

On an open question about the complexity of a dynamic spectrum management problem

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Abstract

In this paper we discuss the complexity of a dynamic spectrum management problem within a multi-user communication system with K users and N available tones. In this problem a common utility function is optimized. In particular, so called min-rate, harmonic mean and geometric mean utility functions are considered. The complexity of the optimization problems with these utility functions is already discussed in [4] for different cases. However, the complexity for the case $N = 2$, K arbitrary is an open question. In this paper we show that these cases are also NP-hard by a reduction from the partitioning problem for the min-rate utility function, and from the independent set problem for the harmonic mean and geometric mean utility functions.

KEYWORDS: Dynamic Spectrum Management, Complexity, Partitioning Problem, Independent Set

1 Introduction

Reducing interference within a multi-user communication system is a quite relevant task. One possible way to accomplish it is by a subdivision of the available spectrum into multiple tones which are then a priori assigned, with no overlap, to the users. We call this approach *static* spectrum management. Its main drawback is that a tone assigned to a user can not be assigned to another user even if the former is idle. The alternative to the static approach is an approach where the entire spectrum is shared by all users, and the users dynamically adjust their transmit power spectral densities over it. In order to maximize the performance of the whole system, a Dynamic Spectrum Management (DSM) problem must be faced. DSM can be formulated either as a noncooperative Nash game (see, e.g., [2, 6]), or as a cooperative utility maximization problem (see, e.g., [1, 7]). In [4] the complexity of the problem under different common utility measures has been established. While the complexity results in that paper cover almost all cases, a few open

questions are left. The aim of this paper is to propose a solution for some of these open questions. Before presenting the new complexity results, in what follows we give a formal description of the DSM problem we are going to consider.

Let us consider a communication system where transmitter-receiver pairs (called in what follows *users*) share a common frequency band $f \in \Omega$. After normalization Ω is assumed to be the unit interval $[0, 1]$. Each user $k \in \{1, \dots, K\}$ has a fixed transmit power budget to be allocated across $[0, 1]$. That results in a function

$$s_k : \Omega \rightarrow [0, \infty),$$

called power spectral density (or power allocation) function of user k . The transmit power budget constraint is

$$\int_{\Omega} s(f) df \leq P_k,$$

where P_k is some positive value. In a multi-user context, the utility function u_k of user k does not only depend on s_k , but on all functions s_j , $j \in \{1, \dots, K\}$

$$u_k = u_k(s_1, \dots, s_K) = \int_{\Omega} R_k(s_1(f), \dots, s_K(f)) df,$$

where R_k is a Lebesgue integrable, possibly nonconcave function. In a dynamic spectrum management problem we aim to find functions $s_k(f)$, $k \in \{1, \dots, K\}$, which satisfy the power transmit budget constraint and at the same time optimizes some function which combines the utility of all the users, i.e., we search for some social optimum of the communication system. Possible common utility functions $H(u_1, \dots, u_k)$ are

$\frac{1}{K} \sum_{k=1}^K u_k$	arithmetic mean
$\left(\prod_{k=1}^K u_k \right)^{\frac{1}{K}}$	geometric mean
$K \left(\sum_{k=1}^K (u_k)^{-1} \right)^{-1}$	harmonic mean
$\min_{k=1, \dots, K} u_k$	min-rate.

In order to find a numerical solution of the problem, the frequency band is discretized with N available tones. This way we are led to a problem with the decision variables s_k^i , the transmitted complex Gaussian signal from user $k \in \{1, \dots, K\}$ at tone $i \in \{1, \dots, N\}$. These variables must satisfy nonnegativity requirements, and the power transmit budget constraints become

$$\frac{1}{N} \sum_{i=1}^N s_k^i \leq P_k, \quad k \in \{1, \dots, K\}.$$

The utility u_k of user k is

$$u_k = \frac{1}{N} \sum_{i=1}^N \ln \left(1 + \frac{s_k^i}{\sigma_k^i + \sum_{j \neq k} \alpha_{kj} s_j^i} \right),$$

where σ_k^i is the normalized background power, while α_{kj}^i is the normalized crosstalk coefficient from transmitter j to receiver k at tone i . We search for a possible assignment to the variables s_k^i , $k \in \{1, \dots, K\}$, $i \in \{1, \dots, N\}$, which satisfy the previously introduced constraints and at the same time optimizes one of the utility functions previously discussed. For K arbitrary and a fixed N value the complexity of these problems has been established in most cases in [4]. In particular, for the arithmetic mean utility function, it has been established that the problem is NP-hard for all N values. For the other utility functions, the case $N = 1$ is polynomially solvable, while the case $N > 2$ has been proved to be NP-hard. The complexity for the case $N = 2$ is an open question. In this paper we establish that NP-hardness holds also for $N = 2$. In Section 2 the result will be established for the min-rate utility function by a reduction from the partitioning problem. In Section 3 we will give the (easier) proof of NP-hardness for the geometric and harmonic mean utility functions.

2 Min-rate utility function

For the min-rate utility function the optimization problem to be solved is

$$\max_{\mathbf{s}^1, \dots, \mathbf{s}^N \in \mathcal{S}_N^K} \min_{k=1, \dots, K} \sum_{i=1}^N \log \left(1 + \frac{s_k^i}{\sigma_k^i + \sum_{j \neq k} \alpha_{jk}^i s_j^i} \right) \quad (1)$$

where

$$\mathcal{S}_N^K = \left\{ \mathbf{s}^1, \dots, \mathbf{s}^N : \frac{1}{N} \sum_{i=1}^N s_k^i \leq P_k, \forall k = 1, \dots, K, \mathbf{s}^1, \dots, \mathbf{s}^N \geq \mathbf{0} \right\}.$$

For $N = 2$ and K arbitrary, the problem is equivalent to

$$\max_{\mathbf{s}^1, \mathbf{s}^2 \in \mathcal{S}_2^K} \min_{k=1, \dots, K} \left(1 + \frac{s_k^1}{\sigma_k^1 + \sum_{j \neq k} \alpha_{jk}^1 s_j^1} \right) \left(1 + \frac{s_k^2}{\sigma_k^2 + \sum_{j \neq k} \alpha_{jk}^2 s_j^2} \right)$$

which is further equivalent to

$$\max_{\mathbf{s}^1, \mathbf{s}^2 \in \mathcal{S}_2^K} \min_{k=1, \dots, K} \frac{s_k^1}{\sigma_k^1 + \sum_{j \neq k} \alpha_{jk}^1 s_j^1} + \frac{s_k^2}{\sigma_k^2 + \sum_{j \neq k} \alpha_{jk}^2 s_j^2} + \frac{s_k^1 s_k^2}{(\sigma_k^1 + \sum_{j \neq k} \alpha_{jk}^1 s_j^1)(\sigma_k^2 + \sum_{j \neq k} \alpha_{jk}^2 s_j^2)} \quad (2)$$

We prove NP-hardness of this problem through a reduction from the partitioning problem. In the *partitioning problem*, we are given a set of K integer positive values N_1, \dots, N_K (possibly with some repetitions) and we are asked to establish whether there exists a partition P_1, P_2 of $\{1, \dots, K\}$ such that $\sum_{j \in P_1} N_j = \sum_{j \in P_2} N_j$, or not. The problem is known to be NP-complete although it admits a pseudo-polynomial algorithm (see, e.g.,

[3, 5]). In this paper we show that we are able to solve the partitioning problem by solving the following special instance of (2)

$$\begin{aligned}
\max \quad & \min_{k=1,\dots,K} \frac{s_k^1}{N_k + \sum_{j \neq k} N_j s_j^1} + \frac{s_k^2}{N_k + \sum_{j \neq k} N_j s_j^2} + \frac{s_k^1 s_k^2}{(N_k + \sum_{j \neq k} N_j s_j^1)(N_k + \sum_{j \neq k} N_j s_j^2)} \\
& s_k^1 + s_k^2 \leq 1 \quad k = 1, \dots, K \\
& s_k^1, s_k^2 \geq 0 \quad k = 1, \dots, K,
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
\sigma_k^1 &= \sigma_k^2 = N_k \quad \forall k \in \{1, \dots, K\} \\
\alpha_{kj}^1 &= \alpha_{kj}^2 = N_j \quad \forall j \in \{1, \dots, K\}, j \neq k,
\end{aligned}$$

and the feasible set is equal to \mathcal{S}_2^K with $2P_k = 1, \forall k$.

We first need the following lemma about optimal solutions of problem (3).

Lemma 2.1 *Let $\mathbf{s}^{1*} = (s_1^{1*}, \dots, s_K^{1*})$, $\mathbf{s}^{2*} = (s_1^{2*}, \dots, s_K^{2*})$ be an optimal solution of problem (3). Then, for any sufficiently small $\rho > 0$, it holds that*

$$\sum_{i=1}^K s_i^{j*} \geq \rho, \quad j = 1, 2,$$

or, equivalently $\mathbf{s}^{1*}, \mathbf{s}^{2*} \neq \mathbf{0}$.

Proof. By contradiction we assume that $\mathbf{s}^{1*} = \mathbf{0}$ (similar for $\mathbf{s}^{2*} = \mathbf{0}$). Then, the objective function value at this solution is

$$\min_{k=1,\dots,K} \frac{s_k^{2*}}{N_k + \sum_{j \neq k} N_j s_j^{2*}}.$$

We first remark that for all $h, t \in \{1, \dots, K\}$

$$\frac{s_t^{2*}}{N_t + \sum_{j \neq t} N_j s_j^{2*}} = \frac{s_h^{2*}}{N_h + \sum_{j \neq h} N_j s_j^{2*}} = \min_{k=1,\dots,K} \frac{s_k^{2*}}{N_k + \sum_{j \neq k} N_j s_j^{2*}}. \tag{4}$$

Indeed, if we assume by contradiction that for some k

$$\frac{s_h^{2*}}{N_h + \sum_{j \neq h} N_j s_j^{2*}} > \min_{k=1,\dots,K} \frac{s_k^{2*}}{N_k + \sum_{j \neq k} N_j s_j^{2*}},$$

we are able to increase the objective function value by decreasing s_h^{2*} , thus contradicting the optimality of the solution. Now, let

$$p \in \arg \min_{i=1,\dots,K} s_i^{2*}.$$

Then, the optimal value is

$$\frac{s_p^{2*}}{N_p + \sum_{j \neq p} N_j s_j^{2*}} = \frac{1}{\frac{1}{s_p^{2*}} N_p + \sum_{j \neq p} N_j \frac{s_j^{2*}}{s_p^{2*}}} \leq \frac{1}{\sum_{j=1}^K N_j}.$$

However, if we consider the feasible solution $\bar{\mathbf{s}}^1 = (\frac{1}{2}, \dots, \frac{1}{2})$, $\bar{\mathbf{s}}^2 = (\frac{1}{2}, \dots, \frac{1}{2})$, the objective function value at this solution is

$$\min_{k=1, \dots, K} \frac{1}{N_k + \sum_{j \neq k} \frac{1}{2} N_j} > \frac{1}{\sum_{j=1}^K N_j},$$

which contradicts the optimality of $\mathbf{s}^{1*} = \mathbf{0}$, \mathbf{s}^{2*} . \square

Thus, we can modify problem (3) as follows

$$\begin{aligned} \max \quad & \min_{k=1, \dots, K} \frac{s_k^1}{N_k + \sum_{j \neq k} N_j s_j^1} + \frac{s_k^2}{N_k + \sum_{j \neq k} N_j s_j^2} + \frac{s_k^1 s_k^2}{(N_k + \sum_{j \neq k} N_j s_j^1)(N_k + \sum_{j \neq k} N_j s_j^2)} \\ & s_k^1 + s_k^2 \leq 1 \quad k = 1, \dots, K \\ & \sum_{i=1}^K s_i^j \geq \rho \quad j = 1, 2 \\ & s_k^1, s_k^2 \geq 0 \quad k = 1, \dots, K. \end{aligned} \tag{5}$$

For each feasible solution $\bar{\mathbf{s}}^1 = (\bar{s}_1^1, \dots, \bar{s}_K^1)$, $\bar{\mathbf{s}}^2 = (\bar{s}_1^2, \dots, \bar{s}_K^2)$ of (5), the objective function value is

$$\min_{k=1, \dots, K} \frac{\bar{s}_k^1}{A_k^1(\bar{\mathbf{s}}^1)} + \frac{\bar{s}_k^2}{A_k^2(\bar{\mathbf{s}}^2)} + \frac{\bar{s}_k^1 \bar{s}_k^2}{A_k^1(\bar{\mathbf{s}}^1) A_k^2(\bar{\mathbf{s}}^2)},$$

where for $i = 1, 2$, $k = 1, \dots, K$

$$A_k^i(\bar{\mathbf{s}}^i) = N_k + \sum_{j \neq k} N_j \bar{s}_j^i. \tag{6}$$

Now, for $i = 1, 2$ let us set

$$B^i(\bar{\mathbf{s}}^i) = \sum_{j=1}^K N_j \bar{s}_j^i. \tag{7}$$

We prove the following lemma.

Lemma 2.2 *For any feasible solution of (5) it holds that*

$$\frac{\bar{s}_k^1}{A_k^1(\bar{\mathbf{s}}^1)} + \frac{\bar{s}_k^2}{A_k^2(\bar{\mathbf{s}}^2)} + \frac{\bar{s}_k^1 \bar{s}_k^2}{A_k^1(\bar{\mathbf{s}}^1) A_k^2(\bar{\mathbf{s}}^2)} \leq \frac{\bar{s}_k^1}{B^1(\bar{\mathbf{s}}^1)} + \frac{\bar{s}_k^2}{B^2(\bar{\mathbf{s}}^2)}, \tag{8}$$

and equality holds only if $\bar{s}_k^1, \bar{s}_k^2 \in \{0, 1\}$.

Proof. In view of the definitions (6) and (7), proving (8) is equivalent to prove

$$\frac{N_k \bar{s}_k^1 (1 - \bar{s}_k^1)}{A_k^1(\bar{\mathbf{s}}^1) B^1(\bar{\mathbf{s}}^1)} + \frac{N_k \bar{s}_k^2 (1 - \bar{s}_k^2)}{A_k^2(\bar{\mathbf{s}}^2) B^2(\bar{\mathbf{s}}^2)} - \frac{\bar{s}_k^1 \bar{s}_k^2}{A_k^1(\bar{\mathbf{s}}^1) A_k^2(\bar{\mathbf{s}}^2)} \geq 0.$$

Now, let us introduce the following function, where \bar{s}_k^2 is replaced by the variable s_k^2 *only in the numerators*

$$F(s_k^2) = \frac{N_k \bar{s}_k^1 (1 - \bar{s}_k^1)}{A_k^1(\bar{\mathbf{s}}^1) B^1(\bar{\mathbf{s}}^1)} + \frac{N_k s_k^2 (1 - s_k^2)}{A_k^2(\bar{\mathbf{s}}^2) B^2(\bar{\mathbf{s}}^2)} - \frac{\bar{s}_k^1 s_k^2}{A_k^1(\bar{\mathbf{s}}^1) A_k^2(\bar{\mathbf{s}}^2)}.$$

The feasible interval for s_k^2 is $[0, 1 - \bar{s}_k^1]$. Then, if we are able to prove that $\min_{s_k^2 \in [0, 1 - \bar{s}_k^1]} F(s_k^2) \geq 0$, we are done. Since F is a concave function, it attains its minimum value at the extremes of the feasible interval. If the minimum is attained for $s_k^2 = 0$, then

$$\min_{s_k^2 \in [0, 1 - \bar{s}_k^1]} F(s_k^2) = \frac{N_k \bar{s}_k^1 (1 - \bar{s}_k^1)}{A_k^1(\bar{\mathbf{s}}^1) B^1(\bar{\mathbf{s}}^1)} \geq 0,$$

and equality only holds for $\bar{s}_k^1 \in \{0, 1\}$. Otherwise, if the minimum is attained for $s_k^2 = 1 - \bar{s}_k^1$, then we need to prove that

$$\frac{N_k \bar{s}_k^1 (1 - \bar{s}_k^1)}{A_k^1(\bar{\mathbf{s}}^1) B^1(\bar{\mathbf{s}}^1)} + \frac{N_k \bar{s}_k^1 (1 - \bar{s}_k^1)}{A_k^2(\bar{\mathbf{s}}^2) B^2(\bar{\mathbf{s}}^2)} - \frac{\bar{s}_k^1 (1 - \bar{s}_k^1)}{A_k^1(\bar{\mathbf{s}}^1) A_k^2(\bar{\mathbf{s}}^2)} \geq 0.$$

If $\bar{s}_k^1 \in \{0, 1\}$ equality holds. Otherwise, we need to prove that

$$\frac{N_k}{A_k^1(\bar{\mathbf{s}}^1) B^1(\bar{\mathbf{s}}^1)} + \frac{N_k}{A_k^2(\bar{\mathbf{s}}^2) B^2(\bar{\mathbf{s}}^2)} - \frac{1}{A_k^1(\bar{\mathbf{s}}^1) A_k^2(\bar{\mathbf{s}}^2)} > 0.$$

Since

$$\frac{1}{A_k^1(\bar{\mathbf{s}}^1) A_k^2(\bar{\mathbf{s}}^2)} \leq \frac{1}{2[A_k^1(\bar{\mathbf{s}}^1)]^2} + \frac{1}{2[A_k^2(\bar{\mathbf{s}}^2)]^2},$$

it is enough to prove that for $i = 1, 2$

$$\frac{N_k}{A_k^i(\bar{\mathbf{s}}^i) B^i(\bar{\mathbf{s}}^i)} - \frac{1}{2[A_k^i(\bar{\mathbf{s}}^i)]^2} > 0,$$

or, equivalently

$$\frac{N_k A_k^i(\bar{\mathbf{s}}^i)}{B^i(\bar{\mathbf{s}}^i)} > \frac{1}{2}.$$

The result immediately follows from $B^i(\bar{\mathbf{s}}^i) \leq A_k^i(\bar{\mathbf{s}}^i)$ and $N_k \geq 1$. \square

Now, let us consider the following optimization problem, whose optimal value, in view of Lemma 2.2, is an upper bound for the optimal value of (2)

$$\begin{aligned} \max \quad & \min_{k=1, \dots, K} \frac{s_k^1}{B^1(\bar{\mathbf{s}}^1)} + \frac{s_k^2}{B^2(\bar{\mathbf{s}}^2)} \\ & s_k^1 + s_k^2 \leq 1 \quad k = 1, \dots, K \\ & \sum_{i=1}^K s_i^j \geq \rho \quad j = 1, 2 \\ & s_k^1, s_k^2 \geq 0 \quad k = 1, \dots, K. \end{aligned} \tag{9}$$

We remark that the objective function of the problem is not defined for $\mathbf{s}^1 = \mathbf{0}$ and/or $\mathbf{s}^2 = \mathbf{0}$. However, such solutions do not belong to the feasible region of problem (9) in view of the constraints $\sum_{i=1}^K s_i^j \geq \rho$, $j = 1, 2$. Next, we need to prove some lemmas about properties of all or some optimal solutions of (9).

Lemma 2.3 *Let $(\mathbf{s}^{1*}, \mathbf{s}^{2*})$ be an optimal solution of (9). Then $\forall p, r \in \{1, \dots, K\}$*

$$\frac{s_p^{1*}}{B^1(\mathbf{s}^{1*})} + \frac{s_p^{2*}}{B^2(\mathbf{s}^{2*})} = \frac{s_r^{1*}}{B^1(\mathbf{s}^{1*})} + \frac{s_r^{2*}}{B^2(\mathbf{s}^{2*})} = \min_{k=1, \dots, K} \frac{s_k^{1*}}{B^1(\mathbf{s}^{1*})} + \frac{s_k^{2*}}{B^2(\mathbf{s}^{2*})}.$$

Proof. The proof is completely analogous to that of (4) in Lemma 2.1. We first remark that the optimal value of (9) is strictly larger than 0, so that $s_p^{1*} + s_p^{2*} > 0$ must hold for all $p = 1, \dots, K$. Then, it is enough to observe that if we assume by contradiction that for some p

$$\frac{s_p^{1*}}{B^1(\mathbf{s}^{1*})} + \frac{s_p^{2*}}{B^2(\mathbf{s}^{2*})} > \min_{k=1, \dots, K} \frac{s_k^{1*}}{B^1(\mathbf{s}^{1*})} + \frac{s_k^{2*}}{B^2(\mathbf{s}^{2*})},$$

then, by a sufficiently small decrease either of s_p^{1*} or of s_p^{2*} , we are able to increase the objective function value, thus contradicting the optimality of $\mathbf{s}^{1*}, \mathbf{s}^{2*}$. \square

Next we prove the existence of an optimal solution with a special feature.

Lemma 2.4 *There always exists an optimal solution $(\mathbf{s}^{1*}, \mathbf{s}^{2*})$ of (9) such that $s_k^{1*} + s_k^{2*} = 1$ for all $k = 1, \dots, K$ and:*

- either $B^1(\mathbf{s}^{1*}) = B^2(\mathbf{s}^{2*})$;
- or

$$s_k^{1*} = s_h^{1*}, \quad s_k^{2*} = s_h^{2*}, \quad \forall k, h.$$

Proof. We first observe that for any α, β such that $(\alpha\mathbf{s}^{1*}, \beta\mathbf{s}^{2*})$ is feasible, it also holds that $(\alpha\mathbf{s}^{1*}, \beta\mathbf{s}^{2*})$ is an optimal solution of (9). Indeed, $(\alpha\mathbf{s}^{1*}, \beta\mathbf{s}^{2*})$ has the same objective function value as $(\mathbf{s}^{1*}, \mathbf{s}^{2*})$. In particular, the values α, β such that $(\alpha\mathbf{s}^{1*}, \beta\mathbf{s}^{2*})$ is feasible satisfy

$$\begin{aligned} \alpha s_k^{1*} + \beta s_k^{2*} &\leq 1 \quad k = 1, \dots, K \\ \alpha, \beta &\geq 0. \end{aligned} \tag{10}$$

This set is a two-dimensional polytope. If for some $p, r \in \{1, \dots, K\}$ we have $s_r^{1*} = s_p^{1*}$ and $s_r^{2*} = s_p^{2*}$, we can eliminate, e.g., the p -th inequality due to redundancy. Otherwise, we observe that if $s_r^{1*} > s_p^{1*}$, then we must have $s_r^{2*} < s_p^{2*}$. Indeed, in view of Lemma 2.3 we must have

$$\frac{\bar{s}_p^{1*} - \bar{s}_r^{1*}}{B^1(\bar{\mathbf{s}}^{1*})} + \frac{\bar{s}_p^{2*} - \bar{s}_r^{2*}}{B^2(\bar{\mathbf{s}}^{2*})} = 0.$$

For each pair r, p such that $s_r^{1*} > s_p^{1*}$ and $s_r^{2*} < s_p^{2*}$, it holds that the solution $(\bar{\alpha}, \bar{\beta})$ of the bivariate linear system

$$\begin{aligned}\alpha s_p^{1*} + \beta s_p^{2*} &= 1 \\ \alpha s_r^{1*} + \beta s_r^{2*} &= 1,\end{aligned}\tag{11}$$

satisfies $\bar{\alpha}, \bar{\beta} > 0$. For some pair p, r $(\bar{\alpha}, \bar{\beta})$ is a vertex of the polytope (10). Now, let us consider the optimal solution $(\bar{\mathbf{s}}^{1*}, \bar{\mathbf{s}}^{2*}) = (\bar{\alpha}\mathbf{s}^{1*}, \bar{\beta}\mathbf{s}^{2*})$. In view of Lemma 2.3

$$\frac{\bar{s}_p^{1*}}{B^1(\bar{\mathbf{s}}^{1*})} + \frac{\bar{s}_p^{2*}}{B^2(\bar{\mathbf{s}}^{2*})} = \frac{\bar{s}_r^{1*}}{B^1(\bar{\mathbf{s}}^{1*})} + \frac{\bar{s}_r^{2*}}{B^2(\bar{\mathbf{s}}^{2*})},$$

and since $(\bar{\alpha}, \bar{\beta})$ is a solution of (11)

$$\frac{\bar{s}_p^{1*}}{B^1(\bar{\mathbf{s}}^{1*})} + \frac{1 - \bar{s}_p^{1*}}{B^2(\bar{\mathbf{s}}^{2*})} = \frac{\bar{s}_r^{1*}}{B^1(\bar{\mathbf{s}}^{1*})} + \frac{1 - \bar{s}_r^{1*}}{B^2(\bar{\mathbf{s}}^{2*})},$$

which is equivalent to

$$\frac{(\bar{s}_p^{1*} - \bar{s}_r^{1*})}{B^1(\bar{\mathbf{s}}^{1*})} - \frac{(\bar{s}_p^{1*} - \bar{s}_r^{1*})}{B^2(\bar{\mathbf{s}}^{2*})} = 0.$$

Then,

$$(\bar{s}_p^{1*} - \bar{s}_r^{1*}) \frac{B^2(\bar{\mathbf{s}}^{2*}) - B^1(\bar{\mathbf{s}}^{1*})}{B^1(\bar{\mathbf{s}}^{1*})B^2(\bar{\mathbf{s}}^{2*})} = 0.$$

The above equality holds either if

$$B^2(\bar{\mathbf{s}}^{2*}) - B^1(\bar{\mathbf{s}}^{1*}) = 0,\tag{12}$$

or if

$$\bar{s}_p^{1*} - \bar{s}_r^{1*} = 0.\tag{13}$$

If (12) holds, then in view of Lemma 2.3 we have for all $k = 1, \dots, K$

$$\frac{\bar{s}_k^{1*} + \bar{s}_k^{2*}}{B^1(\bar{\mathbf{s}}^{1*})} = \frac{1}{B^1(\bar{\mathbf{s}}^{1*})},$$

and, thus,

$$\bar{s}_k^{1*} + \bar{s}_k^{2*} = 1,$$

as we wanted to prove. If (13) holds, then the derivation of the result is more elaborate. Since $(\bar{\alpha}, \bar{\beta})$ is a solution of (11), it also holds that

$$\bar{s}_p^{2*} - \bar{s}_r^{2*} = 0.$$

Thus, if we consider the system of inequalities (10) with $\mathbf{s}^{1*}, \mathbf{s}^{2*}$ replaced by $\bar{\mathbf{s}}^{1*}, \bar{\mathbf{s}}^{2*}$, i.e.,

$$\begin{aligned}\alpha \bar{s}_k^{1*} + \beta \bar{s}_k^{2*} &\leq 1 \quad k = 1, \dots, K \\ \alpha, \beta &\geq 0,\end{aligned}\tag{14}$$

the two inequalities

$$\begin{aligned}\alpha\bar{s}_p^{1*} + \beta\bar{s}_p^{2*} &\leq 1 \\ \alpha\bar{s}_r^{1*} + \beta\bar{s}_r^{2*} &\leq 1\end{aligned}$$

are equivalent and one of them can be eliminated. In the reduced system of inequalities we can search for a further pair u, v such that the solution $(\tilde{\alpha}, \tilde{\beta})$ of the linear system

$$\begin{aligned}\alpha\bar{s}_u^{1*} + \beta\bar{s}_u^{2*} &= 1 \\ \alpha\bar{s}_v^{1*} + \beta\bar{s}_v^{2*} &= 1,\end{aligned}$$

is a vertex of the polytope (14). This way we are able to detect a new optimal solution $\tilde{\mathbf{s}}^{1*}, \tilde{\mathbf{s}}^{2*}$ such that we are able to eliminate one of the inequalities $\alpha\tilde{s}_k^{1*} + \beta\tilde{s}_k^{2*} \leq 1$. By repeating this procedure, in the end we are able to reduce the system of linear inequalities to just a single inequality (besides the nonnegative requirements), and to derive an optimal solution of problem (9) for which

$$s_k^{1*} + s_k^{2*} = 1, \quad \forall k,$$

and, moreover,

$$s_k^{1*} = s_h^{1*}, \quad s_k^{2*} = s_h^{2*}, \quad \forall k, h,$$

as we wanted to prove. □

We finally prove the following lemma about the optimal value of problem (9).

Lemma 2.5 *The optimal value of problem (9) is*

$$\frac{2}{\sum_{j=1}^K N_j}.$$

Proof. In view of Lemma 2.4 let us first assume that there exists an optimal solution such that $s_k^{1*} + s_k^{2*} = 1$ for all $k = 1, \dots, K$ and $B^1(\mathbf{s}^{1*}) = B^2(\mathbf{s}^{2*})$. We remark that

$$B^2(\mathbf{s}^{2*}) = \sum_{j=1}^K N_j s_j^{*2} = \sum_{j=1}^K N_j - \sum_{j=1}^K N_j s_j^{*1} = \sum_{j=1}^K N_j - B^1(\mathbf{s}^{1*}),$$

and, thus,

$$B^1(\mathbf{s}^{1*}) = B^2(\mathbf{s}^{2*}) = \frac{\sum_{j=1}^K N_j}{2}.$$

In this case the optimal value of problem (9) is

$$\frac{2}{\sum_{j=1}^K N_j},$$

as we wanted to prove.

Next let us assume that $s_k^{1*} + s_k^{2*} = 1$ for all $k = 1, \dots, K$ and

$$s_k^{1*} = s_h^{1*}, \quad s_k^{2*} = s_h^{2*}, \quad \forall k, h.$$

Then, for all $k \in \{1, \dots, K\}$

$$\frac{s_k^{1*}}{B^1(\mathbf{s}^{1*})} + \frac{s_k^{2*}}{B^2(\mathbf{s}^{2*})} = \frac{1}{\sum_{j=1}^K N_j} + \frac{1}{\sum_{j=1}^K N_j} = \frac{2}{\sum_{j=1}^K N_j},$$

as we wanted to prove. \square

Thus, in view of Lemma 2.2 and of Lemma 2.5, the optimal value of (5) is bounded from above by

$$\frac{2}{\sum_{j=1}^K N_j},$$

and such upper bound can only be reached at binary solutions, i.e., solutions for which

$$s_k^2 = 1 - s_k^1, \quad s_k^1 \in \{0, 1\} \quad \forall k \in \{1, \dots, K\}.$$

The last step is to show that the above upper bound is reached if and only if the partitioning problem has a **yes** answer.

Theorem 2.1 *The optimal value of (5) is equal to the upper bound*

$$\frac{2}{\sum_{j=1}^K N_j},$$

if and only if there exists a partition P_1, P_2 of $\{1, \dots, K\}$ such that $\sum_{j \in P_1} N_j = \sum_{j \in P_2} N_j$, i.e., if and only if the partitioning problem has a solution.

Proof. As previously remarked, the upper bound can only be reached at binary solutions, for which the objective function value of (5) is

$$\min \left\{ \frac{1}{\sum_{j : s_j^1=1} N_j}, \frac{1}{\sum_{j : s_j^2=1} N_j} \right\} = \frac{1}{\max\{\sum_{j : s_j^1=1} N_j, \sum_{j : s_j^2=1} N_j\}}.$$

Since

$$\max \left\{ \sum_{j : s_j^1=1} N_j, \sum_{j : s_j^2=1} N_j \right\} \geq \frac{\sum_{j=1}^K N_j}{2},$$

we have that at binary solutions the objective function value can be equal to the upper bound if and only if

$$\sum_{j : s_j^1=1} N_j = \sum_{j : s_j^2=1} N_j = \frac{\sum_{j=1}^K N_j}{2},$$

i.e., if and only if the partition of $\{1, \dots, K\}$

$$P_1 = \{j : s_j^1 = 1\}, \quad P_2 = \{j : s_j^2 = 1\},$$

defines a solution of the partitioning problem. \square

3 Harmonic and geometric mean utility functions

The problem where the harmonic mean is maximized and $N = 2$ is equivalent to

$$\begin{aligned} \min \quad & \sum_{k=1, \dots, K} \left[\log \left(\left(1 + \frac{s_k^1}{\sigma_k^1 + \sum_{j \neq k} \alpha_{jk}^1 s_j^1} \right) * \left(1 + \frac{s_k^2}{\sigma_k^2 + \sum_{j \neq k} \alpha_{jk}^2 s_j^2} \right) \right) \right]^{-1} \\ & s_k^1 + s_k^2 \leq 1 \quad k = 1, \dots, K \\ & s_k^1, s_k^2 \geq 0 \quad k = 1, \dots, K. \end{aligned} \quad (15)$$

We prove that this problem is NP-hard by a reduction from the maximum independent set problem over a graph $G = (V, E)$, with $V = \{1, \dots, K\}$. Let us consider the following special instance of problem (15)

$$\begin{aligned} \min \quad & \sum_{k=1, \dots, K} \left[\log \left(\left(1 + \frac{s_k^1}{\left(\frac{1}{\exp(1)-1}\right) + \sum_{j \neq k} R_{jk} s_j^1} \right) * \left(1 + \frac{s_k^2}{\left(\frac{1}{\exp(1/3)-1}\right)} \right) \right) \right]^{-1} \\ & s_k^1 + s_k^2 \leq 1 \quad k = 1, \dots, K \\ & s_k^1, s_k^2 \geq 0 \quad k = 1, \dots, K. \end{aligned}$$

where we set

$$\begin{aligned} \sigma_k^1 &= \frac{1}{\exp(1)-1}, \quad \sigma_k^2 = \frac{1}{\exp(1/3)-1} \quad \forall k \in \{1, \dots, K\} \\ \alpha_{kj}^1 &= R_{kj}, \quad \alpha_{kj}^2 = 0 \quad \forall j \in \{1, \dots, K\}, j \neq k, \end{aligned} \quad (16)$$

and

$$R_{kj} = \begin{cases} +\infty & \text{if } (k, j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

For some feasible solution $(\mathbf{s}^1, \mathbf{s}^2)$, let

$$C = \{k \in \{1, \dots, K\} : s_k^1 > 0, \text{ and } s_j^1 = 0 \quad \forall j : (k, j) \in E\}. \quad (17)$$

The objective function value at $(\mathbf{s}^1, \mathbf{s}^2)$ is equal to

$$\sum_{k \in C} \left[\log \left(\left(1 + \frac{s_k^1}{\left(\frac{1}{\exp(1)-1}\right)} \right) * \left(1 + \frac{s_k^2}{\left(\frac{1}{\exp(1/3)-1}\right)} \right) \right) \right]^{-1} + \sum_{k \notin C} \left[\log \left(1 + \frac{s_k^2}{\left(\frac{1}{\exp(1/3)-1}\right)} \right) \right]^{-1}.$$

An improved feasible solution $(\bar{\mathbf{s}}^1, \bar{\mathbf{s}}^2)$ can be obtained by setting $\bar{s}_k^1 = 1, \bar{s}_k^2 = 0 \forall k \in C$, and $\bar{s}_k^1 = 0, \bar{s}_k^2 = 1 \forall k \notin C$. The objective function value at this solution is

$$|C| + 3 * (K - |C|).$$

This value is minimized if $|C|$ is as large as possible. Thus, the global minimum value is attained when C^* is the maximum cardinality independent set, and the optimal solution is

$$s_k^{1*} = \begin{cases} 1 & k \in C^* \\ 0 & \text{otherwise} \end{cases} \quad s_k^{2*} = 1 - s_k^{1*} \quad \forall k. \quad (18)$$

The problem where the geometric mean is maximized and $N = 2$ is equivalent to

$$\begin{aligned} \max \quad & \prod_{k=1, \dots, K} \left[\log \left(\left(1 + \frac{s_k^1}{\sigma_k^1 + \sum_{j \neq k} \alpha_{jk}^1 s_j^1} \right) * \left(1 + \frac{s_k^2}{\sigma_k^2 + \sum_{j \neq k} \alpha_{jk}^2 s_j^2} \right) \right) \right] \\ & s_k^1 + s_k^2 \leq 1 \quad k = 1, \dots, K \\ & s_k^1, s_k^2 \geq 0 \quad k = 1, \dots, K. \end{aligned} \quad (19)$$

NP-hardness is proved again by a reduction from the maximum independent set problem over a graph $G = (V, E)$, with $V = \{1, \dots, K\}$. The following special instance of problem (19) is considered

$$\begin{aligned} \min \quad & \prod_{k=1, \dots, K} \left[\log \left(\left(1 + \frac{s_k^1}{\left(\frac{1}{\exp(1)-1} \right) + \sum_{j \neq k} R_{jk} s_j^1} \right) * \left(1 + \frac{s_k^2}{\left(\frac{1}{\exp(1/3)-1} \right)} \right) \right) \right] \\ & s_k^1 + s_k^2 \leq 1 \quad k = 1, \dots, K \\ & s_k^1, s_k^2 \geq 0 \quad k = 1, \dots, K. \end{aligned}$$

where the parameters are chosen as in (16). The proof is similar to the previous one. For a feasible solution $(\mathbf{s}^1, \mathbf{s}^2)$, let C be defined as in (17). The objective function value at $(\mathbf{s}^1, \mathbf{s}^2)$ is

$$\prod_{k \in C} \left[\log \left(\left(1 + \frac{s_k^1}{\left(\frac{1}{\exp(1)-1} \right)} \right) * \left(1 + \frac{s_k^2}{\left(\frac{1}{\exp(1/3)-1} \right)} \right) \right) \right] \times \prod_{k \notin C} \left[\log \left(1 + \frac{s_k^2}{\left(\frac{1}{\exp(1/3)-1} \right)} \right) \right].$$

An improved feasible solution $(\bar{\mathbf{s}}^1, \bar{\mathbf{s}}^2)$ is obtained by setting $\bar{s}_k^1 = 1, \bar{s}_k^2 = 0 \forall k \in C$, and $\bar{s}_k^1 = 0, \bar{s}_k^2 = 1 \forall k \notin C$. The objective function value at this solution is $3^{-(K-|C|)}$ and is maximized if $|C|$ is as large as possible. Thus, the global maximum value is reached when C^* is the maximum cardinality independent set, and the optimal solution is given in (18).

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