

# Facing an Arbitrage Opportunity: Trade or Wait?

Robert A. Jarrow, Haitao Li, Lai Wei and Yongpei Guan\*

December 9, 2014

## Abstract

In traditional thinking, an arbitrageur will trade immediately once an arbitrage opportunity appears. Is this the best strategy for the arbitrageur or it is even better to wait for the best time to trade so as to achieve the maximum profit? To answer this question, this paper studies the optimal trading strategies of an arbitrageur in a dynamic economy where the arbitrageur's trades affect prices, and the arbitrageur faces competition from other arbitrageurs exploiting the same mispricings. The proposed model considers fixed and proportional transaction costs, and closed form expressions for the threshold values in the optimal policies are provided. The theoretical and numerical results answer how the timing of the trade and the trade size depend on the magnitudes of the fixed and proportional transaction costs, the dynamics of the arbitrage opportunity, the interest rate, the market impact, and the level of competition. Furthermore, this study numerically studies how the trading horizon and the trader's risk attitude affect the arbitrageurs' decision. With competition, the start-to-trade threshold approaches the proportional cost. Our paper provides a new perspective on the existing empirical literature testing for the existence of arbitrage opportunities.

*Keywords:* Arbitrage opportunity, competition, impulse control, market impact

## 1 Introduction

It is well known that limited arbitrage opportunities can exist in competitive markets due to market frictions such as transaction costs (see Barles and Soner [2], Cvitanic, Pham and Touze [10], Jouini and Kallal [19], and Soner, Shreve and Cvitanic [30]), short sale restrictions (see Lamont and Thaler [21] and Ofek, Richardson and Whitelaw [26]), collateral/margin requirements (see Broadie, Cvitanic and Soner [6], Basak and Croitoru [3], Cuoco and Liu [9], and Naik and Uppal [24]),

---

\*Jarrow is from the Johnson Graduate School of Management at Cornell University. Li is from the Cheung Kong Graduate School of Business. Wei and Guan are from the Department of Industrial and Systems Engineering at the University of Florida.

or irrational noise traders (see Shleifer [29] and Xiong [32]). For example, Ofek, Richardson and Whitelaw [26] provides empirical evidence of deviations from put–call parity in the presence of short sale restrictions. Based on Detemple and Murthy [11] and [12], by adding redundant securities, Basak and Croitoru [3] proves that mispricing generated by portfolio constraints can exist with positive probability. Basak and Croitoru [4], Kondor [20], and Hugonnier and Prieto [16] study more general models and show that arbitrage activity can affect the market price.

The above papers focus on the existence of specific arbitrage opportunities. In this study, a general arbitrage opportunity is exogenously given, and the focus is on exploring how arbitrageurs can optimally exploit it and how arbitrageurs’ behaviors affect the market. With respect to this question, there are a limited number of papers studying optimal trading strategies in the presence of mispricings. For instance, in Tuckman and Vila [31] and Liu and Longstaff [22], optimal trading strategies are studied that are analogous to investing in the securities themselves. One conclusion from this literature is that, in an efficient market, the limited arbitrage opportunities cannot exceed the “transaction cost” band generated by market frictions. This conclusion is based on the assumption that the arbitrageurs’ actions do not affect market prices. However, this assumption is not true in practice. In fact, it is the trades of these arbitrageurs that make market prices almost arbitrage free. This quantity impact on the market price is related to liquidity risk, see Çetin, Jarrow, and Protter [7] and Çetin, Jarrow, Protter and Warachka [8]. Our paper considers a trade’s market impact and provides optimal trading strategies for arbitrageurs in closed form. Although the preliminary concept was described in Jarrow [17], this paper provides a more comprehensive and thorough study, including fundamental insights and rigorous proofs, under more general settings.

Market impacts and transaction costs have been studied with respect to portfolio management using different modeling approaches (see Almgren and Chriss [1], Bertsimas and Lo [5], Gârleanu and Pedersen [14], Lo, Mamaysky, and Wang [23], and Obizhaeva and Wang [25]). For instance, discrete-time models are studied in Bertsimas and Lo [5] and Gârleanu and Pedersen [14]. The first paper considers an optimal control problem under risk-neutrality, with the objective to minimize the expected cost of buying a fixed number of stock shares. For both linear permanent price impacts or linear percentage temporary price impacts, closed form solutions are provided using stochastic dynamic programming. For the general case, approximate dynamic programming is proposed to

solve the problem. The second paper considers a portfolio selection problem with quadratic trading costs (also explained as temporary price impact) based on a mean-variance framework. The main conclusion is that the investor aims to a target portfolio, but trades only partially towards the objective due to the existence of transaction costs.

As compared to the discrete-time models, the continuous-time model is also commonly considered to tackle finance problems, due to its flexibility by allowing trading at any time. A continuous-time model for portfolio management is studied by He and Mamaysky [15], who consider a finite horizon problem with proportional price impact and transactions costs. Both risk-neutral and risk-averse preferences are studied and optimal strategies are numerically obtained without closed form expressions.

In our paper, the continuous-time model is used to obtain the optimal control strategy in closed form for exploiting arbitrage opportunities with market impact and transaction costs. More specifically, this study first solves the problem in a monopolistic setting, and then, extends this case to incorporate competition. In either setting, the optimal policies are derived with closed form expressions for the threshold values to perform transactions. Comparative statics are explored related to how the threshold values change with the interest rate, the market impact, the magnitude of transactions costs, the dynamics of the arbitrage opportunity, and the level of competition.

In our impulse control model, fixed and proportional transaction costs are considered. The corresponding derived profit function is an atypical nonlinear function. Methodologically our study provides closed form expressions for these nonlinear functions, which, to the best of our knowledge, have never been studied before.

The results obtained herein are in contrast to the standard textbook arbitrage trading strategy (see Duffie [13] and Jarrow and Turnbull [18]) that generates infinite wealth. Instead, arbitrageurs' expected discounted trading profits are finite and it is rational for arbitrageurs not to trade immediately when the arbitrage opportunity's value exceeds its proportional transaction cost. In addition, our results show that both the magnitude and the time that the arbitrageur starts to trade depend on the level of competition. Our paper has important implications for the empirical literature testing for the existence of arbitrage opportunities. Violations of this condition have been taken to be proof of an inefficient market. Our paper shows that this conclusion might be false, because with

market impacts, the non-exploitation of arbitrage opportunities that exceed the “transaction cost” band can be an optimal trading strategy for the arbitrageur. In this regard, the implications of the empirical literature investigating the efficiency of financial markets might need to be revisited and perhaps additional tests performed.

An outline for this paper is as follows. Section 2 presents the model structure. Section 3 studies an economy where competitive arbitrageurs are not explicitly modeled. In this section, closed form solutions are presented and the intuition is illustrated through numerically. In addition, the problem when there are trading constraints is studied. Furthermore, sensitivity analysis and the financial insights are explored. Section 4 adds competing arbitrageurs to the economy and compares the optimal strategies with those in Section 3. In Section 5, a finite horizon problem with risk attitude is explored. Section 6 provides a conclusion.

## 2 The Model

In this study, an infinite horizon and continuous trading economy is considered. An infinite horizon model is reasonable since the arbitrageur is often a hedge fund or financial institution.

For analytic convenience and for clarity of the insights, we focus on a particular arbitrage opportunity available to the arbitrageurs. To exploit this arbitrage opportunity, arbitrageurs buy or sell a traded security and hold an offsetting position in a replicating portfolio. The simplest example is where there are two identical securities trading at different prices: buy the one with the lower price and sell the one with the higher price. Another example is the existence of an over-priced call option: sell the call and delta hedge with the underlying stock.

Let  $X_t$  denote the arbitrage opportunity’s value, which measures the magnitude of the mispricing, per unit transaction at time  $t$ . For instance, if an arbitrageur buys a security for  $q_t$  dollars and sells it in another market for  $p_t$  dollars at time  $t$ , then the arbitrage opportunity’s value is  $X_t = p_t - q_t$ . More generally, if the arbitrage opportunity needs a dynamic trading strategy to exploit, then  $X_t$  represents the present value (at time  $t$ ) of the profit generated by this trading strategy. Since we are considering an arbitrage opportunity, a unit trade is defined to be that quantity of the relevant securities for which bid and ask prices are quoted. For example, when trading an arbitrage opportunity’s value involving stocks, the trading unit is one round lot or 100

shares.

In order to characterize the evolution of the arbitrage opportunity's value, we let  $\Omega$  be the space of all real valued continuous functions  $\omega : [0, \infty) \rightarrow \mathbb{R}$  and  $\tilde{W} = \{W_t\}_{t \geq 0}$  be a standard Brownian motion defined on the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$  satisfying the usual conditions (see [28]), where  $P$  is a probability measure,  $\mathcal{F}_t$  is the natural filtration generated by  $\tilde{W}$ , and  $\mathcal{F}$  is the smallest  $\sigma$ -field of  $\Omega$  such that  $\mathcal{F}_t \subseteq \mathcal{F}$  for all  $t \geq 0$ . We assume that the arbitrage opportunity's value evolves as follows:

$$dX_t = b(X_t) dt + \sigma(X_t) dW_t + \int_{\mathbb{R}} \gamma(X_{t-}, z) N(dt, dz) \quad (1)$$

where  $b : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\sigma(x) > 0$  for all  $x$ ,  $z$  is the jump size, and  $N(dt, dz)$  is the jump measure.

Note here that in (1) we do not restrict the arbitrage opportunity's value to be non-negative. There are two interesting cases. If trades in exploiting the arbitrage opportunity are completely unrestricted, then one can impose the condition that  $X_t \geq 0$  for all  $t$ , because arbitrageurs can just change the sign of the position and turn negative profits into positive profits. In some circumstances, however, there are restrictions such as short sale constraints or collateral requirements, and arbitrageurs can only take one direction of a trade. For this fixed direction, the trade could sometimes generate negative profits as in [22]. Our general form of expression (1) applies to both cases.

The evolution of the arbitrage opportunity's value in (1) is determined by all market participants except our arbitrageur. In this regard, we take a partial equilibrium perspective. That is, if there are other arbitrageurs in the market, what the arbitrageur sees is the "remaining" arbitrage opportunity available to him. As such, competing arbitrageurs are implicitly included in the assumed evolution.

## 2.1 The Assumptions

For mispricings to exist in equilibrium, we need to include market frictions, among which we focus on transactions costs, assuming that each trade involves both a proportional cost ( $c$  per unit traded) and a fixed cost ( $c_0$  per transaction) as shown in Assumption 1. These transaction costs include set-up costs, transaction fees (bid/ask spreads), and the holding costs similarly described in [31].

**Assumption 1. (*Transaction Costs*)** Both the fixed transaction cost  $c_0$  and the proportional transaction cost  $c$  are non-negative.

It is well known that in a model which does not consider market impact, an arbitrageur will trade infinite units whenever the arbitrage opportunity's value exceeds the proportional cost and obtain infinite profit. However, when there are market impact and transaction costs, the above situation might change. In fact, when the arbitrageur trades, the arbitrage opportunity's value is reduced. In this study, the market impact of a transaction is characterized by the following assumption.

**Assumption 2. (*Market Impact*)** When the arbitrageur trades  $v$  units at time  $\tau^-$ , his market impact reduces the arbitrage opportunity's value to

$$X_\tau = X_{\tau^-} e^{-\theta v}, \quad (2)$$

where  $\theta > 0$  is a constant, and  $X_{\tau^-} := \lim_{t \rightarrow \tau, t < \tau} X_t$ .

Note here that in the above assumption, after the arbitrageur trades a positive  $v$  units, the mispricing declines by a positive factor  $e^{-\theta v} < 1$ . This assumption is reasonable since selling securities decreases the securities' prices and buying securities increases the corresponding prices. So the arbitrageur's trade, usually including both selling and buying the corresponding securities, decreases the arbitrage opportunity's value. After the trade, the arbitrage opportunity's value jumps downwards, and evolves as in equation (1) until the arbitrageur's next possible transaction.

**Assumption 3. (*Total Revenues*)** When the arbitrageur trades  $v$  units at time  $\tau^-$  with the arbitrage opportunity's value  $X_{\tau^-}$ , after considering his market impact, the total revenue generated is

$$\int_0^v X_{\tau^-} e^{-\theta s} ds = X_{\tau^-} (1 - e^{-\theta v}) / \theta. \quad (3)$$

Similar settings are also adopted in [15] to study portfolio management with price impacts.

Thus, the profit from a transaction at time  $\tau$  with a trading quantity  $v$  is revenue minus transaction costs, which is

$$X_{\tau^-} (1 - e^{-\theta v}) / \theta - cv - c_0. \quad (4)$$

**Assumption 4. (*Interest Rate*)** The compound interest rate  $r > 0$  is a constant.

## 2.2 The Optimization Problem

The arbitrageur's transaction decisions are a sequence of  $\mathcal{F}_t$ -stopping times  $(\tau_1, \tau_2, \dots)$  and corresponding  $\mathcal{F}_{\tau_k}$ -measurable trading quantities  $(v_1, v_2, \dots)$ , denoted as

$$w = ((\tau_1, v_1), (\tau_2, v_2), \dots),$$

where  $\tau_k \leq \tau_{k+1}$  and  $v_k \geq 0$ ,  $k = 1, 2, \dots$ . We denote the set of all such transaction decisions as  $W$ . Note here that, from equations (1) and (2), for a given trading strategy  $w$ , the evolution of the arbitrage opportunity's value is

$$\begin{cases} dX_t = b(X_t) dt + \sigma(X_t) dW_t + \int_{\mathbb{R}} \gamma(X_{t-}, z) N(dt, dz), & \tau_i \leq t < \tau_{i+1}, \quad i \geq 0 \\ X_{\tau_i} = X_{\tau_i-} e^{-\theta v_i}, & i \geq 1, \end{cases}$$

where  $\tau_i$  and  $v_i$  are the time and quantity of the  $i^{\text{th}}$  transaction for  $i = 1, 2, \dots$

For a given trading strategy  $w \in W$ , the arbitrageur's expected discounted profit at time  $t$ , corresponding to the current arbitrage opportunity's value at  $x$ , is

$$J^w(t, x) := E_{(t,x)} \left[ \sum_{k=1}^{\infty} e^{-r\tau_k} \left( X_{\tau_k-} \left( 1 - e^{-\theta v_k} \right) / \theta - cv_k - c_0 \right) \right]. \quad (5)$$

To maximize the expected discounted profit  $J^w(t, x)$ , the arbitrageur tries to find the optimal trading strategy  $w \in W$ , i.e.,

$$\sup_{w \in W} J^w(t, x). \quad (6)$$

For the moment, this completes the model's description. Additional extensions will be discussed in subsequent sections.

## 3 Optimal Trading Policies without Competitors

In this section, we discuss the case when the arbitrageur does not have competitors. We first consider the case when there is no restriction on trade. Accordingly, the arbitrage opportunity's values are nonnegative and assumed to follow a geometric Brownian motion (GBM), i.e.,

$$dX_t = bX_t dt + \sigma X_t dW_t, \quad (7)$$

where the drift speed  $b$  and volatility  $\sigma > 0$  are constants, and the noise term  $W_t$  is a standard Brownian motion on the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ .

### 3.1 The Threshold Policy

We first consider an extreme case, where  $b \geq r$ , and show that it is optimal to wait forever in this circumstance.

**Theorem 1.** *Suppose the arbitrageur is only allowed to trade once. If the drift term  $b > r$ , then it is optimal to wait forever and the expected profit is infinite. If the drift term  $b = r$ , then the optimal policy is also to wait forever, but with a finite expected profit.*

*Proof.* The proof is given in Appendix A.1 □

Because the condition on the drift,  $b \geq r$ , generates cases inconsistent with any economic equilibrium, in the following we only discuss the case where  $b < r$ .

To derive the optimal threshold policy for this case, we first guess that the value function  $\phi(t, x)$  is separable in  $t$  and  $x$ , and described as  $\phi(t, x) = e^{-rt}\psi(x)$ .

Second, we guess there exists a threshold trading policy. More specifically, we trade when the arbitrage opportunity's value  $x$  reaches  $y_2$  and trade the amount such that arbitrage opportunity's value decreases to  $y_1$ , i.e., the optimal trading quantity is  $v^* = [\ln(y_2) - \ln(y_1)] / \theta$ . Accordingly, the continuation region takes the form  $C = \{x : x < y_2\}$ .

Furthermore, we guess the value function is

$$\psi(x) = \begin{cases} Kx^{\gamma^*} & \text{if } x < y_2 \\ Ky_1^{\gamma^*} + (x - y_1) / \theta - c [\ln(x) - \ln(y_1)] / \theta - c_0 & \text{if } x \geq y_2 \end{cases}, \quad (8)$$

where  $\gamma^* = \left[ -(b - \sigma^2/2) + \sqrt{(b - \sigma^2/2)^2 + 2r\sigma^2} \right] / \sigma^2$ , and  $y_1$ ,  $y_2$ , and  $K$  are decided in the following proposition.

**Proposition 1.** *For any  $c > 0$ ,  $c_0 > 0$ ,  $\theta > 0$ , and  $\gamma > 1$ , there exists a unique group of  $(y_1, y_2, K)$ , with  $c < y_1 < c\gamma / (\gamma - 1) < y_2$  and  $K > 0$ , such that*

$$\theta K \gamma y_1^\gamma = y_1 - c, \quad (9)$$

$$\theta K \gamma y_2^\gamma = y_2 - c, \quad (10)$$

and

$$\theta K (y_2^\gamma - y_1^\gamma) = (y_2 - y_1) - c [\ln(y_2) - \ln(y_1)] - \theta c_0. \quad (11)$$



*Proof.* The proof is given in Appendix A.2. □

Next, we show that the chosen  $y_1$ ,  $y_2$ , and  $K$  guarantee that the guessed function  $\psi(x)$  is optimal. We first show that this function  $\psi(x)$  is smooth.

Note that  $\psi(x)$  is smooth in  $x$  if and only if  $\psi(x)$  is smooth at  $x = y_2$  (i.e.,  $\psi(x)$  is continuous at  $x = y_2$  and its left and right derivatives are equal at  $x = y_2$ ), which requires

$$Ky_2^{\gamma^*} = Ky_1^{\gamma^*} + (y_2 - y_1)/\theta - c[\ln(y_2) - \ln(y_1)]/\theta - c_0 \quad (12)$$

and

$$K\gamma^*y_2^{\gamma^*-1} = 1/\theta - c/(\theta y_2). \quad (13)$$

By dividing  $\theta$  on both sides and reorganizing the terms in (11), (12) is satisfied. Similarly, by dividing  $\theta y_2$  on both sides of (10), we have that (13) is satisfied.

Then, in the following main theorem, we verify the guessed function is optimal by proving that it satisfies the Verification Theorems 7 and 8 we developed in Appendix A.3.

**Theorem 2.** *Suppose  $b < r$ ,  $\theta > 0$ ,  $c > 0$ , and  $c_0 > 0$ . The value function is*

$$\phi(t, x) = \sup_{w \in W} J^w(t, x) = \begin{cases} e^{-rt} Kx^{\gamma^*} & \text{if } x < y_2 \\ e^{-rt} \left[ Ky_1^{\gamma^*} + (x - y_1)/\theta - c[\ln(x) - \ln(y_1)]/\theta - c_0 \right] & \text{if } x \geq y_2, \end{cases}$$

where  $\gamma^* = \left[ -(b - \sigma^2/2) + \sqrt{(b - \sigma^2/2)^2 + 2r\sigma^2} \right] / \sigma^2 > 0$ ,  $y_1$ ,  $y_2$ , and  $K$  are as in Proposition 1 with  $\gamma^*$  replacing  $\gamma$ . Accordingly, the optimal trading strategy is to wait until the arbitrage opportunity's value  $x$  reaches  $y_2$ , and then to transact such that  $x$  decreases to  $y_1$ , i.e., the optimal trading quantity is  $v^* = [\ln(y_2) - \ln(y_1)]/\theta$ .

*Proof.* First of all, we know that, based on (7), the generator of the diffusion process  $X_t$  is:

$$A\phi(t, x) = \left( \frac{\partial}{\partial t} + bx\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2} \right) \phi(t, x). \quad (14)$$

And the switch operator

$$M\phi(t, x) = \sup_{y < x} \left\{ \phi(t, y) + e^{-rt} [(x - y)/\theta - c[\ln(x) - \ln(y)]/\theta - c_0] \right\}. \quad (15)$$

Following (14),  $A(\phi(t, x)) = A(e^{-rt}\psi(x)) = e^{-rt}A_0(\psi(x))$ , where

$$A_0\psi(x) = -r\psi + bx\psi'(x) + \frac{1}{2}\sigma^2x^2\psi''(x). \quad (16)$$

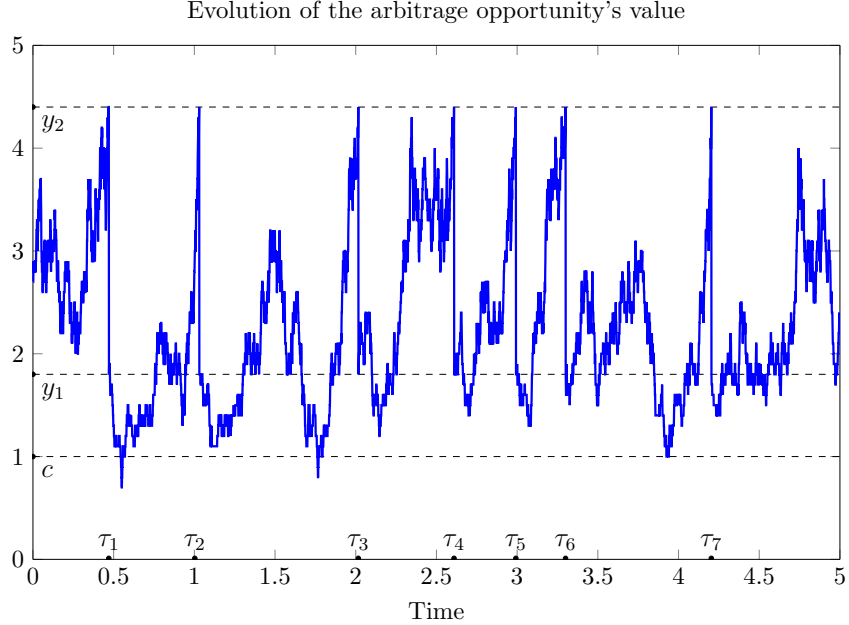


Figure 1: Optimal transaction times for a sample path of the arbitrage opportunity's value

From (15), we have  $M\phi(t, x) = M(e^{-rt}\psi(x)) = e^{-rt}M_0(\psi(x))$ , where

$$M_0\psi(x) = \sup_{y < x} \{ \psi(y) + (x - y) / \theta - c [\ln(x) - \ln(y)] / \theta - c_0 \}. \quad (17)$$

The choice of  $y_1$ ,  $y_2$ , and  $K$  guarantee that  $\psi(x)$  is smooth for  $x$ . In order to show that the guessed value function and trading policy are optimal, based on Theorems 7 and 8 in Appendix A.3, it suffices to verify that the following conditions are satisfied: (i)  $\psi(x) = M_0\psi(x)$  for  $x \geq y_2$ , (ii)  $\psi(x) > M_0\psi(x)$  for  $x < y_2$ , (iii)  $A_0\psi(x) \leq 0$  for  $x \geq y_2$ , (iv)  $A_0\psi(x) = 0$  for  $x < y_2$ , (v) Let  $\tau_k^C := \inf\{t > \tau_{k-1}^C : (t, x_t) \notin C\}$ . For any positive integer  $l$ , we have  $|\phi(\tau_k^C \wedge l, x_{\tau_k^C \wedge l})| \leq G$  a.s. and  $|\phi((\tau_{k+1}^C)^- \wedge l, x_{(\tau_{k+1}^C)^- \wedge l})| \leq G$  for  $k = 1, \dots, n$ , where  $E[G] < \infty$ , and (vi)  $\phi((\tau_k^C)^-, x_{(\tau_k^C)^-}) = 0$  and  $K((\tau_k^C)^-, x_{(\tau_k^C)^-}, 0) = 0$  if  $\tau_k^C = \infty$  for  $k = 1, \dots, n$ .

The detailed proof is given in Appendix A.4.  $\square$

### 3.2 Discussion

In this subsection, we explore the intuition and discuss our theoretical results. In particular, we show how to perform transactions based on the threshold policies derived from the previous

subsections and we explore the start-to-trade threshold and trading amounts, plus their impacts on the arbitrage opportunity's value, as compared to the traditional approach.

### 3.2.1 Graphic Illustration and Numerical Experiments

To understand the intuition behind Theorem 2, a graph on the evolution of the arbitrage opportunity's value is presented in Figure 1. In the graph the jagged line is a realized scenario of the arbitrage opportunity's value  $X_t$ . The proportional transaction cost  $c$ , the start-to-trade threshold  $y_2$ , and the trade down-to target  $y_1$  are illustrated in the vertical axis. The optimal arbitrage trading strategy is, whenever arbitrage opportunity's value reaches the start-to-trade threshold  $y_2$ , to trade such that the arbitrage opportunity's value decreases to  $y_1$ . In addition, in this sample scenario, the first trade happens at time  $\tau_1$ . Due to its market impact, the transaction makes the arbitrage opportunity's value drop down to  $y_1$ . After the decline, the arbitrage opportunity's value continues evolving, until time  $\tau_2$  when the start-to-trade threshold is reached again and another transaction occurs, and so on.

A further illustration of the optimal policy for a numerical example is shown in Figure 2. In the figure, the optimal arbitrage opportunity's value after trading (trade-down-to target) and the optimal expected trading profit  $\psi(x)$  versus the arbitrage opportunity's value before trading  $x$  are reported. The threshold and target values shown in the figure are obtained by setting the interest rate  $r = 0.1$ , the drift speed  $b = 0.05$ , the volatility  $\sigma = 0.2$ , the market impact parameter  $\theta = 0.1$ , the proportional transaction cost  $c = 1$ , and the fixed transaction cost  $c_0 = 1$ . These values are obtained by solving the problem numerically instead of using the closed form derived in the previous sections. Our numerical study provides the threshold values  $y_1 = 1.8$  and  $y_2 = 4.4$ , which confirms the results derived from the closed form solution. Meanwhile, Figure 2 also confirms that the optimal expected trading profit as a function of the arbitrage opportunity's value is monotonically increasing, convex, and smooth at the start-to-trade threshold.

### 3.2.2 Comparison to a Myopic Strategy

It is worthwhile demonstrating the benefit of our proposed optimal trading strategy, as compared to the commonly used myopic strategy which maximizes the current transaction profit.

The myopic strategy works as follows: For the given fixed and proportional transaction costs  $c_0$

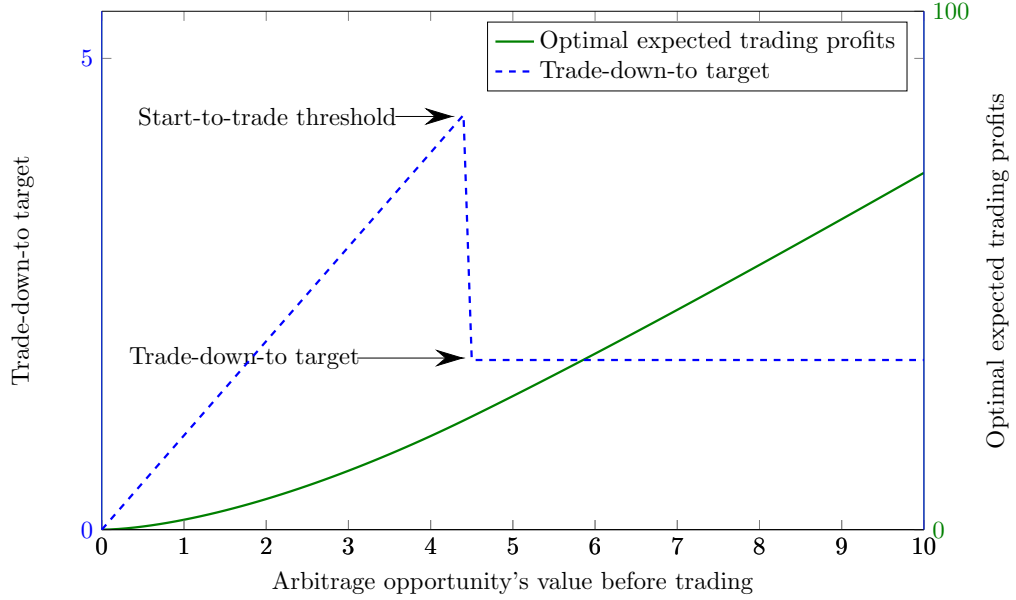


Figure 2: Policy illustration (start-to-trade threshold, trade-down-to target, and optimal expected discounted profits)

and  $c$  and the current arbitrage opportunity's value  $x$ , if the arbitrageur trades (which accordingly makes the arbitrage opportunity's value decreased to  $y = xe^{-\theta v}$  following (2)), then from (4), the profit is

$$P(x, y) = (x - y) / \theta - c [\ln(x) - \ln(y)] / \theta - c_0.$$

Note that  $\partial P / \partial y = -1/\theta + c/(\theta y)$   $\begin{cases} > 0 & \text{for } 0 < y < c \\ = 0 & \text{for } y = c \\ < 0 & \text{for } y > c \end{cases}$ . So there is no trading action when  $x \leq c$ , and when  $x > c$ , the myopic optimal solution is  $y_1^{\text{myop}} = c$  and the corresponding profit is

$$P^{\text{myop}}(x) = (x - c) / \theta - c [\ln(x) - \ln(c)] / \theta - c_0. \quad (18)$$

Since the above myopic profit function  $P^{\text{myop}}(x)$  is increasing in  $x$ , the myopic start-to-trade threshold  $y_2^{\text{myop}}$  can be defined as the point such that

$$P^{\text{myop}}(y_2^{\text{myop}}) = 0. \quad (19)$$

That is,  $y_2^{\text{myop}}$  is the arbitrage opportunity's value above which the arbitrageur can start to make positive profit.

**Proposition 2.** *The optimal start-to-trade threshold  $y_2$  is strictly larger than the myopic start-to-trade threshold  $y_2^{\text{myop}}$ , which is strictly larger than the proportional transaction cost  $c$ .*

*Proof.* The proof is given in Appendix A.5. □

From the above analysis, the optimal start-to-trade threshold  $y_2$  is strictly larger than the myopic threshold  $y_2^{\text{myop}}$ . Meanwhile, the optimal trade-down-to target  $y_1$  is strictly larger than the proportional transaction cost  $c$ . All of these generate a higher probability of a larger arbitrage opportunity's value in the future for our optimal trading strategy, as compared to the myopic one. In other words, when the arbitrage opportunity's value is above the myopic threshold  $y_2^{\text{myop}}$  but below the optimal start-to-trade threshold  $y_2$ , no transaction occurs, which indicates that it is rational not to exploit the arbitrage opportunity as soon as trading is profitable. Furthermore, for realistic drifts (i.e.,  $b < r$ ), the arbitrageur's total expected discounted profit from trading is finite. These results contradict the standard textbook characterization of the arbitrage trading strategy, where without market impact and with transaction cost, the arbitrageur trades infinite units and gets infinite profits once the arbitrage opportunity's value exceeds the proportional transaction cost.

### 3.3 Trading Restrictions

As discussed in Section 2, in some circumstances when there are trading restrictions, the arbitrageur can only exploit one direction of arbitrage opportunity. Under this situation, the arbitrage opportunity's value could be negative. In this case a Brownian motion model can be utilized to capture the evolution of arbitrage opportunity's value, which can be described as follows:

$$dX_t = bdt + \sigma dW_t, \tag{20}$$

where  $b$  and  $\sigma > 0$  are constants, and  $W_t$  is a standard Brownian motion on the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$  as described in Section 2. The optimal trading strategy also exists (shown below in Theorem 3) and the corresponding threshold values are decided following Proposition 3.

**Theorem 3.** *Suppose  $\theta > 0$ ,  $c > 0$ , and  $c_0 > 0$ . The value function is*

$$\phi(t, x) = \sup_{w \in W} J^w(t, x) = \begin{cases} e^{-rt} K e^{\gamma^{BM} x} & \text{if } x < y_2 \\ e^{-rt} \left[ K e^{\gamma^{BM} y_1} + (x - y_1) / \theta - c [\ln(x) - \ln(y_1)] / \theta - c_0 \right] & \text{if } x \geq y_2, \end{cases}$$

where  $\gamma^{BM} = \left[ -b + \sqrt{b^2 + 2r\sigma^2} \right] / \sigma^2 > 0$ ,  $y_1, y_2$ , and  $K$  are as in Proposition 3 with  $\gamma^{BM}$  replacing  $\gamma$ . Accordingly, the optimal trading strategy is to wait until the arbitrage opportunity's value  $x$

reaches  $y_2$ , and then to transact such that  $x$  decreases to  $y_1$ , i.e., the optimal trading quantity is  $v^* = [\ln(y_2) - \ln(y_1)] / \theta$ .

*Proof.* The proof is similar to that for Theorem 2 and so is omitted.  $\square$

**Proposition 3.** *For any  $c > 0$ ,  $c_0 > 0$ ,  $\theta > 0$ , and  $\gamma > 0$ , there exists a unique group of  $(y_1, y_2, K)$ , with  $c < y_1 < c + 1/\gamma < y_2$  and  $K > 0$ , such that*

$$\theta K \gamma e^{\gamma y_1} = y_1 - c, \quad \theta K \gamma e^{\gamma y_2} = y_2 - c,$$

and

$$\theta K (e^{\gamma y_2} - e^{\gamma y_1}) = (y_2 - y_1) - c [\ln(y_2) - \ln(y_1)] - \theta c_0.$$

*Proof.* The proof is similar to that for Proposition 1 and so is omitted.  $\square$

Since the above theorem's representation is analogous to Theorem 2, no further explanation is provided. Since BM model generates similar insights, from now on, we will focus on GBM model.

### 3.4 Sensitivity Analysis

In this subsection, we explore a sensitivity analysis of the basic parameters such as the proportional and fixed transaction costs, the interest rate, the market impact parameter, and the drift and volatility terms of the evolution of the arbitrage opportunity's value. The conditions described in Theorem 2 are assumed to hold for this analysis.

#### 3.4.1 Proportional Transaction Cost

We explore how the proportional transaction cost affects the two thresholds, including qualitative and quantitative analyses and the corresponding financial interpretations. The qualitative analysis is provided in the following proposition.

**Proposition 4.** *Both the start-to-trade threshold and the trade-down-to target values  $y_2$  and  $y_1$  increase as the proportional transaction cost  $c$  increases.*

*Proof.* The proof is given in Appendix A.6.  $\square$

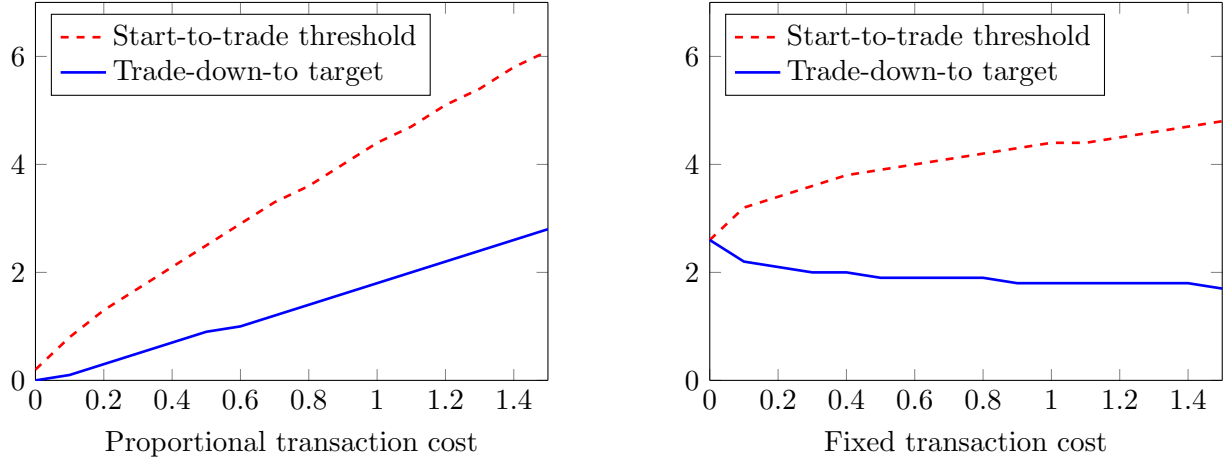


Figure 3: Sensitivity analysis of transaction costs

This proposition implies that with a higher proportional transaction cost, the arbitrageur pays more for each unit traded, and so the arbitrageur waits a longer time to trade (corresponding to a higher  $y_2$ ) and stops the trade sooner (corresponding to a higher  $y_1$  with  $y_1 > c$ ).

The quantitative analysis is given in the left graph of Figure 3 obtained by setting  $r = 0.1$ ,  $b = 0.05$ ,  $\sigma = 0.2$ ,  $\theta = 0.1$ ,  $c_0 = 1$ , and  $c \in [0, 1.5]$ . This numerical result confirms the conclusion obtained in Proposition 4. Meanwhile, it is interesting to observe that the gap between the two thresholds also increases when the proportional transaction cost  $c$  increases.

### 3.4.2 Fixed Transaction Cost

Similarly, we explore qualitative and quantitative analyses on how the fixed transaction cost affects the two thresholds. The qualitative analysis is provided in the following proposition.

**Proposition 5.** *As the fixed transaction cost  $c_0$  increases, the trade-down-to target  $y_1$  decreases while the start-to-trade threshold  $y_2$  increases.*

*Proof.* The proof is given in Appendix A.7. □

With a larger fixed transaction cost, the arbitrageur needs to gain more revenue per trade, and so the arbitrageur waits a longer time to trade (corresponding to a higher  $y_2$ ) and trades more at each transaction (corresponding to a lower  $y_1$ ). The quantitative analysis is given in the right graph of Figure 3 obtained by setting  $r = 0.1$ ,  $b = 0.05$ ,  $\sigma = 0.2$ ,  $\theta = 0.1$ ,  $c = 1$ , and  $c_0 \in [0, 1.5]$ . From the graph, we find that when  $c_0$  goes to 0,  $y_1$  and  $y_2$  coincide.

Now we discuss the case when there is no fixed transaction cost, i.e,  $c_0 = 0$ . This is also known as a singular control problem, because an arbitrageur can trade an infinite number of times in any finite time interval. For this case, from (40), (42), and (44) in the proof of Proposition 1, letting  $c_0 \rightarrow 0$ , we have  $y_1 = y_2 = c\gamma^*/(\gamma^* - 1)$  and  $K = (y_2 - c) / (\theta\gamma^*y_2^{\gamma^*})$ . This gives us a new guess about the value function. Meanwhile, since  $c_0 = 0$ , this is no longer an impulse control problem and we cannot use verification Theorems 7 and 8 in Appendix A.3 to verify that our guessed function is optimal. Instead, we will use the verification theorem for singular control problems to prove the claim.

**Theorem 4.** *Suppose  $b < r$ ,  $\theta > 0$ ,  $c > 0$ , and  $c_0 = 0$ . The value function is*

$$\phi(t, x) = \sup_{w \in W} J^w(t, x) = \begin{cases} e^{-rt} K x^{\gamma^*} & \text{if } x < y_2 \\ e^{-rt} \left[ K y_2^{\gamma^*} + (x - y_2) / \theta - c [\ln(x) - \ln(y_2)] / \theta \right] & \text{if } x \geq y_2, \end{cases} \quad (21)$$

where  $\gamma^* = \left[ -(b - \sigma^2/2) + \sqrt{(b - \sigma^2/2)^2 + 2r\sigma^2} \right] / \sigma^2 > 0$ ,  $y_2 = c\gamma^*/(\gamma^* - 1)$ , and  $K = (y_2 - c) / (\theta\gamma^*y_2^{\gamma^*})$ . Accordingly, the optimal trading strategy is to wait until the arbitrage opportunity's value  $x$  is higher than  $y_2$ , and then to transact such that  $x$  decreases to  $y_2$ .

*Proof.* The detailed proof is given in Appendix A.8. □

From the above analysis, the start-to-trade threshold and trade-down-to target coincide and are both strictly larger than  $c$ . Accordingly, the start-to-trade threshold  $y_2$  acts like a reflecting barrier.

Finally, if both the fixed and proportional transaction costs are 0, we have  $y_1 = y_2 = 0$ , which means it is optimal to trade immediately when there is positive arbitrage opportunity. This is the traditional textbook arbitrage trading strategy. However, this strategy only generates a finite profit since there is market impact.

### 3.4.3 Market Impact

In this part, we explore how market impact affects the two thresholds. First, the qualitative analysis is provided in the following proposition.

**Proposition 6.** *As market impact parameter  $\theta$  increases, the trade-down-to target value  $y_1$  decreases while the start-to-trade threshold value  $y_2$  increases.*



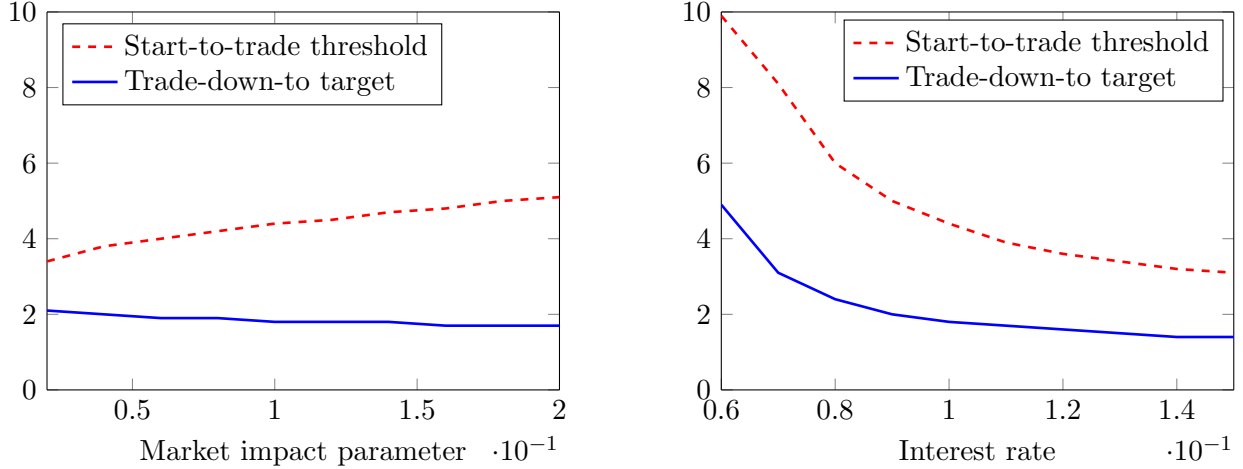


Figure 4: Sensitivity analysis of market impact and interest rate

*Proof.* The proof is given in Appendix A.9. □

From the above analysis, with a larger market impact, the arbitrage opportunity's value decreases faster and the arbitrageur obtains less revenue per trade. So the arbitrageur waits a longer time to trade (corresponding to a higher  $y_2$ ) and the trade results in a lower arbitrage opportunity's value (corresponding to a lower  $y_1$ ). The quantitative analysis is given in the left graph of Figure 4, which is obtained by setting  $r = 0.1$ ,  $b = 0.05$ ,  $\sigma = 0.2$ ,  $c = 1$ ,  $c_0 = 1$ , and  $\theta \in [0.02, 0.2]$  (Note that the  $x$ -axis is scaled by  $10^{-1}$ ).

When  $\theta = 0$ , it is obvious that the optimal trading strategy is to trade infinite units as soon as  $x > c$  and it becomes the standard textbook solution which generates infinite wealth by exploiting this arbitrage opportunity.

### 3.4.4 The Interest Rate

The interest rate also affects the two thresholds. The qualitative analysis is provided in the following proposition.

**Proposition 7.** *As the interest rate  $r$  increases, both the start-to-trade threshold and the trade-down-to target values  $y_2$  and  $y_1$  decrease.*

*Proof.* The proof is given in Appendix A.10. □

From the above proposition, with a larger interest rate, the arbitrage opportunity's value in the

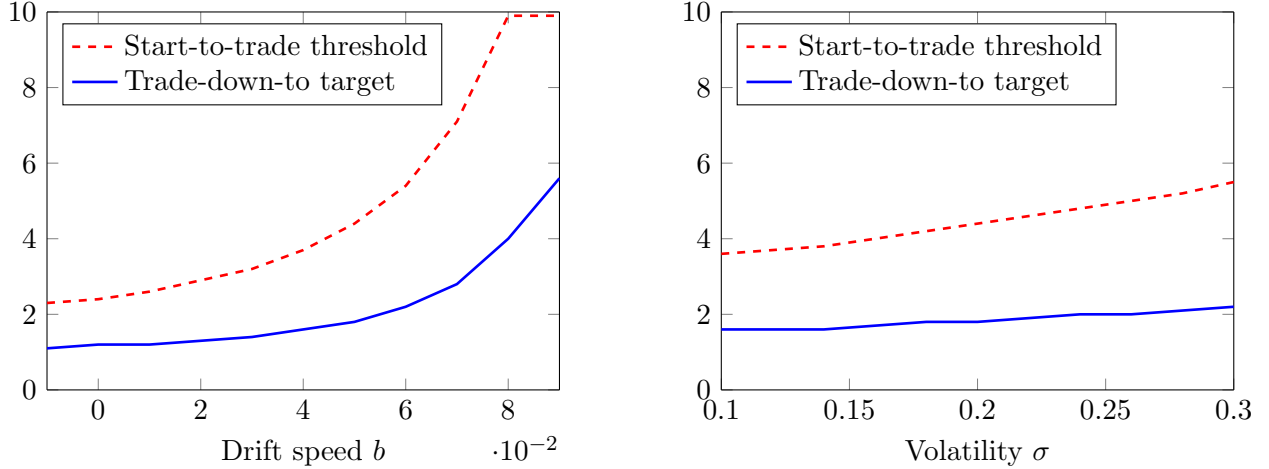


Figure 5: Sensitivity analysis of drift speed and volatility

future is worth less. So the arbitrageur waits a shorter time to trade (corresponding to a higher  $y_2$ ) and reserves less of the arbitrage opportunity for future trading (corresponding to a lower  $y_1$ ). The quantitative analysis is given in the right graph of Figure 4 obtained by setting  $b = 0.05$ ,  $\sigma = 0.2$ ,  $\theta = 0.1$ ,  $c = 1$ ,  $c_0 = 1$ , and  $r \in [0.06, 0.15]$ , which verifies our theoretical results.

### 3.4.5 The Drift Speed

In this part, qualitative and quantitative analyses on how the drift speed affects the two thresholds are explored. The qualitative analysis is provided in the following proposition.

**Proposition 8.** *As the drift speed  $b$  increases, both the start-to-trade threshold and the trade-down-to target values  $y_2$  and  $y_1$  increase.*

*Proof.* The proof is given in Appendix A.11. □

From the above analysis, it can be observed that if the drift term of the arbitrage opportunity's value increases, which means that after trading the arbitrage opportunity's value returns to a higher level sooner, the arbitrageur sets a higher threshold for trading in order to obtain a larger profit per trade. The quantitative analysis is given in the left graph of Figure 5 obtained by setting  $r = 0.1$ ,  $\sigma = 0.2$ ,  $\theta = 0.1$ ,  $c = 1$ ,  $c_0 = 1$ , and  $b \in [-0.01, 0.09]$ , which verifies our theoretical results.

### 3.4.6 The Volatility

In this part, we explore how the volatility affects the two thresholds. The qualitative analysis is provided in the following proposition.

**Proposition 9.** *As the volatility  $\sigma$  increases, both the start-to-trade threshold and the trade-down-to target values  $y_2$  and  $y_1$  increase.*

*Proof.* The proof is given in Appendix A.12. □

If the volatility of the arbitrage opportunity's value increases, which means the arbitrage opportunity's value becomes higher with an increased probability, the arbitrageur again tends to set a higher trading threshold. The corresponding quantitative analysis is given in the right graph of Figure 5 obtained by setting  $r = 0.1$ ,  $b = 0.05$ ,  $\theta = 0.1$ ,  $c = 1$ ,  $c_0 = 1$ , and  $\sigma \in [0.1, 0.3]$ , which verifies our theoretical results.

In sum, with higher transaction costs, larger market impact, lower interest rate, higher profit return (drift), or larger volatility, the arbitrageur becomes more patient and sets a larger start-to-trade threshold, which could be much higher than the proportional transaction cost and significantly violates the traditional textbook arbitrage trading strategy. However, when competitors join in, this changes. We discuss this case in the following section.

## 4 Optimal Trading Policies with Competitors

In the previous analysis, competing arbitrageurs are implicitly captured in the arbitrage opportunity's value evolution available to our arbitrageur. Under the given market structure, we see that our arbitrageur's trades cause a discontinuity in the arbitrage opportunity's value's evolution. It seems natural that, if there are competing arbitrageurs, then their trades would cause similar discontinuities in the arbitrage opportunity's values available to our arbitrageur. Yet, this is inconsistent with the continuous sample path arbitrage evolution contained in (7). Hence, to have the arbitrage opportunity's value evolution consistent with competing arbitrageurs, the arbitrage opportunity's value evolution needs to include unanticipated downward jumps in its evolution.<sup>1</sup>

---

<sup>1</sup>Although we only include downward jumps here, it is not difficult to add upward jumps to model increases in the arbitrage opportunity's value due to random events.

Thus, in this section, our study starts with the GBM plus jump model where the jump intensity is independent of the jump size and arbitrage opportunity's value and the downward jump size is proportional to the magnitude of the arbitrage opportunity's value. The optimal threshold policy for this case is derived in Subsection 4.1. Accordingly, the comparison between the models with and without jumps is illustrated in Subsection 4.2 and the further insights on the relationship between threshold values and jump size/intensity are described in Subsection 4.3. Then, this study is further explored numerically in Subsection 4.4 to consider a more general setting where the jump intensity is proportional to the arbitrage opportunity's value.

#### 4.1 The Threshold Policy

Following the above description, to illustrate the impact of competing arbitrageurs, in this part, a compounded Poisson process is utilized to model the jumps. Meanwhile, the assumption of the downward jump size to be proportional to the magnitude of the arbitrage opportunity's value follows the intuition that the competitors are likely to trade more, when the arbitrage opportunity's value is high. With this consideration in mind, the arbitrage opportunity's value evolves as a geometric Brownian motion with jumps, i.e.,

$$\begin{cases} dX_t = X_t \left( bdt + \sigma dB_t + \int_{-1}^0 z N(dt, dz) \right) & \tau_i \leq t < \tau_{i+1} \\ X_{\tau_i} = X_{\tau_i^-} e^{-\theta v_i}, \end{cases} \quad (22)$$

where  $b$  and  $\sigma > 0$  are constants,  $B_t$  is a standard Brownian motion on the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ ,  $z$  is the relative jump size, and  $N(dt, dz)$  is the jump measure.

Under this evolution, the optimal trading strategy for  $b + \int_{-1}^0 z \nu(dz) \geq r$ , where  $\nu(\cdot) := E[N(1, \cdot)]$  is a Levy measure [27], is as characterized in the following theorem.

**Theorem 5.** *Suppose the arbitrageur is only allowed to trade once. If the drift term  $b + \int_{-1}^0 z \nu(dz) > r$ , then it is optimal to wait forever and the expected profit is infinite. If the drift term  $b + \int_{-1}^0 z \nu(dz) = r$ , then the optimal policy is also to wait forever, but with a finite expected profit.*

*Proof.* See Appendix A.13 for the detailed proof. □

Because these two conditions on the drift generate trading strategies inconsistent with any economic equilibrium, in the following, we only discuss the remaining  $b + \int_{-1}^0 z \nu(dz) < r$  case.

Similar with the case without jumps, we guess that the value function  $\phi_d(t, x)$  is separable in  $t$  and  $x$ , and described as  $\phi_d(t, x) = e^{-rt}\psi_d(x)$ .

Second, we guess there exists a threshold trading policy. More specifically, we trade when the arbitrage opportunity's value  $x$  reaches  $y_2^d$  and trade the amount such that arbitrage opportunity's value decreases to  $y_1^d$ , i.e., the optimal trading quantity is  $v_d^* = [\ln(y_2^d) - \ln(y_1^d)] / \theta$ . Accordingly, the continuation region takes the form  $C = \{x : x < y_2^d\}$ .

Moreover, we guess the value function is

$$\psi_d(x) = \begin{cases} K_d x^{\gamma_d^*} & \text{if } x < y_2^d \\ K_d (y_1^d)^{\gamma_d^*} + (x - y_1^d) / \theta - c [\ln(x) - \ln(y_1^d)] / \theta - c_0 & \text{if } x \geq y_2^d \end{cases}, \quad (23)$$

where  $\gamma_d^*$  is decided in Lemma 1 in Appendix A.14, and  $y_1^d$ ,  $y_2^d$ , and  $K_d$  are chosen as in Proposition 1 with  $\gamma_d^*$  substituting  $\gamma$ .

Furthermore, we can prove that  $\psi_d(x)$  is smooth at  $x = y_2^d$  implied by the following two equations:

$$K_d (y_2^d)^{\gamma_d^*} = K_d (y_1^d)^{\gamma_d^*} + (y_2^d - y_1^d) / \theta - c [\ln(y_2^d) - \ln(y_1^d)] / \theta - c_0 \quad (24)$$

and

$$K_d \gamma_d^* (y_2^d)^{\gamma_d^* - 1} = 1 / \theta - c / (\theta y_2^d), \quad (25)$$

where (24) follows from (11) and (25) follows from (10).

In the following theory, we verify that the guessed function is the value function and the trading strategy is optimal by proving that our guess satisfies the Verification Theorems 7 and 8 in Appendix A.3.

**Theorem 6.** *Suppose  $b + \int_{-1}^0 z \nu(dz) < r$ ,  $\theta > 0$ ,  $c > 0$ , and  $c_0 > 0$ . The value function is*

$$\phi_d(x) = \sup_{w \in W} J^w(t, x) = \begin{cases} e^{-rt} K_d x^{\gamma_d^*} & \text{if } x < y_2^d \\ e^{-rt} [K_d (y_1^d)^{\gamma_d^*} + (x - y_1^d) / \theta - c [\ln(x) - \ln(y_1^d)] / \theta - c_0] & \text{if } x \geq y_2^d, \end{cases}$$

where  $\gamma_d^*$  is decided in Lemma 1 in Appendix A.14,  $y_1^d$ ,  $y_2^d$ , and  $K_d$  are as in Proposition 1 with  $\gamma_d^*$  substituting  $\gamma$ . Accordingly, the optimal trading strategy is to wait until the arbitrage opportunity's value  $x$  reaches  $y_2^d$ , and then to transact such that  $x$  decreases to  $y_1^d$ , i.e., the optimal trading quantity is  $v_d^* = [\ln(y_2^d) - \ln(y_1^d)] / \theta$ .

*Proof.* The proof is given in Appendix A.15. □

The threshold policy is similar to that under monopoly setting except for different threshold values.

## 4.2 Comparison with and without Competitors

In this subsection, we compare the thresholds derived from the models with and without jumps, which correspondingly provide the optimal control policies for the cases with and without competitors.

**Proposition 10.** *If the conditions of Theorem 2 hold, then compared to the case without downward jumps, both values for the threshold policy with jumps decrease, i.e.,  $y_1^d < y_1$  and  $y_2^d < y_2$ , where  $(y_1^d, y_2^d)$  and  $(y_1, y_2)$  are values for the optimal threshold policies for the cases with and without jumps respectively as described in Theorems 6 and 2.*

*Proof.* See Appendix A.16 for the detailed proof. □

Perhaps not surprising, the optimal strategy's thresholds decrease as compared to those in Theorem 2. The arbitrageur trades more quickly in fear that a jump down may occur, which would reduce or even eliminate profitable trades. Through numerical study, we further discuss the intuition in the next subsection.

## 4.3 Numerical Illustration

In this subsection, we conduct numerical experiments investigating how the jump size and the jump intensity affect the two trading thresholds. We assume that the jump size is independent of the jump intensity and for simplicity of illustration we assume that the relative jump size is a constant, i.e.,  $\int_{-1}^0 zN(dt, dz) = -z_0\lambda dt$ , where  $z_0$  is the relative jump size and  $\lambda$  represents the jump intensity.

We consider four cases: (i) no jump as that in Section 3.2.1, (ii) small jumps with the relative jump size 0.05 and the jump intensity 0.1, (iii) frequent jumps with the relative jump size 0.05 and the jump intensity 0.5, and (iv) frequent and large jumps with the relative jump size 0.5 and the jump intensity 0.5. Figure 6 gives the graphs of the optimal trade-down-to target as a function of the arbitrage opportunity's value for different relative jump sizes and intensities. The results follow our intuition. The bigger the relative jump size and the larger the jump intensity, the arbitrageur waits a shorter time between trades, i.e., the start-to-trade threshold is lower.

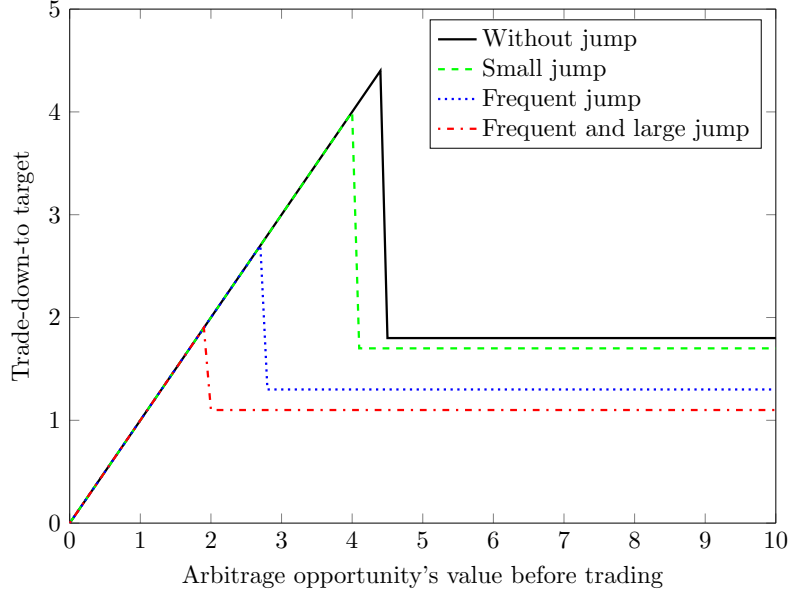


Figure 6: Optimal trade-down-to target with downward jump

#### 4.4 Jump Size and Intensity Dependent on Arbitrage Opportunity's Value

In the above analysis, we assume that the jump intensity is independent of the arbitrage opportunity's value. In practice, the jump intensity can be dependent on the arbitrage opportunity's value. For instance, when the arbitrage opportunity's values are high, competing arbitrageurs trade more (the downward jump size is larger) and downward jumps occur more frequently. To capture this consideration, we set the jump intensity to be proportional to the arbitrage opportunity's value. Figure 7 graphs the start-to-trade threshold and the trade-down-to target as a function of the jump intensity parameter for the case where the jump size is 0.05, where the jump intensity parameter is equal to the jump intensity divided by the arbitrage opportunity's value. From Figure 7, we observe that both the start-to-trade threshold and the trade-down-to target decrease as the jump intensity parameter increases.

These experiments illustrate that with more competition, both the start-to-trade threshold and the trade-down-to target get closer to the proportional transaction cost. This indicates that with sufficiently large competition, arbitrageurs will trade as soon as trading becomes profitable, which is of course the classical solution.

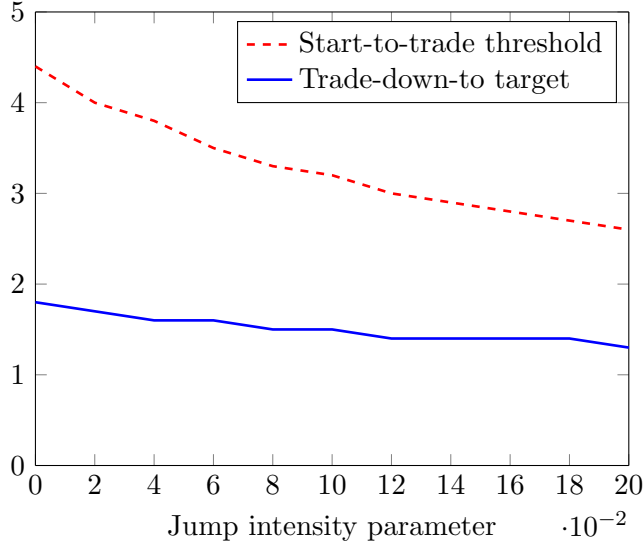


Figure 7: Threshold and target vs jump intensity parameter

## 5 Finite Horizon Problems and Risk Attitude

In Sections 3 and 4, the theoretical analysis explored threshold policies for infinite horizon and risk-neutral arbitrageurs. In practice, arbitrageurs may have different risk attitudes and wealths. In this section, we consider the finite horizon problem with risk attitude.

### 5.1 Finite Horizon

Finite horizon problems commonly exist in practice, such as arbitrage opportunities involving call options and its replicating portfolio. However, it is difficult to derive closed form optimal solutions for finite horizon stochastic control problems, because the value function is not separable in time  $t$  and arbitrage opportunity's value  $x$ . So we numerically solve the finite horizon problem and discuss the insights.

In the numerical experiment, the parameters are set as follows:  $r = 0.1$ ,  $b = 0.05$ ,  $\sigma = 0.2$ ,  $\theta = 0.1$ ,  $c = 1$ , and  $c_0 = 1$ . The arbitrage profit is limited to remain in the interval  $[0, 10]$  and this interval is divided by 100 units to discretize the state space. Meanwhile, we let the time horizon  $T = 10$  years and divide this horizon by  $10 \cdot 250 \cdot 8 \cdot 60$  units with each unit representing one minute.

From the numerical study, the optimal trade-down-to target as a function of arbitrage opportunity's value at the current time is represented in the left graph in Figure 8. These values actually match the ones for the infinite horizon case as described in Figure 2. This result is not surprising,



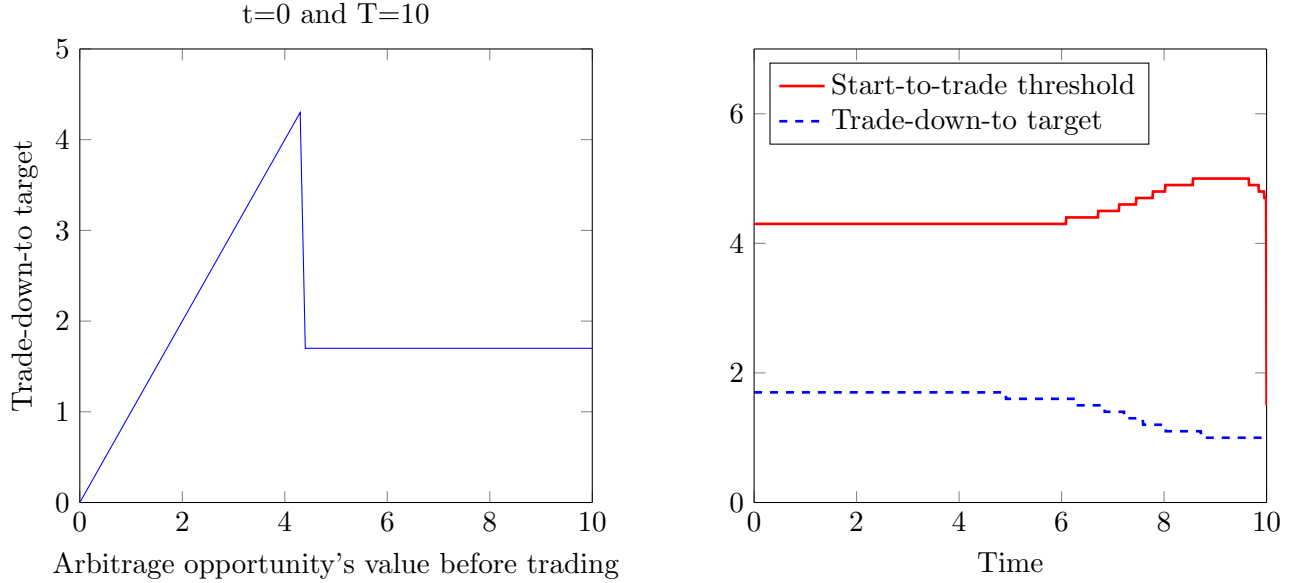


Figure 8: Threshold and target when  $T = 10$

because the time horizon is large.

The optimal policy is still a  $(y_1, y_2)$  policy but both values of  $y_1$  and  $y_2$  change with time. More specifically, it is  $(y_1(t), y_2(t))$  policy. The detailed two threshold values and the start-to-trade threshold and the trade-down-to target levels, as a function of time, are illustrated in the right graph in Figure 8.

The interesting behavior is that with time close to the end of horizon, the start-to-trade thresholds first increase and then decrease. The intuition is: when there is some remaining time left, the arbitrageur can only trade very limited times before the end of the horizon. The arbitrageur needs to consider the tradeoff between the number of possible trades and the profit per trade, due to the market impact. Therefore, if the current arbitrage opportunity's value is not high enough and the arbitrageur trades now, although the arbitrageur can trade more times, the total profit might not be high, because the profit for each trade might be low. Thus the arbitrageur might prefer to wait (corresponding to a higher start-to-trade threshold value) and trade more units per trade (corresponding to a lower trade-down-to target value). Following this strategy, although the arbitrageur may trade less times, but with a higher arbitrage opportunity's value in each trade in the future, the arbitrageur may get a higher total profit.

Meanwhile, when it comes close to the end of horizon, the arbitrageur might be able to trade only

once with a higher probability, and accordingly the arbitrageur becomes less patient (corresponding to a lower start-to-trade threshold) and finally trade as long as it is profitable, i.e., the start-to-trade value decided by the myopic policy.

## 5.2 Risk Attitude and Wealth Effects

The objective function incorporating risk attitude can lead to optimal trading strategies for arbitrageurs with different risk attitudes. To derive a precise model, one additional variable – wealth or money account  $M(t)$  at time  $t$  – is introduced, which leads to the following objective function expression:

$$\max_{\omega \in W} (M(T))^\lambda, \quad (26)$$

where  $\lambda > 0$  is the risk coefficient.

The dynamic of  $M(t)$  is

$$dM(t) = rM(t)dt + \sum_{i=1}^{\infty} 1_{\{\tau_i=t\}} \left\{ X_{\tau_i}^- (1 - e^{-\theta v_i}) / \theta - cv_i - c_0 \right\}, \quad (27)$$

where the first term follows from the compound interest rate formula and the second term is the profit increment due to the arbitrageur's transaction.

In the numerical study, the parameters are set as  $r = 0.1$ ,  $b = 0.05$ ,  $\sigma = 0.2$ ,  $\theta = 0.1$ ,  $c = 0.1$ , and  $c_0 = 0.1$ . The arbitrage opportunity's values are normalized to remain within the interval  $[0, 1]$  and the interval is divided by 50 units to discretize the state space. Similarly, we assume the time horizon  $T = 1$ , which is divided by 100 units to be discretized. Meanwhile, the wealth index is assumed to remain in the interval  $[0, 10]$  and the interval is divided by 1000 units. The results are illustrated in Figures 9 and 10.

From Figure 9, where the initial wealth is set to zero, by observing the optimal trade-down-to target versus the arbitrage opportunity's value at the current time with different risk coefficients including risk-averse (e.g.,  $\lambda = 1/3$  and  $\lambda = 2/3$ ), risk-neutral (e.g.,  $\lambda = 1$ ), and risk-loving ( $\lambda = 4/3$ ), we find the results to be intuitive. That is, the more risk averse the arbitrageur is (i.e., the smaller the  $\lambda$  value is), the sooner the arbitrageur will trade (i.e., the smaller the start-to-trade threshold).

Figure 10 depicts the relationship between the optimal trade-down-to target and the arbitrage opportunity's value with different initial wealths at the current time. The risk coefficient is set as

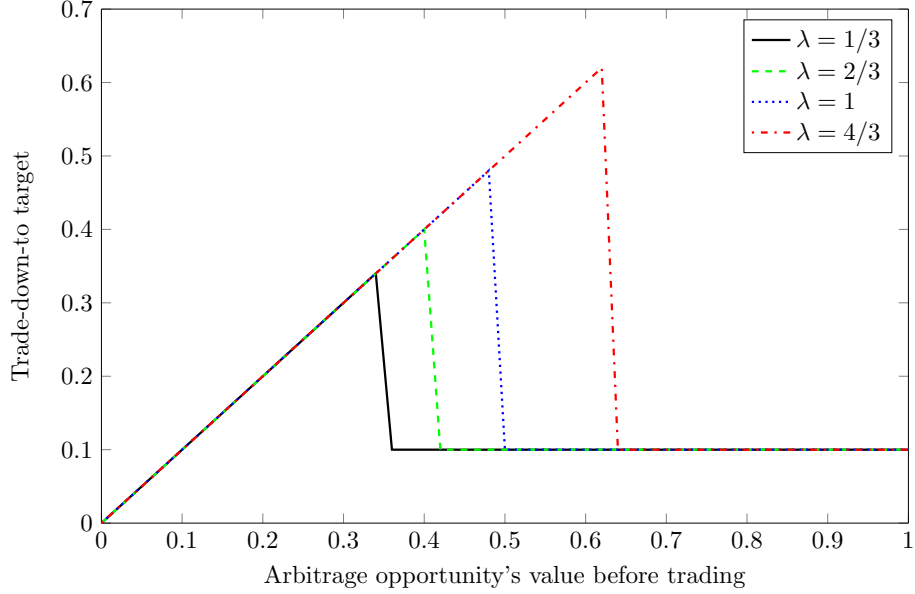


Figure 9: Optimal trade-down-to targets with different risk attitude

2/3. The numerical results indicate that with less wealth, the arbitrageur is more conservative and so waits less time, i.e., a smaller start-to-trade threshold value, which is reasonable in practice.

Based on the studies and discussions in the previous sections, in sum, the arbitrageur who first finds an arbitrage opportunity will not trade immediately when the trade is profitable. Instead, the arbitrageur waits until the arbitrage opportunity's value reaches a certain threshold, which could be significantly larger than that decided by traditional arbitrage theory. Later on, after competitors are observed, arbitrageurs decrease their start-to-trade thresholds. In addition, various lengths of horizons and arbitrageurs' risk attitudes and wealth on hand contribute to customized optimal trading strategies and further dynamics of arbitrage opportunity.

## 6 Conclusion

In this paper, we discuss the optimal trading strategies for an arbitrageur whose transactions incur transaction costs and affect the magnitude of the arbitrage opportunity, and where the arbitrageur may face potential competitors attempting to exploit the same arbitrage opportunity. Accordingly, in our models, we include fixed and proportional transaction costs, arbitrageur's market impact, and competition.

We start with the arbitrage trading problem with market impact in a monopoly environment,

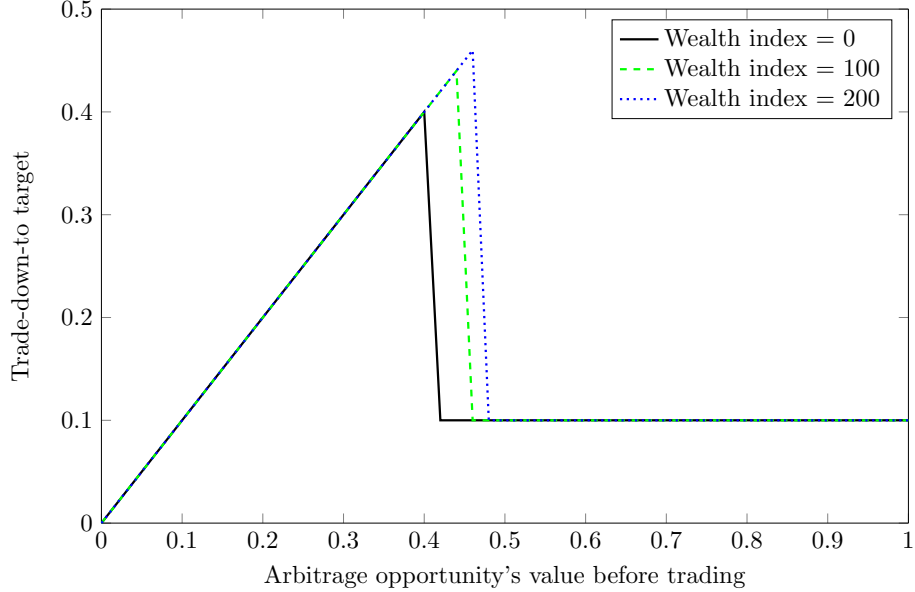


Figure 10: Optimal trade-down-to targets with different wealth

and then, extend it to incorporate competition. For both cases, we explicitly solve the problems. The derived optimal trading policy has a start-to-trade threshold and a trade-down-to target. It is optimal for an arbitrageur to wait until the arbitrage opportunity's value exceeds the start-to-trade threshold and then trade the amount such that the arbitrage opportunity's value decreases to the trade-down-to target. We provide closed form expressions for the threshold values and our proofs are self-contained.

In addition, our results reveal how the timing of the trade and the trade size depend on the magnitude of transactions costs, the dynamics of the arbitrage opportunity's value, the interest rate, the market impact, and the level of competition. Specifically, both the start-to-trade threshold and the trade-down-to target increase as the drift speed or the volatility or the proportional transaction cost increases; the start-to-trade threshold increases while the trade-down-to target decreases as the fixed transaction cost or the market impact increases; both the start-to-trade threshold and the trade-down-to target decrease as the interest rate or the level of competition increases.

Finally, our results show that, with market impact and transaction costs, the expected discounted arbitrage trading profit is finite and it is optimal for the arbitrageur not to trade immediately when trading is profitable. However, when competitors who find the arbitrage opportunity start to trade and decrease the arbitrage opportunity's value, the arbitrageur will trade sooner,

which might in turn make other competitors trade sooner and finally return the traditional solution. We provide a new perspective on the existing empirical literature testing for the existence of arbitrage opportunities. The arbitrage opportunity's value can far exceed the "transaction cost" band, long or short, until some arbitrageur starts to trade, after which the arbitrage opportunity's value might again recover to a high level if this arbitrageur is the only one to find the arbitrage opportunity or gradually jump to the "transaction cost" band and remain nearby due to several competitors' mutually excited trade.

## References

- [1] R. Almgren and N. Chriss. Optimal execution of portfolio transactions. *Journal of Risk*, 3:5–40, 2001.
- [2] G. Barles and H. M. Soner. Option pricing with transaction costs and a nonlinear black-scholes equation. *Finance and Stochastics*, 2(4):369–397, 1998.
- [3] S. Basak and B. Croitoru. Equilibrium mispricing in a capital market with portfolio constraints. *Review of Financial Studies*, 13(3):715–748, 2000.
- [4] S. Basak and B. Croitoru. On the role of arbitrageurs in rational markets. *Journal of Financial Economics*, 81(1):143–173, 2006.
- [5] D. Bertsimas and A. W. Lo. Optimal control of execution costs. *Journal of Financial Markets*, 1(1):1–50, 1998.
- [6] M. Broadie, J. Cvitanic, and H. M. Soner. Optimal replication of contingent claims under portfolio constraints. *Review of Financial Studies*, 11(1):59–79, 1998.
- [7] U. Çetin, R. A. Jarrow, and P. Protter. Liquidity risk and arbitrage pricing theory. *Finance and Stochastics*, 8(3):311–341, 2004.
- [8] U. Çetin, R. A. Jarrow, P. Protter, and M. Warachka. Pricing options in an extended black scholes economy with illiquidity: Theory and empirical evidence. *Review of Financial Studies*, 19(2):493–529, 2006.
- [9] D. Cuoco and H. Liu. A martingale characterization of consumption choices and hedging costs with margin requirements. *Mathematical Finance*, 10(3):355–385, 2000.
- [10] J. Cvitanić, H. Pham, and N. Touzi. A closed-form solution to the problem of super-replication under transaction costs. *Finance and Stochastics*, 3(1):35–54, 1999.
- [11] J. Detemple and S. Murthy. Intertemporal asset pricing with heterogeneous beliefs. *Journal of Economic Theory*, 62(2):294–320, 1994.
- [12] J. Detemple and S. Murthy. Equilibrium asset prices and no-arbitrage with portfolio constraints. *Review of Financial Studies*, pages 1133–1174, 1997.
- [13] D. Duffie. *Dynamic Asset Pricing Theory*. Princeton University Press, 3rd edition, 2001.
- [14] N. Gârleanu and L. H. Pedersen. Dynamic trading with predictable returns and transaction costs. *The Journal of Finance*, 68(6):2309–2340, 2013.

- [15] H. He and H. Mamaysky. Dynamic trading policies with price impact. *Journal of Economic Dynamics and Control*, 29(5):891–930, 2005.
- [16] J. N. Hugonnier and R. Prieto. Asset pricing with arbitrage activity. *Swiss Finance Institute Research Paper*, (13-57), 2013.
- [17] R. A. Jarrow. Optimal trading of arbitrage opportunities with market impact. *Johnson School Research Paper Series*, (20-2010), 2010.
- [18] R. A. Jarrow and S. Turnbull. *Derivative Securities*. South-Western College Pub., 1996.
- [19] E. Jouini and H. Kallal. Martingales and arbitrage in securities markets with transaction costs. *Journal of Economic Theory*, 66(1):178–197, 1995.
- [20] P. Kondor. Risk in dynamic arbitrage: the price effects of convergence trading. *The Journal of Finance*, 64(2):631–655, 2009.
- [21] O. A. Lamont and R. H. Thaler. Can the market add and subtract? mispricing in tech stock carve-outs. *The Journal of Political Economy*, 111(2):227–268, 2003.
- [22] J. Liu and F. A. Longstaff. Losing money on arbitrage: Optimal dynamic portfolio choice in markets with arbitrage opportunities. *Review of Financial Studies*, 17(3):611–641, 2004.
- [23] A. W. Lo, H. Mamaysky, and J. Wang. Asset prices and trading volume under fixed transactions costs. *Journal of Political Economy*, 112(5):1054–1090, 2004.
- [24] V. Naik and R. Uppal. Leverage constraints and the optimal hedging of stock and bond options. *Journal of Financial and Quantitative Analysis*, 29(2), 1994.
- [25] A. A. Obizhaeva and J. Wang. Optimal trading strategy and supply/demand dynamics. *Journal of Financial Markets*, 16(1):1–32, 2013.
- [26] E. Ofek, M. Richardson, and R. F. Whitelaw. Limited arbitrage and short sales restrictions: Evidence from the options markets. *Journal of Financial Economics*, 74(2):305–342, 2004.
- [27] B. Øksendal and A. Sulem. *Applied Stochastic Control of Jump Diffusions*. Springer, 2007.
- [28] P. Protter. *Stochastic Integration and Differential Equations: Version 2.1*, volume 21. Springer Verlag, 2004.
- [29] A. Shleifer. *Inefficient Markets: An Introduction to Behavioral Finance*. Oxford university press, 2000.
- [30] H. M. Soner, S. E. Shreve, and J. Cvitanić. There is no nontrivial hedging portfolio for option pricing with transaction costs. *The Annals of Applied Probability*, 5(2):327–355, 1995.
- [31] B. Tuckman and J. Vila. Arbitrage with holding costs: A utility-based approach. *The Journal of Finance*, 47(4):1283–1302, 1992.
- [32] W. Xiong. Convergence trading with wealth effects: an amplification mechanism in financial markets. *Journal of Financial Economics*, 62(2):247–292, 2001.