mathbb{R}^3$. The set $\{\boldsymbol{q} : \boldsymbol{w}^\top\boldsymbol{q} > 0\}$ is gray-colored with its boundary $\{\boldsymbol{q} : \boldsymbol{w}^\top\boldsymbol{q} = 0\}$ excluded and the line $\{tJ_n\boldsymbol{w} : t \in \mathbb{R}\}$ which is on the boundary of $\{\boldsymbol{q} : \boldsymbol{w}^\top\boldsymbol{q} > 0\}$ is blue-colored. We point out that in this case, $\mathcal{R}(\mathbb{K}^n, M)$ is convex but not closed.

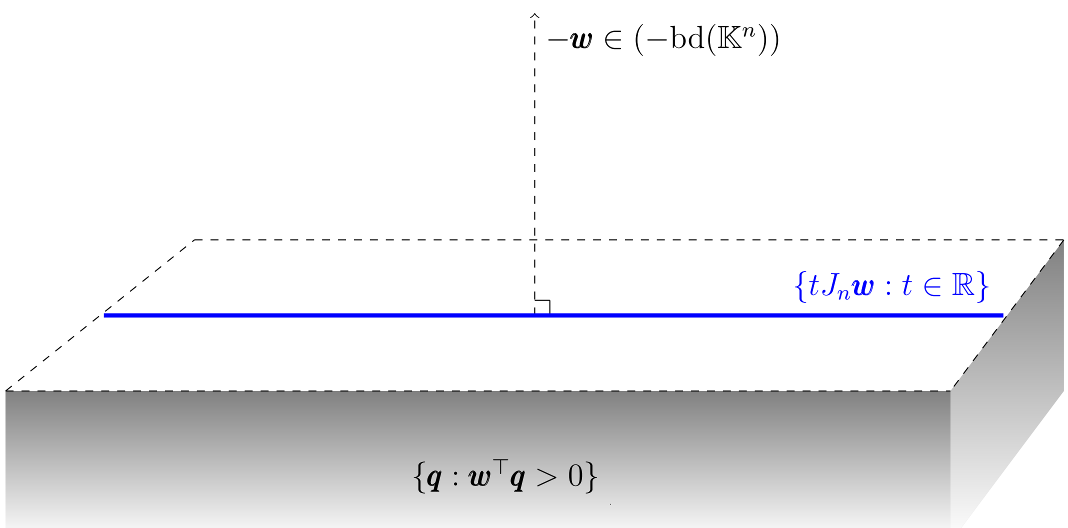

Figure 1. Demonstration of $\mathcal{R}(\mathbb{K}^n, M)$ in Theorem 4.2 for $n = 3$

4.2 The case $\tau = 0$

For the case $\tau = 0$, we will see that $\mathcal{R}(\mathbb{K}^n, M)$ is neither convex nor closed. First, we give two preliminary results.

Lemma 4.4. *If $\tau = 0$, then*

(1) *$M^\top$ is pseudomonotone, and*

(2) *$\mathrm{Ker}(M) \cap \mathbb{K}^n = \mathrm{Ker}(M^\top) \cap \mathbb{K}^n$.*

Proof. For (1) we only need to consider the case $\mathrm{rank}(M) = 1$ based on Lemma 2.4 (1). Note that when $\mathrm{rank}(M) = 1$, by Proposition 3.2, $M = \boldsymbol{u}\boldsymbol{u}^\top$ for some nonzero $\boldsymbol{u} \in \mathbb{R}^n$, and so $M^\top$ is pseudomonotone.

(2). Pick $\boldsymbol{x} \in \mathrm{Ker}(M) \cap \mathbb{K}^n$. From $\boldsymbol{x}^\top M \boldsymbol{x} = 0$ and by Lemma 2.4 (3), we have $M^\top \boldsymbol{x} \in (-\mathbb{K}^n)$. Since $M^\top$ is pseudomonotone and $\tau(M^\top) = 0$, according to Theorem 3.2 and Lemma 3.2 (2), we must have $M^\top \boldsymbol{x} = \boldsymbol{0}$ and so $\mathrm{Ker}(M) \cap \mathbb{K}^n \subseteq \mathrm{Ker}(M^\top) \cap \mathbb{K}^n$. We can substitute M by $M^\top$ to obtain $\mathrm{Ker}(M^\top) \cap \mathbb{K}^n \subseteq \mathrm{Ker}(M) \cap \mathbb{K}^n$, and the two sets coincide. □

Lemma 4.5. *If $\tau = 0$ and $\mathrm{Ker}(M) \cap \mathrm{int}(\mathbb{K}^n) \neq \emptyset$, then*

(1) *$M\mathbb{K}^n = M^\top \mathbb{K}^n = \mathcal{R}(M) = \mathcal{R}(M^\top)$, and*

(2) *M is monotone.*

Proof. Let $\boldsymbol{x} \in \mathrm{Ker}(M) \cap \mathrm{int}(\mathbb{K}^n)$ be arbitrary. We first prove (1). For any $\boldsymbol{z} \in \mathbb{R}^n$, if $|t|$ is sufficiently small, then $\boldsymbol{y} = \boldsymbol{x} + t\boldsymbol{z} \in \mathbb{K}^n$. Thus $M\boldsymbol{z} = M\boldsymbol{y}/t \in M\mathbb{K}^n$, which implies that $\mathcal{R}(M) \subseteq M\mathbb{K}^n$. Since $M\mathbb{K}^n \subseteq \mathcal{R}(M)$ holds trivially, we have $\mathcal{R}(M) = M\mathbb{K}^n$. By Lemma 4.4, $M^\top$ is pseudomonotone and $\mathrm{Ker}(M^\top) \cap \mathrm{int}(\mathbb{K}^n) \neq \emptyset$. Then the preceding proof tells us $\mathcal{R}(M^\top) = M^\top \mathbb{K}^n$.

Pick any $\boldsymbol{r} \in \mathrm{Ker}(M)$, and we know $\boldsymbol{x} + t\boldsymbol{r} \in \mathrm{Ker}(M) \cap \mathrm{int}(\mathbb{K}^n)$ for some $t > 0$. Thus $M^\top(\boldsymbol{x} + t\boldsymbol{r}) = \boldsymbol{0}$, which yields $M^\top \boldsymbol{r} = \boldsymbol{0}$ and so $\mathrm{Ker}(M) \subseteq \mathrm{Ker}(M^\top)$. Since $M^\top$ is pseudomonotone, similarly we have $\mathrm{Ker}(M^\top) \subseteq \mathrm{Ker}(M)$. Thus these two sets coincide, which gives $\mathcal{R}(M) = \mathcal{R}(M^\top)$, and the assertion (1) follows.

(2). Assume that $\mathrm{rank}(M) = r$. Then $M^\top$ has a decomposition $M^\top = UV^\top$, where $U, V \in \mathbb{R}^{n \times r}$ are of full column rank, which together with $\mathrm{Ker}(M) = \mathrm{Ker}(M^\top)$ implies that $V = U\Lambda$, for some nonsingular $\Lambda \in \mathbb{R}^{r \times r}$. Then $M = U\Lambda^\top U^\top$. Furthermore, for any $\boldsymbol{a} \in \mathbb{R}^r$, there exists $\boldsymbol{b} \in \mathbb{R}^n$ such that $\boldsymbol{a} = U^\top \boldsymbol{b}$. Let $t \in \mathbb{R}$ be such that $\boldsymbol{b} + t\boldsymbol{x} \in \mathbb{K}^n$. Thus by $U^\top \boldsymbol{x} = \boldsymbol{0}$ (from $M\boldsymbol{x} = \boldsymbol{0}$ and $\mathrm{rank}(V) = r$) and Lemma 2.4 (3), it follows

$$\boldsymbol{a}^\top \Lambda^\top \boldsymbol{a} = (\boldsymbol{b} + t\boldsymbol{x})^\top U\Lambda^\top U^\top (\boldsymbol{b} + t\boldsymbol{x}) = (\boldsymbol{b} + t\boldsymbol{x})^\top M (\boldsymbol{b} + t\boldsymbol{x}) \geq 0.$$

Since $\boldsymbol{a}$ is arbitrary, $\Lambda^\top$ is monotone, and so is M. □

With the preceding results on hand, we are able to show that if $\tau = 0$ and $\mathrm{Ker}(M) \cap \mathrm{int}(\mathbb{K}^n) \neq \emptyset$, then $\mathcal{R}(\mathbb{K}^n, M)$ is the sum of $\mathbb{K}^n$ and $-\mathcal{R}(M)$.

Theorem 4.3. *If $\tau = 0$ and $\mathrm{Ker}(M) \cap \mathrm{int}(\mathbb{K}^n) \neq \emptyset$, then*

$$\begin{aligned} \mathcal{R}(\mathbb{K}^n, M) &= \mathbb{K}^n \cup \mathcal{R}(M) \cup \big(\mathrm{int}(\mathbb{K}^n) - \mathcal{R}(M)\big) && (4.12) \\ &= \big(\mathrm{Ker}(M^\top J_n) \cap \mathrm{bd}(\mathbb{K}^n)\big) \cup \mathcal{R}(M) \cup \big(\mathrm{int}(\mathbb{K}^n) - \mathcal{R}(M)\big). && (4.13) \end{aligned}$$

Proof. First by Lemma 2.6 and Lemma 4.5 (1), $\mathcal{R}(\mathbb{K}^n, M) \supset \mathbb{K}^n \cup (-M\mathbb{K}^n) = \mathbb{K}^n \cup \mathcal{R}(M)$. Next we will prove that $\mathcal{R}(\mathbb{K}^n, M) \supset \text{int}(\mathbb{K}^n) - \mathcal{R}(M)$. Pick a nonzero $\boldsymbol{h} \in \text{Ker}(M)$. For any $\boldsymbol{x} \in \mathbb{K}^n$, there exists $t \in \mathbb{R}$ such that $\boldsymbol{u} = \boldsymbol{x} + t\boldsymbol{h} \in \text{bd}(\mathbb{K}^n)$, which implies that

$$\mathcal{R}(M) = M\mathbb{K}^n = \cup_{\boldsymbol{u} \in \text{bd}(\mathbb{K}^n)} M\boldsymbol{u}.$$

Then for any $\boldsymbol{p} \in \text{int}(\mathbb{K}^n) - \mathcal{R}(M)$, there exist $\boldsymbol{a} \in \text{int}(\mathbb{K}^n)$ and $\boldsymbol{u} \in \text{bd}(\mathbb{K}^n)$ such that $\boldsymbol{p} = \boldsymbol{a} - M\boldsymbol{u}$. Let $\widehat{\mathcal{C}} = \cup_{s>0} \mathbb{K}_s$. By (3.8) and Proposition 3.1 (2), if $s > t > 0$ then $-\mathbb{K}^n \subset \mathbb{K}_s \subset \mathbb{K}_t$, and therefore, $\widehat{\mathcal{C}}$ is convex and has nonempty interior. Note that $\mathbb{K}_s \ni M_s\boldsymbol{u} \to M\boldsymbol{u}$ as $s \to 0$, we have $M\boldsymbol{u} \in \text{bd}(\widehat{\mathcal{C}})$. By $\boldsymbol{a} \in \text{int}(\mathbb{K}^n)$ and $-\mathbb{K}^n \subset \widehat{\mathcal{C}}$, we know that $\boldsymbol{a} \in \text{int}(\widehat{\mathcal{C}})$, which together with $-\boldsymbol{p} = (-2\boldsymbol{a} + 2M\boldsymbol{u})/2$ and $2M\boldsymbol{u} \in \text{bd}(\widehat{\mathcal{C}})$ implies that $-\boldsymbol{p} \in \widehat{\mathcal{C}}$. In other words, there exists $s > 0$ such that $\boldsymbol{p} \in -\mathbb{K}_s$, which together with Lemma 4.1 yields $\boldsymbol{p} \in \mathcal{R}(\mathbb{K}^n, M)$. Thus $\mathcal{R}(\mathbb{K}^n, M) \supset \text{int}(\mathbb{K}^n) - \mathcal{R}(M)$.

Next we will prove the converse containment. It is not difficult to see by Theorem 3.1 (2) that for $\boldsymbol{q} \in \mathcal{R}(\mathbb{K}^n, M) \backslash (\mathbb{K}^n \cup (-M\mathbb{K}^n))$, there exists $\boldsymbol{x} \in \text{bd}(\mathbb{K}^n)$ and $s > 0$ such that $M_s\boldsymbol{x} = -\boldsymbol{q}$. Moreover, we must have

$$\boldsymbol{x} \notin \text{Ker}(M) \cap \mathbb{K}^n, \tag{4.14}$$

because otherwise, $M_s\boldsymbol{x} = -sJ_n\boldsymbol{x}$ and so $\boldsymbol{q} \in \mathbb{K}^n$. Next we shall show that

$$(J_n\boldsymbol{x} + \mathcal{R}(M)) \cap \text{int}(\mathbb{K}^n) \neq \emptyset.$$

If not, then by the convex separation theorem [15, Theorem 11.3], there exists $\boldsymbol{u} \neq \boldsymbol{0}$ so that

$$\boldsymbol{u}^\top (J_n\boldsymbol{x} + M\boldsymbol{h}) \leq \boldsymbol{u}^\top \boldsymbol{g}, \quad \forall \boldsymbol{h} \in \mathbb{R}^n, \ \forall \boldsymbol{g} \in \mathbb{K}^n, \tag{4.15}$$

from which it is not difficult to see $\boldsymbol{u}^\top \boldsymbol{g} \geq 0$ for all $\boldsymbol{g} \in \mathbb{K}^n$ and thereby $\boldsymbol{u} \in \mathbb{K}^n$. Let $\boldsymbol{h} = \boldsymbol{g} = \boldsymbol{0}$ in (4.15) to get $\boldsymbol{u}^\top J_n\boldsymbol{x} \leq 0$, which together with $\boldsymbol{0} \neq \boldsymbol{u} \in \mathbb{K}^n$ shows $\boldsymbol{u} = t\boldsymbol{x}$ for some $t > 0$. Because (4.15) holds for all $\boldsymbol{h} \in \mathbb{R}^n$, $M^\top \boldsymbol{u} = tM^\top \boldsymbol{x} = \boldsymbol{0}$, and thus $\boldsymbol{x} \in \text{Ker}(M^\top) \cap \mathbb{K}^n = \text{Ker}(M) \cap \mathbb{K}^n$, contradicting (4.14). Consequently, we can choose $\boldsymbol{y}$ such that $J_n\boldsymbol{x} + M\boldsymbol{y} = \boldsymbol{z}$ for some $\boldsymbol{z} \in \text{int}(\mathbb{K}^n)$ to get $-\boldsymbol{q} = M\boldsymbol{x} - sJ_n\boldsymbol{x} = M(\boldsymbol{x} + s\boldsymbol{y}) - s\boldsymbol{z}$, implying $\boldsymbol{q} \in \text{int}(\mathbb{K}^n) - \mathcal{R}(M)$, and thus $\mathcal{R}(\mathbb{K}^n, M)$ is a subset of the righthand of (4.12), and so (4.12) holds.

For (4.13), it suffices to show

$$\mathcal{C}_1 := \mathbb{K}^n \backslash \big(\text{int}(\mathbb{K}^n) - \mathcal{R}(M)\big) = \text{Ker}(M^\top J_n) \cap \text{bd}(\mathbb{K}^n) =: \mathcal{C}_2.$$

To this end, let $\boldsymbol{u} \in \mathcal{C}_1$ which implies $\boldsymbol{u} \in \text{bd}(\mathbb{K}^n)$, and we will further show $M^\top J_n\boldsymbol{u} = \boldsymbol{0}$. Note that $\boldsymbol{u} \notin \text{int}(\mathbb{K}^n) - \mathcal{R}(M)$, and by the convex separation theorem [15, Theorem 11.3] again, there is $\boldsymbol{h} \neq \boldsymbol{0}$ such that

$$\boldsymbol{h}^\top (\boldsymbol{a} - M\boldsymbol{b}) \geq \boldsymbol{h}^\top \boldsymbol{u}, \quad \forall \boldsymbol{a} \in \mathbb{K}^n, \ \forall \boldsymbol{b} \in \mathbb{R}^n, \tag{4.16}$$

which implies that $\boldsymbol{h}^\top \boldsymbol{a} \geq 0$ for all $\boldsymbol{a} \in \mathbb{K}^n$ and hence $\boldsymbol{h} \in \mathbb{K}^n$. Let $\boldsymbol{a} = \boldsymbol{0}$ in (4.16) to get $-(M^\top \boldsymbol{h})^\top \boldsymbol{b} \geq \boldsymbol{h}^\top \boldsymbol{u}$ for all $\boldsymbol{b} \in \mathbb{R}^n$, which leads to $M^\top \boldsymbol{h} = \boldsymbol{0}$ and $\boldsymbol{h}^\top \boldsymbol{u} \leq 0$. Since $\boldsymbol{h}, \boldsymbol{u} \in \mathbb{K}^n$,

$\boldsymbol{h}^\top \boldsymbol{u} \le 0$ implies $\boldsymbol{h} = t_0 J_n \boldsymbol{u}$ for some $t_0 > 0$ and thereby $M^\top J_n \boldsymbol{u} = \boldsymbol{0}$. Thus $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Last, for $\boldsymbol{u} \in \mathcal{C}_2$, if there exist $\boldsymbol{x} \in \text{int}(\mathbb{K}^n)$ and $\boldsymbol{y} \in \mathbb{R}^n$ such that $\boldsymbol{u} = \boldsymbol{x} - M\boldsymbol{y}$, we have

$$0 = (J_n \boldsymbol{u})^\top \boldsymbol{u} = (J_n \boldsymbol{u})^\top \boldsymbol{x} - \boldsymbol{u}^\top J_n M \boldsymbol{y} = (J_n \boldsymbol{u})^\top \boldsymbol{x} > 0,$$

a contradiction. Therefore, we have $\mathcal{C}_1 = \mathcal{C}_2$. □

We make a remark for Theorem 4.3. The condition $\text{Ker}(M) \cap \text{int}(\mathbb{K}^n) \neq \emptyset$ shows that we can choose a $\boldsymbol{w} \in \text{int}(\mathbb{K}^n)$ satisfying $M\boldsymbol{w} = \boldsymbol{0}$. This implies that such a $\boldsymbol{w} \in \text{int}(\mathbb{K}^n)$ is a J_n-eigenvector of M associated with the J_n-eigenvalue $\tau = 0$, and therefore, the assignment of $\boldsymbol{w}$ as a J_n-eigenvector of M holds in this case either.

Before the discussion for the last case $\tau = 0$ and $\text{Ker}(M) \cap \text{int}(\mathbb{K}^n) = \emptyset$, we illustrate in Figure 2 the range of $\mathcal{R}(\mathbb{K}^n, M)$ in Theorem 4.3 with $n = 3$ and $\text{rank}(M) = 1$. We remark that the gray part with its boundary excluded in Figure 2 is $\text{int}(\mathbb{K}^n) - \mathcal{R}(M)$ which can be obtained by moving $\text{int}(\mathbb{K}^n)$ (the red-colored) along the line $\mathcal{R}(M)$ (the brown-colored); moreover, $\text{Ker}(M^\top J_n) \cap \text{bd}(\mathbb{K}^n)$ reduces to two lines (one is blue-colored and the other is hidden on the opposite side), which is attached on the boundary of $\text{int}(\mathbb{K}^n) - \mathcal{R}(M)$ (the gray part in Figure 2). Note that in this case, $\mathcal{R}(\mathbb{K}^n, M)$ is neither convex nor closed.

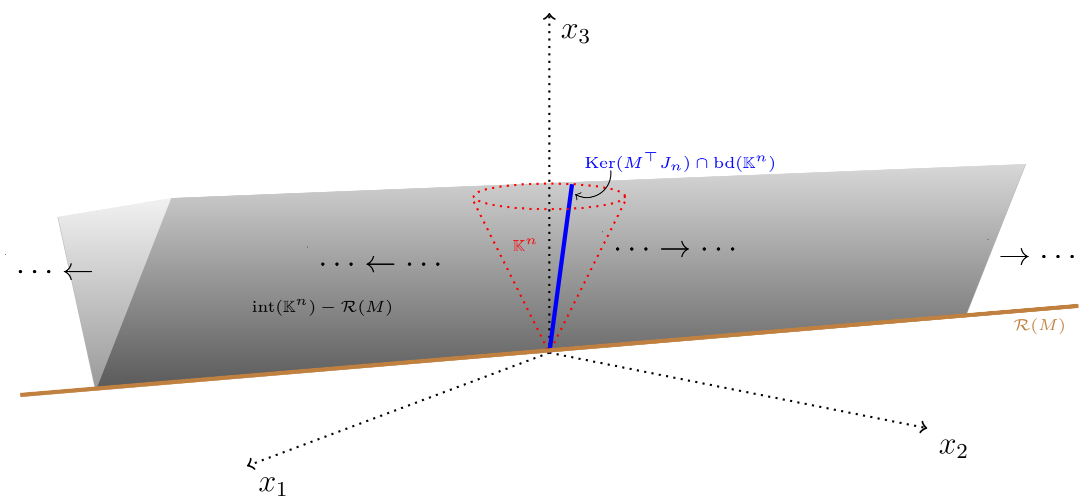

Figure 2. Demonstration of $\mathcal{R}(\mathbb{K}^n, M)$ in Theorem 4.3 for $n = 3$ and $\text{rank}(M) = 1$

We last address the case $\tau = 0$ and $\text{Ker}(M) \cap \text{int}(\mathbb{K}^n) = \emptyset$.

Lemma 4.6. *If $\tau = 0$ and $\text{Ker}(M) \cap \text{int}(\mathbb{K}^n) = \emptyset$, then*

(1) *there exists $\boldsymbol{w} \in \text{bd}(\mathbb{K}^n)$ such that*

$$\text{Ker}(M) \cap \mathbb{K}^n = \text{Ker}(M^\top) \cap \mathbb{K}^n = \{t\boldsymbol{w} : t \ge 0\}, \tag{4.17}$$

(2) $\text{Ker}(M) \subseteq (J_n \boldsymbol{w})^\perp$ *and* $\text{Ker}(M^\top) \subseteq (J_n \boldsymbol{w})^\perp$, *and*

(3) *for any $\epsilon > 0$, there exists $\boldsymbol{x} \in \text{bd}(\mathbb{K}^n)$ such that $\|M\boldsymbol{x} - J_n \boldsymbol{w}\| < \epsilon$.*

Proof. (1). By Lemma 3.3 (2), we have $\mathrm{Ker}(M)\cap\mathbb{K}^n \neq \{\mathbf{0}\}$, which together with $\mathrm{Ker}(M)\cap \mathrm{int}(\mathbb{K}^n) = \emptyset$ implies that $\mathrm{Ker}(M)\cap\mathbb{K}^n = \{t\boldsymbol{w} : t \geq 0\}$ for some $\boldsymbol{w} \in \mathrm{bd}(\mathbb{K}^n)$. From Lemma 4.4 (2), it follows that $\mathrm{Ker}(M) \cap \mathbb{K}^n = \mathrm{Ker}(M^\top) \cap \mathbb{K}^n$.

(2). If $\mathrm{Ker}(M) \subseteq (J_n\boldsymbol{w})^\perp$ does not hold, there exists $\hat{\boldsymbol{z}} \in \mathrm{Ker}(M)$ such that $(J_n\boldsymbol{w})^\top\hat{\boldsymbol{z}} \neq 0$. Without loss of generality, we assume $(J_n\boldsymbol{w})^\top\hat{\boldsymbol{z}} > 0$. From $-J_n\boldsymbol{w} \in \mathbb{N}_{\mathbb{K}^n}(\boldsymbol{w})$ (see Lemma 2.3 (6)) and (2.1), it follows that $\boldsymbol{w} + t\hat{\boldsymbol{z}} \in \mathrm{int}(\mathbb{K}^n)$ for some t. Thus $\boldsymbol{w} + t\hat{\boldsymbol{z}} \in \mathrm{Ker}(M) \cap \mathrm{int}(\mathbb{K}^n)$, a contradiction. The assertion $\mathrm{Ker}(M^\top) \subseteq (J_n\boldsymbol{w})^\perp$ can be proved similarly.

(3). From $\mathrm{Ker}(M^\top) \subseteq (J_n\boldsymbol{w})^\perp$, we know that $J_n\boldsymbol{w} \in \mathcal{R}(M)$ which implies that there exists $\boldsymbol{d} \neq \mathbf{0}$ such that $M\boldsymbol{d} = J_n\boldsymbol{w}$. We next treat the different situations of $\boldsymbol{d}$.

If $\boldsymbol{d} \in \mathcal{H} = \{\boldsymbol{d} : (J_n\boldsymbol{w})^\top\boldsymbol{d} > 0\}$, then by Lemma 4.2, there exists $\hat{\boldsymbol{x}} \in \mathbb{K}^n$ such that $M\hat{\boldsymbol{x}} = M\boldsymbol{d} = J_n\boldsymbol{w}$. Choosing a suitable t so that $\boldsymbol{x} = \hat{\boldsymbol{x}} + t\boldsymbol{w} \in \mathrm{bd}(\mathbb{K}^n)$, we have $M\boldsymbol{x} = M\boldsymbol{d} = J_n\boldsymbol{w}$, and the assertion (3) follows obviously.

If $\boldsymbol{d} \notin \mathcal{H}$, then we can also show $\boldsymbol{d} \notin (-\mathcal{H})$ by contradiction. In fact, when $\boldsymbol{d} \in (-\mathcal{H})$, then by Lemma 4.2 again, there exists $\boldsymbol{y} \in \mathbb{K}^n$ such that $M\boldsymbol{y} = -M\boldsymbol{d} = -J_n\boldsymbol{w}$, which by Lemma 3.2 (2) implies that $\boldsymbol{w}$ is a J_n-eigenvector of M associated with a negative J_n-eigenvalue, contradicting Theorem 3.2 (1). Consequently, $\boldsymbol{d} \notin (-\mathcal{H}) \cup \mathcal{H}$, from which we have $\boldsymbol{d} \in (J_n\boldsymbol{w})^\perp$. Thus for any $\epsilon > 0$, there exists $\boldsymbol{h} \in \mathcal{H}$ so that $\|\boldsymbol{h} - \boldsymbol{d}\|_2 < \frac{\epsilon}{\|M\|_2}$, and hence $\|M\boldsymbol{h} - J_n\boldsymbol{w}\|_2 = \|M\boldsymbol{h} - M\boldsymbol{d}\|_2 < \epsilon$. On the other hand, using Lemma 4.2, for $\boldsymbol{h} \in \mathcal{H}$, there exists $\tilde{\boldsymbol{x}} \in \mathbb{K}^n$ such that $M\tilde{\boldsymbol{x}} = M\boldsymbol{h}$, and by choosing a suitable t, we have $\boldsymbol{x} = \tilde{\boldsymbol{x}} + t\boldsymbol{w} \in \mathrm{bd}(\mathbb{K}^n)$ and $M\boldsymbol{x} = M\boldsymbol{h}$. Hence, $\|M\boldsymbol{h} - M\boldsymbol{d}\|_2 = \|M\boldsymbol{x} - J_n\boldsymbol{w}\|_2 < \epsilon$. □

From Lemma 4.6, we observe that the pair $(\tau = 0, \boldsymbol{w})$ is also a J_n-eigenpair of M and thus, the assignment of $\boldsymbol{w}$ as a J_n-eigenvector of M associated with τ continues for the case $\tau = 0$.

Theorem 4.4. *If $\tau = 0$ and $\mathrm{Ker}(M) \cap \mathrm{int}(\mathbb{K}^n) = \emptyset$, then*

$$\mathcal{R}(\mathbb{K}^n, M) = (-M\mathbb{K}^n) \cup \{\boldsymbol{q} : \boldsymbol{w}^\top\boldsymbol{q} > 0\} \cup \{tJ_n\boldsymbol{w} : t \geq 0\}, \tag{4.18}$$

where $\boldsymbol{w} \in \mathrm{bd}(\mathbb{K}^n)$ is defined in (4.17).

Proof. Obviously, $\mathcal{R}(\mathbb{K}, M) \supseteq (-M\mathbb{K}^n) \cup \{tJ_n\boldsymbol{w} : t \geq 0\}$. Let $\mathcal{F} \triangleq \{\boldsymbol{q} : \boldsymbol{w}^\top\boldsymbol{q} > 0\}$. For any $\boldsymbol{q} \in \mathcal{F}$, by Lemma 2.3 (5), we have $\boldsymbol{q} = -t_0J_n\boldsymbol{w} + \boldsymbol{a}$, where $t_0 > 0$ and $\boldsymbol{a} \in \mathrm{int}(\mathbb{K}^n)$. Let $\epsilon > 0$ be such that if $\|\hat{\boldsymbol{a}} - \boldsymbol{a}\|_2 < \epsilon$ then $\hat{\boldsymbol{a}} \in \mathrm{int}(\mathbb{K}^n)$. By Lemma 4.6 (3), there exists $\boldsymbol{x} \in \mathrm{bd}(\mathbb{K}^n)$ such that $\|M\boldsymbol{x} - t_0J_n\boldsymbol{w}\|_2 < \epsilon$, implying $\|\boldsymbol{q} + M\boldsymbol{x} - \boldsymbol{a}\|_2 < \epsilon$ and so $\boldsymbol{q} + M\boldsymbol{x} \in \mathrm{int}(\mathbb{K}^n)$. Denote $\boldsymbol{z} = \boldsymbol{q} + M\boldsymbol{x}$ and $\widehat{\mathcal{C}} = \cup_{s>0}\mathbb{K}_s$. By (3.8), $\tau = 0$ and Proposition 3.1, we know that $\widehat{\mathcal{C}}$ is convex. Since $-\mathbb{K}^n \subset \mathbb{K}_s$ for $s > 0$, $-\boldsymbol{z} \in \mathrm{int}(\mathbb{K}_s) \subset \mathrm{int}(\widehat{\mathcal{C}})$. From $\mathbb{K}_t \ni M_t\boldsymbol{x} \to M\boldsymbol{x}$ as $t \to 0$, it follows that $M\boldsymbol{x} \in \mathrm{bd}(\widehat{\mathcal{C}})$, which together with $-\boldsymbol{z} \in \mathrm{int}(\widehat{\mathcal{C}})$ implies that $-\boldsymbol{q} = (-2\boldsymbol{z} + 2M\boldsymbol{x})/2 \in \widehat{\mathcal{C}}$. By the definition of $\widehat{\mathcal{C}}$, $-\boldsymbol{q} \in \mathbb{K}_{s_0}$ for some $s_0 > 0$, which based on Lemma 4.1 leads to $\boldsymbol{q} \in \mathcal{R}(\mathbb{K}^n, M)$. Thus $\mathcal{R}(\mathbb{K}^n, M)$ contains the right hand of (4.18).

Lastly, for any $\boldsymbol{q} \in \mathcal{R}(\mathbb{K}^n, M)$, if $\boldsymbol{q} \notin (-M\mathbb{K}^n) \cup \mathbb{K}^n$, by Theorem 3.1 (2), $-\boldsymbol{q} = M\boldsymbol{x} - sJ_n\boldsymbol{x}$ for some $\boldsymbol{x} \in \mathrm{bd}(\mathbb{K}^n)$ and $s > 0$. Since $-\boldsymbol{q} \notin (-\mathbb{K}^n)$, we have $\boldsymbol{x} \neq t\boldsymbol{w}$ for any $t \geq 0$ where $\boldsymbol{w}$ is given by (4.17). It is obvious that $\boldsymbol{w}^\top\boldsymbol{q} = s\boldsymbol{w}^\top J_n\boldsymbol{x} > 0$, and so $\boldsymbol{q} \in \mathcal{F}$. Thus $\mathcal{R}(\mathbb{K}^n, M)$ is a subset of the right hand of (4.18), which completes the proof. □

We present an illustration for the range $\mathcal{R}(\mathbb{K}^n, M)$ in Theorem 4.4 with $n = 3$ and $\text{rank}(M) = 2$. In Figure 3, the purple-colored $(-M\mathbb{K}^n)$ is a half space on $\boldsymbol{w}^{\perp}$, the gray part is $\{\boldsymbol{q} : \boldsymbol{w}^{\top}\boldsymbol{q} > 0\}$ and the blue-colored is the ray $\{tJ_n\boldsymbol{w} : t \geq 0\}$. Note that in this case, $\mathcal{R}(\mathbb{K}^n, M)$ is neither convex nor closed.

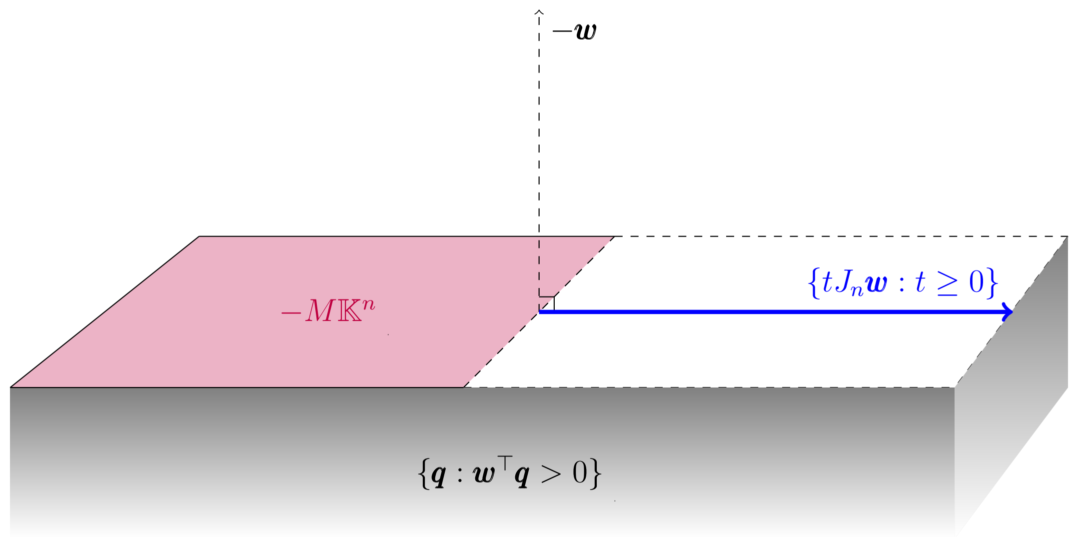

Figure 3. Demonstration of $\mathcal{R}(\mathbb{K}^n, M)$ in Theorem 4.4 for $n = 3$ and $\text{rank}(M) = 2$

To conclude this section, we finally summarize the four mutually exclusive cases of the range $\mathcal{R}(\mathbb{K}^n, M)$ in terms of the J_n-eigenpair $(\tau, \boldsymbol{w})$ of M in Table 1.

Table 1: $\mathcal{R}(\mathbb{K}^n, M)$ in terms of J_n-eigenpair $(\tau, \boldsymbol{w})$

J_n-eigenvalue	$\mathcal{R}(\mathbb{K}^n, M)$	
$\tau > 0$	$\boldsymbol{w} \in \text{int}(\mathbb{K}^n)$	$\mathbb{R}^n$
	$\boldsymbol{w} \in \text{bd}(\mathbb{K}^n)$	$\{\boldsymbol{q} : \boldsymbol{w}^{\top}\boldsymbol{q} > 0\} \cup \{tJ_n\boldsymbol{w} : t \in \mathbb{R}\}$
$\tau = 0$	$\boldsymbol{w} \in \text{int}(\mathbb{K}^n)$	$\mathbb{K}^n \cup \mathcal{R}(M) \cup \big(\text{int}(\mathbb{K}^n) - \mathcal{R}(M)\big)$
	$\boldsymbol{w} \in \text{bd}(\mathbb{K}^n)$	$(-M\mathbb{K}^n) \cup \{\boldsymbol{q} : \boldsymbol{w}^{\top}\boldsymbol{q} > 0\} \cup \{tJ_n\boldsymbol{w} : t \geq 0\}$

5 Concluding remarks and future topics

In this paper, we have performed a thorough analysis related to the solutions for the pseudomonotone SOCLCP. Along the line, two important notions, namely, the J_n-eigenpair and the $\mathbb{K}_s$-nested-cone-property, have been proposed, which are vital in delinating the geometry pictures of the set $\text{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$ of solutions as well as of the range $\mathcal{R}(\mathbb{K}^n, M)$. Moreover, both notions are helpful in designing efficient algorithms and the bisection-Newton iteration [21] for the GUS property is potentially one of possibilities to solve the pseudomonotone SOCLCP. We leave the study of the numerical methods as our future topic.

Our analysis for the set $\mathrm{SOL}(M, \mathbb{K}^n, \boldsymbol{q})$ generalizes the results [13, Theorem 3.2] established for the monotone SOCLCP. For the range $\mathcal{R}(\mathbb{K}^n, M)$ on the other hand, we have explicitly characterized and formulated $\mathcal{R}(\mathbb{K}^n, M)$ into four mutually exclusive cases in terms of the J_n-eigenpair $(\tau, \boldsymbol{w})$ of M. These four cases are finally summarized in Table 1. The complete characterization is helpful in analyzing the continuity and perturbation of solution map of the pseudomonotone SOCLCP, which is another future research topic.

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