CERTIFICATES OF OPTIMALITY AND SENSITIVITY ANALYSIS USING GENERALIZED SUBADDITIVE GENERATOR FUNCTIONS: A TEST STUDY ON KNAPSACK PROBLEMS

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Abstract. We introduce a family of subadditive functions called Generator Functions for mixed integer linear programs. These functions were previously defined for pure integer programs with non-negative entries by Klabjan [13]. They are feasible in the subadditive dual and we show that they are enough to achieve strong duality. Several properties of the functions are shown. We then use this class of functions for generating certificates of optimality for MILPs. We have done a test study on Knapsack problems to see how good the certificates can be.

1. Introduction

An important aspect of computational research is the verifiability of reported benchmark results [18]. Generating certificates of optimality for mixed integer programming problems is getting more important every day. Especially when a very hard problem of large size is solved, checking the optimality of the solution at hand might not be easy. For example, see [1] where the authors provide a certificate of optimality for an optimal TSP tour through 85,900 cities. This certificate is mostly going over the branch and bound tree and checking optimality at each node, and also checking the validity of each cutting plane that has been added. The time needed to check the optimality using this certificate might take up to 568.9 hours (24 days). The actual time for solving the problem is 286.2 days so this is a good certificate (we will define what "good" means later in this paper).

Today most of the optimization problems in science and industry are solved using major commercial solvers such as Gurobi and/or IBM ILOG CPLEX. They are reliable to some extent. However, there are instances of mixed integer programming problems that both these solvers fail to solve correctly. See [4] for such examples and related discussions.

Subadditive Generator Functions defined by Klabjan [13] can serve as certificates of optimality for specific families of integer programs such as knapsack or set covering problems. However, they are restricted to pure integer programs with non-negative entries. In this paper, we first generalize these functions to any mixed integer linear program and then we will show that these functions are feasible in the subadditive dual and actually they are enough to get strong duality; i.e. if we consider this family only, strong duality still holds. Then we will make use of these functions for generating certificates of optimality.

Sensitivity analysis is another important topic in MILP studies. It has been studied in the work of Wolsey [19] using Chvátal functions and value functions of integer programs and also by John Hooker in [10] using branch and bound tree and inference duality (also see [9] for a survey). However, in both cases, for a large-size problem, it is not easy to perform sensitivity analysis. In this paper, we will see that if we have an optimal dual feasible function, we can generalize the tools for sensitivity analysis in linear programming to mixed integer programming.

The first section of this article is some preliminaries about subadditive duality. We generalize Klabjan's functions in the second part. In the third section, we show some more properties of these functions and in the last section, we present some of our computational experiments and numerical results.

Date: December 9, 2014.

Key words and phrases. Subadditive Functions , Subadditive Duality , Certificate of Optimality , Sensitivity Analysis , Knapsack Problem , Mixed Integer Programming , Optimization.

2. Preliminaries

Let m and n be positive integers. Let $A \in \mathbb{Q}^{m\times n}, b \in \mathbb{Q}^m, c \in \mathbb{Q}^n, N = \{1, ..., n\}, I \subseteq N$, and $C = N\backslash I$. Let $M(A) = \{v \in \mathbb{R}^m : v = Ax, x_i \in \mathbb{Z}_+ \forall i \in I, x_i \in \mathbb{R}_+ \forall i \in C\}$. Consider the following Mixed Integer Linear Program :

$$\begin{array}{lll} \min & c^T x & \\ \text{s.t.} & Ax = b & \qquad (MILP) \\ & x \geq 0 & \\ & x_i \in \mathbb{Z} \forall i \in I. & \end{array}$$

The subadditive dual of (MILP) is given by:

$$\begin{array}{llll} \max & F(b) & & \\ \text{s.t.} & F(a_j) \leq c_j & \text{for all } j \in I & \qquad (SD) \\ & \overline{F}(a_j) \leq c_j & \text{for all } j \in C & \\ & F \in \Gamma^m & & \end{array}$$

where a_j denotes the jth column of A, Γ^m is the set of subadditive functions $F : M(A) \to \mathbb{R} \cup \{-\infty\}$ with $F(0) = 0$ and $\overline{F}(d) = \limsup_{\delta\to 0^+} \frac{F(\delta d)}{\delta}$. (A function $F : \Delta \to \mathbb{R} \cup \{-\infty\}$ where $\Delta \subset \mathbb{R}^m$ is closed under addition (i.e. a monoid) is said to be subadditive if $F(x + y) \leq F(x) + F(y)$ for all $x, y \in \Delta$ with the convention that $-\infty - \infty = -\infty$.)

Lemma 2.1. *If $F \in \Gamma^m$, then for any $d \in \mathbb{R}^m$ with $\overline{F}(d) < \infty$ and $\lambda \geq 0$, $F(\lambda d) \leq \lambda \overline{F}(d)$.*

Proof. see [9]. □

Theorem 2.2. (Weak Duality) *Suppose x is a feasible solution to MILP and F is a feasible solution to the subadditive dual (SD). Then $F(b) \leq cx$.*

Proof.

$$F(b) = F(Ax) = F\left(\sum_{i=1}^{n} a_i x_i\right) = F\left(\sum_{i\in I} a_i x_i + \sum_{i\in C} a_i x_i\right) \leq F\left(\sum_{i\in I} a_i x_i\right) + F\left(\sum_{i\in C} a_i x_i\right)$$
$$\leq \sum_{i\in I} F(a_i)x_i + \sum_{i\in C} \overline{F}(a_i)x_i \leq c^T x.$$

The inequalities follow from subadditivity of F and Lemma 2.1 . □

Theorem 2.3. [11] *If the primal problem (MILP) has a finite optimum, then so does the dual problem (SD) and they are equal.*

Unlike linear programming, finding a dual optimal subadditive function does not seem to be straightforward. Two well-known families are known [2] though they appear difficult to work with. However, for the case when all the entries of A and b are nonnegative and all variables are required to be integers, Klabjan [13] defined a family of subadditive functions sufficient for strong duality that are computationally attractive.

In the case of linear programming, it is easy to see that the function $F_{LP}(d) = \max_{v\in\mathbb{R}^m}\{vd : vA \leq c\}$ is a feasible subadditive function and using this function, the subadditive dual will reduce to the LP duality.

The subadditive dual plays a very important role in the study of mixed integer programming. Any feasible solution to the dual gives a lower bound to the (MILP). A dual feasible function F with $F(b) = z_{IP}$ is a certificate of optimality for the (MILP). It will be equivalent to the dual vector in LP and the reduced cost of a column i can be defined as $c_i - F(a_i)$ and most of the other properties from LP can be extended to MILP, for example complementary slackness, and the fact that all optimal solutions can be found only among the columns i with $c_i = F(a_i)$, if F is optimal.

3. Generalized Generator Subadditive Functions

Klabjan [13] defined Subadditive Generator Functions for pure integer programming problems with nonnegative entries.

For a pure integer program

$$
\begin{array}{ll}
\min & cx \\
\text{s.t.} & Ax = b \\
& x \geq 0, \text{ integer}
\end{array}
\tag{3.1}
$$

with non-negative A and b, a *generator subadditive function* is defined for a given $\alpha \in \mathbb{R}^m$ as:

$$F_\alpha(\ell) = \alpha\ell - \max\left\{ \sum_{i\in E(\alpha)} (\alpha a_i - c_i)x_i : \sum_{i\in E(\alpha)} a_i x_i \leq \ell, x \geq 0, \text{ integer} \right\}$$

where $E(\alpha) = \{i \in N \ : \ \alpha a_i > c_i\}$.

Also a *ray generator subadditive function* is defined for some given $\beta \in \mathbb{R}^m$ as:

$$G_\beta(\ell) = \beta\ell - \max\left\{ \sum_{i\in E(\beta)} (\beta a_i)x_i \ : \sum_{E(\beta)} a_i x_i \leq \ell \ , x \geq 0, \text{ integer} \right\}$$

where $E(\beta) = \{i \in N \ : \ \beta a_i > 0\}$.

In this paper, we extend Klabjans work to (MILP). Let $\alpha \in \mathbb{R}^m$ and define $E(\alpha) = \{i \in N : \alpha a_i > c_i\}$. Let $E \subseteq N$ be such that $E(\alpha) \subseteq E$. Also let $K \subset \mathbb{R}^m$ be such that

$$\left\{ \sum_{i\notin E} x_i a_i : x \geq 0, x_i \in \mathbb{Z} \forall i \in I \backslash E \right\} \subseteq K. \tag{3.2}$$

Definition 3.1. For $\alpha \in \mathbb{R}^m, E$ and K define the Subadditive Generator Function $F_{\alpha,E,K} : M(A) \to \mathbb{R} \cup \{-\infty\}$ by

$$F_{\alpha,E,K}(\ell) \quad = \quad \alpha^T \ell - \max\left\{ \sum_{i\in E} (\alpha a_i - c_i)x_i : \ell - \sum_{i\in E} a_i x_i \in K, x \geq 0, x_i \in \mathbb{Z} \forall i \in E \cap I \right\}.$$

Also for $E(\beta) = \{i \in N : \beta a_i > 0\}$ a Ray Generator Subadditive Function $G_{\beta,E,K} : M(A) \to \mathbb{R} \cup \{-\infty\}$ is defined as:

$$G_{\beta,E,K}(\ell) = \beta\ell - \max\left\{ \sum_{i\in E} (\beta a_i)x_i \ : \ \ell - \sum_{i\in E} a_i x_i \in K \ , x \geq 0, x_i \in \mathbb{Z} \forall i \in E \cap I \right\}.$$

Here we state a result by Meyer [14] that we will use later.

Theorem 3.2. *(Meyer [14]) Given a rational matrix A, G and a rational vector b, let $P := \{(x, y) : Ax + Gy \leq b\}$ and let $S := \{(x, y) \in P : x$ integral $\}$.*

1. *There exist rational matrices A', G' and a rational vector b' such that $conv(S) = \{(x, y) : A'x + G'y \leq b'\}$.*
2. *If S is nonempty, the recession cones of $conv(S)$ and P coincide.*

Theorem 3.3. *$F_{\alpha,E,K}(a_i) \leq c_i$ for all $i \in I$ and $\overline{F}_{\alpha,E,K}(a_i) \leq c_i$ for all $i \in C$.*

Proof. First we will show that $F_{\alpha,E,K}(a_i) \leq c_i$ for all $i \in I$. We have

$$F_{\alpha,E,K}(a_i) \quad = \quad \alpha a_i - \max\left\{ \sum_{i\in E} (\alpha a_i - c_i)x_i : a_i - \sum_{i\in E} a_i x_i \in K, x_i \in \mathbb{Z} \forall i \in E \cap I \right\}.$$

There are two cases: for $i \in I$, i is either in E or not. If $i \in E$, then $x = \delta^i$ is a feasible solution to the maximization problem where δ^i is the unit vector with 1 as the i-th component and zero otherwise. This gives us $F_{\alpha,E,K}(a_i) \leq c_i$. If $i \notin E$, then $x = 0$ is a feasible solution to the maximization problem and gives us $F_{\alpha,E,K}(a_i) \leq \alpha a_i \leq c_i$.

To show that $\overline{F}_{\alpha,E,K}(a_i) \le c_i$ for all $i \in C$, first note that Gomory and Johnson [7] show that if $\overline{F}_{\alpha,E,K}(\ell)$ is finite, then the limsup and the ordinary limit coincide. Now we have:

$$\begin{aligned}\overline{F}_{\alpha,E,K}(a_i) &= \lim_{h\to 0^+} \frac{F_{\alpha,E,K}(ha_i)}{h} \\ &= \alpha a_i - \lim_{h\to 0^+} \frac{1}{h} \max\left\{\sum_{i\in E}(\alpha a_i - c_i)x_i : ha_i - \sum_{i\in E} a_i x_i \in K, x_i \in \mathbb{Z}\forall i \in E\cap I\right\}.\end{aligned}$$

If i is in E, then $x = h\delta^i$ is a feasible solution to the maximization problem where δ^i is the unit vector. Then the maximum will be greater than or equal to $h(\alpha a_i - c_i)$, so the limit will be greater than or equal to $\alpha a_i - c_i$ since $h > 0$. This gives us $\overline{F}_{\alpha,E,K}(a_i) \le c_i$. If $i \notin E$, then $x = 0$ is a feasible solution to the maximization problem and a similar argument gives us $\overline{F}_{\alpha,E,K}(a_i) \le \alpha a_i \le c_i$. □

Theorem 3.4. *Let K be a finitely generated convex cone satisfying (3.2). Then $F_{\alpha,E,K}(\ell)$ is subadditive for any choice of α and E.*

Proof. Let $\ell_1, \ell_2 \in M(A)$. To show subadditivity i.e. $F_{\alpha,E,K}(\ell_1) + F_{\alpha,E,K}(\ell_1) \le F_{\alpha,E,K}(\ell_1 + \ell_2)$, it suffices to show

$$\begin{aligned}&\max\left\{\sum_{i\in E}(\alpha a_i - c_i)x_i : \ell_1 - \sum_{i\in E} a_i x_i \in K, x_i \in \mathbb{Z}\forall i \in E\cap I\right\} + \\ &\max\left\{\sum_{i\in E}(\alpha a_i - c_i)x_i : \ell_2 - \sum_{i\in E} a_i x_i \in K, x_i \in \mathbb{Z}\forall i \in E\cap I\right\} \le \\ &\max\left\{\sum_{i\in E}(\alpha a_i - c_i)x_i : \ell_1 + \ell_2 - \sum_{i\in E} a_i x_i \in K, x_i \in \mathbb{Z}\forall i \in E\cap I\right\}.\end{aligned}$$

If x_1^* and x_2^* are optimal solutions for two maximization problems on the left, then $x_1^* + x_2^*$ is a feasible solution to the problem on the right and the result follows. □

The following lemma shows that $F_{\alpha,E,K}(0) = 0$ for any $\alpha \in \mathbb{R}^m$.

Lemma 3.5. *If $F_{\alpha,E,K}$ is not identically $-\infty$, then $F_{\alpha,E,K}(0) = 0$.*

Proof. First note that

$$F_{\alpha,E,K}(0) = -\max\left\{\sum_{i\in E}(\alpha^T a_i - c_i)x_i : -\sum_{i\in E} a_i x_i \in K\right\}.$$

Since $x = 0$ satisfies $-\sum_{i\in E} a_i x_i \in K$, we get $F_{\alpha,E,K}(0) \le 0$.

Suppose that $F_{\alpha,E,K}(0) < 0$. If $F_{\alpha,E,K}(0) = -\infty$, then by subadditivity,

$$F_{\alpha,E,K}(\ell) \le F_{\alpha,E,K}(\ell) + F_{\alpha,E,K}(0) = -\infty \qquad \forall \ell \in M(A).$$

Hence, $F_{\alpha,E,K}(0)$ is identically $-\infty$, contradicting the assumption. If $F_{\alpha,E,K}(0) > -\infty$, then by subadditivity,

$$F_{\alpha,E,K}(0) \le F_{\alpha,E,K}(0) + F_{\alpha,E,K}(0)$$

implying that $F_{\alpha,E,K}(0) \ge 0$. The result now follows. □

Theorem 3.6. *(Strong Duality) If (MILP) is feasible, then there exist $\alpha \in \mathbb{R}^m, E$ and a finitely generated convex cone K with $F_{\alpha,E,K}(b) = z^*_{MILP}$ and $F_{\alpha,E,K}(0) = 0$. If (MILP) is infeasible, then there exist $\beta \in \mathbb{R}^m, E$ and a finitely generated convex cone K with $G_{\alpha,E,K}(b) > 0$.*

Proof. Let $K = \{x : x \ge 0\}$. Let (MILP) be feasible and let $\pi^j x \le \pi_0^j,\ j \in J$ be valid inequalities for the set

$$V = \{x : b - Ax \in K, x \ge 0, x_i \in \mathbb{Z}\forall i \in I\}$$

such that

$$\begin{aligned} z^*_{MILP} = \quad &\min \quad cx \\ &\text{s.t.} \quad Ax = b \\ &\qquad\ \ \pi^j x \le \pi_0^j \quad j \in J \\ &\qquad\ \ x \ge 0. \end{aligned}$$

This is possible by Theorem 3.2 where it is shown that if A and b in (MILP) are rational, then the convex hull of the feasible points is a polyhedron (finitely generated). Let (α, γ) be an optimal dual vector where γ corresponds to constraints $\pi^j x \leq \pi_0^j,\ j \in J$. Let $E = E(\alpha) \cup \{i \in N : a_i \ngeq 0\}$. Then we have

$$K = \{x : x \geq 0\} \supseteq \left\{ \sum_{i \notin E} a_i x_i : x \geq 0, x_i \in \mathbb{Z} \forall i \in I \backslash E \right\}.$$

We show that $F_{\alpha,E,K}(b) \geq z^*_{MILP}$.

The dual program of the above LP is

$$\begin{array}{ll} \max & b\alpha - \sum_{j \in J} \pi_0^j \gamma_j \\ \text{s.t.} & \alpha a_i - \sum_{j \in J} \pi_i^j \gamma_j \leq c_i \quad i \in N \\ & \alpha \text{ unrestricted}, \gamma \geq 0. \end{array}$$

The optimal value of this problem is z^*_{MILP}. Let x be a feasible solution to (MILP) such that $\sum_{i \in E} a_i x_i \leq b$. Such a vector exists since we have assumed that (MILP) is feasible and we know that E contains all the columns which are not entirely non-negative. So any column not in E will be entirely non-negative. We have

$$\begin{aligned} \sum_{i \in E} (\alpha a_i - c_i) x_i &\leq \sum_{i \in E} \sum_{j \in J} x_i \pi_i^j \gamma_j \\ &= \sum_{j \in J} \gamma_j (\sum_{i \in E} x_i \pi_i^j) \\ &\leq \sum_{j \in J} \gamma_j \pi_0^j = b\alpha - z^*_{MILP}. \end{aligned}$$

The last inequality holds because $\gamma \geq 0$ and the fact that $\pi^j x \leq \pi_0^j$ is a valid inequality for the set V. So we get that

$$\max \left\{ \sum_{i \in E} (\alpha a_i - c_i) x_i : b - \sum_{i \in E} a_i x_i \in K, x \geq 0, x_i \in \mathbb{Z} \forall i \in E \cap I \right\} \leq b\alpha - z^*_{MILP}.$$

So we have $F_{\alpha,E,K}(b) \geq z^*_{MILP}$. Also, we know that $F_{\alpha,E,K}(b) \leq z^*_{MILP}$ since $F_{\alpha,E,K}$ is feasible to subadditive dual problem for (MILP). So $F_{\alpha,E,K}(b) = z^*_{MILP}$. Also, by Lemma 3.5, $F_{\alpha,E,K}(0) = 0$.

Without loss of generality we can assume that $b \geq 0$ since otherwise one can multiply rows of $Ax = b$ corresponding to negative right hand side by negative constants to make b non-negative and then consider the new problem. If (MILP) is infeasible, with the same E and K as above, the problem

$$\begin{array}{ll} \min & \mathbb{1} u \\ \text{s.t.} & Ax + u = b \\ & x, u \geq 0 \\ & x_j \text{ integer for } j \in I \end{array}$$

has optimal value $z^* > 0$. So by the last case, there exists some β such that:

$$\begin{aligned} z^* &= b\beta - \max\{\beta A^E \tilde{x} + \sum_{i \in \tilde{E}} (\beta_i - 1) \tilde{u}_i : \\ & \quad b - A^E \tilde{x} - \tilde{u} \in K, x, u \geq 0, x_i \in \mathbb{Z} \forall i \in I \cap E\} \end{aligned}$$

where $\tilde{E} = \{i \in N : \beta_i > 1\} \cup \{i \in N : a_i \ngeq 0\}$ and $E = \{i \in N : \beta a_i > 0\} \cup \{i \in N : a_i \ngeq 0\}$. So we have:

$$0 < z^* \leq b\beta - \max\{\beta A^E \tilde{x} : b - A^E \tilde{x} \in K, \tilde{x} \geq 0, \tilde{x}_i \in Z \forall i \in E \cap I\}$$

and the proof is complete. $\square$

4. Properties of Generalized Subadditive Generator Functions

If F is any subadditive function with $F(0) = 0$ and dual feasible, then

$$\sum_{j \in I} F(a_j)x_j + \sum_{j \in C} \overline{F}(a_j)x_j \geq F(b)$$

is a valid inequality for (MILP).

It is enough to restrict our attention to a subset of generator functions called basic generator functions. These functions are enough to describe the convex hull of an MILP finitely.

Theorem 4.1. *The optimum value of (MILP) is equal to* $\max\{\eta : (\eta, \alpha) \in Q_b(E)\}$ *where*

$$\begin{aligned} Q_b(E) = \{(\eta, \alpha) \in (\mathbb{R} \times \mathbb{R}^m) : \alpha a_i \leq c_i \textit{ for } i \in N \backslash E, \\ \eta + \alpha(A^E x - b) \leq c^E x, \\ b - A^E x \in K, x \in \mathbb{Z}_+^{E \cap I} \times \mathbb{R}_+^{E \cap C}\} \end{aligned} \tag{4.1}$$

for some E, K.

Proof. By Theorem 3.6 there exists $\alpha \in \mathbb{R}^m$ with $F_{\alpha,E,K}(b) = z^*_{MILP}$. Choose $E = E(\alpha) \cup \{i \in N : a_i \not\geq 0\}$, let $K = \{x : x \geq 0\}$ and consider $Q_b(E)$. Then

$$\begin{aligned} \eta^* &= \max\{\eta : (\eta, \alpha) \in Q_b(E)\} \\ &= \max\{\eta : \eta \leq c^E x - \alpha(A^E x - b) : b - A^E x \in K, x \geq 0, x_i \in \mathbb{Z} \forall i \in I \cap E\} \\ &= \alpha b + \max\{(c^E - \alpha A^E)x : b - A^E x \in K, x_i \in \mathbb{Z} \forall i \in I \cap E\} \\ &= F_{\alpha,E,K}(b) = z^*_{MILP}. \end{aligned}$$

□

Theorem 4.2. *If K is a finitely generated convex cone, then $Q_b(E)$ is a polyhedron.*

Proof. K has a finite number of extreme rays. Let M denote the set of extreme rays of K. Then since M is finite, we have

$$\begin{aligned} Q_b(E) = \{(\eta, \alpha) \in (\mathbb{R} \times \mathbb{R}^m) : \quad & \alpha a_i \leq c_i \text{ for } i \in N \backslash E \\ & \eta + \alpha(A^E x - b) \leq c^E x \text{ for } x \in M\}. \end{aligned}$$

which is obviously a polyhedron (finitely generated). □

Definition 4.3. A generator function $F_{\alpha,E,K}$ is called basic if $(F_{\alpha,E,K}(b), \alpha)$ is an extreme point of (4.1).

Since there are finite choices for E and for each E, $Q_b(E)$ has a finite number of extreme points, there are only a finite number of basic generator functions. It is obvious that we only need basic generator subadditive functions. Since there are only a finite number of them, the following theorem holds.

Theorem 4.4. *Given A, b and c, there exists a finite set of subadditive generator functions $K(b)$ such that the linear program*

$$\begin{array}{ll} \min & cx \\ & Ax = b \\ & \sum_{j \in I} F_{\alpha,E,K}(a_j)x_j + \sum_{j \in C} \overline{F}_{\alpha,E,K}(a_j)x_j \geq F_{\alpha,E,K}(b) \quad F_{\alpha,E,K} \in K(b) \\ & x_j \geq 0 \quad \forall j \in N \end{array} \tag{4.2}$$

has the following properties:

1. *LP (4.2) is infeasible if and only if (MILP) is infeasible.*
2. *LP (4.2) has unbounded optimum value if and only if (MILP) has unbounded optimum value.*
3. *Otherwise LP (4.2) has an optimal extreme point solution which is also optimal for (MILP).*

Also, since we know that the convex hull of feasible solutions to (MILP) is polyhedral, i.e. it can be described by a finite set of facet defining valid inequalities, we have the following corollary.

Corollary 4.5. For A and b rational, there exists a finite set of subadditive generator functions $\{F_{\alpha_i,E,K}\}_{i=1}^R$ such that

$$\begin{array}{lll} & Ax = b & \\ (4.3) & \sum_{j\in I} F_{\alpha_i,E,K}(a_j)x_j + \sum_{j\in C} \overline{F}_{\alpha_i,E,K}(a_j)x_j \geq F_{\alpha_i,E,K}(b) & i = 1,...,R \\ & x_j \geq 0 & \forall j \in N \end{array}$$

is the convex hull of solutions to (MILP) with right hand side b.

5. Certificates of Optimality ans Sensitivity Analysis

In this section we show that subadditive generator functions can be used as a cerificate of optimality for mixed integer programming problems.

Definition 5.1. A *Certificate of Optimality* for an MILP is information that can be used to check optimality without having to solve the MILP itself.

Ideally we are interested in types of certificates that allow us to perform the checking in (much) shorter time. By Theorem 3.6 any subadditive generator function with $F_{\alpha,E,K}(b) = z^*_{MILP}$ for which E is smaller than N can be used as a certificate of optimality.

Definition 5.2. For (MILP) with optimal solution x^*, α is a certificate of optimality if $F_{\alpha,E,K}(b) = cx^*$. We call $\alpha \in \mathbb{R}^m$ a "good" certificate if $|E| << n$. α^* is called minimal if we have:

$$\alpha^* \in \operatorname*{argmin}_{\alpha\in\mathbb{R}^m} \{|E| : F_{\alpha,E,K}(b) = z^*_{MILP}\}.$$

The following example shows that optimal α may not be unique.

Example 5.3. Consider the following IP:

$$\begin{array}{ll} \min & 6x_1 + 6x_2 + 9x_3 + 10x_4 + 7x_5 \\ \text{s.t.} & x_4 + x_5 \geq 1 \\ & x_1 + x_2 + x_3 + x_4 \geq 1 \\ & x_1 + x_5 \geq 1 \\ & x \in \{0,1\}^5. \end{array}$$

Any $[\alpha_1, \alpha_2, \alpha_3]^T$ with $\alpha_1 = 7$ and $\alpha_2 = 6$ and $\alpha_3 \in [3, 7]$ gives $F_{\alpha,E,K}(b) = z^*_{MILP}$ with $E = E(\alpha) = \{1, 4, 5\}$. Any α with $|E| \leq 2$ will give $F_{\alpha,E,K}(b) < z^*_{MILP}$.

Note that if the size of E is much smaller than N, then the certificate that we have is very much easier to check since the number of variables is remarkably reduced. However, there still might be instances that $E = N$ or size of E is comparable to N. In this case obviously we don't have a good certificate. But our computational experiment will show that this is not the case for specific families of problems. For example consider the following pure ILP:

$$\begin{array}{ll} \min & x_1 + 2x_2 + \cdots + nx_n \\ \text{s.t.} & 2x_1 + 2x_2 + \cdots + 2x_{n-1} + x_n = 3 \\ & x \in \mathbb{Z}^n_+. \end{array}$$

Clearly, there is a unique optimal solution. But the set E will have to include $\{1, \ldots, n-1\}$ no matter what α we use that gives an optimal subadditive dual function. Note that for each variable x_i, there is a feasible solution with $x_i \neq 0$. So none of the variables can be eliminated right away. (If we make the coefficients of $x_1, \ldots, x_{n-1}$ in the objective function to be 1, we will have $n-1$ optimal solutions.)

Theorem 5.4. *Assume that $E = E(\alpha) \cup \{i \in N : a_i \not\geq 0\}$. Let $E(\alpha) = \emptyset$. Then α is an optimal certificate of optimality for (MILP) if and only if the optimal solution of linear programming relaxation of (MILP) solves (MILP).*

Proof. Without loss of generality assume that $b \geq 0$. If α is such that $E(\alpha) = \emptyset$, then $F_{\alpha,E,K}(b) = \alpha b$ with $\alpha A \leq c$. This means that α is the optimal solution to the linear programming dual of the LP relaxation of (MILP). Conversely suppose that α is the optimal solution to the linear programming dual of the LP relaxation. Then we have $F_{\alpha,E,K}(b) = \alpha b$ with $\alpha A \leq c$ and obviously $E(\alpha) =$. □

Remark 5.5. [8] For (MILP) with optimal solution x^*, α with $F_{\alpha,E,K}(b) = cx^*$ can be used for sensitivity analysis.

We refer the reader to [6] and [19] where Geoffrion and Nauss and Wolsey state the conditions under which primal feasibility, dual feasibility and optimality still hold for x^* and F^* (optimal primal vector and subadditive dual function) after changes made to the input of (MILP).

From now on we only focus on subadditive generator functions with $E = E(\alpha) \cup \{i \in N : a_i \ngeq 0\}$ and $K = \left\{\sum_{i \notin E} a_i x_i : x \geq 0, x_i \in \mathbb{Z} \forall i \in I\right\}$.

5.1. **Obtaining** α**.** Assume that (MILP) has been solved to optimality using cutting plane method. So we have access to all cutting planes that when added to the LP relaxation of (MILP), it gives us the optimal solution. However these cuts need to be lifted to become valid for the set V of Theorem 3.6. The following theorem shows that this is in fact possible.

Theorem 5.6. *The polyhedral set*

$$conv(\{x \in \mathbb{Z}_+^I \times \mathbb{R}_+^C : Ax = b\})$$

is a face of the polyhedral set

$$conv(\{x \in \mathbb{Z}_+^I \times \mathbb{R}_+^C : Ax \leq b\}).$$

Proof. See Appendix A. □

Now we can use a standard method mentioned in [5] by Espinoza to lift all the cuts that we need and then by Theorem 3.6, the optimal dual vector of the LP relaxation will give us the desired α. See [15] for more details.

6. Computational Studies and Numerical Results

When we allow entries of the original matrix A be any number, we have to include in E all columns with at least one negative element. (In case of knapsack problems, a column has only one element.) This will increase the size of E compared to non-negative entries case. Hence in case that we have negative elements, we only report the ratio of number of non-negative columns in $|E|$ ($|E_+|$) over the number of variables with non-negative entries N_+.

0-1 Knapsack problems are probably the easiest problems that one could find a certificate for since α has length one. Our computational experiments show that in most 0-1 mixed integer knapsack problems, the size of E_+ is significantly smaller than N_+. In lower dimensions size of E_+ is usually about 10% of the size of N_+, but when $|N_+|$ is large, this ratio will decrease to 1% in average and even less depending on the problem type. In the best case, we had $|E_+|/|N_+| = 0.01$.

Also the results are even better when we are working with non-negative entries i.e. when A and b in (MILP) are non-negative. This is obvious since all columns with at least one negative element should be put in E. However if we have a problem with lots of negative entries, we can multiply rows of A by -1 to get a better structure of the problem.

A 0-1 knapsack problem is an optimization problem of the form

$$\begin{array}{ll} \max & \sum_{i=1}^n c_i x_i \\ \text{s.t.} & \sum_{i=1}^n a_i x_i \leq b \\ & x \in \{0,1\}^n. \end{array}$$

Also if we have continuous variables in the problem, then it is called a mixed integer 0-1 knapsack problem. All the instances that we work with were generated randomly. These instances include problems with only 0,1 coefficients and also instances where coefficents vary as the size of the problem increases.

In this sections, we represent our numerical experiments for each family of problems. Coin-OR Cbc [3] has been used as MILP solver. Solving times have been only reported for the problems for which it takes Cbc more than 0.1 seconds to solve.

All the instances used for computational experiments are available in MPS or LP file format from `http://cs.ucdavis.edu/~bmoazzez/P1.html` along with the generated optimal certificates.

For pure integer knapsack problems with non-negative coefficients, we get the best results. We have several examples with 100,000 variables with $|E|/|N|$ about 0.02. Figure 1 shows that as the number of variables increase, certificates become better and better. Orange bars show the number of variables in the problem (logarithmic) and blue bars show $|E|/|N|$.

Problem	# of Variables	$\|E\|$	Ratio $\|E\|/\|N\|$	MILP time (s)	Certificate time (s)
ipkn1.mps	100	11	0.11		
ipkn3.mps	500	62	0.124		
ipkn4.mps	1000	104	0.104		
ipkn5.mps	1000	24	0.024		
ipkn6.mps	5000	109	0.0218	0.4	0.08
ipkn7.mps	10000	212	0.0212	3.42	0.04
ipkn8.mps	50000	2131	0.04262	14.23	0.29
ipkn9.mps	100000	2227	0.02227	72.83	0.16

Table 1. Pure Integer Knapsack Problems with Non-negative Coefficients

Problem	$\|I\|, \|C\|$	$\|E\|$	Ratio $\|E\|/\|N\|$	MILP time (s)	Certificate time (s)
mipkn1.mps	100,100	8	0.04		
mipkn2.mps	500,500	28	0.028		
mipkn3.mps	1000,1000	91	0.0455		
mipkn4.mps	5000,5000	155	0.0155	29.97	0.01
mipkn5.mps	10000,10000	998	0.0499	1944.64	0.01
mipkn6.mps	50000,50000	5022	0.05022	2449.36	0.05

Table 2. Mixed Integer Knapsack Problems with Non-negative Coefficients

Problem	# of Variables	$\|E_+\|$	Ratio $\|E_+\|/\|N_+\|$	MILP time (s)	Certificate time (s)
nipkn1.mps	200	15	0.13		
nipkn2.mps	1000	40	0.077		
nipkn3.mps	2000	45	0.044		
nipkn4.mps	10000	1006	0.19	0.23	0.19
nipkn5.mps	20000	1974	0.193	0.95	0.56
nipkn6.mps	50000	1136	0.04	3.19	0.79

Table 3. Pure Integer Knapsack Problems

Problem	$\|I\|, \|C\|$	$\|E_+\|$	Ratio $\|E_+\|/\|N_+\|$	MILP time (s)	Certificate time (s)
nmipkn1.mps	100,100	10	0.07		
nmipkn3.mps	500,500	198	0.3		
nmipkn4.mps	1000,1000	270	0.19	0.86	0.03
nmipkn5.mps	5000,5000	687	0.1	4.13	0.06
nmipkn6.mps	10000,10000	1278	0.09	68.44	0.76

Table 4. Mixed Integer Knapsack Problems

7. Summary and Conclusion

We have generalized the definition of Subadditive Generator Functions and Ray Generator Functions to mixed integer programming problems without any restriction in the input data. We have shown that this family of functions is very strong in the sense that one can achieve strong duality considering this class solely in the subadditive dual. Also it is shown that the convex hull of a MILP could be represented using finite number of these functions. Our main result states that these functions can be used as a certificate of

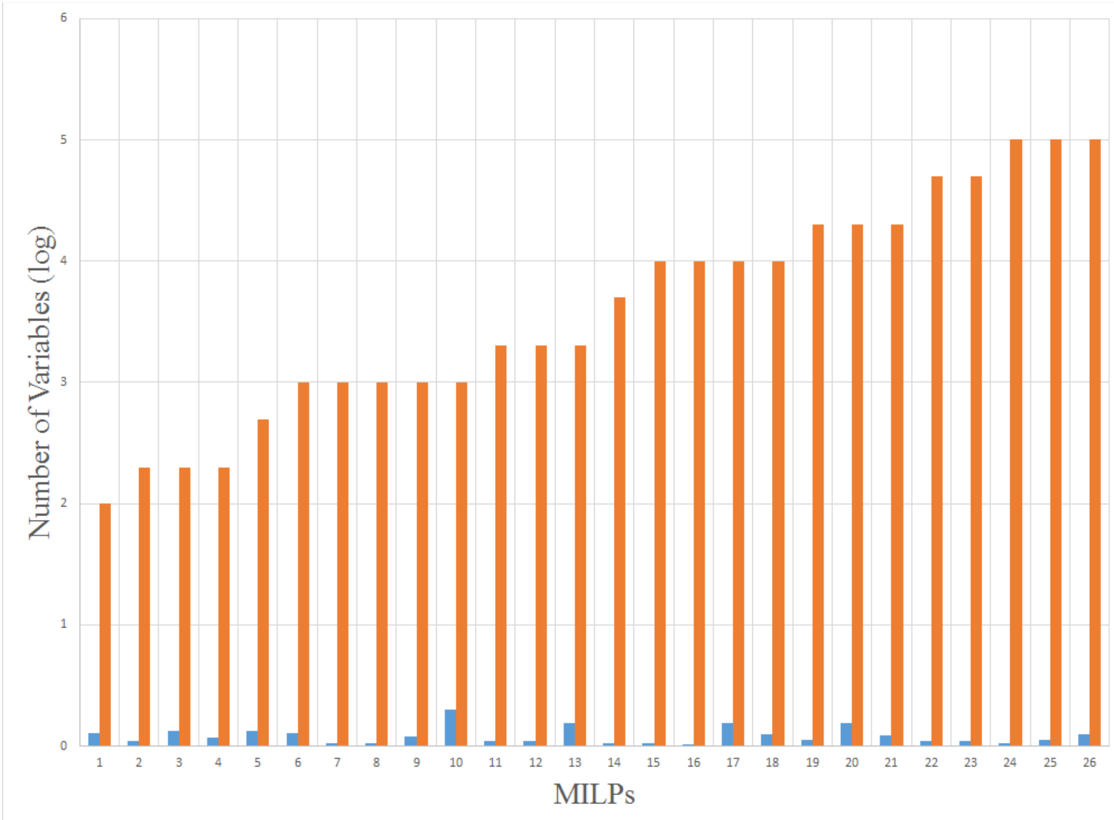

FIGURE 1. Camparison of the number of variables in the original MILP and in the certificate

optimality for specific classes of MILP problems. Checking certificates generated using this method requires much less time and is less expensive. Our computational experiments show that in case of knapsack problems (pure or mixed) the ratio of E over N in the best case is 1 percent. The efficiency and generation of these certificates on other families of problems such as set covering is the subject of an ongoing study.

APPENDIX A

We will show that the set $\{(x,y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : Ax+By = b\} \neq \emptyset$ is a face of $\{(x,y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : Ax+By \leq b\}$ which means that this type of lifting can be used to retrieve α.

Lemma A.1. *Let $P = \{x : Ax \leq b\}$ be a polyhedron and $T = \{x : \alpha x \leq \alpha_0\}$ be a supporting hyperplane. Then $P \cap \{x : \alpha x = \alpha_0\}$ is a face of P.*

Let $P_1 = \{(x,y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : Ax + By = b\} \neq \emptyset$ and $P_2 = \{(x,y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : Ax + By \leq b\}$. Also let $P_1' = conv(P_1)$ and $P_2' = conv(P_2)$. Let $\pi^1 = \mathbb{1}A$ and $\pi^2 = \mathbb{1}B$ and $\pi_0 = \mathbb{1}b$ where $\mathbb{1}$ is vector of ones. Obviously $\pi^1 x + \pi^2 y \leq \pi_0$ is a supporting hyperplane of P_2'. (Let (x^*, y^*) be a feasible point of P_1. This point is on the hyperplane $\pi^1 x + \pi^2 y = \pi_0$ and is also in P_2 and so in P_2'. Also any point in P_2' will satisfy $Ax + By \leq b$ [1] and hence $\pi^1 x + \pi^2 y \leq \pi_0$.)

Lemma A.2. *$ext(P_1') \subseteq P_2' \cap \{(x,y) : \pi^1 x + \pi^2 y = \pi_0\}$ where ext is the set of extreme points.*

Proof. Suppose that $(x,y) \in ext(P_1')$. This means that $(x,y) \in P_1'$ and x is integral. So we have $Ax+By = b$. Now we conclude that $\pi^1 x + \pi^2 y = \pi_0$ and $Ax + By \leq b$. Since x is integral, $(x,y) \in P_2$ and so in P_2' and we are done. □

Lemma A.3. *$ext(P_2') \cap \{(x,y) : \pi^1 x + \pi^2 y = \pi_0\} \subseteq P_1'$.*

[1] Note that $P_1 \subseteq P_1' \subseteq \{(x,y) \in \mathbb{R}_+^{n+p} : Ax + By = b\}$ and $P_2 \subseteq P_2' \subseteq \{(x,y) \in \mathbb{R}_+^{n+p} : Ax + By \leq b\}$

Proof. Let (x, y) be an extreme point of P_2' with $\pi^1 x + \pi^2 y = \pi_0$. Since $(x, y) \in P_2'$, we have $Ax + By \leq b$ and also x is integral. Since $\pi^1 x + \pi^2 y = \pi_0$ and $Ax + By \leq b$, we conclude that $Ax + By = b$[2]. Since x is integral, $(x, y) \in P_1$ and so $(x, y) \in P_1'$. □

Theorem A.4. *Let $P_1 = \{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : Ax + By = b\} \neq \emptyset$ and $P_2 = \{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : Ax + By \leq b\}$ and A, B and b have rational entries. Then $conv(P_1)$ is a face of $conv(P_2)$.*

Proof. From Theorem 3.2 we know that

$$rec(P_1) = rec(P_1') \text{ and } rec(P_2) = rec(P_2').$$

Also it is obvious that since $P_1 \subseteq P_2$, $rec(P_1) \subseteq rec(P_2)$ so

$$rec(P_1') \subseteq rec(P_2'). \tag{A.1}$$

Moreover we know that

$$rec(P_1) = \{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : Ax + By = 0\}$$

and

$$rec(P_2) = \{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^n : Ax + By \leq 0\}.$$

Now let $(x, y) \in P_1'$. Then by theorem of Minkowski and Weyl (see [17] page 88), we can write

$$(x, y) = \sum \lambda_i (x_i, y_i) + \sum \mu_i (z_i, w_i)$$

where $\sum \lambda_i = 1$, (x_i, y_i) are extreme points and (z_i, w_i) are extreme rays of P_1'. By Lemma A.2, $(x_i, y_i) \in P_2' \cap \{(x, y) : \pi^1 x + \pi^2 y = \pi_0\}$. Also we have:

$$\pi^1 x + \pi^2 y = \sum \lambda_i (\pi^1 x_i + \pi^2 y_i) + \sum \mu_i (\pi^1 z_i + \pi^2 w_i).$$

But $\pi^1 z_i + \pi^2 w_i = 0$ since (z_i, w_i) are extreme rays of P_1' and $\pi^1 = \mathbb{1}A$ and $\pi^2 = \mathbb{1}B$ and $\sum \lambda_i (\pi^1 x_i + \pi^2 y_i) = \sum \lambda_i \pi_0 = \pi_0$. This shows that $\pi^1 x + \pi^2 y = \pi_0$. Now using (A.1) and the fact that P_2' and $\{(x, y) : \pi^1 x + \pi^2 y = \pi_0\}$ are both convex sets, we can conclude that $(x, y) \in P_2' \cap \{(x, y) : \pi^1 x + \pi^2 y = \pi_0\}$.

Conversely let $(x, y) \in P_2' \cap \{(x, y) : \pi^1 x + \pi^2 y = \pi_0\}$. We can write

$$(x, y) = \sum \lambda_i (x_i, y_i) + \sum \mu_i (z_i, w_i)$$

where $\sum \lambda_i = 1$, (x_i, y_i) are extreme points and (z_i, w_i) are extreme rays of P_2'. By Lemma A.1 $(x_i, y_i) \in \{(x, y) : \pi^1 x + \pi^2 y = \pi_0\}$ and by Lemma A.3 $(x_i, y_i) \in P_1'$.

Claim A.5. $(z_i, w_i) \in rec(P_1')$ for every i.

Proof of the claim: We know that $(z_i, w_i) \in rec(P_2')$ so we have $Az_i + Bw_i \leq 0$. Also we know that

$$\pi^1 x + \pi^2 y = \sum \lambda_i (\pi^1 x_i + \pi^2 y_i) + \sum \mu_i (\pi^1 z_i + \pi^2 w_i) = \pi_0.$$

But we know that $\sum \lambda_i (\pi^1 x_i + \pi^2 y_i) = \pi_0$. So we must have $\sum \mu_i (\pi^1 z_i + \pi^2 w_i) = 0$. Since $\mu_i \geq 0$ and $\pi^1 z_i + \pi^2 w_i \leq 0$, we conclude that $\pi^1 z_i + \pi^2 w_i = 0$ for all i and so $(z_i, w_i) \in rec(P_1')$ for every i.

Now By Claim A.5 and the fact that P_1' is a convex set, we get the desired result.

□

References

[1] David L. Applegate, Robert E. Bixby, Vašek Chvátal, William Cook, Daniel G. Espinoza, Marcos Goycoolea, and Keld Helsgaun, *Certification of an optimal TSP tour through 85,900 cities*, Oper. Res. Lett. **37** (2009), no. 1, 11–15.

[2] C. E. Blair and R. G. Jeroslow, *The value function of an integer program*, Math. Programming **23** (1982), no. 3, 237–273, DOI 10.1007/BF01583794. MR662494 (84h:90040)

[3] COIN-OR CBC. `https://projects.coin-or.org/Cbc`.

[4] William Cook, Thorsten Koch, Daniel E. Steffy, and Kati Wolter, *A hybrid branch-and-bound approach for exact rational mixed-integer programming*, Math. Program. Comput. **5** (2013), no. 3, 305–344.

[5] Daniel Espinoza, Ricardo Fukasawa, and Marcos Goycoolea, *Lifting, tilting and fractional programming revisited*, Oper. Res. Lett. **38** (2010), no. 6, 559–563.

[2] We can take proper combinations of the rows of the two systems.

[6] L. N. Kozeratskaya, T. T. Lebedeva, and I. V. Sergienko, *Stability, parametric, and postoptimality analysis of discrete optimization problems*, Kibernetika (Kiev) **4** (1983), 71–80 (Russian, with English summary).
[7] Ralph E. Gomory, *Some polyhedra related to combinatorial problems*, Linear Algebra and Appl. **2** (1969), 451–558.
[8] M. Guzelsoy, *Dual Methods in Mixed Integer Linear Programming*, 2010. Thesis (Ph.D.)–Lehigh University.
[9] M. Guzelsoy and T. K. Ralphs, *Duality for mixed-integer linear programs*, Int. J. Oper. Res. (Taichung) **4** (2007), no. 3, 118–137.
[10] John N. Hooker, *Integrated methods for optimization*, 2nd ed., International Series in Operations Research & Management Science, 170, Springer, New York, 2012.
[11] Robert G. Jeroslow, *Minimal inequalities*, Math. Programming **17** (1979), no. 1, 1–15, DOI 10.1007/BF01588222. MR538120 (81e:90067)
[12] Ellis L. Johnson, *Cyclic groups, cutting planes, shortest paths*, Mathematical programming (Proc. Advanced Sem., Univ. Wisconsin, Madison, Wis., 1972), 1973, pp. 185–211.
[13] Diego Klabjan, *Subadditive approaches in integer programming*, European J. Oper. Res. **183** (2007), no. 2, 525–545, DOI 10.1016/j.ejor.2006.10.009. MR2334010 (2008d:90019)
[14] R. R. Meyer, *On the existence of optimal solutions to integer and mixed-integer programming problems*, Math. Programming **7** (1974), 223–235. MR0373602 (51 #9802)
[15] B. Moazzez, *Subadditive Approaches to Mixed Integer Programming*, Thesis (Ph.D.)–Carleton University, 2014. `https://curve.carleton.ca/theses/31724`.
[16] R. Tyrrell Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J., 1970. MR0274683 (43 #445)
[17] Alexander Schrijver, *Theory of linear and integer programming*, Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons Ltd., Chichester, 1986. A Wiley-Interscience Publication.
[18] M. Trick, *Is most published operations research false?*, Michael Tricks Operations Research Blog. http://mat.tepper.cmu.edu/blog/?p=161.
[19] Laurence A. Wolsey, *Integer programming duality: price functions and sensitivity analysis*, Math. Programming **20** (1981), no. 2, 173–195, DOI 10.1007/BF01589344. MR607405 (83f:90091)

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