On the polyhedrality of cross and quadrilateral closures

Sanjeeb Dash
IBM Research
sanjeebd@us.ibm.com

Oktay Günlük
IBM Research
gunluk@us.ibm.com

Diego A. Morán R.
Department of Industrial & Systems Engineering,
Virginia Polytechnic Institute & State University,
Blacksburg, VA 24061,
and School of Business,
Universidad Adolfo Ibañez,
Diagonal Las Torres 2640, Santiago, Chile
damr@vt.edu

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Abstract

Split cuts form a well-known class of valid inequalities for mixed-integer programming problems. Cook, Kannan and Schrijver (1990) showed that the split closure of a rational polyhedron P is again a polyhedron. In this paper, we extend this result from a single rational polyhedron to the union of a finite number of rational polyhedra. We then use this result to prove that cross cuts yield closures that are rational polyhedra. Cross cuts are a generalization of split cuts introduced by Dash, Dey and Günlük (2012). Finally, we show that the quadrilateral closure of the two-row continuous group relaxation is a polyhedron, answering an open question in Basu, Bonami, Cornuéjols and Margot (2011).

1 Introduction

Cutting planes (or *cuts*, for short) are crucial for solving mixed-integer programs (MIPs), and currently the most effective cuts for general MIPs are special cases of *split cuts*, which are two-term disjunctive cuts. Given a family of cuts, an important theoretical question is whether only finitely many cuts from the family imply the rest for a polyhedron. Cook, Kannan and Schrijver [10] proved such a result for split cuts, by showing that the split closure of a rational polyhedron – that is, the set of points in the polyhedron that satisfy all split cuts – is again a polyhedron. Earlier, Schrijver [20] showed that the set of points in a rational polyhedron satisfying all Gomory-Chvátal cuts is a polyhedron, and Dunkel and Schulz [18] and Dadush, Dey and Vielma [12] proved that this result holds, respectively, for arbitrary polytopes, and compact convex sets.

Recently there has been substantial work on generalizing split cuts in different ways to obtain new and more effective classes of cutting planes, and analogues of the polyhedrality of the split closure result have been obtained for some of these classes. Andersen, Louveaux, Weismantel and Wolsey [3] studied cuts for the two-row continuous group relaxation that are obtained from two dimensional convex lattice-free sets, and Andersen, Louveaux and Weismantel [2] showed that the set of points in a rational polyhedron satisfying all cuts from convex lattice-free sets with bounded max-facet-width is a polyhedron. Averkov [4] gave a short proof of this latter result. Del Pia and Weismantel [15] showed that the closure with respect to cuts obtained from integral lattice-free sets is a polyhedron. In a recent paper, Basu, Hildebrand and Köeppe [9] showed that the triangle closure (points satisfying cuts obtained from maximal lattice-free triangles) of the two-row continuous group relaxation is a polyhedron.

As a different generalization of split cuts, Dash, Dey and Günlük [13] studied cuts that are obtained by considering two split sets simultaneously. These cuts are called *cross cuts* and are equivalent to the 2-branch split cuts of Li and Richard [19]. Dash, Dey and Günlük also define a subclass of cross cuts called *unimodular cross cuts* and show that the unimodular cross cut closure of the two-row continuous group relaxation equals its quadrilateral closure [13, Theorem 3.1]. The question of whether the quadrilateral closure is a polyhedron or not was posed by Basu, Bonami, Cornuéjols and Margot [8].

In this paper, we generalize the polyhedrality of the split closure result to the union of a finite number of rational polyhedra, i.e., we show that the split closure of the union of a finite number of rational polyhedra is generated by a finite collection of split disjunctions. We use this result to show that given any list of cross disjunctions, only finitely many yield nonredundant cross cuts for a rational polyhedron. From this we conclude that the cross cut closure of a rational polyhedron is a polyhedron. Furthermore, we also use this result to conclude that the unimodular cross closure of the two-row continuous group relaxation is a polyhedron. This implies that the quadrilateral closure of this relaxation is also a polyhedron, thus answering a question in Basu et. al. [8].

1.1 Summary of earlier results, proof techniques, and our contribution

We next formally define split sets, split cuts for a given polyhedron (all polyhedra in this paper are assumed to be rational) and the split closure of a polyhedron. For a given set $X \subseteq \mathbb{R}^n$, we denote its convex hull by $\operatorname{conv}(X)$. Given $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, the *split set* associated with (π, π_0) is defined to be

$$S(\pi, \pi_0) = \{x \in \mathbb{R}^n : \pi_0 < \pi^T x < \pi_0 + 1\}.$$

Clearly, $S(\pi, \pi_0) \cap \mathbb{Z}^n = \emptyset$ and consequently the integer points contained in a polyhedron $P \subseteq \mathbb{R}^n$ are the same as the ones contained in $\operatorname{conv}(P \setminus S(\pi, \pi_0)) \subseteq P$. Linear inequalities that are valid for $\operatorname{conv}(P \setminus S(\pi, \pi_0))$ are called *split cuts* derived from the split set $S(\pi, \pi_0)$. Many authors define split cuts in terms of *split disjunctions* instead of split sets. A split disjunction derived from $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ can be viewed as the set $\{x \in \mathbb{R}^n : \pi^T x \leq \pi_0 \text{ or } \pi^T x \geq \pi_0 + 1\} = \mathbb{R}^n \setminus S(\pi, \pi_0)$.

Let $\mathcal{S}^* = \{S(\pi, \pi_0) : (\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}\}$ denote the collection of all split sets and let $\mathcal{S} \subseteq \mathcal{S}^*$ be given. The split closure of a set $P \subseteq \mathbb{R}^n$, with respect to $\mathcal{S}$, is defined as

$$\mathrm{SC}(P, \mathcal{S}) = \bigcap_{S \in \mathcal{S}} \mathrm{conv}\,(P \setminus S)\,.$$

We refer to $\mathrm{SC}(P, \mathcal{S}^*)$ as the split closure of P and denote it as $\mathrm{SC}(P)$.

Cook, Kannan and Schrijver [10] proved that $\mathrm{SC}(P, \mathcal{S}^*) = \mathrm{SC}(P, \mathcal{S})$ for some finite set $\mathcal{S} \subset \mathcal{S}^*$.

Theorem 1 ([10])**.** *For any rational polyhedron P, there is a finite collection of split sets $\mathcal{S} \subseteq \mathcal{S}^*$ such that any split cut derived from a split set $S \in \mathcal{S}^*$ is implied by split cuts derived from split sets in $\mathcal{S}$. In other words,* $\mathrm{SC}(P, \mathcal{S}^*) = \mathrm{SC}(P, \mathcal{S})$.

This result also implies that $\mathrm{SC}(P)$ is a polyhedron as $\mathrm{conv}(P \setminus S)$ is polyhedral for all $S \in \mathcal{S}^*$, see [1, 2] and also Lemma 6 for a generalization.

Andersen, Cornuéjols and Li [1] proved the following stronger version of this result. Recall that a basic relaxation of a polyhedral set is obtained by relaxing all but a linearly independent subset of inequalities defining it. Here a collection of inequalities $a_i x \leq b_i$ for $i \in I$ is called linearly independent if the vectors a_i for $i \in I$ are linearly independent.

Theorem 2 ([1])**.** *For any rational polyhedron P, there is a finite collection of split sets $\mathcal{S} \subset \mathcal{S}^*$ such that any split cut for P is implied by split cuts obtained from basic relaxations of P using split sets in $\mathcal{S}$.*

Furthermore, Andersen, Cornuéjols and Li's proof technique is substantially different from that of Cook, Kannan and Schrijver [10], and is based on an analysis of the possible points of intersection of edges of a rational, pointed polyhedron with the hyperplanes bounding split sets. Other proofs of the polyhedrality of $\mathrm{SC}(P)$ can be found in [21], [14] and [2]. The first two of the above three papers give explicit bounds on the sizes of coefficients defining 'nonredundant' split sets thereby implying that only finitely many split sets can be nonredundant. The last paper builds on the proof technique in [1] to show that the closure of a polyhedron with respect to cuts from lattice-free sets having bounded max-facet-width (split sets have max-facet-width 1) is polyhedral. In [4], Averkov builds on proof techniques in [1] and [2]; his results imply the following strong generalization of Cook, Kannan and Schrijver's result.

Theorem 3 ([4])**.** *Given a polyhedron P and any collection of split sets $\mathcal{S}$, there is a finite collection of split sets $\mathcal{S}' \subseteq \mathcal{S}$ such that any split set $S \in \mathcal{S}$ is dominated by a split set $S' \in \mathcal{S}'$ in the sense that* $\mathrm{conv}(P \setminus S') \subseteq \mathrm{conv}(P \setminus S)$.

In other words, Averkov's result implies that each split cut derived from one split set is implied by a nonnegative linear combination of split cuts obtained from a single split set from a finite list of sets.

One can view the above results as proving that there exists a finite set $\hat{\mathcal{S}} \subseteq \mathcal{S}^*$ such that $\mathrm{SC}(P, \mathcal{S}^*) = \mathrm{SC}(P, \hat{\mathcal{S}})$. When such $\hat{\mathcal{S}}$ exists, we say that the split closure is *finitely generated.* For a nonpolyhedral set the split closure is not necessarily polyhedral. Even then, in some cases it may be finitely generated (see for example [11]).

As a generalization of split cuts, recently Dash, Dey and Günlük [13] studied *cross cuts*. A cross set is the union of two split sets $\{S_1, S_2\}$, where $S_1, S_2 \in \mathcal{S}^*$. Let

$$\mathcal{C}^* = \big\{\{S_1, S_2\} \,:\, S_1, S_2 \in \mathcal{S}^*\big\}$$

denote the collection of all unordered pairs of split sets from $\mathcal{S}^*$ and let $\mathcal{C} \subseteq \mathcal{C}^*$. The cross closure of a set $P \subseteq \mathbb{R}^n$, with respect to $\mathcal{C}$, is defined as

$$\mathrm{CC}(P, \mathcal{C}) = \bigcap_{\{S_1,S_2\}\in\mathcal{C}} \mathrm{conv}\left(P \setminus (S_1 \cup S_2)\right), \tag{1}$$

and the cross closure of P is $\mathrm{CC}(P, \mathcal{C}^*)$, denoted simply by $\mathrm{CC}(P)$. In Section 4, we give our main result, which generalizes Cook, Kannan and Schrijver's result to cross cuts and also to an arbitrary list of cross sets instead of all cross sets.

Theorem 4. *Let P be a rational polyhedron and let $\mathcal{C} \subseteq \mathcal{C}^*$ be given. Then*

$$\mathrm{CC}(P, \mathcal{C}) = \bigcap_{\{S_1,S_2\}\in\hat{\mathcal{C}}} \mathrm{conv}\left(P \setminus (S_1 \cup S_2)\right)$$

where $\hat{\mathcal{C}} \subseteq \mathcal{C}$ is a finite set. Consequently, $\mathrm{CC}(P, \mathcal{C})$ is a polyhedron.

We use this theorem, along with results from [13], to prove the following result, which closes an open problem from [8].

Theorem 5. *The quadrilateral closure of the two-row continuous group relaxation is a polyhedron.*

Our proof draws on techniques from the proofs of the highlighted results above, namely from [10], [1] and [4]. An important intermediate result we prove is the following generalization of Averkov's result to a finite union of rational polyhedra.

Theorem 6. *Let $P = \bigcup_{k\in K} P_k$ be a finite union of rational polyhedra and $\mathcal{S} \subseteq \mathcal{S}^*$. Then, there exists a finite set $\hat{\mathcal{S}} \subseteq \mathcal{S}$ such that for all $S_1 \in \mathcal{S}$ there exists $S_2 \in \hat{\mathcal{S}}$ such that*

$$\mathrm{conv}\left(P \setminus S_2\right) \subseteq \mathrm{conv}\left(P \setminus S_1\right).$$

Consequently, $\mathrm{SC}(P, \mathcal{S}) = \mathrm{SC}(P, \hat{\mathcal{S}})$, and is finitely generated.

Note that Theorem 6 does not always imply that $\mathrm{SC}\,(P, \mathcal{S})$ is polyhedral as it is easy to see that for $P_1 = \{(0, 0)\}$ and $P_2 = \{x \in \mathbb{R}^2 \,:\, x_2 = 1\}$ we have $\mathrm{SC}\,(P_1 \cup P_2, \mathcal{S}^*) = \mathrm{conv}(P_1 \cup P_2)$ which is not a polyhedron.

2 Preliminaries

We let $\mathbb{R}^n, \mathbb{Q}^n$ and $\mathbb{Z}^n$ stand for, respectively, the n-dimensional Euclidean space, the set of all rational points in $\mathbb{R}^n$ and the set of all integer points in $\mathbb{R}^n$. We let $\mathbb{R}^n_+, \mathbb{Q}^n_+$ and $\mathbb{Z}^n_+$ stand for the points in, respectively, $\mathbb{R}^n, \mathbb{Q}^n$ and $\mathbb{Z}^n$ with all components nonnegative. For any subsets A and B of $\mathbb{R}^n$, we let $A + B$ stand for the set $\{a + b : a \in A, b \in B\}$ and for a scalar $c \in \mathbb{R}$, we let $cA = \{ca : a \in A\}$. For a convex set $K \subseteq \mathbb{R}^n$, we denote its recession cone by $\mathrm{rec}(K)$ where

$$\mathrm{rec}(K) = \{d \in \mathbb{R}^n : x + \lambda d \in K \ \forall x \in K \text{ and } \forall \lambda \geq 0\}.$$

The lineality space of K is

$$\mathrm{ls}(K) := \mathrm{rec}(K) \cap -\mathrm{rec}(K) = \{d \in \mathbb{R}^n : x + \lambda d \in K \ \forall x \in K, \forall \lambda \in \mathbb{R}\}.$$

The affine hull of K is denoted by $\mathrm{aff}(K)$, and it dimension by $\dim(K)$. Let K^I, stand for the convex hull of integer points in K, also called the integer hull of K. Note that given a split set $S(\pi, \pi_0)$, its lineality space is $\{d \in \mathbb{R}^n : \pi^T d = 0\}$.

For a linear subspace L of $\mathbb{R}^n$, we denote its orthogonal linear subspace as $L^\perp$. The orthogonal projection of a set K onto L is denoted as $\mathrm{Proj}_L(K) := \{x \in L : \exists y \in L^\perp \text{ such that } x + y \in K\}$. The following result will be used in the next section.

Lemma 1. *Let $L \subseteq \mathbb{R}^n$ be a linear subspace and let $A, B \subseteq L^\perp$. Then*

1. $\mathbb{R}^n \setminus (A + L) = (L^\perp \setminus A) + L$.
2. $(A + L) \cap (B + L) = (A \cap B) + L$.

Properties (1) and (2) in the above Lemma do not necessarily hold when A and B are not contained in $L^\perp$. Note that if L is a linear subspace, then L is contained in the lineality subspace of a split set $S(\pi, \pi_0)$ if and only if $\pi \in L^\perp$.

For a rational polyhedron P, we denote its set of vertices by $V(P) \subseteq \mathbb{Q}^n$ and its set of extreme rays by $E(P) \subseteq \mathbb{Q}^n$. When $V(P) \neq \emptyset$, we say that the polyhedron is pointed (equivalently $\mathrm{ls}(P) = \{0\}$). Recall that every rational polyhedron $P \subseteq \mathbb{R}^n$ can be written in the form

$$P = Q + L,$$

where $L = \mathrm{ls}(P)$ is a rational linear subspace and $Q \subseteq L^\perp$ is a pointed rational polyhedron.

A *unimodular* matrix is a square matrix with determinant ± 1. If U is an $n \times n$ unimodular matrix, and $v \in \mathbb{Z}^n$, the affine transformation $\sigma(x) = Ux + v$ is called a unimodular transformation and is a one-to-one, invertible, mapping of $\mathbb{R}^n$ to $\mathbb{R}^n$ with $\sigma^{-1}(x) = U^{-1}(x - v)$ and is also a one-to-one invertible mapping of $\mathbb{Z}^n$ to $\mathbb{Z}^n$. If U is an integral unimodular matrix, then so is U^{-1} Further, if $a \in \mathbb{Z}^n, b \in \mathbb{Z}$, the set $\{x \in \mathbb{R}^n : a^T x = b\}$ is mapped by σ to the set

$$\{x' \in \mathbb{R}^n : a^T U^{-1}(x' - v) = b\} = \{x' \in \mathbb{R}^n : a^T U^{-1} x' = b + a^T U^{-1} v\},$$

where $a^T U^{-1} \in \mathbb{Z}^n$. Therefore, given a split set $S(a, b)$, $\sigma(S(a, b))$ and $\sigma^{-1}(S(a, b))$ are both split sets. Finally, given a k-dimensional rational affine subspace A of $\mathbb{R}^n$ with $0 < k < n$, there exists a unimodular transformation σ such that $\sigma(A) = \mathbb{R}^k \times \alpha$ where $\alpha \in \mathbb{Q}^{n-k}$ is a rational vector. If in addition, A contains an integer point, then the transformation can be chosen such that α is zero.

2.1 Subtracting split sets from a convex set

In this section we analyze the effect of subtracting multiple split sets from a non-pointed polyhedron and convexifying the remaining points. More precisely, we show that if the lineality space of a split set does not contain that of the polyhedron, then the split set does not affect the resulting convex hull. Using this observation, we subsequently show that in order to obtain the convex hull one can work with the pointed polyhedron given by projecting the original polyhedron onto the orthogonal complement of its lineality space.

Proposition 2. *Let $S_1, S_2, \ldots, S_m \in \mathcal{S}^*$, and let $P = Q + L$, where L is a linear subspace and $Q \subseteq L^\perp$. Let $I = \{1, \ldots, m\}$ and let $J = \{i \in I : L \subseteq \mathrm{ls}(S_i)\}$. Then*

$$\mathrm{conv}(P \setminus (\bigcup_{i \in I} S_i)) = \mathrm{conv}(P \setminus (\bigcup_{i \in J} S_i)).$$

Furthermore, if it is not empty, then

$$\mathrm{conv}(P \setminus (\bigcup_{i \in J} S_i)) = \mathrm{conv}(Q \setminus (\bigcup_{i \in J} S_i)) + L.$$

Proof. If $L = \emptyset$, both parts of the claim hold trivially and therefore we assume that $L \neq \emptyset$. Further, if $I = J$, there is nothing to prove for the first part of the proposition, so assume $I \neq J$. The inclusion $\mathrm{conv}(P \setminus (\bigcup_{i \in I} S_i)) \subseteq \mathrm{conv}(P \setminus \bigcup_{i \in J} S_i)$ is straighforward.

We next prove that $\mathrm{conv}(P \setminus (\bigcup_{i \in J} S_i)) \subseteq \mathrm{conv}(P \setminus (\bigcup_{i \in I} S_i))$. For all $i \in I \setminus J$, as $L \nsubseteq \mathrm{ls}(S_i)$, we have $L \cap \mathrm{ls}(S_i)$ is a linear subspace of $\mathbb{R}^n$ with dimension less than that of L. Therefore

$$L \nsubseteq \bigcup_{i \in I \setminus J} \mathrm{ls}(S_i).$$

If this were not the case, then

$$L \subseteq \bigcup_{i \in I \setminus J} \mathrm{ls}(S_i) \Rightarrow L = \bigcup_{i \in I \setminus J} (L \cap \mathrm{ls}(S_i)),$$

which would imply that L equals the finite union of some sets, each with a lower dimension than that of L, which is not possible. Therefore, there exists some $v_0 \in L \setminus \cup_{i \in I \setminus J} \mathrm{ls}(S_i)$. Let $S_i = \{x \in \mathbb{R}^n : \pi_0^i < (\pi^i)^T x < \pi_0^i + 1\}$ for $i \in I$. Note that $(\pi^i)^T v_0 \neq 0$ for $i \in I \setminus J$ and $(\pi^i)^T v_0 = 0$ for $i \in J$.

Let $x_0 \in P \setminus (\bigcup_{i \in J} S_i)$. We can choose an $\alpha > 0$ large enough such that

$$x_0 + \alpha v_0,\ x_0 - \alpha v_0 \in P \setminus S_i \text{ for all } i \in I \setminus J. \tag{2}$$

Further, we have $(\pi^i)^T (x_0 + \alpha v_0) = (\pi^i)^T x_0$ for $i \in J$. Therefore,

$$x_0 + \alpha v_0,\ x_0 - \alpha v_0 \in P \setminus S_i, \text{ for all } i \in J. \tag{3}$$

Now, by using (2) and (3) we obtain

$$x_0 + \alpha v_0,\ x_0 - \alpha v_0 \in P \setminus (\bigcup_{i \in I} S_i). \tag{4}$$

As $x_0 \in \text{conv}(\{x_0 + \alpha v_0, x_0 - \alpha v_0\})$, (4) implies that $x_0 \in \text{conv}(P \setminus (\bigcup_{i \in I} S_i))$. Therefore,

$$P \setminus \bigcup_{i \in J} S_i \subseteq \text{conv}(P \setminus \bigcup_{i \in I} S_i)$$

and we conclude that $\text{conv}(P \setminus \bigcup_{i \in J} S_i) \subseteq \text{conv}(P \setminus \bigcup_{i \in I} S_i)$.

For the second part of the proposition, note that since $L \subseteq \text{ls}(S_i)$ for $i \in J$, we can write

$$S_i = \widehat{S}_i + L \text{ where } \widehat{S}_i \subseteq L^{\perp}, \text{ for all } i \in J.$$

Using this equality and the properties in Lemma 1, we obtain

$$\begin{aligned}
P \setminus (\bigcup_{i \in J} S_i) &= (Q + L) \setminus [\bigcup_{i \in J} (\widehat{S}_i + L)] \\
&= (Q + L) \cap [\bigcap_{i \in J} \mathbb{R}^n \setminus (\widehat{S}_i + L)] \\
&= (Q + L) \cap [\bigcap_{i \in J} ((L^{\perp} \setminus \widehat{S}_i) + L)] \\
&= [Q \cap \bigcap_{i \in J} (L^{\perp} \setminus \widehat{S}_i)] + L \\
&= [Q \setminus (\bigcup_{i \in J} \widehat{S}_i)] + L \\
&= [Q \setminus (\bigcup_{i \in J} S_i)] + L.
\end{aligned}$$

For any two convex sets $A, B \subseteq \mathbb{R}^n$, it is well-known that $\text{conv}(A + B) = \text{conv}(A) + \text{conv}(B)$. Therefore, $\text{conv}(P \setminus (\bigcup_{i \in J} S_i)) = \text{conv}(Q \setminus (\bigcup_{i \in J} S_i) + L) = \text{conv}(Q \setminus (\bigcup_{i \in J} S_i)) + L$, as desired.
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2.2 Intersection points and Gordan-Dickson Lemma

In [1], Anderson, Cornuejols and Li give an alternate proof of the polyhedrality of the split closure of polyhedra using a new proof technique. An important ingredient of the proof is the analysis of intersection points of (closed) split sets and half-lines. We start with defining the point where a rational half-line intersects for the first time the complement of a split set that contains the end point of the half-line.

Definition 3 (Intersection point step size)**.** *Let $v, r \in \mathbb{Q}^n$ and $S \in \mathcal{S}^*$ such that $v \in S$, then*

$$\lambda_{vr}(S) = \min\{\lambda : v + \lambda r \notin S,\ \lambda > 0\}.$$

Given a split set $S = S(\pi, \pi_0)$, the step size can be explicitly computed as follows:

$$\lambda_{vr}(S) = \begin{cases} (\pi_0 + 1 - \pi^T v)/(\pi^T r) & \pi^T r > 0 \\ (\pi^T v - \pi_0)/(-\pi^T r) & \pi^T r < 0 \\ +\infty & \pi^T r = 0. \end{cases}$$

Let $p = v + \lambda_{vr}(S)\, r$, then it is easy to see that if $\pi^T r > 0$, then p is the point where the half-line $H = \{v + \lambda r : \lambda \geq 0\}$ intersects the hyperplane $\{x \in \mathbb{R}^n : \pi^T x = \pi_0 + 1\}$. Similarly, if $\pi^T r < 0$ then p is the intersection point of H with the hyperplane $\{x \in \mathbb{R}^n : \pi^T x = \pi_0\}$. Moreover, it is possible to bound the intersection point step size when it is finite.

Lemma 4 (Lemma 5 in [1])**.** *Let $v, r \in \mathbb{Q}^n$ and $S \in \mathcal{S}^*$ such that $v \in S$. If $\lambda_{vr}(S) < +\infty$, then $\lambda_{vr}(S) < \min\{z \in \mathbb{Z}_+ : zr \in \mathbb{Z}^n\}$.*

Consequently, if $t \in \mathbb{Z}_+$ and $t \cdot r$ is integral, then $\lambda_{vr}(S) \leq t$ provided that it is finite. Note that for rational r, there always is a finite t such that $t \cdot r$ is integral. We next review some properties of $\lambda_{vr}(S)$ presented in [1] and [2].

Lemma 5 (Lemma 6 in [1])**.** *Let $v, r \in \mathbb{Q}^n$, then the set*

$$\Lambda(\lambda^*) = \big\{\lambda \in \mathbb{R} \,:\, \lambda = \lambda_{vr}(S),\ \infty > \lambda_{vr}(S) \geq \lambda^*,\ v \in S,\ S \in \mathcal{S}^*\big\},$$

is finite for all $\lambda^* > 0$.

In other words for any fixed number $\lambda^* > 0$, there are only a finite number of step sizes that are larger than λ^*. This observation can be used to conclude that there are only a finite number of possible points on the half-line H that can intersect with the complement of a split set provided that the intersection point is not very close to v. Based on this observation, it is easy to associate an index with each intersection point with step size $\hat{\lambda}$ that corresponds to the cardinality of the set $\Lambda(\hat{\lambda})$. Therefore, for a given half-line $H = \{v + \lambda r : \lambda \geq 0\}$ where $v, r \in \mathbb{Q}^n$ we can define the following function

$$h_{vr}(S) = \begin{cases} 0, & \lambda_{vr}(S) = +\infty \\ |\Lambda(\lambda_{vr}(S))|, & \lambda_{vr}(S) < +\infty \end{cases}$$

that maps any given split set $S \in \mathcal{S}$ with $v \in S$ to a finite integer.

Next we summarize some of the results originally presented in [1] for polyhedral cones and later generalized by Andersen, Louveaux, and Weismantel [2] to general polyhedra.

Lemma 6 ([1, 2])**.** *Let Q be a pointed rational polyhedron and let $S \in \mathcal{S}^*$. If $Q \setminus S \neq \emptyset$, then*

1. $\operatorname{conv}(Q \setminus S)$ *is a rational polyhedron with the same recession cone as Q.*
2. *If u is a vertex of $\operatorname{conv}(Q \setminus S)$, then either $u \in V(Q) \setminus S$, or, $u = v + \lambda_{vr}(S)\, r$, where $v \in V(Q) \cap S$ and r satisfies one of the following:*
   (a) *$r \in E(Q)$ such that $\{v + \lambda r : \lambda \geq 0\}$ is an edge of Q and $\lambda_{vr}(S) < +\infty$, or,*
   (b) *$r = v' - v$ for some $v' \in V(Q) \setminus S$ such that $\operatorname{conv}(v, v')$ is an edge of Q*

This result essentially asserts that $\operatorname{conv}(Q \setminus S)$ is completely determined by the intersections of the edges of Q with the two hyperplanes bounding the split set S; more precisely it equals the convex hull of the portions of the edges of Q which are not contained in S. We next define the "relevant" edge directions associated with a given vertex of a polyhedron.

Definition 7 (Relevant directions)**.** *Let Q be a pointed polyhedron and $v \in V(Q)$. Let*

$$\begin{aligned} D_v(Q) = &\big\{v' - v : v' \in V(Q), \text{ and } \mathrm{conv}(v, v') \text{ is a 1-dimensional face of } Q\big\} \\ &\cup \ \big\{r \in E(Q) : \{v + \lambda r : \lambda \geq 0\} \text{ is a 1-dimensional face of } Q\big\} \end{aligned}$$

denote the set of relevant directions for the vertex v.

The following is a simple observation based on Lemma 6.

Lemma 8. *Let Q be a pointed rational polyhedron and $S_1, S_2 \in \mathcal{S}^*$. Let $V' \subseteq V(Q)$ be such that $V' = V(Q) \cap S_i$ for $i = 1, 2$. If*

$$h_{vr}(S_2) \leq h_{vr}(S_1)$$

holds for all $v \in V'$ and $r \in D_v(Q)$, then

$$\mathrm{conv}\,(Q \setminus S_2) \subseteq \mathrm{conv}\,(Q \setminus S_1)\,.$$

Proof. First note that if $Q \setminus S_2 = \emptyset$, the claim holds and therefore we only need to consider the case when $Q \setminus S_2 \neq \emptyset$. Notice that by Lemma 6 $\mathrm{conv}(Q \setminus S_2)$ and $\mathrm{conv}(Q \setminus S_1)$ are polyhedral and have the same recession cone. We will next argue that the vertices of $\mathrm{conv}(Q \setminus S_2)$ belong to $\mathrm{conv}(Q \setminus S_1)$. As Q is pointed and $\mathrm{conv}(Q \setminus S_2)$ is contained in Q, it has to be pointed as well. Let u be a vertex of $\mathrm{conv}(Q \setminus S_2)$. Clearly, $u \in Q$. If $u \in V(Q) \setminus V'$, then $u \notin S_1$ and therefore $u \in Q \setminus S_1$.

If, on the other hand, $u \notin V(Q) \setminus V'$, then it is a 'new' vertex and by Lemma 6 we have $u = v + \lambda_{vr}(S_2)\, r$ for some $v \in V'$ and $r \in D_v(Q)$. In addition, $\lambda_{vr}(S_2)$ is finite. In this case,

$$h_{vr}(S_2) \leq h_{vr}(S_1) \Rightarrow \lambda_{vr}(S_2) \geq \lambda_{vr}(S_1).$$

Further, as $\lambda_{vr}(S_2) \geq \lambda_{vr}(S_1)$, we have $u \notin S_1$. Therefore, $u \in Q \setminus S_1$. ∎ ∎

By Proposition 2, we have the following corollary.

Corollary 9. *Let $P = Q + L$ be a polyhedron where L is a rational linear subspace and $Q \subseteq L^\perp$ is a pointed rational polyhedron. Let $V' \subseteq V(Q)$ and let $S_1, S_2 \in \mathcal{S}^*$ be such that $P \neq \mathrm{conv}(P \setminus S_i)$ and $V' = V(Q) \cap S_i$ for $i = 1, 2$. If*

$$h_{vr}(S_2) \leq h_{vr}(S_1),$$

for all $v \in V'$ and $r \in D_v(Q)$, then

$$\mathrm{conv}\,(P \setminus S_2) \subseteq \mathrm{conv}\,(P \setminus S_1)\,.$$

In Figure 1, the first picture depicts intersection points of split set boundaries with rays incident with the vertex v. The second one shows two splits S_1 and S_2 with $h_{vr_1}(S_1) = 3, h_{vr_2}(S_1) = 1, h_{vr_1}(S_2) = 1, h_{vr_2}(S_2) = 2$. The third picture shows the split set S_1 and S_3, where $h_{vr_1}(S_3) = 4, h_{vr_2}(S_3) = 6$. As $h_{vr_i}(S_1) \leq h_{vr_i}(S_3)$ for $i = 1, 2$, the intersection points of S_1 with the rays r_1 and r_2 are further away from v then the corresponding intersection points of S_3, and Lemma 8 implies

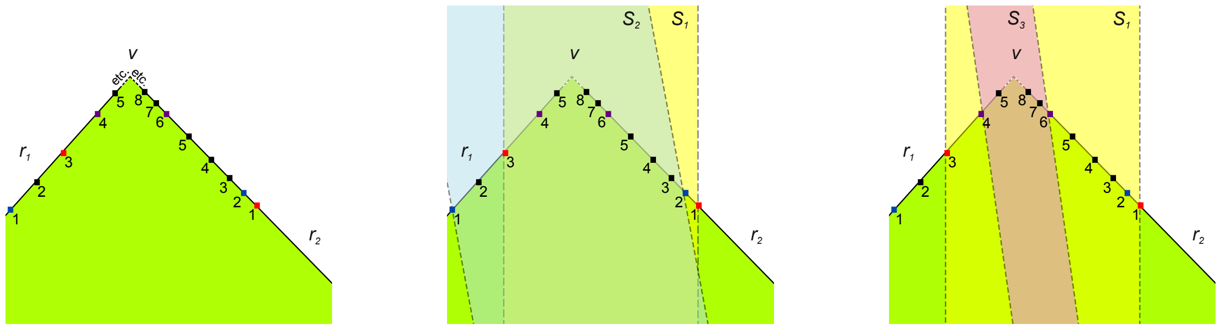

Figure 1: Indices of intersection points and three split sets S_1, S_2, S_3.

that $\operatorname{conv}(Q \setminus S_1) \subseteq \operatorname{conv}(Q \setminus S_3)$ as can be seen in the figure. On the other hand, we se that $h_{vr_1}(S_1) > h_{vr_1}(S_2)$ but $h_{vr_2}(S_1) < h_{vr_2}(S_2)$, therefore neither of $\operatorname{conv}(Q \setminus S_1)$ or $\operatorname{conv}(Q \setminus S_2)$ is a subset of the other set.

We next state a very simple and useful lemma that shows that for any positive integer p, every set of p-tuples of natural numbers has finitely many minimal elements.

Lemma 10 (Gordan-Dickson Lemma)**.** *Let $X \subseteq \mathbb{Z}_+^p$. Then there exists a finite set $Y \subseteq X$ such that for every $x \in X$ there exists $y \in Y$ satisfying $x \geq y$.*

This observation together with Corollary 9 can be used to show that the split closure of a polyhedron is again a polyhedron. In [4], Averkov uses a similar argument to show the polyhedrality of more general closures that include the split closure.

3 Split Closure of a Finite Collection of Polyhedral Sets

In this section, we prove Theorem 6, namely we show that given a finite collection of rational polyhedra, there exists a finite set of splits that define the split closure of their union. Let $P_k \subset \mathbb{R}^n$ for $k \in K$ be a finite collection of rational polyhedra where $P_k = Q_k + L_k$, and L_k is a rational linear subspace and $Q_k \subseteq L_k^\perp$ is a pointed rational polyhedron. In addition, let $\mathcal{S} \subseteq \mathcal{S}^*$ be a collection of split sets of appropriate dimension. We are interested in the split closure of $P = \bigcup_{k \in K} P_k$ with respect to $\mathcal{S}$:

$$\operatorname{SC}(P, \mathcal{S}) = \bigcap_{S \in \mathcal{S}} \operatorname{conv}(P \setminus S).$$

Note that for any $S \in \mathcal{S}^*$

$$\operatorname{conv}(P \setminus S) = \operatorname{conv}\left(\bigcup_{k \in K} \operatorname{conv}(P_k \setminus S)\right). \tag{5}$$

Furthermore, note that by Proposition 2 we have

$$\operatorname{conv}(P_k \setminus S) = \begin{cases} \operatorname{conv}(Q_k \setminus S) + L_k & \text{if } \; L_k \subseteq \operatorname{ls}(S) \\ P_k & \text{otherwise.} \end{cases}$$

for any $k \in K$. Therefore, in the context of split cuts for P_k, it suffices to consider splits sets whose lineality space contains the lineality space of P_k.

A natural question (and one posed to us by a referee) is whether the split sets that define the split closure of P_k for each $k \in K$ can be combined to give the split closure of P. More formally, if $\mathcal{S}_k$ is a finite collection of split sets such that $\mathrm{SC}(P_k, \mathcal{S}^*) = \mathrm{SC}(P_k, \mathcal{S}_k)$ for each $k \in K$, is it the case that $\mathrm{SC}(P, \mathcal{S}^*) = \mathrm{SC}(P, \bigcup_{k \in K} \mathcal{S}_k)$? If the answer were affirmative, then the split closure of P would be polyhedral as each $\mathcal{S}_k$ is a finite collection of split sets. However, we next show that this is not the case with the following example.

Example 1. *Let $P_1 = \{(x, y) \in \mathbb{R}^2 : 1 \le x \le 5, 1 \le y \le 1.5\}$ and $P_2 = \{(x, y) \in \mathbb{R}^2 : 2 \le x \le 5, 2 \le y \le 2.5\}$ be two polyhedra depicted in Figure 2(a) by the shaded regions. The bold lines represent the respective integer hulls; the integer hull of P_1 is the set $\{(x, y) : 1 \le x \le 5, y = 1\}$. It is easy to see that the split closure of P_1 is given by the split set $S_1 = \{(x, y) \in \mathbb{R}^2 : 1 < y < 2\}$ as $\mathrm{conv}(P_1 \setminus S_1)$ equals the integer hull of P_1. Similarly, if $S_2 = \{(x, y) \in \mathbb{R}^2 : 2 < y < 3\}$, then $\mathrm{conv}(P_2 \setminus S_2)$ equals the integer hull of P_2 and thus $\mathrm{SC}(P_i, \{S_i\}) = \mathrm{SC}(P_i, \mathcal{S}^*)$ for $i = 1, 2$. Furthermore, S_i is the only nonredundant split set for P_i.*

However, when we let $P = P_1 \cup P_2$, then $\mathrm{conv}(P \setminus S_1) \cap \mathrm{conv}(P \setminus S_2)$ does not equal the integer hull of P. In Figure 2, $\mathrm{conv}(P \setminus S_1)$ is depicted by the polyhedron with the blue boundary, and $\mathrm{conv}(P \setminus S_2)$ is the polyhedron with the red boundary. If we let $S_3 = \{(x, y) \in \mathbb{R}^2 : 0 < y - x < 1\}$, then it is easy to see that $\mathrm{conv}(P \setminus S_2) \cap \mathrm{conv}(P \setminus S_3)$ gives the integer hull of P, shown in grey. To see this note that $y - x \le 1.5$ is a valid inequality for P, and therefore the inequality $y - x \le 1$ is a split cut for P derived from S_3. See Figure 2(b). Therefore the split closure of P equals its integer hull and $\mathrm{SC}(P, \mathcal{S}^*) = \mathrm{SC}(P, \{S_2, S_3\}) \neq \mathrm{SC}(P, \{S_1, S_2\})$.

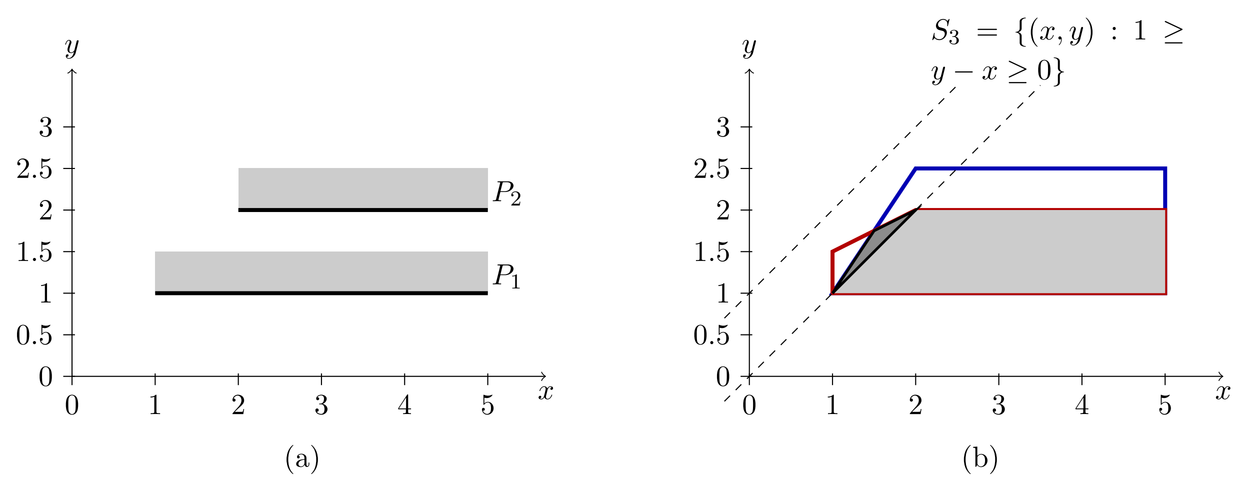

Figure 2: The split sets giving the split closures of P_1, P_2 do not give the split closure of $P_1 \cup P_2$

3.1 Split closure of a union of polyhedra

We start by partitioning the split sets in $\mathcal{S}$ into subcollections based on which polyhedra P_k they yield nontrivial split cuts for. More precisely, for all $K' \subseteq K$ including $K' = \emptyset$, we define

$$\mathcal{S}(K') \;=\; \{S \in \mathcal{S} \;:\; \mathrm{conv}(P_k \setminus S) \neq P_k \text{ for } k \in K',\; \mathrm{conv}(P_k \setminus S) = P_k \text{ for } k \notin K'\}.$$

Clearly, $\mathcal{S} = \bigcup_{K' \subseteq K} \mathcal{S}(K')$. Also notice that if $L_k \not\subseteq \mathrm{ls}(S)$, for some $S \in \mathcal{S}$, then $S \notin \mathcal{S}(K')$ whenever $k \in K'$.

We next partition $\mathcal{S}(K')$ further into smaller subsets depending on which vertices of the polyhedra Q_k they contain. For a fixed set $K' \subseteq K$, let $V' \subseteq \bigcup_{k \in K'} V(Q_k)$ be given, then we define

$$\mathcal{S}(K', V') \;=\; \{S \in \mathcal{S}(K') \;:\; S \cap V = V'\}$$

where $V = \bigcup_{k \in K'} V(Q_k)$. Note that for any $K' \subseteq K$

$$\mathcal{S}(K') = \bigcup_{V' \subseteq \bigcup_{k \in K'} V(Q_k)} \mathcal{S}(K', V').$$

Consequently,

$$\mathcal{S} = \bigcup_{K' \subseteq K} \; \bigcup_{V' \subseteq \bigcup_{k \in K'} V(Q_k)} \mathcal{S}(K', V'). \tag{6}$$

Also note that given $K', K'' \subseteq K$ and $V', V'' \subseteq V$, we have $\mathcal{S}(K', V') \bigcap \mathcal{S}(K'', V'') = \emptyset$ unless $K' = K''$ and $V' = V''$. We next show that $\mathrm{SC}(P, \mathcal{S}(K', V'))$ is finitely generated for any $K' \subseteq K$ and $V' \subseteq \bigcup_{k \in K'} V(Q_k)$.

Proposition 11. *Let $\mathcal{S} \subseteq \mathcal{S}^*$, $K' \subseteq K$ and $V' \subseteq \bigcup_{k \in K'} V(Q_k)$ be given. Then, there exists a finite set $\mathcal{S}_Y \subseteq \mathcal{S}(K', V')$ such that for all $S_1 \in \mathcal{S}(K', V')$ there exists $S_2 \in \mathcal{S}_Y$ such that*

$$\mathrm{conv}\,(P_k \setminus S_2) \subseteq \mathrm{conv}\,(P_k \setminus S_1) \qquad \textit{for all } k \in K'.$$

Proof. If $\mathcal{S}(K', V') = \emptyset$, there is nothing to prove so we assume that $\mathcal{S}(K', V') \neq \emptyset$. In this case, if $K' = \emptyset$, then it is easy to see that $\mathrm{conv}(P_k \setminus S) = P_k$ for all $k \in K$. Thus, we can take take $S_Y = \{S\}$, where $S \in \mathcal{S}(K', V')$ can be chosen arbitrarily.

We now consider the case when $K' \neq \emptyset$. As we assumed $\mathcal{S}(K', V')$ is nonempty, V' must be nonempty too. Let $V'_k = V' \cap V(Q_k)$ for $k \in K'$. Using the function h_{rv} defined earlier, we now define a function $H : \mathcal{S} \to \mathbb{Z}^p$ where $p = \sum_{k \in K'} \sum_{v \in V'_k} |D_v(Q_k)|$. More precisely, for $S \in \mathcal{S}(K', V')$, the p-tuple $H(S)$ has a component for each $k \in K'$, $v \in V'_k$, and $r \in D_v(Q_k)$ that equals $h_{vr}(S)$.

Now consider the following set that contains all possible values of $H(S)$ for $S \in \mathcal{S}(K', V')$:

$$X = \Big\{t \in \mathbb{Z}^p_+ \;:\; t = H(S),\; S \in \mathcal{S}(K', V')\Big\}.$$

By Lemma 10, the set X contains a finite set of minimal elements. Let $Y \subseteq X$ be a finite set such that for every $x \in X$ there exists $y \in Y$ satisfying $x \geq y$. For each $y \in Y$, let $S_y \in \mathcal{S}(K', V')$ be a split set such that $H(S_y) = y$ and let $\mathcal{S}_Y = \{S_y \,:\, y \in Y\}$. In other words, for each $y \in Y$ the set $\mathcal{S}_Y$ contains a set S_y such that $H(S_y) = y$.

Now let $x = H(S_1)$ and $y \leq x$ be such that $y \in Y$. Further let $S_2 \in \mathcal{S}_Y$ be such that $H(S_2) = y$. Clearly we$K' \subseteq K$ have $H(S_2) \leq H(S_1)$ and therefore

$$h_{vr}(S_1) \geq h_{vr}(S_2), \text{ for all } r \in D_v(Q_k) \text{ and } v \in V_k'$$

for all $k \in K'$. Consequently, by Corollary 9 we obtain that

$$\operatorname{conv}(P_k \setminus S_2) \subseteq \operatorname{conv}(P_k \setminus S_1)$$

for all $k \in K'$. ∎ ∎

Using Proposition 11 we now generalize Theorem 3 for the split closure of a union of polyhedra.

Theorem 6. *Let $P = \bigcup_{k \in K} P_k$ be a finite union of rational polyhedra and $\mathcal{S} \subseteq \mathcal{S}^*$. Then, there exists a finite set $\hat{\mathcal{S}} \subseteq \mathcal{S}$ such that for all $S_1 \in \mathcal{S}$ there exists $S_2 \in \hat{\mathcal{S}}$ such that*

$$\operatorname{conv}(P \setminus S_2) \subseteq \operatorname{conv}(P \setminus S_1).$$

Consequently, $\operatorname{SC}(P, \mathcal{S})$ *is finitely generated.*

Proof. For $k \in K$ let $P_k = Q_k + L_k$, where L_k is a rational linear subspace and $Q_k \subseteq L_k^\perp$ is a pointed rational polyhedron and let

$$\mathcal{S} = \bigcup_{K' \subseteq K} \; \bigcup_{V' \subseteq \bigcup_{k \in K'} V(Q_k)} \mathcal{S}(K', V') \tag{7}$$

be the finite partition of $\mathcal{S}$ as defined in (6). By Proposition 11, there exists a finite set $\mathcal{S}_Y(K', V') \subseteq \mathcal{S}(K', V')$ for each $K' \subseteq K$ and $V' \subseteq \bigcup_{k \in K'} V(Q_k)$ with the property that for each for $S_1 \in \mathcal{S}(K', V')$ there exists $S_2 \in \mathcal{S}_Y(K', V')$ such that

$$\operatorname{conv}(P_k \setminus S_2) \subseteq \operatorname{conv}(P_k \setminus S_1) \qquad \text{for all } k \in K',$$

implying $\operatorname{conv}(P \setminus S_2) \subseteq \operatorname{conv}(P \setminus S_1)$ by equation (5). Consequently, taking

$$\hat{\mathcal{S}} = \bigcup_{K' \subseteq K} \; \bigcup_{V' \subseteq \bigcup_{k \in K'} V(Q_k)} \mathcal{S}_Y(K', V')$$

completes the proof. ∎ ∎

4 Cross Closure of a Polyhedral Set

Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron of the form $P = Q + L$, where $L := \operatorname{ls}(P)$ is a rational linear subspace of $\mathbb{R}^n$ and $Q \subseteq L^\perp$ is a pointed, rational polyhedron. Recall that $\mathcal{C}^* = \mathcal{S}^* \times \mathcal{S}^*$ denotes the set of all pairs of split sets and a cross set is simply the union of two split sets. In this section we will show that for a given $\mathcal{C} \subseteq \mathcal{C}^*$

$$\mathrm{CC}(P, \mathcal{C}) = \bigcap_{\{S_1,S_2\}\in\mathcal{C},} \mathrm{conv}\,(P \setminus (S_1 \cup S_2))$$

is a polyhedron. Let $\mathcal{S} = \{S \in \mathcal{S}^* : S \in C, \text{ for some } C \in \mathcal{C}\}$ denote the collection of split sets that appear in at least one of the cross sets defined by $\mathcal{C}$. Furthermore, let $\mathcal{S} = \mathcal{S}^L \cup \mathcal{S}^o$ be a partition of $\mathcal{S}$ such that $\mathcal{S}^L$ contains $S \in \mathcal{S}$ with $L \subseteq \mathrm{ls}(S)$ and $\mathcal{S}^o$ contains $S \in \mathcal{S}$ with $L \not\subseteq \mathrm{ls}(S)$. Clearly,

$$\mathrm{CC}(P, \mathcal{C}) \subseteq \mathrm{SC}(P, \mathcal{S}) = \mathrm{SC}(P, \mathcal{S}^L).$$

Furthermore, by Proposition 2, for a given $C \in \mathcal{C}$ we have

$$\mathrm{conv}\,(P \setminus C) = \begin{cases} \mathrm{conv}(Q \setminus C) + L & if \;\; |C \cap \mathcal{S}^L| = 2, \\ \mathrm{conv}(Q \setminus S) + L & if \;\; C \cap \mathcal{S}^L = \{S\}, \\ \mathrm{conv}(P) & if \;\; C \cap \mathcal{S}^L = \emptyset. \end{cases}$$

Consequently, for $C \in \mathcal{C}$,

$$\mathrm{conv}\,(P \setminus C) \not\supseteq \mathrm{SC}(P, \mathcal{S}) \Rightarrow |C \cap \mathcal{S}^L| = 2.$$

Therefore, if we let $\mathcal{C}^L \subseteq \mathcal{C}$ to denote set of $C \in \mathcal{C}$ with the property that $|C \cap \mathcal{S}^L| = 2$, we have the following observation

$$\mathrm{CC}(P, \mathcal{C}) = \mathrm{SC}(P, \mathcal{S}^L) \cap \left(\bigcap_{C\in\mathcal{C}^L} \mathrm{conv}(P \setminus C) \right), \tag{8}$$

where $\mathrm{SC}(P, \mathcal{S}^L)$ is a polyhedral set. Furthermore, as $\mathrm{conv}(P \setminus C) = \mathrm{conv}(Q \setminus C) + L$ for all $C \in \mathcal{C}^L$, we also have

$$\bigcap_{C\in\mathcal{C}^L} \mathrm{conv}(P \setminus C) = L + \bigcap_{C\in\mathcal{C}^L} \mathrm{conv}(Q \setminus C) = L + \mathrm{CC}(Q, \mathcal{C}^L). \tag{9}$$

Consequently, $\mathrm{CC}(P, \mathcal{C})$ is polyhedral, if and only if $\mathrm{CC}(Q, \mathcal{C}^L)$ is a polyhedron. Therefore, we conclude that to prove the polyhedrality of the cross closure for any rational polyhedron, it is sufficient to study the special case when the polyhedron is pointed.

4.1 Cross closure of pointed polyhedra

In this section, we show that the cross closure of a pointed, rational polyhedron is again a polyhedron. We combine the proof technique of Cook, Kannan and Schrijver [10] for showing that the split closure of a polyhedron is polyhedral along with the results we derived in earlier sections based on proof techniques of Anderson, Cornuéjols, Li [1], and Averkov [4]. We need some definitions to discuss the overall techniques used. Let $\|\cdot\|$ denote the usual Euclidean norm. Define the width of a split set $S(\pi, \pi_0)$ as $w(S(\pi, \pi_0)) = 1/\|\pi\|$ (this is the geometric distance between the parallel hyperplanes bounding the split set). Then $w(S(\pi, \pi_0)) > \eta$ for some $\eta > 0$ implies that $\|\pi\| < 1/\eta$. Therefore, for any fixed $\eta > 0$ and $\pi_0 \in \mathbb{Z}^n$, there are only finitely many $\pi \in \mathbb{Z}^n$ such that $w(S(\pi, \pi_0)) > \eta$.

For a given pointed polyhedron P, Cook, Kannan, Schrijver prove the polyhedrality of $\mathrm{SC}(P, \mathcal{S}^*)$ using the following idea. Let $\mathcal{S} \subseteq \mathcal{S}^*$ be a finite list of split sets and consider the set $\mathrm{SC}(P, \mathcal{S}) = \bigcap_{S \in \mathcal{S}} \mathrm{conv}(P \setminus S)$. Suppose that for every face F of P, $\mathrm{SC}(P, \mathcal{S}) \cap F = \mathrm{SC}(F, \mathcal{S}^*)$. Then (i) there are only finitely many split sets beyond the ones contained in $\mathcal{S}$ which yield split cuts cutting off points of $\mathrm{SC}(P, \mathcal{S})$ (they show that if $S(\pi, \pi_0)$ is such a split set, then π must have bounded norm). Therefore, (ii) if one assumes (by induction on dimension) that the number of split sets needed to define the split closure of each face of a polyhedron is finite, then so is the number of split sets needed to define the split closure of the polyhedron.

Santanu Dey [17] observed that idea (i) in the Cook, Kannan, Schrijver proof technique can also be used in the case of some disjunctive cuts which generalize split cuts. We apply a modification of idea (i) to cross cuts. Let $\mathcal{C} \subseteq \mathcal{C}^*$ be a finite list of pairs of split sets (that define cross sets) and recall that $\mathrm{CC}(P, \mathcal{C}) = \bigcap_{\{S_1, S_2\} \in \mathcal{C}} \mathrm{conv}(P \setminus (S_1 \cup S_2))$. We show in Lemma 16 that if $CC(P, \mathcal{C})$ intersected with each face of P equals the cross closure of the face, then there exists a number $\eta > 0$ such that any cross set $S_1 \cup S_2$, where both $w(S_1)$ and $w(S_2)$ are at most η, only yields cross cuts valid for $\mathrm{CC}(P, \mathcal{C})$, and are therefore not needed to define the cross closure of P. We then only need to consider cross sets $S_1 \cup S_2$ where at least one of $w(S_1), w(S_2)$ is greater than η (such cross sets are still infinitely many in number).

We first need a generalization of Lemma 6, property (1). ■

Lemma 12. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron and let $S_i \in \mathcal{S}^*$ be split sets for $i \in \{1, \ldots, m\}$ where m is a positive integer. Then,* $\mathrm{conv}\,(P \setminus (\bigcup_{i=1}^m S_i))$ *is a polyhedron, which, if nonempty, has the same recession cone as P.*

Proof. Let P_∞ denote the recession cone of P. Let $S_i = \{x \in \mathbb{R}^n \,:\, \pi_0^i < (\pi^i)^T x < \pi_0^i + 1\}$ for $i \in \{1, \ldots, m\}$. Then $\mathbb{R}^n \setminus S_i = D_i^0 \cup D_i^1$ where $D_i^0 = \{x \in \mathbb{R}^n \,:\, (\pi^i)^T x \leq \pi_0^i\}$ and $D_i^1 = \{x \in \mathbb{R}^n \,:\, (\pi^i)^T x \geq \pi_0^i + 1\}$.

We first write $P \setminus (\bigcup_{i=1}^m S_i)$ as a union of polyhedral sets. To this end, let $B = \{0,1\}^m$ and consider sets $P^b = P \cap D^b$ for $b \in B$ where

$$D^b = D_1^{b_1} \cap D_2^{b_2} \cap \ldots \cap D_m^{b_m}.$$

We can then write

$$P \setminus (\bigcup_{i=1}^m S_i) = \bigcup_{b \in B} P^b. \tag{10}$$

To prove the Lemma, we will show that the convex hull of the right-hand-side of equation (10) is polyhedral. If $P_\infty = \{0\}$, then this claim is trivially true. Therefore, assume $P_\infty \neq \{0\}$ and let $P_\infty^b = P^b + P_\infty$. Note that, by definition, $P_\infty^b = \emptyset$ if $P^b = \emptyset$.

We first show that the recession cone of $\text{conv}(P \setminus (\bigcup_{i=1}^m S_i))$ equals P_∞ when $P \setminus (\bigcup_{i=1}^m S_i)$ is nonempty. Let $x \in \text{conv}(P \setminus (\bigcup_{i=1}^m S_i))$. Then, for some $t > 0$ we have

$$x = \sum_{j=1}^{t} \lambda_j x_j \quad \text{where} \quad \begin{array}{l} x_1, \ldots, x_t \in P \setminus (\bigcup_{i=1}^{m} S_i), \\ 0 \le \lambda_1, \ldots, \lambda_t \le 1 \text{ and } \lambda_1 + \cdots \lambda_t = 1. \end{array}$$

Let $d \in P_\infty$. Then, for all $j = 1, \ldots, t$ we have $x_j + \alpha d \in P$ for all $\alpha \ge 0$. Furthermore, $x_j + \alpha d \notin S_i$ for all $\alpha \ge \alpha_{ij}$ where

$$\alpha_{ij} = \begin{cases} 0 & \text{if } d^T\pi^i = 0 \\ (\pi_0^i + 1 - x_j^T\pi^i)/d^T\pi^i & \text{if } d^T\pi^i > 0 \\ (\pi_0^i - x_j^T\pi^i)/d^T\pi^i & \text{if } d^T\pi^i < 0. \end{cases}$$

Consequently, letting $\alpha^* \ge 0$ be an upper bound on all α_{ij}s, we conclude that

$$\sum_{j=1}^{t} \lambda_j (x_j + \alpha d) = x + \alpha d \in \text{conv}(P \setminus (\bigcup_{i=1}^{m} S_i)) \quad \text{for all } \alpha \ge \alpha^*.$$

As x is contained in $\text{conv}(P \setminus (\bigcup_{i=1}^m S_i))$, it follows that so is $x + \alpha d$ for all $\alpha \ge 0$. Therefore $\text{rec}(\text{conv}(P \setminus (\bigcup_{i=1}^m S_i))) = \text{rec}(\text{conv}(\bigcup_{b\in B} P^b)) = P_\infty$. Then

$$\begin{aligned} \text{conv}(\bigcup_{b\in B} P^b) = \text{conv}(\bigcup_{b\in B} P^b) + P_\infty &= \text{conv}(\bigcup_{b\in B} P^b) + \text{conv}(P_\infty) \\ &= \text{conv}((\bigcup_{b\in B} P^b) + P_\infty) = \text{conv}(\bigcup_{b\in B} (P^b + P_\infty)). \end{aligned}$$

But the last convex hull is a polyhedron as each $P^b + P_\infty$ is a polyhedron with the same recession cone P_∞. The result follows. ▪ ▪

We next make an elementary observation, the proof of which follows from [10].

Lemma 13. *Let P be a polyhedron and let F be a face of P. For any set B,* $\text{conv}(P \setminus B) \cap F = \text{conv}(F \setminus B)$.

We next extend a result of Cook, Kannan and Schrijver [10] to handle polyhedra which are not full-dimensional. We denote the dimension of a polyhedron P by $\dim(P)$. We let $B(u, r) = \{x \in \mathbb{R}^n : ||x - u|| \le r\}$ stand for a closed ball of radius r centered at the point u. If the set K is not full-dimensional, then we refer to a ball in K as a set of the form $B(u, r) \cap \text{aff}(K)$ for some $u \in K$; its radius is the distance of its boundary from u.

Lemma 14. *Let $Q, W \subset \mathbb{R}^n$ be pointed polyhedra such that $W \subset Q$. Let $\hat{x} \in \text{relint}(Q) \cap V(W)$ and $c^Tx \le \gamma$ be a valid inequality for W but not for Q such that $c^T\hat{x} = \gamma$. Then there exists a $\dim(Q)-$dimensional ball $B \subset Q \setminus W$ of radius $r > 0$ (where r is independent of $\hat{x}$ and c) such that $c^Tx > \gamma$ for all $x \in B$.*

Proof. Let $V = \text{relint}(Q) \cap V(W)$. Further, let L denote the linear subspace parallel to the affine hull of Q and let $L^\perp$ be the orthogonal linear subspace. Without loss of generality,

$$Q = \{x : a_i^T x \leq b_i \text{ for } i = 1, \ldots, m\} \cap \text{aff}(Q)$$

where $a_i \in L$ and $||a_i|| = 1$ for all $i = 1, \ldots, m$. Note that $m > 0$, as Q is pointed and has a nonempty relative interior. Furthermore, as $a_i^T v < b_i$ for all $v \in V$ and $i = 1, \ldots, m$, there exists an $\epsilon > 0$ such that

$$b_i - a_i^T v \geq \epsilon$$

for all $v \in V$ and $i = 1, \ldots, m$.

The vector c defining the valid inequality can be written as $c = \hat{c} + \bar{c}$ where $\bar{c} \in L$ and $\hat{c} \in L^\perp$. By scaling, we assume that $||\bar{c}|| = 1$. Clearly, for any point $x \in \text{aff}(Q)$ we have $\hat{c}^T x = \mu$ for some constant $\mu \in \mathbb{R}$, and consequently $c^T x - \mu = \bar{c}^T x$ for all $x \in Q$. As $c^T x \leq \gamma$ is valid for W but not for Q, the same holds for the inequality $\bar{c}^T x \leq \gamma - \mu$ and therefore

$$z^W = \max\{\bar{c}^T x : x \in W\} < \max\{\bar{c}^T x : x \in Q\} = z^Q.$$

Let the first maximum be obtained at a vertex v^W, and let the second maximum be obtained at a vertex v^Q.

By LP duality, there exists multipliers $\lambda \in \mathbb{R}^m_+$ such that $\bar{c} = \sum_{i=1}^m \lambda_i a_i$ and $z^Q = \sum_{i=1}^m \lambda_i b_i$. Furthermore as $||\bar{c}|| = ||a_i||$ for all $i = 1, \ldots, m$, we have $\sum_{i=1}^m \lambda_i \geq 1$ and therefore,

$$z^Q - z^W = z^Q - \bar{c}^T v^W = \sum_{i=1}^m \lambda_i (b_i - a_i^T v^W) \geq \epsilon.$$

As $||\bar{c}|| = 1$, , the distance between the hyperplanes $\bar{c}^T x = z^W$ and $\bar{c}^T x = z^Q$ is at least ϵ. Therefore, any point x in the n-dimensional ball $B(v^Q, \epsilon/2)$ satisfies $\bar{c}^T x > \bar{c}^T v^W$ (and also $c^T x > c^T v^W$).

Note that the diameter of the ball $B(v^Q, \epsilon/2)$ does not depend on the vector $\bar{c}$ but only on Q and W. Now consider $B(v^Q, \epsilon/2) \cap Q$ which has dimension $\dim(Q)$ and notice that it must contain a $\dim(Q)$-dimensional ball in $\text{aff}(Q)$ of radius $\delta(v^Q) > 0$. Letting r denote the smallest $\delta(v)$ over vertices $v \in V(Q)$, it follows that there exists a collection of balls $\mathcal{B}$ of common radius r, one per each vertex of Q, such that any inequality that satisfies the conditions of the lemma must separate one of the balls in $\mathcal{B}$ from W. ∎ ∎

A (open) *strip* in $\mathbb{R}^n$ is the set of points (strictly) between a pair of parallel hyperplanes, i.e., a set of the form $\{x \in \mathbb{R}^n : b \leq a^T x \leq b'\}$ for some $a \in \mathbb{R}^n$ and $b < b' \in \mathbb{R}$. The width of a strip is the distance between its bounding hyperplanes. The topological closure of a split set is a strip. The *minimum width* of a bounded closed convex set A is defined as the minimum width of a strip containing A and is denoted by $w(A)$. It is known (Bang [7]) that the sum of widths of a collection of strips containing A must exceed its minimum width. The following statement is a trivial consequence of Bang's result.

Lemma 15. *Let B be a ball of radius $r > 0$, and let S_1, S_2 be split sets such that $B \subseteq S_1 \cup S_2$. Then,*

$$w(S_1) + w(S_2) \geq 2r.$$

The next result generalizes a result in Cook, Kannan and Schrijver [10] on the action of split sets on full-dimensional polyhedra to the action of cross sets on polyhedra which may not be full-dimensional. We need some definitions. Given a list of pairs of split sets $\mathcal{C} \subseteq \mathcal{C}^*$ we define

$$\text{splits}(\mathcal{C}) = \bigcup_{\{S_1,S_2\}\in\mathcal{C}} \{S_1, S_2\}.$$

Thus splits($\mathcal{C}$) is simply the list of all split sets contained in the sets in $\mathcal{C}$.

Lemma 16. *Let $Q \subset \mathbb{R}^n$ be a pointed polyhedron and let $\mathcal{C} \subseteq \mathcal{C}^*$ be such that $Q \not\subseteq S_1 \cup S_2$ for any $\{S_1, S_2\} \in \mathcal{C}$. Let $W \subseteq Q$ be a polyhedron such that $W \cap F = \mathrm{CC}(F, \mathcal{C})$ for each proper face F of Q. Then there exists a finite set $\mathcal{S} \subseteq \text{splits}(\mathcal{C})$ such that if a cross cut derived from $\{S_1, S_2\} \in \mathcal{C}$ is not valid for W, then there is an $S' \in \mathcal{S}$ such that either $S_1 \cap \text{aff}(Q)$ or $S_2 \cap \text{aff}(Q)$ equals $S' \cap \text{aff}(Q)$.*

Proof. Let $c^T x \leq \mu$ be a cross cut derived from $\{S_1, S_2\} \in \mathcal{C}$. Suppose $c^T x \leq \mu$ is not valid for W. Then

$$\gamma = \max\{c^T x : x \in \text{conv}(Q \setminus (S_1 \cup S_2))\} \leq \mu < \max\{c^T x : x \in W\} < \infty. \tag{11}$$

The last inequality above follows from Lemma 12 which implies that $\text{rec}(Q) = \text{rec}(\text{conv}(Q \setminus (S_1 \cup S_2)))$ and from the fact that $\text{rec}(W) \subseteq \text{rec}(Q)$.

Suppose the second maximum above is obtained at a vertex v of W. If v is contained in a proper face F of Q, then using Lemma 13,

$$\begin{aligned} c^T v = \max\{c^T x : x \in W\} &= \max\{c^T x : x \in W \cap F\} \\ &= \max\{c^T x : x \in \mathrm{CC}(Q, \mathcal{C}) \cap F\} \\ &\leq \max\{c^T x : x \in \mathrm{CC}(Q, \mathcal{C})\} \\ &\leq \max\{c^T x : x \in \text{conv}(Q \setminus (S_1 \cup S_2))\} = \gamma, \end{aligned}$$

a contradiction to the fact that $\gamma < c^T v$, stated in equation (11).

Therefore, we can assume v is in the relative interior of Q. As W is contained in Q and is a pointed polyhedron, Lemma 14 implies that there exists a ball B of radius r (for some fixed $r > 0$) in the relative interior of Q with all points in the ball satisying $c^T x > c^T v$.

Case 1: Q is full-dimensional. In this case, for any $S \in \mathcal{S}^*$, $S \cap \text{aff}(Q) = S$. Clearly $B \subseteq S_1 \cup S_2$, otherwise there exists an $x \in B \setminus (S_1 \cup S_2) \subseteq \text{conv}(Q \setminus (S_1 \cup S_2))$; such an x satisfies $c^T x > c^T v > \gamma$, a contradiction to (11). Let $S_i = S(\pi^i, \pi^i_0)$. Lemma 15 implies that $w(S_1) + w(S_2) \geq 2r$ which implies that for either $i = 1$ or $i = 2$ we have (a) $S_i \cap B \neq \emptyset$ and (b) $w(S_i) \geq r$, i.e., $||\pi^i|| \leq 1/r$. There are only finitely many split sets in $\mathcal{S}^*$ which satisfy properties (a) and (b). Let $\mathcal{S}$ stand for the set of such split sets in splits($\mathcal{C}$). Thus either $S_1 \in \mathcal{S}$ or $S_2 \in \mathcal{S}$.

Case 2: Let $k = \dim(Q) < n$. By applying a unimodular transformation to Q, we can assume that $\text{aff}(Q) = \{x \in \mathbb{R}^n : x_{k+1} = \alpha_1, \ldots, x_n = \alpha_{n-k}\}$, where $\alpha \in \mathbb{R}^{n-k}$ is a rational vector, and $\alpha_i = p_i/\Delta$ where p_i is an integer, for $i = 1, \ldots, n-k$ and Δ is a positive integer. Recall that any unimodular transformation maps split sets to split sets and therefore the cross sets are mapped to cross sets as well. Assume $S_1 = S(\pi, \pi_0)$. For $\pi \in \mathbb{Z}^n$ we define $\bar{\pi} \in \mathbb{Z}^k$ to denote the first k components of π and $\hat{\pi} \in \mathbb{Z}^{n-k}$ to denote the remaining components. Then

$$S_1 \cap \text{aff}(Q) = \bar{S}_1 \times \{\alpha\} \text{ where } \bar{S}_1 = \{x \in \mathbb{R}^k : \pi_0 - \alpha^T\hat{\pi} < \bar{\pi}^T x < \pi_0 + 1 - \alpha^T\hat{\pi}\}.$$

Therefore $\bar{S}_1$ is a strip in $\mathbb{R}^k$ with $w(\bar{S}_1) = 1/||\bar{\pi}||$. Let $\bar{S}_2$ be defined similarly in terms of $S_2 = S(\pi', \pi'_0)$.

Since B is contained in the relative interior of Q, we can write $B = \bar{B} \times \{\alpha\}$, where $\bar{B} \subseteq \mathbb{R}^k$ is a full-dimensional ball of radius r. Now any point x in $\bar{B} \setminus (\bar{S}_1 \cup \bar{S}_2)$ satisfies $c^T x > \gamma$, and therefore $\bar{B} \subseteq \bar{S}_1 \cup \bar{S}_2$, which implies that either (i) $w(\bar{S}_1) \geq r$ and $\bar{S}_1 \cap \bar{B} \neq \emptyset$ or (ii) $w(\bar{S}_2) \geq r$ and $\bar{S}_2 \cap \bar{B} \neq \emptyset$. Without loss of generality, assume (i) holds and therefore $||\bar{\pi}|| \leq 1/r$. Consequently, the first k coefficients of π come from a bounded set, but the last $n-k$ coefficients can be arbitrary integers. However, as α is rational, $\alpha^T\hat{\pi}$ is an integral multiple of $1/\Delta$. In other words, the set $\bar{S}_1$ has the form

$$S(\tau, \tau_0, \delta) = \{x \in \mathbb{R}^k : \tau_0 - \delta < \tau^T x < \tau_0 + 1 - \delta\}, \tag{12}$$

where $\tau \in \mathbb{Z}^k$, $\tau_0 \in \mathbb{Z}$ and $1 > \delta \geq 0$ is an integral multiple of $1/\Delta$. Furthermore, there are only finitely many choices of tuples (τ, τ_0, δ). Let

$$\Phi = \left\{(\tau, \tau_0, \delta) \in \mathbb{Z}^k \times \mathbb{Z} \times \frac{1}{\Delta}\mathbb{Z} \;:\; S(\tau, \tau_0, \delta) \cap \bar{B} \neq \emptyset,\; w(S(\tau, \tau_0, \delta)) \geq r\right\}$$

be the collection of such tuples. Each $S(\pi, \pi_0) \in \text{splits}(\mathcal{C})$ is associated with a unique tuple $(\tau, \tau_0, \delta) \in \Phi$, where $\delta = a^T\pi - \lfloor a^T\pi \rfloor$, $\tau_0 = \pi_0 - \lfloor a^T\pi \rfloor$ and $\tau = \bar{\pi}$ (i.e the first k components of π). Let $\bar{\Phi} \subseteq \Phi$ be the collection of tuples that have an associated split set in splits($\mathcal{C}$). Now construct a set $\mathcal{S} \subseteq \text{splits}(\mathcal{C})$ that contains exactly one split set S for each $\phi \in \bar{\Phi}$. Clearly, $\mathcal{S}$ is a finite set and it contains a split set $S \in \mathcal{S}$ for the tuple ϕ associated with $\bar{S}_1$. In other words, $S_1 \cap \text{aff}(Q) = S \cap \text{aff}(Q)$ and the proof is complete. ∎ ∎

We now prove our main result using the previous Lemma.

Theorem 4. *Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron and let $\mathcal{C} \subseteq \mathcal{C}^*$ be given. Then*

$$\text{CC}(P, \mathcal{C}) = \bigcap_{\{S_1, S_2\} \in \hat{\mathcal{C}}} \text{conv}\left(P \setminus (S_1 \cup S_2)\right)$$

where $\hat{\mathcal{C}} \subseteq \mathcal{C}$ is a finite set. Consequently, $\text{CC}(P, \mathcal{C})$ *is a polyhedron.*

Proof. If $P = \emptyset$, the claim clearly holds and therefore we consider the case when $P \neq \emptyset$. In addition, by equations (8) and (9), we only need to show the result for pointed polyhedra and therefore we can assume that P is pointed. Furthermore, if P is contained in $S_1 \cup S_2$ for some $\{S_1, S_2\} \in \mathcal{C}$, then $\text{CC}(P, \mathcal{C}) = \text{CC}(P, \{\{S_1, S_2\}\}) = \emptyset$ and the result follows. We therefore assume that this condition does not hold.

The proof is by induction on $\dim(P) \leq n$. For the base case, let $\dim(P) = 0$, i.e., P has a single point. As P is not contained in $S_1 \cup S_2$ for any $\{S_1, S_2\} \in \mathcal{C}$, we have $\text{CC}(P, \mathcal{C}) = P = CC(P, \{C'\})$ where C' is an arbitrary element in $\mathcal{C}$.

For the inductive step, we assume that $\mathrm{CC}(Q,\mathcal{C})$ is defined by a finite set of cross sets for all polyhedra Q of dimension strictly less than $\dim(P)$. Let F be a proper face of P. Since $\dim(F) < \dim(P)$, by the induction hypothesis we infer that there exists a finite set $\mathcal{C}(F) \subseteq \mathcal{C}$ such that $\mathrm{CC}(F,\mathcal{C}) = \mathrm{CC}(F,\mathcal{C}(F))$. Let

$$\tilde{\mathcal{C}} = \bigcup_{F \text{ is a proper face of } P} \mathcal{C}(F),$$

and note that $\tilde{\mathcal{C}}$ is finite. Then for any proper face F of P, we have

$$\mathrm{CC}(P,\tilde{\mathcal{C}}) \cap F = \mathrm{CC}(F,\mathcal{C})$$

as $\mathrm{CC}(P,\tilde{\mathcal{C}}) \cap F = \mathrm{CC}(F,\tilde{\mathcal{C}})$ by Lemma 13 and $\mathrm{CC}(F,\tilde{\mathcal{C}}) = \mathrm{CC}(F,\mathcal{C})$ follows from $\mathrm{CC}(F,\mathcal{C}(F)) = \mathrm{CC}(F,\mathcal{C})$ and $\mathcal{C}(F) \subseteq \tilde{\mathcal{C}} \subseteq \mathcal{C}$. Note that $\mathrm{CC}(P,\tilde{\mathcal{C}}) \subseteq P$ is a pointed polyhedron.

Applying Lemma 16 with P in place of Q and $\mathrm{CC}(P,\tilde{\mathcal{C}})$ in place of W, we infer that there is a finite set $\mathcal{S} \subseteq \mathrm{splits}(\mathcal{C})$ such that if a cross cut derived from $\{S_1, S_2\} \in \mathcal{C}$ is not valid for $\mathrm{CC}(P,\tilde{\mathcal{C}})$, then there is an $S' \in \mathcal{S}$ such that either $S_1 \cap \mathrm{aff}(P)$ or $S_2 \cap \mathrm{aff}(P)$ equals $S' \cap \mathrm{aff}(P)$.

Therefore,

$$\begin{aligned}\mathrm{CC}(P,\mathcal{C}) &= \bigcap_{\{S_1,S_2\}\in\mathcal{C}} \mathrm{conv}\,(P \setminus (S_1 \cup S_2)) \\ &= \mathrm{CC}(P,\tilde{\mathcal{C}}) \cap \bigcap_{\substack{S\in\mathcal{S} \\ \{S_1,S_2\}\in\mathrm{match}(S,\mathcal{C})}} \mathrm{conv}\,(P \setminus (S_1 \cup S_2)) \qquad (13)\end{aligned}$$

where

$$\mathrm{match}(S,\mathcal{C}) = \{\{S_1, S_2\} \in \mathcal{C} : S_1 \cap \mathrm{aff}(P) = S \cap \mathrm{aff}(P) \text{ or } S_2 \cap \mathrm{aff}(P) = S \cap \mathrm{aff}(P)\}.$$

Note that the last convex hull in (13) satisfies

$$\mathrm{conv}\,(P \setminus (S_1 \cup S_2)) = \begin{cases} \mathrm{conv}\,((P \setminus S) \setminus S_2)) & \text{if } S_1 \cap \mathrm{aff}(P) = S \cap \mathrm{aff}(P), \\ \mathrm{conv}\,((P \setminus S) \setminus S_1)) & \text{if } S_2 \cap \mathrm{aff}(P) = S \cap \mathrm{aff}(P). \end{cases}$$

Therefore, for any $S \in \mathcal{S}$,

$$\bigcap_{\{S_1,S_2\}\in\mathrm{match}(S,\mathcal{C})} \mathrm{conv}\,(P \setminus (S_1 \cup S_2)) = \mathrm{SC}(P \setminus S, \mathrm{partner}(S,\mathcal{C}))$$

where

$$\mathrm{partner}(S,\mathcal{C}) \;=\; \{S_1 \in \mathcal{S}^* : \exists S_2 \in \mathcal{S}^* \text{ such that } \{S_1, S_2\} \in \mathcal{C} \text{ and } S_2 \cap \mathrm{aff}(P) = S \cap \mathrm{aff}(P)\}.$$

For any split set $S \in \mathcal{S}$, the set $P \setminus S$ is a union of two pointed rational polyhedra (possibly empty), and therefore Theorem 6 implies that $\mathrm{SC}(P \setminus S, \mathrm{partner}(S,\mathcal{C}))$ is finitely generated. We can conclude from this that

$$\bigcap_{\{S_1,S_2\}\in\mathrm{match}(S,\mathcal{C})} \mathrm{conv}\,(P \setminus (S_1 \cup S_2)) = \bigcap_{\{S_1,S_2\}\in\mathcal{C}(S)} \mathrm{conv}\,(P \setminus (S_1 \cup S_2))$$

for some finite $\mathcal{C}(S) \subseteq \text{match}(S, \mathcal{C})$. Finally, as the set $\mathcal{S}$ is finite, we conclude that $\text{CC}(P, \mathcal{C})$ is finitely generated and

$$\text{CC}(P, \mathcal{C}) = \text{CC}\left(P, \tilde{\mathcal{C}} \cup \bigcup_{S \in \mathcal{S}} \mathcal{C}(S)\right).$$

Furthermore, by Lemma 12, $\text{CC}(P, \mathcal{C})$ is a polyhedron. ■ ■

5 Mixed-integer Sets

Consider a mixed-integer set defined by a polyhedron $P^{LP} \subseteq \mathbb{R}^{n+l}$ and the mixed-integer lattice $\mathbb{Z}^n \times \mathbb{R}^l$ where n and l are positive integers:

$$P^I = P^{LP} \cap (\mathbb{Z}^n \times \mathbb{R}^l). \tag{14}$$

We next present the extension of the main results from the earlier sections to mixed-integer sets.

5.1 Split closure of a union of mixed-integer sets

An inequality is called a split cut for P^{LP} with respect to the lattice $\mathbb{Z}^n \times \mathbb{R}^l$ if it is valid for $\text{conv}(P^{LP} \setminus S)$ for some $S \in \mathcal{S}^*_{n,l}$ where

$$\mathcal{S}^*_{n,l} = \{S(\pi, \pi_0) \in \mathcal{S}^* \ : \ \pi \in \mathbb{Z}^n \times \{0\}^l\}.$$

The split closure is then defined in the usual way as the intersection of all such split cuts. A straightforward extension of Theorem 6 is the following:

Corollary 17. *Let $P_k \in \mathbb{R}^{n+l}$ be a rational polyhedron for $k \in K$ where K is a finite set and let $P = \bigcup_{k \in K} P_k$. Then $\text{SC}(P, \mathcal{S})$ is finitely generated for any $\mathcal{S} \subseteq \mathcal{S}^*_{n,l}$.*

5.2 Cross closure of mixed-integer sets

Similarly, an inequality is called a cross cut for P^{LP} with respect to the lattice $\mathbb{Z}^n \times \mathbb{R}^l$ if it is valid for $\text{conv}(P^{LP} \setminus (S_1 \cup S_2))$ for some $S_1, S_2 \in \mathcal{C}^*_{n,l}$, where $\mathcal{C}^*_{n,l}$ denotes the collection of pairs of split sets from $\mathcal{S}^*_{n,l}$. For any given subset of $\mathcal{C}^*_{n,l}$, the cross closure is then defined in the usual way. A straightforward extension of Theorem 4 is the following:

Corollary 18. *Let $P \in \mathbb{R}^{n+l}$ be a rational polyhedron, then $\text{CC}(P, \mathcal{C})$ is finitely generated for any $\mathcal{C} \subseteq \mathcal{C}^*_{n,l}$.*

6 The quadrilateral closure of the two-row continuous group set

A recent topic of interest is the generation of cutting planes for mixed-integer programs from canonical k-row mixed-integer sets where k is a small integer. These sets resemble the simplex tableau of a k-row MIP where all basic variables are free integer variables and all nonbasic variables are nonnegative continuous variables. Clearly, these sets can be obtained simply by selecting some of the rows of the simplex tableau associated with the LP relaxation of an MIP. In this case, the resulting relaxation can be viewed as a relaxation of the corner polyhedron associated with the basis defining the tableau. These relaxations are also called *k-row continuous group* relaxations. All the nontrivial valid inequalities for the canonical k-row set are intersection cuts (a concept introduced by Balas [6]) that are derived using maximal lattice-free convex sets in $\mathbb{R}^k$. We next discuss the case when $k = 2$ in more detail.

6.1 The two-row continuous group relaxation

Andersen, Louveaux, Weismantel and Wolsey [3] studied the *two-row continuous group relaxation*

$$T = \Big\{(x, s) \in \mathbb{Z}^2 \times \mathbb{R}^n : \ x - rs = f, \ s \geq 0\Big\}, \tag{15}$$

where $r = [r_1, r_2, ..., r_n] \in \mathbb{R}^{2\times n}$ and $f \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ and both r and f are rational. Furthermore, assume that no column of r is equal to the zero vector. Let T^{LP} denote the continuous relaxation of T. Andersen et. al. showed that all facet-defining inequalities of $\text{conv}(T)$ are *two-dimensional lattice-free* cuts for T, and we discuss this family of cuts shortly.

Such a set T can be viewed as a relaxation of an integer program obtained by taking two rows of an optimal simplex tableau of its LP relaxation corresponding to basic integer variables, and then relaxing all bounds on the basic variables, and relaxing the upper bounds and integrality restrictions on the nonbasic variables.

A *lattice-free* convex set in $\mathbb{R}^2$ is one which contains no integer point in its interior. We denote the interior of a convex set B by $\text{int}(B)$, the boundary by $\text{bnd}(B)$, and also recall that recession cone if B is denoted by $\text{rec}(B)$. A set B is called a maximal lattice-free convex set if B is lattice-free and there does not exist a convex set B' such that B' is lattice-free and $B' \supseteq B$. Let B be any lattice-free convex set in $\mathbb{R}^2$ containing f in its interior. The set B yields the cut $\sum_{i=1}^n \alpha_i s_i \geq 1$, valid for T, where the coefficients α_i are computed as follows:

$$\alpha_i = \begin{cases} 0 & \text{if } r_i \in \text{rec}(B), \\ 1/\lambda : \lambda > 0 \text{ and } f + \lambda r_i \in \text{bnd}(B) & \text{if } r_i \notin \text{rec}(B). \end{cases} \tag{16}$$

More precisely, $\sum_{i=1}^n \alpha_i s_i \geq 1$ is a valid inequality for the set $T^{LP} \setminus (\text{int}(B) \times \mathbb{R}^n)$; see [3].

In $\mathbb{R}^2$, any maximal, full-dimensional, lattice-free convex set is a polyhedron with at most 4 facets and is one of the following sets [16], see Figure 3.

1. A split set $\{(x_1, x_2) : b \leq a_1x_1 + a_2x_2 \leq b + 1\}$ where a_1 and a_2 are coprime integers and b is an integer.

2. A triangle with at least one integral point in the relative interior of each of its sides which in is either

   (a) A type 1 triangle, i.e., a triangle with integral vertices and exactly one integral point in the relative interior of each side;

(b) A type 2 triangle, i.e., one with at least one fractional vertex v, exactly one integral point in the relative interior of the two sides incident to v and at least two integral points on the third side;

(c) A type 3 triangle, i.e., a triangle with exactly three integral points on the boundary.

3. A quadrilateral containing exactly one integral point in the relative interior of each of its sides.

If a maximal lattice-free convex set B with f in its interior is a quadrilateral, then the cut generated using B via (16) is called a *quadrilateral cut*. Similarly, if B is a maximal lattice-free triangle of type 1, 2, or 3, the associated cut is called a *triangle cut* of type 1, 2, or 3, respectively. Dash, Dey and Günlük [13] show that the quadrilateral cuts and triangle cuts of type 1 or 2 are cross cuts, based on the fact that lattice-free sets of these types are contained in an appropriately chosen cross set, see Figure 3. Andersen et. al. [3] showed that the convex hull of T is given by split cuts, quadrilateral cuts, and triangle cuts.

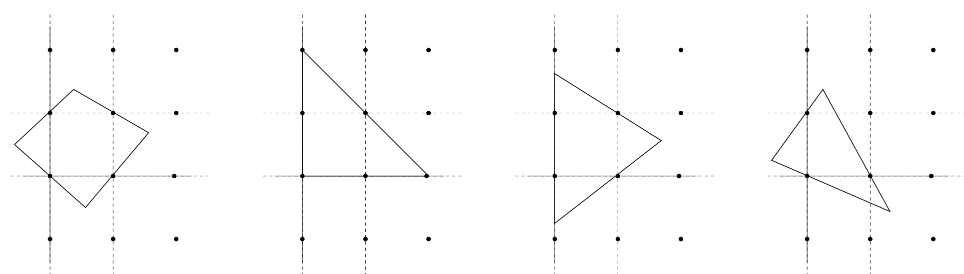

Figure 3: Maximal lattice free sets in $\mathbb{R}^2$: a quadrilateral and triangles of type 1, 2, and 3.

6.2 The quadrilateral closure

Basu, Bonami, Cornuéjols and Margot [8] define the *triangle closure* of T as the set of points in T^{LP} satisfying all triangle cuts, and the *quadrilateral closure* of T as the set of points in T^{LP} satifying all quadrilateral cuts for T, and ask whether these sets are polyhedra. Basu, Bonami, Cornuéjols and Margot [8] show that the quadrilateral closure of T satisfies all split cuts for T. Awate, Cornuéjols, Guenin and Tuncel [22, Theorems 1.3,1.4] further show that the quadrilateral closure also satisfies all triangle cuts of types 1 and 2. Basu, Hildebrand and Köeppe [9] also show that the triangle closure of T is a polyhedron. We next show that the quadrilateral closure of T is a polyhedron as well.

Theorem 5. *The quadrilateral closure of T is a polyhedron.*

Proof. In [13], Dash, Dey and Günlük show that the quadrilateral closure is equivalent to the closure with respect to unimodular cross cuts, which they define to be the family of cross cuts defined by two split sets $S(\pi^1, \pi_0^1)$ and $S(\pi^2, \pi_0^2)$ where $\pi^1, \pi^2 \in \mathbb{Z}^2$ and the 2×2 matrix with columns consisting of π^1 and π^2 has determinant ± 1. Let

$$\mathcal{C}^U = \Big\{ \{S_1, S_2\} \in \mathcal{C} : \; S_i = S(\pi^i, \pi_0^i), \; i = 1, 2; \; \det([\pi^1, \pi^2]) = \pm 1 \Big\}$$

and notice that the quadrilateral closure of T equals $\mathrm{CC}(T^{LP}, \mathcal{C}^U)$ and therefore it is a polyhedron by Theorem 4. ∎ ∎

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