Characterizations of the Lagrange-Karush-Kuhn-Tucker Property

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Abstract

In this note, we revisit the classical first order necessary condition in mathematical programming in infinite dimension. The constraint set being defined by $C = g^{-1}(K)$ where g is a smooth map between Banach spaces, and K a closed convex cone, we show that existence of Lagrange-Karush-Kuhn-Tucker multipliers is equivalent to metric subregularity of the multifunction defining the constraint, and is also equivalent to a generalized Abadie's qualification condition. These results extend widely previous ones like [10, 11, 12, 16] by removing convexity type assumptions on the data.

Key-words: Lagrange and Karush-Kuhn-Tucker multipliers, metric subregularrity, error bounds, Abadie's qualification condition.

1 Introduction and Notations

We consider a $\mathcal{C}^1$ mapping $g : U \to Y$ where U is an open subset of a Banach space X and Y is a Banach space and we denote by g' its Fréchet derivative. Given a closed convex cone $K \subset Y$ we consider

(1) $$C = \{x \in U : g(x) \in K\} = \varphi^{-1}(]-\infty, 0]) = \varphi^{-1}(0),$$

where the function $\varphi : U \to \mathbb{R} \cup \{+\infty\}$ is defined by

(2) $$\varphi(x) = d_K(g(x)),$$

with $d_K(z) = \inf_{u \in K} \|z - u\|$. If $K = \{0\}$, we recover the Lagrange case in which the constraint set is defined by an equality.

In section 2 we give our main result in general Banach spaces. It is shown the equivalence between existence of a local error bound for the constraint set C near $\bar{x} \in C$, and existence of a Lagrange-Karush-Kuhn-Tucker multiplier for a local minimum on C near $\bar{x}$ of any function of the type $f + h$ with f of class $\mathcal{C}^1$ and h convex and Lipschitzian

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near $\bar{x}$. In section 3, we give another characterization of the Lagrange-Karush-Kuhn-Tucker property in Asplund spaces via a generalized Abadie's qualification condition. To this end we give a technical result in order to compute the normal cone to a sublevel set of a function admiting a local error bound for the distance to this sublevel. Some equivalences of this kind where obtained in particular cases. Let us mention [10] relying on convexity type assumptions, [11, Proposition 2.2.1] see also [12] and [16] in the convex case. To our knowledge there is no such general result in the nonconvex case. Finally, in section 4 we consider the convex case in which more general results are obtained.

We shall denote by $B_r(x)$ (resp. $B_r[x]$) the open (resp. closed) ball with center x and radius r. Let us recall that the Clarke's penalization principle ([6, Proposition 2.4.3]) says that if a function $h : U \to \mathbb{R}$ has a minimum on a subset C of a metric space U at $\bar{x} \in C$, and if h is Lipschitzian with rank κ on U, then the function $h + \kappa\, d_C$ attains its minimum on U at $\bar{x}$. Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, its domain $\operatorname{dom} f$ is the set $f^{-1}(\mathbb{R})$ and the function is said to be proper whenever $\operatorname{dom} f \neq \emptyset$. The Ekeland's variational principle will be used under the following form: for any lower semicontinuous and bounded from below function $f : X \to \mathbb{R} \cup \{+\infty\}$ defined on a complete metric space (X, d), and for any $x \in X$, we can find $z \in X$ such that $f(z) < f(y) + d(y, z)$ for all $y \in X \setminus \{z\}$ and $f(z) + d(z, x) \leq f(x)$. The indicator function ι_S of a subset $S \subset X$ will be defined as $\iota_S(x) = 0$ if $x \in S$ and $\iota_S(x) = +\infty$ if $x \in X \setminus S$.

2 Characterizations of the Lagrange-Karush-Kuhn-Tucker Property by Error Bounds

In the sequel ∂h will denote the convex subdifferential of a convex function $h : X \to \mathbb{R} \cup \{+\infty\}$ defined on a normed linear space. We shall use the fact, for which we refer for example to [8, 11, 19], that for every $y \in Y$,

$$\partial d_K(y) = \{\xi \in K^- \cap \bar{B}_* : \langle \xi, y \rangle = d_K(y)\}, \tag{3}$$

where $\bar{B}_*$ is the closed unit ball of Y^* and $K^- = \{\xi \in Y^* : \langle \xi, y \rangle \leq 0 : \text{ for all } y \in K\}$. Observe that the function d_K is regular in the sense of Clarke since it is Lipschitzian and convex (see [6, Proposition 2.3.6]), thus we derive from [6, Theorem 2.3.10] that

$$\partial^{\uparrow} \varphi(x) = g'(x)^* \big(\partial d_K(g(x))\big) \tag{4}$$

where $\partial^{\uparrow}$ is the Clarke's subdifferential.

Let us begin by the definition of the Lagrange-Karush-Kuhn-Tucker property.

Definition 2.1 *We say that the pair (g, K) has the Lagrange-Karush-Kuhn-Tucker property near $\bar{x} \in C$ whenever there exists $\theta > 0$ and $\rho > 0$ such that for any function $\psi = f + h$ with f of class $\mathcal{C}^1$ and h convex continuous which has a local minimum on C at some $x \in C \cap B_\rho(\bar{x})$ and is κ-Lipschitzian near x, then we can find $\xi \in K^-$ such* *that* $\|\xi\|_* \leq \theta\kappa$ *and,*

$$(5) \qquad \begin{cases} 0 \in f'(x) + \partial h(x) + g'(x)^*(\xi) = 0 \\ \\ \langle \xi, g(x) \rangle = 0 \end{cases} .$$

In the Lagrange case, corresponding to $K = \{0\}$, and if $h = 0$, then property (5) reduces to $f'(x) + g'(x)^*(\xi) = 0$, that is ξ is a classical Lagrange multiplier. If $K \neq \{0\}$ we recover that ξ is a Karush-Kuhn-Tucker multiplier. It is pleasant that these two cases may be treated in a unified way. Our main result is the following:

Theorem 2.1 *The two following properties are equivalent.*

(a) The set C has the following error bound property near $\bar{x}$: there exist $\tau > 0$ and $\rho > 0$ such that

$$(6) \qquad \tau\, d_C(x) \leq \varphi(x) \ \text{ for every } x \in B_\rho(\bar{x}),$$

where φ is defined in (2).

(b) The pair (g, K) has the Lagrange-Karush-Kuhn-Tucker property near $\bar{x} \in C$.

Proof. (a) $\Rightarrow$ (b) Let $\psi = f + h$ with f of class $\mathcal{C}^1$ and h convex continuous be a function which attains a local minimum on C at $x \in C \cap B_\rho(x)$ and is κ-Lipschitzian near x. We can find $r > 0$ such that ψ is κ-Lipschitzian on $B_{2r}(x) \subset B_\rho(\bar{x})$ and attains its minimum on $C \cap B_{2r}(x)$ at x. Applying the Clarke's penalization principle, we obtain that for every $z \in B_r(x)$,

$$\psi(z) + \tau^{-1}\kappa\, d_K(g(z)) \geq \psi(z) + \kappa\, d_C(z) = \psi(z) + \kappa\, d_{C \cap B_{2r}(x)}(z) \geq \psi(x).$$

It follows that the function $\psi + \gamma\,\varphi = \psi + \gamma\,(d_K \circ g)$ with $\gamma = \tau^{-1}\kappa$ has a local minimum at x, thus, using for example the Clarke's subdifferential, we get $0 \in \partial^\uparrow \psi(x) + \gamma\, \partial^\uparrow \varphi(x)$ leading to (5) via (4) for some $\xi \in K^-$ satisfying $\|\xi\|_* \leq \theta\kappa$ with $\theta = \tau^{-1}$.

(b) $\Rightarrow$ (a) Let $\sigma \in (0, \theta^{-1})$ and let $\varepsilon \in (0, \theta^{-1} - \sigma)$. We can find $\rho > 0$ such that

$$\|g(x) - g(z) - g'(z)(x - z)\| \leq \varepsilon \|x - z\|,$$

for every x, $z \in B_{2\rho}(\bar{x})$. Given $x \in (X \setminus C) \cap B_\rho(\bar{x})$ and a sequence $(\delta_n)_{n \in \mathbb{N}}$ of positive numbers converging to 0, let $x_n \in C$ be such that $\chi(x_n) \leq d_C(x) + \delta_n$ where

$$\chi(z) = \iota_C(z) + \|z - x\|,$$

so that $\lim_{n \to \infty} \|x - x_n\| = d_C(x)$. From Ekeland's principle applied with the metric $d(x, y) = \sqrt{\delta_n}\|x - y\|$, we can find $z_n \in C$ such that

$$(7) \qquad \chi(z_n) + \sqrt{\delta_n}\|x_n - z_n\| \leq \chi(x_n),$$

and that z_n minimizes the function $\chi(\cdot) + \sqrt{\delta_n}\|z_n - \cdot\|$ on X. Observe that (7) yields $\lim_{n\to\infty} \|x_n - z_n\| = 0$. Setting

$$h(z) = \|z - x\| + \sqrt{\delta_n}\|z - z_n\|,$$

this function is convex Lipschitzian with rank $\kappa := 1 + \sqrt{\delta_n}$ and has a global minimum on C at z_n. Thus we can find, from (b) vectors $\zeta_n \in \partial\|\cdot\|(z_n - x)$, $\eta_n \in \sqrt{\delta_n}\,\partial\|\cdot\|(0)$, $\xi_n \in K^-$ such that $\|\xi_n\|_* \le \theta(1 + \sqrt{\delta_n})$ and

$$0 = \zeta_n + \eta_n + g'(z_n)^*(\xi_n).$$

Thus we get $\|\zeta_n\|_* = 1$, $\|\eta_n\|_* \le \sqrt{\delta_n}$, $\langle \zeta_n, z_n - x\rangle = \|x - z_n\|$, and

$$\begin{cases} -\zeta_n - \eta_n = g'(z_n)^*(\xi_n) \\ \\ \langle \xi_n, g(z_n)\rangle = 0 \end{cases},$$

and then $\tau_n\xi_n \in \partial^\uparrow d_K(g(z_n))$ with $\tau_n = \big(\theta(1 + \sqrt{\delta_n})\big)^{-1}$. We have

$$\lim_{n\to\infty} \|x - z_n\| = \lim_{n\to\infty} \|x - x_n\| = d_C(x) \le \|x - \bar{x}\| < \rho,$$

and then $z_n \in B_{2\rho}(\bar{x})$ eventually. Observing that $d_K(g(z_n)) = 0$ we get,

$$\begin{aligned}
\tau_n^{-1} d_K(g(x)) &\ge \langle \xi_n, g(x) - g(z_n)\rangle \\
&\ge \langle g'(z_n)^*(\xi_n), x - z_n\rangle - \varepsilon\,\tau_n^{-1}\|x - z_n\| \\
&\ge \langle -\zeta_n, x - z_n\rangle - (\varepsilon\,\tau_n^{-1} + \sqrt{\delta_n})\|x - z_n\| \\
&\ge (1 - \varepsilon\,\tau_n^{-1} - \sqrt{\delta_n})\|x - z_n\|.
\end{aligned}$$

Multipying on both sides by τ_n and letting n go to $+\infty$, we get the desired conclusion:

$$\varphi(x) = d_K(g(x)) \ge (\theta^{-1} - \varepsilon)\, d_C(x) \ge \sigma\, d_C(x) \ \text{ for every } \ x \in B_\rho(\bar{x}).$$

■

It follows from Theorem 2.1 that the error bound property (6) is the ultimate qualification condition in order to derive existence of multipliers.

Remark 2.1

(a) *Theorem 2.1 extends widely [13, Theorem 2] in which the same conclusion was obtained in the particular case of finite dimensions with* $K = \{0\}^p \times \mathbb{R}^q_-$.

(b) Observe that the error bound property (6) is nothing else than metric subregularity at $(\bar{x}, 0)$ *of the multifunction* $G \subset X \times Y$ *defined by* $G(x) := g(x) - K$ *.*

(c) One could replace the closed convex cone K *by a closed convex set* $Q \subset Y$ *with (5) replaced by* $\|\xi\|_* \leq \theta\kappa$ *and,*

$$\begin{cases} 0 \in f'(x) + \partial h(x) + g'(x)^*(\xi) = 0 \\ \\ \xi \in N_Q(g(x)) \end{cases} .$$

(d) In the implication (a) $\Rightarrow$ *(b), the mapping* g *could be only Lipschitzian. It is required to be* $\mathcal{C}^1$ *only in the reverse implication. In fact, in this reverse implication, it is possible to suppose only that the function* φ *has a uniform subdifferentiability property like in [17, Definition 1].*

Remark 2.2 *Let us consider the qualification condition near* $\bar{x} \in C$*:*

$$\liminf_{\begin{cases} x \to \bar{x} \\ g(x) \notin K \end{cases}} d_*\big(0, g'(x)^*(\partial d_K(g(x)))\big) > 0. \tag{8}$$

Observe that (8) is in force whenever the following stronger condition holds:

$$d_*(0, g'(\bar{x})^*(K^- \cap \bar{S}_*)) > 0, \tag{9}$$

where $\bar{S}_*$ *is the unit sphere of* Y^**, due to the continuity of* g' *and to the fact that* $\partial d_K(g(x)) \subset K^- \cap \bar{S}_*$ *whenever* $g(x) \notin K$*. Then, observing that* $g'(x)^*(\partial d_K(g(x)) = \partial\varphi(x)$ *where* ∂ *is, for example the Clarke's subdifferential, we deduce from [4, Proposition 4.1 and Theorem 5.1] or [18, Theorem 2.2] that condition (b) of Theorem 2.1 is satisfied. Thus if a* $\mathcal{C}^1$ *function* f *admits a local minimum on* C *at* $\bar{x}$*, we can find a Lagrange-Karush-Kuhn-Tucker multiplier, that is an element* $\xi \in K^-$ *such that*

$$\begin{cases} f'(\bar{x}) + g'(\bar{x})^*(\xi) = 0 \\ \\ \langle \xi, g(\bar{x}) \rangle = 0 \end{cases} .$$

In the Lagrange case, that is $K = \{0\}$*, condition (8) holds true if*

$$d_*(0, g'(\bar{x})^*(\bar{S}_*)) > 0,$$

which is the case whenever $g'(\bar{x})$ *is surjective. If* Y *is finite dimensional, then compactness of the unit sphere leads to the fact that (8) is satisfied whenever*

$$ker(g'(\bar{x})^*) \cap K^- \cap (g(\bar{x}))^\perp = \{0\}. \tag{10}$$

For example if $Y = \mathbb{R}^p \times \mathbb{R}^m$ *with* $K = \{0_p\} \times \mathbb{R}^m_-$, *condition (10) becomes*

$$\begin{cases} \mu \geq 0 \text{ and } \sum_{i=1}^{p} \lambda_i g_i'(\bar{x}) + \sum_{j=p+1}^{p+m} \mu_j g_j'(\bar{x}) = 0 \\ \\ \sum_{j=p+1}^{p+m} \mu_j g_j(\bar{x}) = 0 \end{cases} \quad \text{implies } (\lambda, \mu) = (0_p, 0_m),$$

that is the classical Mangasarian-Fromowitz condition.

Remark 2.3 *It is also interesting to consider the case where the constraint set is defined by* $C \cap S$ *where* C *is defined in (1) and* S *is a closed subset of* X. *Assume now that the following local error bound holds near* $\bar{x} \in C \cap S$: *there exist* $\tau > 0$ *and* $\rho > 0$ *such that*

$$\tau\, d_C(x) \leq \varphi(x) \;\; \text{for every } x \in S \cap B_\rho(\bar{x}).$$

If a $\mathcal{C}^1$ *function* f *attains a local minimum on* $S \cap C$ *at* $\bar{x}$ *and if the subdifferential* ∂ *satisfies the exact Fermat rule, then the Clarke's penalization principle applied in* S *instead of* X *leads to the existence of* $\xi \in K^-$ *such that*

$$\begin{cases} 0 \in N_S(\bar{x}) + f'(\bar{x}) + \; g'(\bar{x})^*(\xi) = 0 \\ \\ \langle \xi, g(\bar{x}) \rangle = 0 \end{cases}.$$

3 Characterizations of the Lagrange-Karush-Kuhn-Tucker Property via Abadie's Condition

In the sequel, we shall make use of a *subdifferential operator*, that is, an operator associating to any lower semicontinuous $f : X \to \mathbb{R} \cup \{+\infty\}$, and to any $x \in \operatorname{dom} f$, a subset $\partial f(x)$ of X^* which coincides with the the convex subdifferential when f is convex, and which satisfies the following fuzzy Fermat rule: if $h : X \to \mathbb{R}$ is convex and Lipschitz continuous, and if $\bar{x} \in \operatorname{dom} f$ is a local minimum point of $f + h$, then for every $\delta > 0$ there exist x, $y \in X$, $\xi \in \partial f(x)$, and $\zeta \in \partial h(y)$ such that,

$$\|x - \bar{x}\| \leq \delta\,, \quad \|y - \bar{x}\| \leq \delta\,, \quad f(x) \leq f(\bar{x}) + \delta\,, \quad \text{and} \quad \|\xi + \zeta\|_* \leq \delta.$$

The exact Fermat rule corresponds to the cas $\delta = 0$ that is $0 \in \partial f(\bar{x}) + \partial h(\bar{x})$. For example the Fréchet subdifferential in Asplund spaces satisfies the fuzzy Fermat rule; and the Clarke-Rockafellar subdifferential in general Banach spaces along with the limiting Fréchet (or Mordukhovich) subdifferential in Asplund spaces satisfies the exact Fermat rule. Given such a subdifferential, the normal cone to a closed subset $S \subset X$ at the point $\bar{x} \in S$ is defined by $N_S(\bar{x}) = \partial \iota_S(\bar{x})$.

3.1 Error Bounds and Normal Cones

Let us recall now some different notions of subdifferential and normal cones that will be needed in the sequel. We refer to [15] for the details. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function defined on a Banach space X.

- Given $\varepsilon > 0$ and $\bar{x} \in \operatorname{dom} f$, the ε-Fréchet subdifferential $\partial_\varepsilon^F f(\bar{x})$ of the function f at $\bar{x}$ will be the set of those $\xi \in X^*$ for which there exists $r > 0$ such that

$$f(x) - f(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - \varepsilon \|x - \bar{x}\| \text{ for all } x \in B_r(\bar{x}).$$

- We shall also use the Fréchet subdifferential $\partial^F f(\bar{x})$ defined by

$$\partial^F f(x) = \bigcap_{\varepsilon > 0} \partial_\varepsilon^F f(\bar{x}).$$

  In other words, one has $\xi \in \partial^F f(x)$ if and only if, for all $\varepsilon > 0$, there exists $r > 0$ such that $f(x) - f(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - \varepsilon \|x - \bar{x}\|$ for all $x \in B_r(\bar{x})$.

- The limiting Fréchet (or Mordukhovich) subdifferential of f at $x \in \operatorname{dom} f$ is the set $\partial f(x)$ of $\xi \in X^*$ such that there exist sequences $((x_n, f(x_n), \varepsilon_n))_{n \in \mathbb{N}} \subset X \times \mathbb{R} \times (0, +\infty)$ converging to $(x, f(x), 0)$ and $(\xi_n)_{n \in \mathbb{N}} \subset X^*$ converging $*$-weakly to ξ such that $\xi_n \in \partial_{\varepsilon_n}^F f(x_n)$ eventually.

- The ε-Fréchet normal cone ε-$N_C^F(x)$ to a closed set $C \subset X$ at $\bar{x} \in C$ is the set $\partial_\varepsilon^F(i_C)(\bar{x})$, that is the set of $\xi \in X^*$ such that there exists $r > 0$ such that $\langle \xi, x - \bar{x} \rangle \leq \varepsilon \|x - \bar{x}\|$ for all $x \in B_r(\bar{x})$.

- As usual, the Fréchet normal cone $N_C^F(\bar{x})$ of C to $\bar{x}$ is the set

$$N_C^F(\bar{x}) = \bigcap_{\varepsilon > 0} \varepsilon\text{-}N_C^F(\bar{x}) = \partial^F i_C(\bar{x}).$$

- The Mordukhovich normal cone $N_C(x)$ to a closed set $C \subset X$ at $\bar{x} \in C$ is defined as $\partial \iota_C(x)$ where ∂ is the Mordukhovich subdifferential. Equivalently, it is the set of $\xi \in X^*$ for which there exist sequence $(x_n)_{n \in \mathbb{N}} \subset C$ converging to $\bar{x}$, $(\varepsilon_n)_{n \in \mathbb{N}} \subset (0, +\infty)$ converging to 0, and a sequence $(\xi_n)_{n \in \mathbb{N}}$ in X^* converging $*$-weakly to ξ such that $\xi_n \in \varepsilon_n$-$N_C^F(x_n)$ eventually.

In the following lemma we point out the fact that the existence of a local error bound relative to a sublevel set allows to compute the normal cone to this sublevel set.

Lemma 3.1 *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a closed proper function defined on a Banach space X. Let $\bar{x} \in [f \leq 0] := f^{-1}(]-\infty, 0])$ be such that there exists $\tau > 0$ and $r \in (0, +\infty]$ such that*

$$\tau\, d(x, [f \leq 0]) \leq f^+(x) \quad \textit{for all} \quad x \in B_r(\bar{x}).$$

Let $x \in B_r(\bar{x}) \cap [f \leq 0]$, then, for all $\varepsilon > 0$,

(a)

$$\varepsilon\text{-}N_{[f \leq 0]}^F(x) \cap \bar{B}_* \subset (1 + \varepsilon)\tau^{-1} \partial_\varepsilon^F f^+(x).$$

(b)

$$N^F_{[f\leq 0]}(x)\cap \tau B_* \subset \partial^F f^+(x) \text{ and } N^F_{[f\leq 0]}(x) = \mathbb{R}_+\partial^F f^+(x)$$

where B_* is the open unit ball in X^*.

(c)

$$N_{[f\leq 0]}(x) \subset \mathbb{R}_+\partial f^+(x).$$

Proof. (a) Let $\zeta \in \varepsilon\text{-}N^F_{[f\leq 0]}(x)\cap \bar{B}_*$, so that the function $z \mapsto -\langle\zeta, z\rangle + \varepsilon\|z-x\|$ admits a local minimum on $S = [f\leq 0]$ at x. Thus, by the Clarke's penalization principle (Proposition 2.4.3 in [6]), the function $z \mapsto -\langle\zeta, z\rangle + \varepsilon\|z-x\| + (1+\varepsilon)d(z,S)$ has a local minimum at x. From the local error bound estimate, it follows that the function $z \mapsto -\langle\zeta, z\rangle + \varepsilon\|z-x\| + (1+\varepsilon)\tau^{-1}f^+(z)$ also has a local minimum at x, leading to $\zeta \in (1+\varepsilon)\tau^{-1}\partial^F_\varepsilon f^+(x)$.

(b) Let $\zeta \in N^F_{[f\leq 0]}(x)\cap B_*$, so that for all $\varepsilon > 0$ small enough, we have $(1+\varepsilon)\|\zeta\|_* < 1$. As $(1+\varepsilon)\zeta \in N^F_{[f\leq 0]}(x)\cap B_* \subset \varepsilon\text{-}N^F_{[f\leq 0]}(x)\cap B_*$, we derive from (a) that $\tau\zeta \in \partial^F_\varepsilon f^+(x)$, hence the first part of (b). For the second part, let us simply observe that $\partial^F_\varepsilon f^+(x) \subset \varepsilon\text{-}N^F_{[f\leq 0]}(x)$, thus

$$\mathbb{R}_+\partial^F f^+(x) \subset \mathbb{R}_+\partial^F_\varepsilon f^+(x) \subset \varepsilon\text{-}N^F_{[f\leq 0]}(x) \text{ for all } \varepsilon > 0,$$

and then $\mathbb{R}_+\partial^F f^+(x) \subset N^F_{[f\leq 0]}(x)$.

(c) Let $\zeta \in N_{[f\leq 0]}(x)$, so that there exist sequences $(x_n)_{n\in\mathbb{N}}$ converging to x in $[f\leq 0]$, $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0,+\infty)$ converging to 0 and $(\zeta_n)_{n\in\mathbb{N}} \subset X^*$ converging $*$-weakly to ζ and such that $\zeta_n \in \varepsilon_n\text{-}N^F_{[f\leq 0]}(x_n)$. Let $c > 1$ be such that $\|\zeta_n\|_* < c$ and let $\eta_n = c^{-1}\zeta_n$ so that $\|\eta_n\|_* < 1$ and $\eta_n \in \varepsilon_n\text{-}N^F_{[f\leq 0]}(x_n)$ due to $c > 1$. Thus we get from (a) that $\tau(1+\varepsilon_n)^{-1}\eta_n \in \partial^F_{\varepsilon_n} f^+(x_n)$ hence $\tau c^{-1}\zeta \in \partial f^+(x)$, leading to $N_{[f\leq 0]}(x) \subset \mathbb{R}_+\partial f^+(x)$. ■

3.2 Another Characterization of the Lagrange-Karush-Kuhn-Tucker Property

Let us consider the following definition in which the normal cone $N_C(x) = \partial i_C(x)$ where ∂ is a subdifferential operator defined at the begining of this section:

Definition 3.1 *We say that the pair (g,K) satifies the generalized Abadie's qualification condition near $\bar{x}\in C$ if we can find $\tau > 0$ and $\rho > 0$ such that*

$$N_C(x)\cap \tau B_* \subset g'(x)^*\big(\partial d_K(g(x))\big) \text{ for all } x \in C\cap B_\rho(\bar{x}). \tag{11}$$

Remark 3.1 In the case where $g = (g_1,\cdots,g_m)$ with $g_1,\cdots,g_m$ convex differentiable and $K = \mathbb{R}^m_-$, we have $\varphi(x) = d_K(g(x)) = \sum_{k=1}^m g_k^+(x)$ whenever $\mathbb{R}^m$ is equiped with the ℓ_1 norm. It follows that, for any $x\in C$, we have

$$g'(x)^*\big(\partial d_K(g(x))\big) = \sum_{k\in K(x)} [0,1]g'_k(x),$$

where $K(x) = \{k \in [1,m] : g_k(x) = 0\}$. Thus condition (3.1) implies the dual form of the usual Abadie's qualification condition that is $N_C(x) = \sum_{k\in K(x)} \mathbb{R}_+ g'_k(x)$ near $\bar{x}$. Conversely, it is shown in the proof of [16, Theorem 3] that this condition implies (11). The dual form of Abadie's condition dates back to [10] and is sometimes taken as a definition of Abadie's condition, for example in [16].

Theorem 3.1 *Let us consider the following properties in which subdifferentials are assumed to satisfy the fuzzy Fermat rule:*

(a) The pair (g, K) satisfies the generalized Abadie's qualification condition near $\bar{x} \in C$.

(b) The set C has the following error bound property near $\bar{x}$: there exist $\tau > 0$ and $\rho > 0$ such that

$$\tau\, d_C(x) \leq \varphi(x) \ \text{ for every } \ x \in B_\rho(\bar{x}),$$

where φ is defined in (2).

(c) The set C has the Lagrange-Karush-Kuhn-Tucker property near $\bar{x} \in C$.

Then (a) ⇒ (b) and (c) ⇒ (a) if ∂ is the Fréchet subdifferential. In that case the three properties are equivalent whenever X is Asplund.

Proof. We know by Theorem 2.1 that (b) is equivalent to (c).

(a) ⇒ (b) Let $\varepsilon \in (0, \tau)$ and let $\rho > 0$ be such that for every x, $z \in B_{2\rho}(\bar{x})$, we have

$$\|g(x) - g(z) - g'(z)(x - z)\| \leq \varepsilon \|x - z\|,$$

and

$$N_C(z) \cap \tau B_* \subset g'(z)^* \big(\partial d_K(g(z))\big) \ \text{ for all } \ z \in C \cap B_{2\rho}(\bar{x}). \tag{12}$$

Let $x \in (X \setminus C) \cap B_\rho(\bar{x})$ and let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to 0. For each $n \in \mathbb{N}$, we can find $y_n \in C$ such that $\chi(y_n) \leq d_C(x) + \delta_n$ where $\chi(z) = \iota_C(z) + \|z - x\|$. From Ekeland's principle, there exists $z_n \in C$ such that

$$\chi(z_n) + \sqrt{\delta_n}\|y_n - z_n\| \leq \chi(y_n) \tag{13}$$

and that z_n minimizes the function $\chi(\cdot) + \sqrt{\delta_n}\|z_n - \cdot\|$ on X. Observe that (13) implies $\lim_{n\to\infty} \|x_n - z_n\| = 0$. From the fuzzy Fermat rule, for every $n \in \mathbb{N}$, we can find $x_n \in X$, ζ_n, $\eta_n \in X^*$ such that

$$\|x_n - z_n\| \leq \delta_n, \ \|\zeta_n\|_* \leq 1, \ \|\eta_n\|_* \leq \sqrt{\delta_n}, \ \langle \zeta_n, x_n - x\rangle = \|x - x_n\|,$$

and $c_n \in C$ with $\|c_n - z_n\| \leq \delta_n$ and $\omega_n \in N_C(c_n)$ such that

$$\|\zeta_n + \eta_n + \omega_n\|_* \leq \delta_n.$$

As $\lim_{n\to\infty}\|x-c_n\| = d_C(x)$, we have $c_n \in B_{2\rho}(\bar{x})$ eventually. One has

$$\|\omega_n\|_* \le (1+\sqrt{\delta_n}+\delta_n),$$

thus we get from (12) that there exists $\xi_n \in \partial d_K(g(c_n))$ such that

$$\tau(1+\sqrt{\delta_n}+\delta_n)^{-1}\omega_n = g'(c_n)^*(\xi_n).$$

Observing that $d_K(g(c_n)) = 0$ and setting $k_n = (1+\sqrt{\delta_n}+\delta_n)^{-1}$ we get,

$$\begin{aligned} d_K(g(x)) &\ge \langle \xi_n, g(x)-g(c_n)\rangle \\ &\ge \langle g'(c_n)^*(\xi_n), x-c_n\rangle - \varepsilon\|x-c_n\| \\ &\ge \tau k_n\langle \omega_n, x-x_n\rangle - 2\tau k_n\delta_n - \varepsilon\|x-c_n\| \\ &\ge \tau k_n\langle -\zeta_n, x-x_n\rangle - \tau k_n(\sqrt{\delta_n}+\delta_n)\|x-x_n\| - 2\tau k_n\delta_n - \varepsilon\|x-c_n\|. \end{aligned}$$

As $\langle -\zeta_n, x-x_n\rangle = \|x-x_n\|$ and $\lim_{n\to\infty}\|x-x_n\| = d_C(x)$, we obtain by letting n go to $+\infty$,

$$\varphi(x) = d_K(g(x)) \ge (\tau-\varepsilon)\, d_C(x) \ \text{ for every } \ x \in B_\rho(\bar{x}).$$

(c) $\Rightarrow$ (a) Assuming that ∂ is the Fréchet subdifferential, let $x \in C \cap B_\rho(\bar{x})$ and let $\zeta \in N_C^F(x) \cap \tau\bar{B}_*$ with $0 < \tau < \theta^{-1}$. Given $0 < \varepsilon < \theta^{-1} - \tau$, the function $f(z) = -\langle \zeta, z\rangle + \varepsilon\|z-x\|$ attains a local minimum on C at x. Thus we can find $\xi_\varepsilon \in K^-$ such that $\zeta \in \varepsilon\bar{B}_* + g'(x)^*(\xi_\varepsilon)$ along with $\|\xi_\varepsilon\|_* \le (\tau+\varepsilon)\theta < 1$ and $\langle \xi_\varepsilon, g(x)\rangle = 0$, so that $\xi_\varepsilon \in \partial d_K(g(x))$, yielding

$$N_C^F(x) \cap \tau\bar{B}_* \subset \mathrm{cl}\big(g'(x)^*\big(\partial d_K(g(x))\big)\big) = g'(x)^*\big(\partial d_K(g(x))\big),$$

(we use the fact that $g'(x)^*\big(\partial d_K(g(x))\big)$ is closed since $(\partial d_K(g(x))$ is $*$-weakly compact and $g'(x)^*$ is $*$-weakly continuous).

Now if $\partial = \partial^F$ and if X is Asplund, then ∂ satisfies the fuzzy Fermat rule (see [9]) and then (a) $\Rightarrow$ (b). On the other hand we have (b) $\Rightarrow$ (c) from Theorem 2.1 and, from what precedes, we obtained (c) $\Rightarrow$ (a), thus the three properties are equivalent in that setting. ■

4 The Convex Case

In the convex case, it is possible to extend some results of section 3 to the case of general constraint sets $C = [\varphi \le 0]$ where $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is any closed convex proper function defined on a Banach space. In this sequel, we shall make use of the convention $t\emptyset = \{0\}$ for any $t \in \mathbb{R}$. The following elementary lemma will be useful.

Lemma 4.1 *Let $\varphi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a convex function defined on a normed space X and let $\bar{x} \in [\varphi \le 0]$. Then,*

$$\partial\varphi^+(\bar{x}) = \begin{cases} N_{dom\,\varphi}(\bar{x}) + [0,1]\partial\varphi(\bar{x}) & \text{if } \varphi(\bar{x}) = 0 \\ \\ N_{dom\,\varphi}(\bar{x}) & \text{if } \varphi(\bar{x}) < 0 \end{cases},$$

with the convention $[0,1]\emptyset = \{0\}$.

Proof. Assume that $\varphi(\bar{x}) = 0$. The inclusion $\partial\varphi^+(\bar{x}) \supset N_{\mathrm{dom}\,\varphi}(\bar{x}) + [0,1]\partial\varphi(\bar{x})$ is obvious. On the other hand, given $\zeta \in \partial\varphi^+(\bar{x})$, we have $(\zeta, -1) \in N_{\mathrm{epi}\,\varphi^+}(\bar{x}, 0)$. Observe that $\mathrm{epi}\,\varphi^+ = \mathrm{epi}\,\varphi \cap (X \times \mathbb{R}_+)$ and that the interior of $X \times \mathbb{R}_+$ meets $\mathrm{epi}\,\varphi$, so that $N_{\mathrm{epi}\,\varphi^+}(\bar{x}, 0) = N_{\mathrm{epi}\,\varphi}(\bar{x}, 0) + (\{0\} \times \mathbb{R}_-)$ and then we get $(\zeta, -1) = (\zeta, -s - t)$ for some $s,\ t \ge 0$ with $(\zeta, -s) \in N_{\mathrm{epi}\,\varphi}(\bar{x}, 0)$ and $s + t = 1$. If $s = 0$, then $\zeta \in N_{\mathrm{dom}\,\varphi}(\bar{x})$ and if $s > 0$ then $s \le 1$ and $\zeta \in s\partial\varphi(\bar{x})$ thus $\zeta \in N_{\mathrm{dom}\,\varphi}(\bar{x}) + [0,1]\partial\varphi(\bar{x})$. Assume now that $\varphi(\bar{x}) < 0$, thus $(\zeta, -s) \in N_{\mathrm{epi}\,\varphi}(\bar{x}, 0)$ implies $s = 0$, so that $\zeta \in N_{\mathrm{dom}\,\varphi}(\bar{x})$, then $\partial\varphi^+(\bar{x}) \subset N_{\mathrm{dom}\,\varphi}(\bar{x})$, the opposite inclusion being evident. ■

Lemma 4.2 *Let $\varphi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a closed convex proper function defined on a Banach space X. Let $\bar{x} \in [\varphi \le 0]$ be such that there exists $\tau > 0$ and $r \in (0, +\infty]$ such that $\tau\, d(x, [\varphi \le 0]) \le \varphi^+(x)$ for all $x \in B_r(\bar{x})$. Then, for all $x \in B_r(\bar{x}) \cap [\varphi \le 0]$, we have*

$$N_{[\varphi \le 0]}(x) \cap \tau\bar{B}_* \subset \partial\varphi^+(x), \tag{14}$$

and

$$N_{[\varphi \le 0]}(x) = \begin{cases} N_{dom\,\varphi}(x) + \mathbb{R}_+\partial\varphi(x) & \text{if } \varphi(x) = 0 \\ \\ N_{dom\,\varphi}(x) & \text{if } \varphi(x) < 0 \end{cases}. \tag{15}$$

Proof. Given $\xi \in N_{[\varphi \le 0]}(x) \cap \tau\bar{B}_*$, let $\zeta = \tau^{-1}\xi$, so that $\zeta \in N_{[\varphi \le 0]}(x) \cap \bar{B}_*$, the function $z \mapsto -\langle \zeta, z \rangle$ admits a global minimum on $S = [\varphi \le 0]$ at x. Thus the function $z \mapsto -\langle \zeta, z \rangle + d(z, S)$ attains its minimum over X in x. From the error bound estimate, it follows that the function $z \mapsto -\langle \zeta, z \rangle + \tau^{-1}\varphi^+(z)$ has a local minimum at x, leading to $\zeta \in \tau^{-1}\partial\varphi^+(x)$, and then to (14) and to an inclusion in (15), the converse inclusion being evident. ■

Here we give a new characterization of the existence of an error bound for lower semicontinuous convex functions defined on Banach spaces. Observe that this characterization concerns both local and global error bounds and extends widely the main result of [16].

Theorem 4.1 *Let $\varphi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a closed convex proper function defined on a Banach space X, let $\bar{x} \in [\varphi \le 0]$, and let the two following properties:*

(a) for some $\tau > 0$, $\rho \in (0, +\infty]$ we have

$$\tau\, d(x, [\varphi \le 0]) \le \varphi^+(x) \text{ for all } x \in B_\rho(\bar{x}). \tag{16}$$

(b) For some $\tau > 0$, $r \in (0, +\infty]$ we have

$$N_{[\varphi \leq 0]}(x) \cap \tau \bar{B}_* \subset \partial \varphi^+(x) = \begin{cases} N_{dom\,\varphi}(x) + [0,1]\partial\varphi(x) & \text{if } \varphi(x) = 0 \\ \\ N_{dom\,\varphi}(x) & \text{if } \varphi(x) < 0 \end{cases} \tag{17}$$

for all $x \in [\varphi {\leq} 0] \cap B_r(\bar{x})$.

Then (a) implies (b) with $r = \rho$ and (b) implies (a) with $\rho = r/2$.

Proof. The fact that (a) implies (b) is lemma 4.2. Conversely, assume that (b) is in force and let $x \in B_{r/2}(\bar{x})$ be such that $\varphi(x) > 0$. Using Lemma 1 of [16] we can find a minimizing sequence $(z_n)_{n\in\mathbb{N}} \subset C$ for $d(x, C)$ and a sequence $(\zeta_n)_{n\in\mathbb{N}} \subset X^*$ such that $\zeta_n \in N_C(z_n)$, along with $d_C(x) = \lim_{n\to+\infty}\langle \zeta_n, x - z_n\rangle$ and $\lim_{n\to+\infty} \|\zeta_n\|_* = 1$. For any $\varepsilon > 0$ we have $z_n \in B_r(\bar{x})$ and $\zeta_n \in (1+\varepsilon)\bar{B}_*$ eventually, yielding

$$(1+\varepsilon)^{-1}\zeta \in \tau^{-1}\partial\varphi^+(z_n).$$

We derive that,

$$\langle \tau(1+\varepsilon)^{-1}\zeta_n, x - z_n\rangle \leq \varphi^+(x) - \varphi^+(z_n) = \varphi(x),$$

and then, passing to the limit, we get

$$\tau(1+\varepsilon)^{-1}\, d(x, S) \leq \varphi(x) \text{ for all } x \in B_{r/2}(\bar{x}) \cap [\varphi{>}0],$$

which yields (a) with $\rho = r/2$ by letting ε go to 0. ■

Remark 4.1 *One may observe that (16) implies the stronger property:*

$$\tau\, d(x, [\varphi{\leq}\lambda]) \leq (\varphi(x) - \lambda)^+ \text{ for all } x \in B_\rho(\bar{x}).$$

for every $\lambda > 0$. This follows staightforwardly from the proof of [3, Proposition 2.5] taking into account that $B_\rho(\bar{x})$ is convex.

References

[1] J. Abadie, *On the Kuhn-Tucker theorem*, Nonlinear Programming, J. Abadie ed., pp. 19–36, North-Holland, Amsterdam, (1967).

[2] D. Azé, *A survey on error bounds for lower semicontinuous functions*, ESAIM Proc. **13** (2003), 1–17.

[3] D. Azé, J.-N. Corvellec, *On the Sensitivity Analysis of Hoffman Constants for Systems of Linear Inequalities*, SIAM J. Optimization, **12**, 913–927 (2002).

[4] D. Azé, J.-N. Corvellec, *Characterizations of error bounds for lower semicontinuous functions on metric spaces*, ESAIM Control Optim. Calc. Var. **10**, 409–425 (2004).

[5] J. F. Bonnans and A. Shapiro, Perturbation Analysis of Optimization Problems, *Springer-Verlag, New York, (2000).*

[6] F.H. Clarke, Optimization and nonsmooth analysis. *John Wiley & Sons, Inc., New York, (1983).*

[7] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. **47**, 324–353 (1974).

[8] I. Ekeland, R. Temam, Convex Analysis and Variational Problems, Amsterdam-Oxford, *North-Holland Publ. Company, (1976).*

[9] M. Fabian, *Subdifferentiability and trustworthiness in the light of the new variational principle of Borwein and Preiss*, Acta Univ. Carolin. **30**, 51–56 (1989).

[10] M. Guignard, *Generalized KuhnTucker Conditions for Mathematical Programming Problems in a Banach Space*, SIAM J. Control **7**, 232–241 (1969).

[11] J.-B. Hiriart-Urruty, C. Lemaréchal, Convex analysis and minimization algorithms, *Fundamental Principles of Mathematical Sciences,* **305** *Springer-Verlag, Berlin, (1993).*

[12] W. Li, *Abadie's constraint qualification, metric regularity, and error bounds for differentiable convex inequalities*, SIAM J. Optim. **7**, no. 4, 966–978 (1997).

[13] L. Minchenko, S. Stakhovski, *On relaxed constant rank regularity condition in mathematical programming*, Optimization, **60**, no 4, 429–440 (2011).

[14] O. L. Mangasarian, S. Fromovitz, *The Fritz John necessary optimality conditions in the presence of equality and inequality constraints*, J. Math. Anal. Appl. **17**, 37–47 (1967).

[15] B. S. Mordukhovich, Variational Analysis and Generalized Differentiation I: Basic Theory, *Grundlehren der Mathematischen Wissenschaften, Vol.* **330**, *Springer, Berlin, (2005).*

[16] H.V. Ngai, M. Théra, *Error bounds for convex differentiable inequality systems in Banach spaces*, Math. Program. **104**, no. 2-3, Ser. B, 465–482 (2005).

[17] J.-P. Penot, *Delineating nice classes of nonsmooth functions*, Pac. J. Optim. **4**, no. 3, 605–619 (2008).

[18] Z. Wu, J. Ye, *First-order and second-order conditions for error bounds*, SIAM J. Optim. **14**, no. 3, 621–645 (2003).

[19] C. Zălinescu, Convex Analysis in General Vector Spaces, *World Scientific Publ. Co., River Edge, NJ, (2002).*