Lower Bounding Procedures for the Single Allocation Hub Location Problem

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Abstract

This paper proposes a new lower bounding procedure for the Uncapacitated Single Allocation p-Hub Median Problem based on Lagrangean relaxation. For solving the resulting Lagrangean subproblem, the given problem structure is exploited: it can be decomposed into smaller subproblems that can be solved efficiently by combinatorial algorithms. Our computational experiments for some benchmark instances demonstrate the strength of the new approach.

Keywords: Hub Location, Lagrangian relaxation, Lower bounds.

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1 Introduction

Consider a complete graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$ corresponds to origins, destinations and possible hub locations, and E is the edge set. Let b_{ij} be the transport cost per unit of flow from node i to node j, and W_{ij} be the amount of flow from node i to node j. The cost per unit of flow for each path P_{ij}^{kl} from an origin node i to a destination node j which passes hubs k and l respectively, is $\beta_1 b_{ik} + \alpha b_{kl} + \beta_2 b_{lj}$, where β_1, α, and β_2 are the collection, transfer and distribution costs respectively. The Uncapacitated Single Allocation p-Hub Median Problem (USApHMP) consists of selecting p nodes as hubs and assigning the remaining nodes to these p hubs such that each non-hub node is assigned to exactly one hub node with the minimum overall cost.

The quadratic binary programming formulation for the (USApHMP) is:

$$\min \quad \sum_i \sum_j \sum_k b_{ik}(\beta_1 W_{ij} + \beta_2 W_{ji}) x_{ik} + \sum_i \sum_j \sum_k \sum_l \alpha b_{kl} W_{ij} x_{ik} x_{jl}$$

$$\text{s.t.} \quad \sum_k x_{ik} = 1 \qquad \forall i \tag{1}$$

$$x_{ik} \leq x_{kk} \qquad \forall i, k \tag{2}$$

$$\sum_k x_{kk} = p \tag{3}$$

$$x_{ik} \in \{0, 1\} \qquad \forall i, k, \tag{4}$$

where the binary variable x_{ik} indicates the allocation of node i to the hub located at node k. Constraints (1) indicate that non-hub node i is allocated to precisely one hub node. Constraints (2) enforce that node i is allocated to a hub node at k only if a hub is located at node k. Constraint (3) restricts the number of selected hubs to p.

To ease the argumentation, we define $C_{ik} = b_{ik}(\beta_1 \sum_j W_{ij} + \beta_2 \sum_j W_{ji})$ and $Q_{ikjl} = \alpha b_{kl} W_{ij}$. This allows us to write down the objective function in a more condensed form:

$$\sum_i \sum_k C_{ik} x_{ik} + \sum_i \sum_j \sum_k \sum_l Q_{ikjl} x_{ik} x_{jl}.$$

The USApHMP was first introduced in [9] as a quadratic binary program. Since then, many exact and heuristic algorithms have been proposed in the literature, e.g., by Campbell [3], Ernst and Krishnamoorthy [5], Skorin-Kapov et al. [10], and Ilić et al. [8]. Due to the quadratic nature of the problem,

many attempts have been made to linearize the objective function so that the resulting lower bound is strong enough to be used in a branch-and-bound algorithm. Skorin-Kapov et al. [10] and Ernst and Krishnamoorthy [5] proposed Mixed Integer Linear Programming (MILP) formulations with $O(n^4)$ and $O(n^3)$ variables, repectively. The lower bound obtained from the continuous relaxation of the four index MILP formulation of Skorin-Kapov et al. [10] is tighter than the one obtained using the three index MILP formulation of Ernst and Krishnamoorthy [5]. However, it requires considerably more running time to be computed.

In this paper we consider two new lower bounds for the USApHMP. The first bound trivially builds a new p-median problem from the quadratic cost matrix and solves the resulting problem to obtain a bound. This bounding procedure is then shown to be equivalent to the continuous relaxation of an MILP. Due to the large duality gap implied by this approach, we develop a new MILP formulation and show how to handle it via a Lagrangian relaxation approach to obtain a Lagrangian function with block-diagonal structure.

2 A relaxed Gilmore-Lawler type bound

The Gilmore-Lawler procedure, shortly denoted by GL, is one of the most popular approaches to find a lower bound for the Quadratic Assignment Problem. The new bound we consider is derived from a simple observation on the structure of the USApHMP, in a similar spirit to the GL procedure: If we define $C_{ik}^{\mathrm{TOT}} = C_{ik} + \sum_{j,l} Q_{ikjl} x_{jl}$, this allows us to rewrite USApHMP as

$$z^* = \min \left\{ \sum_i \sum_k C_{ik}^{\mathrm{TOT}} x_{ik} : (1) - (4) \right\}. \tag{5}$$

A lower bound for the problem can, therefore, be obtained if we replace each C_{ik}^{TOT} with its minimum value over the set of possible feasible solutions which contain the assignment of node i to node k. In other words, for each arc (i, k), potentially in the solution, we consider the best cumulation providing the minimum interaction cost with (i, k). This can be done by solving a set of subproblems with a linear objective function, one for each possible assignment. Let P_{ik} represent such a subproblem for a given arc $(i, k) \in E$:

$$P_{ik}: \quad \min \left\{ \sum_j \sum_l Q_{ikjl} x_{jl} : (1) - (4),\ x_{ik} = 1 \right\}. \tag{6}$$

The idea is thus to solve, for each P_{ik}, a p-median problem which contains arc (i,k), using the ik-th column of the quadratic cost matrix as the cost vector. This yields a lower bound for the value of C_{ik}^{TOT} in any feasible solution containing (i,k). However, problem P_{ik} is a p-median problem which is well-known to be NP-hard [6]. Therefore, we consider a relaxation of P_{ik} called P'_{ik} which only requires the allocation of each non-hub node $j \neq i$ to precisely one hub node, i.e.,

$$P'_{ik}: \quad z'_{ik} = \min\Big\{\sum_j \sum_l Q_{ikjl} x_{jl} : (1), (4),\ x_{ik} = 1\Big\}. \tag{7}$$

The problem P'_{ik} is a semi-assignment problem and can be solved in $O\big(n^2\big)$ time. The value of z'_{ik} combined with the linear cost C_{ik} of arc (i,k) yields a lower bound for C_{ik}^{TOT}, which can then be integrated into (5), resulting in

$$\overline{P1}: \quad \overline{z1} = \min\left\{\sum_i \sum_k (C_{ik} + z'_{ik}) x_{ik} : (1) - (3)\right\}. \tag{8}$$

Note that we do not require (4) here, as otherwise solving $\overline{P1}$ would be NP-hard again. We obtain

Theorem 2.1 *Solving $\overline{P1}$ yields a lower bound for USApHMP, i.e., $\overline{z1} \leq z^*$.*

Although the main part of the bounding procedure that we just described is combinatorial, the same bounds can be obtained by solving a linear program. More precisely, we introduce the non-negative continuous variables y_{ikjl} for all $i,k,j,l \in V$, and a set of constraints as

$$\sum_l y_{ikjl} = x_{ik} \qquad \forall i,j,k;\ i \neq j. \tag{9}$$

Now we consider the following MILP formulation:

$$\text{P1: } \min\left\{\sum_i \sum_k C_{ik} x_{ik} + \sum_i \sum_k \sum_j \sum_l Q_{ikjl} y_{ikjl} : (1) - (4), (9), y \geq 0\right\}.$$

For this MILP, we can show the following result. The proof is omitted due to space restrictions.

Theorem 2.2 *The optimal objective value for the continuous relaxation of problem P1 agrees with $\overline{z1}$.*

3 New formulation and Lagrangian relaxation

In order to improve the bound presented in the previous section we follow the idea proposed in [4]. We consider problem P1 with separate variables y_{ikjl} and y_{jlik} for all $(i,k),(j,l) \in E$ and introduce an additional set of constraints:

$$y_{ikjl} = y_{jlik} \quad \forall (i,k),(j,l),\ i<j. \tag{10}$$

We refer to this new formulation as problem P2. We can prove that problems P2 and USApHMP are equivalent in the sense that for any x feasible for USApHMP there exists a y such that (x,y) is feasible for P2, and conversely, for any (x,y) feasible for P2, x is feasible for USApHMP with the same objective value.

Consider the continuous relaxation of P2. Due to the large number of variables and constraints, and also degeneracy of the problem, solving this relaxation in order to obtain a lower bound for USApHMP is too time consuming. Therefore we consider the Lagrangian dual obtained from relaxing constraints (1) and (10), using a set of Lagrangian multipliers μ_i, $i \in V$, and λ_{ikjl} for all (i,k,j,l), $i<j$. For convenience we assume that $\lambda_{jlik} = -\lambda_{ikjl}$ for all (i,k,j,l), $i<j$. The resulting Lagrangian function is as follows:

$$\begin{aligned} \mathrm{P}(\mu,\lambda): \quad \min \quad & \sum_i \sum_k (C_{ik}-\mu_i)x_{ik} + \sum_i \sum_k \sum_j \sum_l (Q_{ikjl}-\lambda_{ikjl})y_{ikjl} \\ \text{s.t.} \quad & (2)-(3),(9) \quad \text{and} \quad \boldsymbol{x},\boldsymbol{y} \geq 0. \end{aligned}$$

Note that $P(\mu,\lambda)$ is the continuous relaxation of P1 where constraints (1) have been relaxed into the objective function. Therefore, we have:

Theorem 3.1 *For any given values of μ and λ, an optimal solution $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ to $P(\mu,\lambda)$ is given by*

$$\boldsymbol{x}^* = \hat{\boldsymbol{x}} \quad \text{and} \quad y^*_{ikjl} = \hat{y}_{ikjl}\hat{x}_{ik} \quad \forall (i,k,j,l),$$

where $\hat{\boldsymbol{y}}$ is the optimal solution of subproblem P'_{ik} (with $Q_{ikjl} = Q_{ikjl} - \lambda_{ikjl}$) while $\hat{\boldsymbol{x}}$ is the optimal solution of the following problem:

$$MP: \quad \min \left\{ \sum_i \sum_k (C_{ik}-\mu_i)x_{ik} : (2)-(3) \right\}. \tag{11}$$

Observe that MP can be solved by a simple inspection: Let $\overline{C}_{ik} = C_{ik} - \mu_i$ for all i,k, and define $d_k = \overline{C}_{kk} + \sum_{i \neq k} \min\{0, \overline{C}_{ik}\}$. Then Problem MP is reduced to selecting the p hubs with smallest cost d_k.

4 Computational results

In this section we present computational experiments on the CAB instances from OR Library [2]. We consider the biggest instances, having $n = 20, 25$ nodes, and different discount factors $\alpha \in \{0.2, 0.4, 0.6, 0.8, 1\}$ while always keeping $\beta_1 = \beta_2 = 1$.

In order to develop a procedure to find the optimal (or near-optimal) dual multipliers of the Lagrangain dual problem, we use the subgradient method [7] with initial step size of 1 and maximum number of allowed iterations of 1500.

Table 1 presents the relative gap between the lower bounds and the optimal objective values, and CPU execution times (in seconds) of using different approaches. The first four columns of the table indicate the problem size (n), the number of desired hubs (P), the discount factor (α), and the optimal objective values obtained from [5]. The next three columns, from left to right, give the relative gaps obtained using the relaxed GL procedure described in Section 2 (RGL), the subgradient implementation for P2 (SubP2), and the linear programming relaxation obtained with the three index formulation of Ernst and Krishnamoorthy [5] (LPEK). The CPU execution times of each approach are given in the last three columns of the tables. The formula we used to compute the relative gaps is $100 \times (Opt - Lb)/Lb$, where Opt and Lb stand for the optimal value and the value of the lower bound, respectively.

As we can observe from the table for all problems tested, the Lagrangian dual procedure significantly outperforms the LPEK in terms of the strength of the bounds. In fact, in 21 out of the 30 instances even optimality was proven using our Lagrangian dual. For the remaining instances, the relative gap of the Lagrangian dual is less than 1%, except for the single case $n = 20$, $p = 3$, $\alpha = 0.8$, for which the gap is 1.37%. These results confirm the strength of the four index MILP formulations as shown already by Skorin-Kapov et al. [10].

However, our new four index MILP formulation presented in this paper is favorable, since its continuous relaxation possesses a special structure with lends itself to decomposition techniques. Even if the time needed to compute our bounds is considerably longer than for computing the LPEK bound, the significantly stronger bounds will likely lead to competitive or even faster running times when solving USApHMP to optimality. Moreover, within a branch-and-bound scheme, the number of iterations of the subgradient method can be reduced significantly by initializing it with the optimal multipliers of the parent node, as already observed for other Lagrangian relaxation based branch-and-bound methods, e.g., [1].

Table 1
Comparison of different lower bounding approaches on TestSet CAB for problem size $n = 20, 25$.

Instance				Gap(%)			CPU Time		
n	P	α	Opt.	RGL	SubP2	LPEK	RGL	SubP2	LPEK
20	2	0.2	979.09	7.92	0.00	0.05	0.0	14.6	0.2
20	2	0.4	1042.57	13.44	0.00	0.59	0.0	12.7	0.2
20	2	0.6	1106.04	18.33	0.00	1.47	0.1	11.6	0.2
20	2	0.8	1169.52	22.69	0.00	2.60	0.1	15.4	0.2
20	2	1	1210.08	25.22	0.00	2.49	0.1	12.3	0.2
20	3	0.2	724.54	17.15	0.00	0.45	0.0	12.7	0.2
20	3	0.4	847.77	28.87	0.00	1.55	0.1	15.3	0.2
20	3	0.6	971.00	37.62	0.00	3.26	0.1	14.1	0.2
20	3	0.8	1091.05	44.23	1.35	5.30	0.0	17.4	0.2
20	3	1	1156.07	47.13	0.37	4.36	0.1	13.3	0.2
20	4	0.2	577.62	27.28	0.00	0.42	0.0	17.1	0.2
20	4	0.4	727.10	41.51	0.00	2.05	0.0	14.3	0.2
20	4	0.6	869.16	50.47	0.00	3.50	0.1	14.2	0.2
20	4	0.8	1008.49	56.79	0.18	5.19	0.1	19.5	0.2
20	4	1	1111.02	60.31	0.65	5.74	0.1	19.5	0.2
25	2	0.2	1000.91	9.98	0.00	0.00	0.1	23.6	0.5
25	2	0.4	1101.63	18.14	0.00	0.39	0.1	26.5	0.5
25	2	0.6	1201.21	24.86	0.00	2.04	0.1	27.1	0.9
25	2	0.8	1294.08	30.19	0.46	3.54	0.1	36.6	0.9
25	2	1	1359.19	33.47	0.72	3.91	0.1	37.3	0.9
25	3	0.2	767.35	17.88	0.00	1.33	0.1	26.4	0.5
25	3	0.4	901.70	29.88	0.00	2.47	0.1	26.3	0.6
25	3	0.6	1033.56	38.61	0.00	3.39	0.1	25.2	0.6
25	3	0.8	1158.83	45.08	0.27	4.01	0.1	36.5	0.9
25	3	1	1256.63	49.20	0.11	3.71	0.1	34.1	1.0
25	4	0.2	629.63	26.09	0.00	0.79	0.1	26.8	0.6
25	4	0.4	787.51	40.37	0.00	1.90	0.1	28.6	0.6
25	4	0.6	939.21	49.55	0.00	3.06	0.1	30.2	0.8
25	4	0.8	1087.66	56.05	0.00	4.61	0.1	31.7	0.9
25	4	1	1211.23	60.18	0.76	4.94	0.1	39.5	0.9

5 Conclusion

We presented a new MIP formulation of the Uncapacitated Single Allocation p-Hub Median Problem. Due to the size of the resulting model, we developed a Lagrangean relaxation approach to compute a strong lower bound. For solving the Lagrangian subproblem, we used a simple observation on the structure of the USApHMP combined with the Gilmore-Lawler procedure. Our future work will concentrate on the integration of our lower bounds into a branch-and-bound scheme and on extending our ideas to the Capacitated SApHMP.

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