Min-max-min robustness: a new approach to combinatorial optimization under uncertainty based on multiple solutions [1]

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Abstract

In the classical min-max approach to robust combinatorial optimization, a single feasible solution is computed that optimizes the worst case over a given set of considered scenarios. As is well known, this approach is very conservative, leading to solutions that in the average case are far from being optimal. In this paper, we present a different approach: the objective is to compute k feasible solutions such that the best of these solutions for each given scenario is worst-case optimal, i.e., we model the problem as a min-max-min problem. In practice, these k solutions can be computed once in a preprocessing phase, while choosing the best of the k solutions can be done in real time for each scenario occuring.

Using a polynomial-time oracle algorithm, we show that the problem of choosing k min-max-min optimal solutions is as easy as the underlying combinatorial problem if $k \geq n+1$ and if the uncertainty set is a polytope or an ellipsoid. On contrary, in the discrete scenario case, many tractable problems such as the shortest path problem or the minimum spanning tree problem turn NP-hard in the new approach.

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1 Introduction

Data uncertainty is inherent in many optimization problems. Measurement or rounding errors can lead to uncertainty in the parameters of an optimization model, but also external influences such as traffic or changing regulations may lead to unknown costs. In recent decades, both robust and stochastic optimization approaches have been developed to address such uncertainties.

In this paper, we consider combinatorial optimization problems of the form

$$\min_{x \in X} c^\top x, \tag{1}$$

where $X \subseteq \{0,1\}^n$ contains the incidence vectors of all feasible solutions of the given problem, e.g., all s-t-paths in a given network in case of the shortest path problem. We assume that uncertainty only occurs in the cost vector c. All probable scenarios, i.e., all cost vectors to be considered, are contained in an uncertainty set $U \subseteq \mathbb{R}^n$. In the literature, different classes of uncertainty sets are discussed, mainly discrete, ellipsoidal, polytopal or interval uncertainty.

The idea of robust optimization is to calculate a solution which is feasible in every possible scenario and optimal in the worst case [4]. This leads to the so called *strictly* or *min-max robust* approach, addressing the problem

$$\min_{x \in X} \max_{c \in U} c^\top x. \tag{2}$$

The resulting class of problems has been studied intensively in the literature. Even in the case of a tractable certain problem (1), Problem (2) often turns out to be NP-hard. This was shown for many classical combinatorial optimization problems, both in case of discrete and hence polytopal uncertainty [9] and in the case of general ellipsoidal uncertainty [11]. Apart from creating hard problems in general, the main drawback of the min-max robust approach is the so-called *price of robustness* [6]: the robust optimal value can differ substantially from the actual optimal value for a specific scenario. Recently, many alternative approaches to robust optimization have been proposed in order to decrease the price of robustness [7,10,3].

In this paper, we propose a novel approach to deal with this drawback. The objective is to calculate k different solutions which minimize the objective value in the worst-case scenario if we consider the best solution of all k solutions in every scenario. Formally, this leads to the problem

$$\min_{x^{(1)},\dots,x^{(k)} \in X} \max_{c \in U} \min_{i=1,\dots,k} c^\top x^{(i)} \tag{3}$$

where $U \subseteq \mathbb{R}^n$ and $X \subseteq \{0,1\}^n$ as before and $k \in \mathbb{N}$. The idea behind this approach is to calculate a set of solutions once in a (potentially expensive) preprocessing phase and from then on choose the best of the calculated solutions each time a scenario occurs. The latter can be done in $O(kn)$ time, independently of the set X.

As an application, imagine a parcel service delivering to the same customers every day. Each morning, the company needs to determine a tour depending on the current traffic situation. However, computing an optimal tour from scratch may take too long in a real-time setting. Instead, in our approach a set of candidate tours is computed once and the company can choose the best one out of these solutions every morning. Apart from yielding better solutions in general compared to the min-max approach, the new approach has the advantage that the solutions are more easily accepted by a human user if they do not change each time but are taken from a relatively small set of candidate solutions.

The main result of this paper is that Problem (3) is not harder than the certain problem (1) provided that we choose $k \geq n+1$ and that the uncertainty set is polytopal or ellipsoidal. This is in contrast to the fact that Problem (2) is generally NP-hard in these cases even if (1) is tractable. On contrary, we show that in the discrete scenario case Problem (3) turns out to be NP-hard for the shortest path and the minimum spanning tree problem, for any k.

In Section 2, we first state some basic results for the case of convex uncertainty sets and describe the general idea of the resulting algorithm for solving Problem (3). In Section 3, we present our main results for the polytopal, ellipsoidal and discrete scenario case.

2 General Results

In this section we reformulate Problem (3) by dualizing the inner maximization problem, resulting in a convex optimization problem if we choose $k \geq n+1$. This is the basis of the complexity results presented in Sections 3.1 and 3.2.

Proposition 2.1 *Let $U \subseteq \mathbb{R}^n$ be a non-empty convex set. Then*

$$\min_{x^{(1)},\ldots,x^{(k)} \in X} \max_{c \in U} \min_{i=1,\ldots,k} c^\top x^{(i)} = \min_{x \in X(k)} \max_{c \in U} c^\top x$$

where

$$X(k) := \left\{ \sum_{i=1}^k \lambda_i x^{(i)} \mid \lambda_i \geq 0,\ \sum_{i=1}^k \lambda_i = 1,\ x^{(i)} \in X \ \ \textit{for } i = 1, \ldots k \right\}.$$

Proof. Using Lagrangian duality and verifying that Slater's condition holds by our assumptions, the inner maximization problem in (3) can be replaced equivalently by

$$\min_{\lambda\geq 0} \max_{(c,z)\in U\times\mathbb{R}} z\big(1-\textstyle\sum_{i=1}^k \lambda_i\big) + c^\top \sum_{i=1}^k \lambda_i x^{(i)}.$$

For any λ with $\sum_{i=1}^k \lambda_i \neq 1$, the inner maximization problem is unbounded, hence the problem above reduces to

$$\min_{\substack{\lambda\geq 0\\ \sum_{i=1}^k \lambda_i=1}} \max_{c\in U} c^\top \textstyle\sum_{i=1}^k \lambda_i x^{(i)} ,$$

showing the result. □

Corollary 2.2 *For $k \geq n+1$ and for each non-empty convex set U we have*

$$\min_{x^{(1)},\dots,x^{(k)}\in X} \max_{c\in U} \min_{i=1,\dots,k} c^\top x^{(i)} = \min_{x\in \mathrm{conv}(X)} \max_{c\in U} c^\top x.$$

Proof. If $k \geq n+1$, then Carathéodory's theorem provides $X(k) = \mathrm{conv}\,(X)$. The assertion thus follows from Proposition 2.1. □

Corollary 2.2 also implies that considering more than $n+1$ solutions will not lead to any further improvement in the objective value of Problem (3). On the other hand, $k = n+1$ is a reasonable choice for our purpose as an optimal solution for a given scenario can be computed in $O\,(n^2)$ time then.

Based on Corollary 2.2, we propose the following two-stage algorithmic scheme for solving (3): in the first stage, we calculate an optimal solution x^* of the continuous problem

$$\min_{x\in \mathrm{conv}(X)} \max_{c\in U} c^\top x .$$

This step depends on the uncertainty set U and the underlying problem X. In the second stage, we calculate a corresponding set of solutions $x^{(1)},\dots,x^{(n+1)}$. The next result, following directly from Theorem 6.5.11 in [8], shows that the latter task can be done in polynomial time if the underlying certain problem can be solved in polynomial time.

Proposition 2.3 *Assume we are given an optimization oracle for the certain problem $c \mapsto \min_{x\in X} c^\top x$. If $x^* \in \mathrm{conv}\,(X)$ is rational, then, in polynomial time, we can compute affinely independent vectors $x^{(1)},\dots x^{(m)} \in X$ and rational coefficients $\lambda_1,\dots,\lambda_m \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$ such that $x^* = \sum_{i=1}^m \lambda_i x^{(i)}$.*

3 Complexity for different uncertainty classes

The min-max problem (2) is known to be NP-hard for most classical optimization problems such as the shortest path problem or the minimum spanning tree problem when U is a general polytope, an ellipsoid or a finite set. In fact, this even remains true in the unconstrained binary case $X = \{0, 1\}^n$. Only few cases are known where (2) remains tractable, e.g., if U is an axis-parallel ellipsoid and $X = \{0, 1\}^n$ [2], if X corresponds to a matroid [12] or for Γ-uncertainty [5].

In contrast to this, we can show that Problem (3) is solvable in polynomial time for polytopal and ellipsoidal uncertainty sets whenever the underlying certain problem is solvable in polynomial time and $k \geq n + 1$. The common case of interval uncertainty is a special case of polyhedral uncertainty, but can also be reduced easily to the certain case as in the min-max approach. Finally, we show that the discrete case is NP-hard in general also for Problem (3).

3.1 Polytopal uncertainty

We first consider a polytopal uncertainty set U. In this case we can show that Problem (3) is tractable provided that the underlying certain problem is tractable and $k \geq n + 1$. More precisely, we have

Theorem 3.1 *Given an optimization oracle for the problem $c \mapsto \min_{x \in X} c^\top x$, for any polytope $U = \{c \in \mathbb{R}^n \mid Ac \leq b\}$ with A and b rational and $k \geq n + 1$ we can solve (3) in polynomial time in the input (n, A, b).*

After using Corollary 2.2 and dualizing the inner maximization problem, the main ingredient in the proof is Theorem 6.4.9 in [8], stating that optimization and separation are equivalent over a rational polytope. Together with Proposition 2.3 we obtain the result.

3.2 Ellipsoidal uncertainty

Next we consider the case of an ellipsoidal uncertainty set

$$U = \{c \in \mathbb{R}^n \mid (c - c_0)^\top \Sigma^{-1} (c - c_0) \leq 1\}$$

with $c_0 \in \mathbb{R}^n$ denoting the center of the ellipsoid and $\Sigma \in \mathbb{R}^{n \times n}$ being a positive definite symmetric matrix. We then have

$$\max_{c \in U} c^\top x = c_0^\top x + \sqrt{x^\top \Sigma x}$$

so that Problem (3) in case $k \geq n+1$ becomes

$$\min_{x \in \mathrm{conv}(X)} \; c_0^\top x + \sqrt{x^\top \Sigma x} \tag{4}$$

by Corollary 2.2. If we have a complete description of polynomial size for conv (X), i.e., if there exist a rational matrix A and a rational vector b with a polynomial number of entries such that conv $(X) = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, then Problem (4) can be transformed into a second-order-cone problem and hence solved in polynomial time up to arbitrary accuracy, e.g., by interior point methods [1]. In general, we can show the following result; the proof is omitted due to space restrictions, it is again mainly based on [8].

Theorem 3.2 *Let* conv (X) *be full-dimensional,* $\varepsilon \in (0,1)$ *and* U *as above. Given an optimization oracle for the problem* $c \mapsto \min_{x \in X} c^\top x$*, we can solve Problem* (3) *up to an accuracy of at most* ε*, in time polynomial in* $(n, \Sigma, c_0, \varepsilon)$ *if* $k \geq n+1$*.*

In Figure 1 we illustrate the solution of a min-max-min robust shortest path problem with ellipsoidal uncertainty for $k = n+1$. The darkness of an edge indicates how often it is used in the solutions $x^{(1)}, \ldots, x^{(k)}$; red edges represent the corresponding min-max solution. Figure 2 shows the solution of a min-max-min robust spanning tree problem.

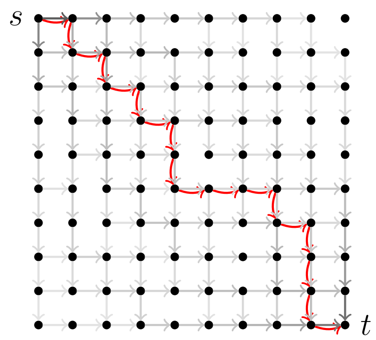

Fig. 1. Min-max-min robust shortest path subject to ellipsoidal uncertainty. The optimum of (3) is 191.153 for all $k \geq 65$. The optimum of (2) is 205.926; the one of the certain problem corresponding to the ellipsoid center is 180.

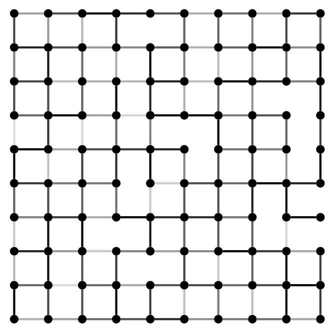

Fig. 2. Min-max-min robust spanning tree subject to ellipsoidal uncertainty. The optimum of (3) is 1025.7 for all $k \geq 120$. Problem (2) could not be solved in reasonable time; the optimum for the certain ellipsoid center is 990.

3.3 Discrete scenario uncertainty

We now consider the so-called discrete scenario case where the uncertainty set $U = \{c_1, \dots, c_m\}$ is finite. Note that in this case U is not convex so that Proposition 2.1 is not applicable. Moreover, replacing U by its convex hull does not yield an equivalent problem. In fact, we now show that Problem (3) becomes NP-hard in the discrete scenario case for the shortest path problem and the spanning tree problem. To this end, we will polynomially reduce the common min-max problem (2) to the min-max-min problem (3) for the latter problems. For the proofs of the following theorems we define G_k^{st} and G_k^{sp} as the graphs arising from G by adding nodes and edges as illustrated in Fig. 3 and 4, respectively.

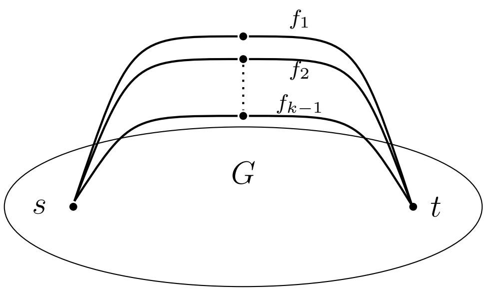

Fig. 3. The graph G_k^{sp}.

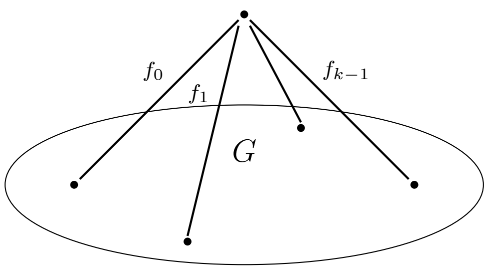

Fig. 4. The graph G_k^{st}.

Theorem 3.3 *For the shortest path problem on a graph G and for a discrete uncertainty set $U = \{c_1, \dots, c_m\}$ we can polynomially reduce Problem (2) to the min-max-min problem (3) with the solution set $\bar{X}$ defined on G_k^{sp} and an uncertainty set $\bar{U}$ with $|\bar{U}| \geq m + k - 1$.*

Proof. Consider the graph G_k^{sp} that arises from G by adding $k-1$ new paths from s to t of length two; see Fig. 3. The idea of the proof is to define scenarios on G_k^{sp} that force the min-max-min problem to choose each such path as one solution. The remaining k-th solution then must be the optimal min-max solution in G. To this end, we extend every scenario $c_i \in U$ by ∞ on the edges $f_1, \dots, f_{k-1}$ and add scenarios $d_1, \dots d_{k-1}$ such that in scenario d_i all edges in G and all edges $f_1, \dots, f_{k-1}$ have costs ∞, except edge f_i which has costs 0. The first edge of each new path has costs 0 in all scenarios. $\square$

Using a similar construction based on the graph G_k^{st}, one can show

Theorem 3.4 *The assertion of Theorem 3.3 also holds for the minimum spanning tree problem on graphs with at least k nodes.*

Corollary 3.5 *Problem (3) is NP-hard for the shortest path problem and the minimum spanning tree problem if $|U|$ is finite and at least $k+1$.*

Proof. The min-max variants of these problems are NP-hard for $m = 2$ by [9]. From Theorem 3.3 and 3.4, we derive that the min-max-min variants of the same problems are NP-hard if the number of scenarios is at least $k+1$. □

The condition $|U| \geq k+1$ in Corollary 3.5 is necessary, as otherwise the problem can be solved by computing an optimal solution for each scenario.

References

[1] Alizadeh, F. and D. Goldfarb, *Second-order cone programming*, Mathematical Programming **95** (2003), pp. 3–51.

[2] Baumann, F., C. Buchheim and A. Ilyina, *Lagrangean decomposition for mean-variance combinatorial optimization*, in: *ISCO 2014*, 2014 pp. 62–74.

[3] Ben-Tal, A., A. Goryashko, E. Guslitzer and A. Nemirovski, *Adjustable robust solutions of uncertain linear programs*, Mathematical Programming **99** (2004), pp. 351–376.

[4] Ben-Tal, A. and A. Nemirovski, *Robust solutions of uncertain linear programs*, Operations Research Letters **25** (1999), pp. 1–13.

[5] Bertsimas, D. and M. Sim, *Robust discrete optimization and network flows*, Mathematical programming **98** (2003), pp. 49–71.

[6] Bertsimas, D. and M. Sim, *The price of robustness*, Operations Research **52** (2004), pp. 35–53.

[7] Fischetti, M. and M. Monaci, *Light robustness*, in: *Robust and online large-scale optimization*, Springer, 2009 pp. 61–84.

[8] Grötschel, M., L. Lovász and A. Schrijver, "Geometric Algorithms and Combinatorial Optimization," Springer Verlag, 1993.

[9] Kouvelis, P. and G. Yu, "Robust Discrete Optimization and Its Applications," Springer, 1996.

[10] Liebchen, C., M. Lübbecke, R. Möhring and S. Stiller, *The concept of recoverable robustness, linear programming recovery, and railway applications*, in: *Robust and online large-scale optimization*, Springer, 2009 pp. 1–27.

[11] Melvyn, S., "Robust optimization," Ph.D. thesis, Massachusetts Institute of Technology (2004).

[12] Nikolova, E., *Approximation algorithms for offline risk-averse combinatorial optimization* (2010).