Technical Report

A close look at auxiliary problem principles for equilibria

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The auxiliary problem principle allows solving a given equilibrium problem (EP) through an equivalent auxiliary problem with better properties. The paper investigates two families of auxiliary EPs: the classical auxiliary problems, in which a regularizing term is added to the equilibrium bifunction, and the regularized Minty EPs. The conditions that ensure the equivalence of a given EP with each of these auxiliary problems are investigated. This analysis leads to extending some known results for variational inequalities and linear EPs to the general case; moreover, new results are obtained as well. In particular, both new results on the existence and uniqueness of solutions and new error bounds based on gap functions with good convexity properties are obtained under weak quasimonotonicity or weak concavity assumptions.

Keywords: Equilibrium problem; auxiliary problem; Minty equilibrium problem; existence of solutions; gap function; error bound.

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1. Introduction

In computational optimization solving a given problem through a family of auxiliary problems, which enjoy better properties and are therefore easier to be solved, is a widespread approach. Among others, penalty and proximal point algorithms for nonlinear programs (see, for instance, [1, 2]) fall within this scheme. A quite general auxiliary principle has been developed in [3, 4] within the framework of decomposition and coordination algorithms, and it provides sufficient conditions for an optimal solution of one suitable auxiliary problem to solve the given problem. This principle was later extended to variational inequalities [5], while the full equivalence with the auxiliary variational inequality was investigated in [6] and exploited to develop solution methods through optimization techniques, for instance, in [6–8]. Quite recently, this kind of approach has been considered also for more general equilibrium problems (see [9–12]), which include optimization, variational inequalities, saddle point problems and Nash equilibria in noncooperative games as particular cases.

In this paper we aim at deepening the analysis of auxiliary principles for equilibria and we focus on the well-known format (see [13–15]) of an equilibrium problem

$$\text{find } \ x^* \in C \ \text{ s.t. } \ f(x^*, y) \geq 0, \qquad \forall \ y \in C, \qquad\qquad (EP)$$

where $C \subseteq \mathrm{R}^n$ is a nonempty, closed and convex set and the equilibrium bifunction $f : \mathrm{R}^n \times \mathrm{R}^n \to \mathrm{R}$ is continuous, satisfies $f(x,x) = 0$ and $f(x, \cdot)$ is convex for all

$x \in \mathrm{R}^n$. Given any $\alpha > 0$, the bifunction

$$f_\alpha(x, y) := f(x, y) + \alpha \, \|y - x\|^2 / 2$$

leads to the following auxiliary equilibrium problem

$$\text{find} \ \ x^* \in C \ \ \text{s.t.} \ \ f_\alpha(x^*, y) \geq 0, \qquad \forall \, y \in C, \qquad\qquad (EP_\alpha)$$

which is equivalent to (EP) in the sense that their solution sets coincide [12].

The so-called Minty equilibrium problem (see [13, 16])

$$\text{find} \ \ x^* \in C \ \ \text{s.t.} \ \ f(y, x^*) \leq 0, \qquad \forall \, y \in C, \qquad\qquad (MEP)$$

can be viewed as an auxiliary problem as well, since any of its solutions solves also (EP) and the solution sets coincide if f is pseudomonotone [16]. Therefore, it is interesting to consider also the equilibrium problem

$$\text{find} \ \ x^* \in C \ \ \text{s.t.} \ \ f_\alpha(y, x^*) \leq 0, \qquad \forall \, y \in C, \qquad\qquad (MEP_\alpha)$$

in connection with (MEP) and (EP), though it did not receive, to the best of our knowledge, any consideration except for a few papers dealing with variational inequalities only [6, 17].

The goal of the paper is to analyse in details the conditions that guarantee the equivalence between (EP) and each of the above auxiliary problems together with the properties and advantages that each equivalence brings. A rather novel feature is the analysis of the case $\alpha < 0$ for which, up to now, (EP_α) has been considered only for the so-called linear equilibrium problem in [18] while (MEP_α) only for variational inequalities in [6, 17].

Section 2 explores the connections between the convexity and monotonicity properties of the bifunctions f and f_α. Section 3 and Section 4 investigate the relationships of (EP) with (EP_α) and (MEP_α), respectively, and the properties of the corresponding gap functions, which allow to reformulate the equilibrium problems as optimization programs. Exploiting negative values for α, also new existence results and error bounds are achieved under weak monotonicity or weak concavity assumptions on f.

2.   Convexity and monotonicity

Both convexity and monotonicity play an important role in the study of equilibrium problems and in the development of solution methods. In order to analyse strong and weak concepts in a unified way, suitable parametric definitions can be introduced.

Definition 1   Given $\gamma \in \mathrm{R}$, a function $g : \mathrm{R}^n \to \mathrm{R}$ is called γ-convex on C if any $u, v \in C$ and $t \in [0, 1]$ satisfy

$$g(t\,u + (1 - t)\,v) \leq t\,g(u) + (1 - t)\,g(v) - \frac{\gamma}{2}\,t\,(1 - t)\,\|u - v\|^2.$$

g is called γ-concave on C if $-g$ is γ-convex on C.

If $\gamma = 0$, the above inequality provides the usual definition of a convex function. If $\gamma > 0$, the inequality is strengthened and f is also called *strongly convex*; similarly, f is called *weakly convex* if $\gamma < 0$. Indeed, g is γ-convex on C if and only if $g(x) - \gamma \|x\|^2/2$ is convex on C (see [19]).

Definition 2   Given $\mu \in \mathrm{R}$, the bifunction f is called

- μ-*monotone* on C if any $x, y \in C$ satisfy the inequality

$$f(x,y) + f(y,x) \leq -\mu \|y - x\|^2;$$

- μ-*pseudomonotone* on C if any $x, y \in C$ satisfy the implication

$$f(x,y) \geq 0 \quad \Longrightarrow \quad f(y,x) \leq -\mu \|y - x\|^2;$$

- μ-*quasimonotone* on C if any $x, y \in C$ satisfy the implication

$$f(x,y) > 0 \quad \Longrightarrow \quad f(y,x) \leq -\mu \|y - x\|^2.$$

Clearly, μ-monotonicity implies μ-pseudomonotonicity, which in turn implies μ-quasimonotonicity. If $\mu = 0$, the well-known concepts of monotonicity, pseudomonotonicity and quasimonotonicity are recovered (see [14, 20–22]). If $\mu > 0$, the requirements are strengthened and the strong counterparts of the above monotonicity concepts defined: strong monotonicity has been often exploited in algorithmic frameworks (see [13]) while strong pseudomonotonicity has been considered mainly for variational inequalities (see [23, 24]) and only very recently for more general equilibrium problems [25]. Similarly, if $\mu < 0$ weaker concepts are introduced: weak monotonicity has been exploited in a few papers [26–29], while weak pseudomonotonicity and weak quasimonotonicity, to the best of our knowledge, are fairly new.

Throughout all the paper we suppose the bifunction f to satisfy the following convexity assumption:

$$\exists \tau \geq 0 \;\; \text{such that } f(x, \cdot) \text{ is } \tau\text{-convex for any } x \in C. \tag{1}$$

If $\tau = 0$, then (1) collapses to the standard assumption of convexity for equilibrium problems. Sections 3 and 4 show that new results and features of equilibrium problems together with improvements of known results can be achieved considering also the case $\tau > 0$.

Note that variational inequalities, that is (EP) with

$$f(x,y) = \langle F(x), y - x \rangle$$

for some mapping $F : \mathrm{R}^n \to \mathrm{R}^n$, satisfy (1) only for $\tau = 0$, while the so-called linear equilibrium problems (see [18]), that is (EP) with

$$f(x,y) = \langle Px + Qy + r, y - x \rangle \tag{2}$$

for some $r \in \mathrm{R}^n$ and some $P, Q \in \mathrm{R}^{n \times n}$, with Q positive semidefinite, satisfy (1) with τ equal to the minimum eigenvalue of $Q + Q^T$. Nash equilibrium problems, in which each player i selects one strategy from the set $C_i \subseteq \mathrm{R}^{n_i}$ to minimize a cost

function $c_i : C_1 \times \cdots \times C_N \to \mathrm{R}$, that is (EP) with $C = C_1 \times \cdots \times C_N$ and

$$f(x, y) = \sum_{i=1}^{N} [c_i(x_{-i}, y_i) - c_i(x))]$$

where $y_i \in C_i$ and $(x_{-i}, y_i) \in C$, satisfy (1) with $\tau = \min\{\tau_i : i = 1, \ldots, N\}$ for τ_i's such that $c_i(x_{-i}, \cdot)$ is a τ_i-convex function on C_i.

The auxiliary bifunction f_α inherits convexity and monotonicity properties from f in the following way.

PROPOSITION 2.1 *(Convexity and monotonicity properties of f_α).*

    a) $f_\alpha(x, \cdot)$ *is $(\tau + \alpha)$-convex for any $x \in C$.*
    b) If f is μ-monotone on C, then f_α is $(\mu - \alpha)$-monotone on C.
    c) If f is μ-quasimonotone on C and $\alpha < 2\mu$, then f_α is $(-\alpha/2)$-pseudomono-
    tone on C.

Proof.
a) Since $\alpha \|y - x\|^2/2$ is α-convex, it follows directly from [19, Proposition 4.1].

b) $f_\alpha(x, y) + f_\alpha(y, x) = f(x, y) + f(y, x) + \alpha\|y - x\|^2 \leq (\alpha - \mu)\|y - x\|^2.$

c) Take any $x, y \in C$ with $x \neq y$ such that $f_\alpha(x, y) \geq 0$. Then,

$$f(x, y) \geq -\alpha \|y - x\|^2/2 > -\mu \|y - x\|^2$$

implies $f(y, x) \leq 0$ since f is μ-quasimonotone on C. Hence, the thesis follows immediately since $f_\alpha(y, x) = f(y, x) + \alpha \|y - x\|^2/2 \leq \alpha \|y - x\|^2/2.$ □

The choice of α should be driven by getting f_α satisfy better properties than f. Anyway, there is some kind of tradeoff between convexity and monotonicity: if $\alpha > 0$, then $f_\alpha(x, \cdot)$ satisfies a stronger convexity condition than $f(x, \cdot)$ but f has stronger monotonicity properties; vice versa, if $\alpha < 0$, then $f(x, \cdot)$ satisfies a stronger convexity condition than $f_\alpha(x, \cdot)$ while f_α has stronger monotonicity properties. For example, consider the case f is monotone and $f(x, \cdot)$ is convex: f_α is just weakly monotone but $f_\alpha(x, \cdot)$ is strongly convex if $\alpha > 0$, while f_α is strongly monotone but $f_\alpha(x, \cdot)$ is just weakly convex if $\alpha < 0$.

Note that if f is quasimonotone, then Proposition 2.1 guarantees that f_α is strongly pseudomonotone if $\alpha < 0$, but it is not necessarily weakly monotone as the following example shows.

Example 1 Consider $f(x, y) = (x^2 + 1)(y - x)$ and $C = (-\infty, 0]$. Clearly, f is 0-quasimonotone since $f(x, y) > 0$ implies $y > x$ and thus $f(y, x) < 0$. However, both f and f_α are not μ-monotone for any $\mu \in \mathrm{R}$ since

$$[f_\alpha(x, y) + f_\alpha(y, x)]/(y - x)^2 = -(x + y - \alpha) \to +\infty$$

as $x \to -\infty$ and $y \to -\infty$ with $x \neq y$.

3. Classical auxiliary problems

The exploitation of (EP_α) as an auxiliary problem is rooted in proximal point algorithms for nonlinear optimization. At first it has been often used in the framework of variational inequalities and afterwards also for other equilibrium problems (see, for instance, Section 3.1 in [13] and the references therein). Generally, just positive values of α have been considered. Anyhow, if strong convexity assumptions are met, i.e., (1) holds for some $\tau > 0$, then it is possible to consider also negative values as shown by the following auxiliary problem principle.

THEOREM 3.1 *(Classical auxiliary problem principle).*
(EP) is equivalent to (EP_α) for any $\alpha \geq -\tau$.

Proof. Since $f(x, \cdot)$ and $f_\alpha(x, \cdot)$ are convex for any $x \in C$, the following equivalences hold:

$$
\begin{aligned}
x^* \text{ solves } (EP) \iff & \; x^* \in \arg\min\{\; f(x^*, y) \; : \; y \in C \;\} \\
\iff & \; \exists \; g^* \in \partial_y f(x^*, x^*) \text{ s.t. } \langle g^*, y - x^* \rangle \geq 0, \; \forall \; y \in C \\
\iff & \; \exists \; g^* \in \partial_y f_\alpha(x^*, x^*) \text{ s.t. } \langle g^*, y - x^* \rangle \geq 0, \; \forall \; y \in C \\
\iff & \; x^* \in \arg\min\{\; f_\alpha(x^*, y) \; : \; y \in C \;\} \\
\iff & \; x^* \text{ solves } (EP_\alpha)
\end{aligned}
$$

where $\partial_y f(x^*, x^*)$ denotes the subdifferential of the convex function $f(x^*, \cdot)$ at x^*. The first and the last equivalence are obvious consequences of the definition of (EP) as $f(x, x) = 0$ for any $x \in C$, while the second and the fourth are the optimality conditions of convex programming. Finally, the third equivalence is due to the equality $\partial_y f_\alpha(x, y) = \partial_y f(x, y) + \alpha(y - x)$ (see [30, Theorem 4.1.1]). $\qquad \square$

If $\tau = 0$, then the above principle collapses to the well-known one given by Lemma 3.1 in [12], while it has been considered with $\tau \geq 0$ only for linear equilibrium problems [18, Lemma 5]. Furthermore, it can be exploited to obtain new existence and uniqueness results for (EP), relying on appropriate values of α.

THEOREM 3.2 *(Existence and uniqueness of solutions).*
If any of the following conditions holds:

    a) f is μ-monotone on C with $\mu > -\tau$,
    b) $\tau > 0$ and f is μ-quasimonotone on C with $\mu > -\tau/2$,

then there exists a unique solution of (EP).

Proof. By Theorem 3.1 it is enough to prove the thesis for (EP_α) exploiting a suitable value of α.

a) Choosing any $\alpha \in [-\tau, \mu)$, then Proposition 2.1 guarantees that $f_\alpha(x, \cdot)$ is convex and f_α is strongly monotone on C. Therefore, (EP_α) has a unique solution (see Section 2.2 in [13]).

b) Choosing $\alpha \in [-\tau, \min\{0, 2\mu\})$, then Proposition 2.1 guarantees that $f_\alpha(x, \cdot)$ is convex and f_α is strongly pseudomonotone on C. Therefore, (EP_α) has a unique solution by Proposition 2.1 in [25]. $\qquad \square$

If $\tau = 0$, then case a) collapses to the well-known existence and uniqueness

result that exploits the strong monotonicity of f (see, for instance, [13]). It is worth noting that the weak quasimonotonicity of f is enough to guarantee the existence of a unique solution of (EP) if $\tau > 0$. Although the quasimonotonicity of f has already been exploited to prove existence results (see, for instance, [20]), to the best of our knowledge no result based on weak quasimonotonicity had been given in the literature up to now.

Equilibrium problems can be reformulated as optimization programs through suitable gap functions. Indeed, it is well-known (see, for instance, [10, 11, 31]) that

$$\varphi_\alpha(x) := \sup\{ -f_\alpha(x, y) \ : \ y \in C \}$$

is a *gap function* for (EP) for any given $\alpha \geq 0$, i.e., φ_α is non-negative on C and x^* solves (EP) if and only if $x^* \in C$ and $\varphi_\alpha(x^*) = 0$. The auxiliary problem principle given by Theorem 3.1 allows extending this reformulation of equilibria for suitable negative values of α if $\tau > 0$.

THEOREM 3.3   *(Properties of φ_α).*

  a) φ_α *is a gap function for (EP) for any $\alpha \geq -\tau$.*
  b) $\varphi_\alpha(x) < +\infty$ *for any $x \in C$ and $\alpha > -\tau$.*
  c) *If $f(\cdot, y)$ is γ-concave on C for any $y \in C$, then φ_α is $(\gamma - \alpha)$-convex on C.*

Proof.
a) It follows directly from Theorem 3.1.

b) Since $\alpha > -\tau$, $f_\alpha(x, \cdot)$ is strongly convex by Proposition 2.1. Hence, its minimum value over C, that is $-\varphi_\alpha(x)$, is finite.

c) Since $\alpha \|y - x\|^2/2$ is $(-\alpha)$-concave on C for any $y \in C$, $f_\alpha(\cdot, y)$ is $(\gamma - \alpha)$-concave on C by Proposition 2.1. As it is the pointwise supremum of a family of $(\gamma - \alpha)$-convex functions, the gap function φ_α is $(\gamma - \alpha)$-convex on C as well (see [19, Proposition 4.1]). □

Notice that it is more likely for φ_α to be convex if $\alpha < 0$: for instance, whenever $f(\cdot, y)$ is weakly concave or concave on C, that is $\gamma \leq 0$, φ_α is convex if $\alpha \leq \gamma$, which does not hold when $\alpha > 0$. The following example shows a case in which φ_α is not convex for any $\alpha > 0$ while convex for some negative values of α.

Example 2   Consider $f(x, y) = (-x + 2y + 1)(y - x)$ and $C = [0, +\infty)$: $f(x, \cdot)$ is τ-convex with $\tau = 4$, thus φ_α is a gap function for any $\alpha \geq -4$. Moreover, it holds

$$\varphi_\alpha(x) = (-1 - \alpha/2)\, x^2 + x$$

for $\alpha \in [-4, -3]$, while

$$\varphi_\alpha(x) = \begin{cases} (-1 - \alpha/2)\, x^2 + x, & \text{if } x \in \left[0, (3 + \alpha)^{-1}\right), \\[2mm] (x + 1)^2/(8 + 2\alpha), & \text{if } x \in \left[(3 + \alpha)^{-1}, +\infty\right), \end{cases}$$

for $\alpha > -3$. Therefore, φ_α is convex on C for any $\alpha \in [-4, -2]$, while it is not convex on C if $\alpha > -2$. Indeed, $f(\cdot, y)$ is γ-concave with $\gamma = -2$ for any $y \in C$.

If f is strongly monotone on C, the gap function φ_α provides an error bound for (EP) whenever $\alpha > 0$ is small enough (see [11, Proposition 4.2]). If $\tau > 0$, the use of negative values for α brings further improvements: error bounds can be established even under weak monotonicity or weak concavity assumptions.

THEOREM 3.4 *(Error bound).*

a) *If f is μ-monotone on C with $\mu > -\tau$ and $\alpha \in [-\tau, \mu)$, then*

$$\varphi_\alpha(x) \geq (\mu - \alpha)\|x - x^*\|^2, \qquad \forall\, x \in C,$$

where x^* is the unique solution of (EP).

b) *If $f(\cdot, y)$ is γ-concave on C for any $y \in C$ with $\gamma > -\tau$ and $\alpha \in [-\tau, \gamma)$, then*

$$\varphi_\alpha(x) \geq (\gamma - \alpha)\|x - x^*\|^2/2, \qquad \forall\, x \in C,$$

where x^* is the unique solution of (EP).

Proof.
a) By Theorem 3.2 a) there exists a unique solution x^* of (EP). Since Theorem 3.1 guarantees the equivalence between (EP) and (EP_α), x^* solves (EP_α) as well. Therefore, any $x \in C$ satisfies

$$
\begin{aligned}
\varphi_\alpha(x) &\geq -f_\alpha(x, x^*) \\
&= -f_\alpha(x, x^*) - f_\alpha(x^*, x) + f_\alpha(x^*, x) \\
&\geq -f_\alpha(x, x^*) - f_\alpha(x^*, x) \\
&\geq (\mu - \alpha)\|x - x^*\|^2,
\end{aligned}
$$

where the last inequality holds since f_α is $(\mu - \alpha)$-monotone on C by Proposition 2.1.

b) By Theorem 4.2 below there exists a unique solution x^* of (EP), and since $\alpha \geq -\tau$ it minimizes φ_α over C by Theorem 3.3 a). Note that φ_α is strongly convex on C since it is $(\gamma - \alpha)$-convex on C (Theorem 3.3 c)) and $\alpha < \gamma$. Therefore, Theorem 6.1.2 in [30] guarantees that the inequality

$$\varphi_\alpha(x) \geq \varphi_\alpha(x^*) + \langle g^*, x - x^* \rangle + (\gamma - \alpha)\|x - x^*\|^2/2$$

holds for any $x \in C$ and any $g^* \in \partial\varphi_\alpha(x^*)$. The optimality of x^* implies $\varphi_\alpha(x^*) = 0$ and the existence of some $g^* \in \partial\varphi_\alpha(x^*)$ such that $\langle g^*, x - x^* \rangle \geq 0$ for any $x \in C$. Therefore, the thesis follows.

$\square$

Theorem 3.4 a) with $\tau = 0$ and $\mu > 0$ has been proved for variational inequalities in [32] and extended to equilibrium problems in [11]. If $\tau > 0$, then the above result provides new error bounds under weaker assumptions than the usual ones.

In the case of linear equilibrium problems, that is when f is given by (2), f is μ-monotone and $f(\cdot, y)$ is γ-concave and the moduli of monotonicity and concavity are explicitly known: μ and γ are the minimum eigenvalues of $(P - Q + (P - Q)^T)/2$ and $P + P^T$, respectively. As τ is the minimum eigenvalue of $Q + Q^T$, the inequality $\tau \geq \gamma + 2\mu$ always holds as well.

4. Minty auxiliary problems

The Minty equilibrium problem (MEP) has been extensively used as an auxiliary problem for (EP) both in order to obtain existence results (see, for instance, [16, 20, 33, 34]) and within algorithmic frameworks (see, for instance, [35–39]). On the contrary, the analogous equilibrium problem with the regularized bifunction f_α did not receive much attention. Indeed, the class of problems (MEP_α) has been explicitly considered only for variational inequalities with $\alpha < 0$ in [6, 17] while indirectly through gap functions with $\alpha > 0$ in [18, 40]. The following auxiliary principle provides the relationships between (EP) and (MEP_α) for any value of α.

THEOREM 4.1   *(Minty auxiliary problem principle).*
Any solution of (MEP_α) is a solution of (EP). The vice versa is true if any of the following conditions holds:

a) *f is μ-pseudomonotone on C and $\alpha \le 2\,\mu$,*
b) *f is μ-monotone on C and $\alpha \le 2\,\mu + \tau$,*
c) *$f(\cdot, y)$ is γ-concave on C for any $y \in C$ with $\gamma \ge -\tau$ and $\alpha \le \gamma$.*

Proof. Let x^* be a solution of (MEP_α). Given any $y \in C$, consider the point $y_t = t\,y + (1 - t)\,x^*$ for $t \in (0, 1)$ so that $y_t \in C$. The following inequalities hold

$$
\begin{aligned}
0 = f(y_t, y_t) \\
\le t\, f(y_t, y) + (1 - t)\, f(y_t, x^*) \\
\le t\, f(y_t, y) - \alpha\,(1 - t)\, \|y_t - x^*\|^2/2 \\
= t\, f(y_t, y) - \alpha\, t^2 (1 - t)\, \|y - x^*\|^2/2.
\end{aligned}
$$

The first is due to the convexity of $f(y_t, \cdot)$ while the second holds since x^* solves (MEP_α). As a consequence, the inequality

$$
f(y_t, y) \ge \alpha\, t(1 - t)\, \|y - x^*\|^2/2
$$

holds as well. Taking the limit as $t \downarrow 0$, the continuity of f guarantees $f(x^*, y) \ge 0$. Hence, x^* solves (EP).

Now, suppose x^* solves (EP). The three cases have to be analysed separately.

a) Any $y \in C$ satisfies

$$
f_\alpha(y, x^*) = f(y, x^*) + \alpha\|y - x^*\|^2/2 \le (\alpha/2 - \mu)\,\|y - x^*\|^2 \le 0
$$

where the first inequality follows from the μ-pseudomonotonicity of f.

b) Any $y \in C$ satisfies

$$
\begin{aligned}
f_\alpha(y, x^*) = f(y, x^*) + \alpha\|y - x^*\|^2/2 \\
\le -f(x^*, y) + (\alpha/2 - \mu)\,\|y - x^*\|^2 \\
\le (\alpha/2 - \mu - \tau/2)\,\|y - x^*\|^2 \\
\le 0,
\end{aligned}
$$

where the first inequality is due to the μ-monotonicity of f while the second holds

since x^* solves $(EP_{-\tau})$ by Theorem 3.1 and therefore $f(x^*, y) \geq \tau \|y - x^*\|^2 / 2$.

c) Consider the bifunction $g(x, y) := -f_\gamma(y, x)$: $g(x, \cdot)$ is convex for any $x \in C$ since Proposition 2.1 guarantees that $f_\gamma(\cdot, x)$ is concave, and obviously $g(x, x) = 0$. Given any $y \in C$, $g(y, x^*) = -f_\gamma(x^*, y) \leq 0$ since x^* solves (EP_γ) by Theorem 3.1. The same argument of the very first part of the proof shows $g(x^*, y) \geq 0$ and therefore $f_\gamma(y, x^*) \leq 0$ holds. Since $\alpha \leq \gamma$, then $f_\alpha(y, x^*) \leq f_\gamma(y, x^*)$ and thus x^* solves (MEP_α). $\qquad\square$

Theorem 4.1 subsumes some known particular cases together with new results. Case a) with $\mu = 0$ and $\alpha = 0$ is the well-known Minty Lemma about the equivalence between (EP) e (MEP) (see, for instance, [16]). Case b) with $\tau = 0$, $\mu > 0$ and $\alpha = 0$ or $\alpha = \mu$ was considered in [18]. Considering just variational inequalities (and therefore having $\tau = 0$), case a) with $\mu = 0$ and $\alpha \leq 0$ has been proved in [6, 17], while case b) with $\mu > 0$ and $\alpha \geq 0$ in [40, Lemma 2.3]. To the best of our knowledge, up to now case c) has not been considered in any framework. Moreover, since μ and γ can be negative, Theorem 4.1 provides further equivalence results, which just require weak pseudomonotonicity or weak concavity assumptions.

Notice that the auxiliary principles of Theorem 3.1 and Theorem 4.1 are somehow symmetric: the former needs α to be bounded by below while the latter requires α to be bounded by above and some monotonicity conditions.

As (MEP) has been widely used to obtain existence results for (EP), the auxiliary problem (MEP_α) can be exploited in the same fashion.

THEOREM 4.2   *(Existence and uniqueness of solutions).*
If $f(\cdot, y)$ is γ-concave on C for any $y \in C$ with $\gamma > -\tau$, then there exists a unique solution of (EP).

Proof. Choosing $\alpha \in (-\tau, \gamma]$, Proposition 2.1 guarantees that $f_\alpha(x, \cdot)$ is strongly convex on C and $f_\alpha(\cdot, y)$ is concave on C. By Theorem 4.1 it is enough to prove that there exists a unique solution of (MEP_α).

Suppose x^* solves (MEP_α). By Theorem 4.1 and Theorem 3.1 x^* solves also (EP_α), that is $f_\alpha(x^*, y) \geq 0$ for any $y \in C$. Since $f_\alpha(x^*, \cdot)$ is strongly convex on C and $f_\alpha(x^*, x^*) = 0$, any $y \in C$ with $y \neq x^*$ satisfies $f_\alpha(x^*, y) > 0$ and thus it does not solve (MEP_α). Therefore, the solution of (MEP_α), if any exists, is unique.

Considering the set-valued map $M(x) := \{y \in C : f_\alpha(x, y) \leq 0\}$, $x^* \in C$ solves (MEP_α) if and only if $x^* \in M(x)$ for any $x \in C$. Therefore, it is enough to prove that the intersection of the sets $M(x)$ over all $x \in C$ is nonempty.

Given any $x \in C$, $M(x)$ is nonempty, closed and bounded since $x \in M(x)$, f is continuous and $f_\alpha(x, \cdot)$ is strongly convex on C. Therefore, the Knaster-Kuratowski-Mazurkiewicz Theorem [41] guarantees the desired nonemptiness if

$$\sum_{i=1}^{k} \beta_i x^i \in \bigcup_{i=1}^{k} M(x^i) \tag{3}$$

holds for any $x^1, \ldots, x^k \in C$ and any $\beta_1, \ldots, \beta_k \geq 0$ such that $\beta_1 + \cdots + \beta_k = 1$. Setting $z = \beta_1 x^1 + \cdots + \beta_k x^k$, the concavity of $f_\alpha(\cdot, z)$ implies

$$0 = f_\alpha(z, z) \geq \sum_{i=1}^{k} \beta_i \, f_\alpha(x^i, z)$$

which entails $f_\alpha(x^j, z) \leq 0$, i.e., $z \in M(x^j)$, for some j. Thus, (3) holds. $\qquad \square$

In the same way gap function for (EP) can be introduced through (EP_α), the auxiliary problem (MEP_α) can be exploited to introduce another family of gap functions. Indeed, the function

$$\psi_\alpha(x) := \sup\{\ f_\alpha(y,x)\ :\ y \in C\ \}$$

is non-negative on C and x^* solves (MEP_α) if and only if $x^* \in C$ and $\psi_\alpha(x^*) = 0$. Therefore, ψ_α is a gap function for (EP) under the assumptions of Theorem 4.1. Moreover, unlike the gap function φ_α of the previous section, it is always convex and this property makes it attractive: in fact, it has been exploited in algorithmic frameworks both for variational inequalities [6, 17, 35, 37] and for more general equilibrium problems [18, 38]. The following result groups its main properties.

THEOREM 4.3　*(Properties of ψ_α).*

　　a) ψ_α is a gap function for (EP) if any of the assumptions of Theorem 4.1 holds.
　　b) $\psi_\alpha(x) < +\infty$ for any $x \in C$ if any of the following conditions holds:
　　　　b1) $f(\cdot, y)$ is γ-concave on C for any $y \in C$ and $\alpha < \gamma$,
　　　　b2) f is μ-monotone on C and $\alpha < 2\mu + \tau$.
　　c) ψ_α is $(\tau + \alpha)$-convex on C for any $\alpha \geq -\tau$.

Proof.
a) ψ_α is a gap function for (EP) whenever (MEP_α) is equivalent to (EP).

b1) Proposition 2.1 guarantees that $f_\alpha(\cdot, x)$ is $(\gamma - \alpha)$-concave for any $x \in C$. Since $\gamma > \alpha$, $f_\alpha(\cdot, x)$ is strongly concave, thus its maximum value over C, that is $\psi_\alpha(x)$, is finite.

b2) The μ-monotonicity of f implies

$$f_\alpha(y,x) \leq -f(x,y) + (\alpha/2 - \mu)\,\|y - x\|^2 = -f_{2\mu - \alpha}(x,y).$$

Proposition 2.1 guarantees that $f_{2\mu - \alpha}(x, \cdot)$ is $(\tau + 2\mu - \alpha)$-convex, hence $-f_{2\mu - \alpha}(x, \cdot)$ is $(\tau + 2\mu - \alpha)$-concave. Since $\tau + 2\mu > \alpha$, it is actually strongly concave and thus its maximum value over C is finite, which implies that $\psi_\alpha(x)$ is finite as well.

c) Since $\alpha\,\|y - x\|^2/2$ is α-convex on C for any $y \in C$, $f_\alpha(y, \cdot)$ is $(\tau + \alpha)$-convex on C by Proposition 2.1. As it is the pointwise supremum of a family of $(\tau + \alpha)$-convex functions, the gap function ψ_α is $(\tau + \alpha)$-convex on C as well (see [19, Proposition 4.1]). $\qquad \square$

Theorem 4.3 a) has been already given in [6, 17, 18, 40] for the same particular cases of Theorem 4.1 that have been previously recalled. Also Theorem 3.3 and Theorem 4.3 are somehow symmetric: considering any $\alpha > -\tau$, φ_α is a finite gap function for (EP) but it is convex only under additional assumptions, while ψ_α is convex but it is a finite gap function for (EP) only under additional assumptions; moreover, γ-concavity provides a common assumption to get convexity in Theorem 3.3 and finiteness in Theorem 4.3.

Also the gap function ψ_α can be exploited to obtain error bounds for (EP). Since it is always convex, some bounds can be achieved also under pseudomonotonicity assumptions.

THEOREM 4.4 *(Error bound).*
If any of the following conditions holds:

   a) f is μ-pseudomonotone on C with $\mu > -\tau/2$ and $\alpha \in (-\tau, 2\mu]$,
   b) f is μ-monotone on C with $\mu > -\tau$ and $\alpha \in (-\tau, 2\mu + \tau]$,
   c) $f(\cdot, y)$ is γ-concave on C for any $y \in C$ with $\gamma > -\tau$ and $\alpha \in (-\tau, \gamma]$.

then ψ_α is a strongly convex gap function for (EP) and

$$\psi_\alpha(x) \geq (\tau + \alpha)\|x - x^*\|^2/2, \qquad \forall\, x \in C,$$

where x^* is the unique solution of (EP).

Proof. In case c) the existence of a unique solution of (EP) is guaranteed by Theorem 4.2. In the cases a) and b) it is guaranteed by Proposition 2.1 in [25] if $\tau = 0$ and by Theorem 3.2 if $\tau > 0$.

By Theorem 4.3 ψ_α is a $(\tau + \alpha)$-convex gap function for (EP), hence it is strongly convex and $\psi_\alpha(x^*) = 0$. Any $x \in C$ satisfies

$$\psi_\alpha(x) \geq f_\alpha(x^*, x) = f(x^*, x) + \alpha\|x - x^*\|^2/2 \geq (\tau + \alpha)\|x - x^*\|^2/2,$$

where the second inequality holds since x^* solves $(EP_{-\tau})$ by Theorem 3.1 and therefore $f(x^*, x) \geq \tau\|x - x^*\|^2/2$. $\qquad\square$

Theorem 4.4 b) with $\tau = 0$ and $\mu > 0$ has been proved for variational inequalities in [40, Lemma 4.2], while all the other cases are, to the best of our knowledge, new. Notice that the use of negative values for α (which is possible if $\tau > 0$) provides error bounds also under weak (pseudo)monotonicity or weak concavity assumptions.

5. Conclusions

The paper investigates in depth the relations of the equilibrium problem (EP) with two different families of auxiliary problems: classical auxiliary problems (EP_α) and Minty auxiliary problems (MEP_α), which both depend upon a regularization parameter α.

Exploiting parametric definitions of strong/weak convexity and monotonicity, results are presented in a unified form that allows subsuming known particular cases together with new results. Indeed, the results require precise relations between α and the moduli of monotonoticity and convexity/concavity. This kind of analysis has been the key tool for improvements: for instance, only $\tau = 0$ holds for variational inequalities and therefore only positive values for α are useful in their particular framework; anyway, other equilibrium problems may satisfy $\tau > 0$, thus allowing α to take negative values that lead to existence results and error bounds under weaker assumptions.

Relying on this approach, auxiliary problem principles are investigated both for positive and negative values of α. The equivalence between (EP) and (EP_α) was already well-known for $\alpha > 0$, yet the analysis for negative values is new, while the equivalence between (EP) and (MEP_α) was already well-known just for $\alpha = 0$ in the pseudomonotone case. These principles lead to new existence results for (EP)

under weak quasimonotonicity or weak concavity assumptions. Furthermore, the auxiliary problems (EP_α) and (MEP_α) with suitable choices of α bring in gap functions φ_α and ψ_α with good convexity properties and new error bounds follow as well. Table 1 provides an overview of the monotonicity or concavity conditions that guarantee the properties of the gap functions and the corresponding error bounds.

Table 1. Conditions that guarantee properties of the gap functions φ_α and ψ_α.

	φ_α	ψ_α
gap function	$\alpha \geq -\tau$	f μ-pseudomonotone, $\alpha \leq 2\mu$ or f μ-monotone, $\alpha \leq 2\mu + \tau$ or $f(\cdot, y)$ γ-concave with $\gamma \geq -\tau$, $\alpha \leq \gamma$
convex	$f(\cdot, y)$ γ-concave, $\alpha \leq \gamma$	$\alpha \geq -\tau$
strongly convex	$f(\cdot, y)$ γ-concave, $\alpha < \gamma$	$\alpha > -\tau$
error bound	f is μ-monotone, $\alpha \in [-\tau, \mu)$ or $f(\cdot, y)$ γ-concave, $\alpha \in [-\tau, \gamma)$	f μ-pseudomonotone, $\alpha \in (-\tau, 2\mu]$ or f μ-monotone, $\alpha \in (-\tau, 2\mu + \tau]$ or $f(\cdot, y)$ γ-concave, $\alpha \in (-\tau, \gamma]$

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