GENERALIZED DUAL FACE ALGORITHM
FOR LINEAR PROGRAMMING

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ABSTRACT. As a natural extension of the dual simplex algorithm, the dual face algorithm performed remarkably in computational experiments with a set of Netlib standard problems. In this paper, we generalize it to bounded-variable LP problems via local duality.

1. INTRODUCTION

Unlike the simplex algorithm or the interior point algorithm, some nonconventional algorithms for solving LP problems attempt to move on the underlying polyhedron more flexibly, not confining iterates to vertices or interior points. For example, criss-cross algorithms [16, 13, 14, 15, 1], switch between primal and dual simplex iterations in their solution process. On the other hand, some approaches [2]–[5] produce iterates that are no longer necessarily vertices, resulting in non-monotone changes in the objective value.

The introduction of deficient basis ([6]-[11]) reflects further efforts along this line. The deficient basis is defined as a submatrix B from the coefficient matrix, whose range space includes the right hand side of the system (in the standard problem). Seeming to be a "fly in the ointment", such a concept can not accommodate the case when B does not satisfy the condition, although the associated algorithms defeated the simplex algorithm significantly in computational experiments.

Independently developed, the concept of dual face [12](pp.595) turns out to be a good answer in this respect. The associated algorithm seems to be a natural extension of the dual simplex method. It proceeds from dual face to dual face until reaching an optimal dual face, together with an optimal dual solution on it. A preliminary empirical evaluation of the algorithm against the revised simplex algorithm with a set of 26 small Netlib standard problems indicates that the former outperformed the latter in dense computation with overall time ratio 10.04 (though with overall iterations ratio 1.19)— an incredible outcome never heard before! This should not be surprising, however, as most systems handled in its solution process are far smaller than the $m \times m$ systems in the latter; and in each iteration the former solves only a single triangular system, compared with the latter solving four.

The aim of this work is to generalize the dual face algorithm in order to apply it to bounded-variable LP problems, a wider class of practical LP models. Thanks to local duality [12] (pp. 668), it is generalized with its essential advantages unchanged.

The paper is organized as follows. Firstly, section 2 introduces the local program via local duality. Then section 3 derives the search direction. Section 4 addresses

the updating of dual feasible solution, and the associated dual face contraction. Section 5 highlights the optimality test, and the dual face expansion conducted when failing with the test. Section 6 formally describes the generalized algorithm.

2. Local Program

Consider the bounded-variable LP problem

$$
\begin{array}{ll}
\min & c^T x \\
\text{s.t.} & Ax = b, \quad l \le x \le u,
\end{array}
\tag{2.1}
$$

where $Ax = b$ is consistent and $A \in \mathcal{R}^{m \times n}$, $m < n$. Note that A is not necessarily of full row rank, in contrast to the assumption in conventional simplex variants.

We will use a_j to denoted the jth column of A, and a_{ij} the entry of the ith row and jth column. e_i denotes the unit vector with the ith component 1.

Introducing slack variables s and t, problem (2.1) can be transformed to

$$
\begin{array}{llll}
\min & c^T x & & \\
& Ax & = & b, \\
\text{s.t.} & x \quad -s & = & l, \,, \qquad s, t \ge 0. \\
& x \quad\quad +t & = & u,
\end{array}
$$

Thereby, the associated dual problem is easily derived, i.e.,

$$
\begin{array}{ll}
\max & b^T y + l^T v + u^T w \\
\text{s.t.} & A^T y + v + w = c, \quad v \ge 0, w \le 0,
\end{array}
\tag{2.2}
$$

Assume that (B, N) is partition of matrix A, where $B \in \mathcal{R}^{m \times k}$ is of full column rank, i.e.,

$$
\operatorname{rank} B = k, \qquad 1 \le k \le m.
$$

The associated variables (or indices) are termed *dual face* and *dual nonface* variables (or indices) respectively, and matrices termed *dual face* and *dual nonface* matrices.

Let (Γ, Π) be any partition to N. Without confusion, (Γ, Π), as a whole, will also be denoted by N simply, and associated quantities will be denoted likewise. Consider $(\bar{y}, \bar{v}, \bar{w})$ such that

$$
\begin{array}{rclcrcl}
B^T \bar{y} & = & c_B, & \quad & \bar{v}_B, \bar{w}_B & = & 0, \\
\bar{v}_\Gamma & = & c_\Gamma - \Gamma^T \bar{y} \;\ge\; 0, & \quad & \bar{w}_\Gamma & = & 0, \\
\bar{w}_\Pi & = & c_\Pi - \Pi^T \bar{y} \;\le\; 0, & \quad & \bar{v}_\Pi & = & 0.
\end{array}
\tag{2.3}
$$

It can be verified that $(\bar{y}, \bar{v}, \bar{w})$ is a feasible solution to the dual problem (2.2) on dual face

$$
(2.4) \quad D(y, v, w) = \{(y, v, w) \mid A^T y + v + w = c,\, v \ge 0,\, w \le 0,\, v_B, w_B, v_\Pi, w_\Gamma = 0\}.
$$

A point on a dual face is said to be *(dual) face solution*. 0-dimensional face $(k = m)$ has a unique dual face solution, i.e., vertex. A face is said to be *(dual) level face* if the dual objective value is constant over it, and said to be *(dual) optimal* if the constant is equal to the optimal value.

The so-called "local dual program" [12] associated with (2.3) is

$$
\begin{array}{lllll}
\max & b^T y + l_\Gamma^T v_\Gamma + u_\Pi^T w_\Pi & & & \\
\text{s.t.} & B^T y & & = & c_B, \\
& \Gamma^T y + v_\Gamma & & = & c_\Gamma, \\
& \Pi^T y & + \; w_\Pi & = & c_\Pi, \\
& v_\Gamma \ge 0, w_\Pi \le 0. & & &
\end{array}
$$

Variables v_Γ, w_Π in the objective function can be eliminated via the last two equality constraints. Thereby, using notation

$$z_\Gamma = v_\Gamma, \qquad z_\Pi = w_\Pi,$$

the preceding local dual program can be transformed to the following compact form:

$$
\begin{array}{lll}
(2.5) & \max & \bar{b}^\mathrm{T} y \\
& \text{s.t.} & B^\mathrm{T} y = c_B, \\
& & N^\mathrm{T} y + z_N = c_N, \ z_\Gamma \geq 0, \ z_\Pi \leq 0,
\end{array}
$$

where constant $l_\Gamma^T c_\Gamma + u_\Pi^T c_\Pi$ was omitted from the objective function, and

$$(2.6) \qquad \bar{b} = b - \Gamma l_\Gamma - \Pi u_\Pi,$$

which is assumed to be nonzero for nontrivialness. It is noted that solution $(\bar{y}, \bar{z}_N)$ is included in the feasible region of the preceding local program, i.e.,

$$(2.7) \qquad D(y, z_N) = \{(y, z_N) \mid B^\mathrm{T} y = c_B, \ N^\mathrm{T} y + z_N = c_N, \ z_\Gamma \geq 0, \ z_\Pi \leq 0\},$$

which and $(\bar{y}, \bar{z}_N)$ correspond to dual face $D(y, v, w)$ and dual face solution $(\bar{y}, \bar{v}, \bar{w})$, respectively; and so do dual level faces. So, we will not distinguish between them.

On the other hand, the local primal program associated with (2.5) is

$$
\begin{array}{lll}
(2.8) & \min & c^T x, \\
& \text{s.t.} & B x_B = \bar{b}, \\
& & x_\Gamma \geq l_\Gamma, \quad x_\Pi \leq u_\Pi.
\end{array}
$$

In principle, it is always possible to set nonface primal variables such that

$$(2.9) \qquad \bar{x}_\Gamma = l_\Gamma, \qquad \bar{x}_\Pi = u_\Pi,$$

so that $(\bar{x}_B, \bar{x}_N)$ and $(\bar{z}_B = 0, \bar{z}_N)$ exhibit complementarity, provided that $\bar{x}_B$ is a solution to $B x_B = \bar{b}$. Nevertheless, it will be clear that such an x_B does not exist unless the associated dual face is a level face (see the proof of Lemma 5.1).

3. The Steepest Uphill

In this section, the orthogonal projection of the dual objective gradient $\bar{b}$ onto the null space of B^T will be computed as the search direction in y-space.

Let P be the associated orthogonal projection, i.e.,

$$(3.1) \qquad P = I - B(B^\mathrm{T} B)^{-1} B^\mathrm{T}.$$

Taking $\Delta y = P\bar{b}$ as the search direction in y-space leads to the following set of search vectors:

$$(3.2) \qquad \Delta y = \bar{b} - B\bar{x}_B, \quad B^\mathrm{T} B \bar{x}_B = B^\mathrm{T} \bar{b},$$

$$(3.3) \qquad \Delta z_N = -N^T \Delta y.$$

Note that the preceding is well-defined. In fact, there always exists an unique solution to the second system in (3.2), since B is of full column rank and $B^\mathrm{T} B$ is hence positive definite. Moreover, Δy vanishes if and only if $\bar{b}$ is included in the range space of B; it is the case if, in particular, B is a square matrix, i.e., $k = m$.

Proposition 3.1. *For Δy defined by (3.2), it holds that $B^\mathrm{T} \Delta y = 0$.*

Proof. It is easy to verify the validity. $\qquad \square$

Proposition 3.2. *The following statements are equivalent*

$$(i) \; \Delta y \neq 0; \qquad (ii) \; \bar{b}^T \Delta y > 0; \qquad (iii) \; \bar{b} \notin \text{range}(B).$$

Proof. From $\Delta y = P\bar{b}$, $P = P^2$ and $P = P^T$, it follows that

$$\bar{b}^T \Delta y = \bar{b}^T P \bar{b} = (P\bar{b})^T (P\bar{b}) = (\Delta y)^T (\Delta y),$$

which implies equivalence of (i) and (ii). If $\Delta y = 0$, then it is known by the first expression of (3.2) that $\bar{b} = B\bar{x}_B$, hence $\bar{b} \in \text{range}(B)$, so (iii) implies (i). Conversely, if $\bar{b} \in \text{range}(B)$, there is some vector u such that

$$(3.4) \qquad\qquad\qquad\qquad \bar{b} = Bu.$$

Premultiplying the preceding equality by B^{T} gives

$$B^T \bar{b} = B^{\mathrm{T}} Bu.$$

Then comparing the preceding and the second expression of (3.2) leads to $u = \bar{x}_B$. Thus, (3.4) implies

$$\bar{b} = B\bar{x}_B,$$

substituting which to the first expression of (3.2) leads to $\Delta y = 0$. So (i) implies (iii), and hence it can be asserted that (i) and (iii) are equivalent. $\qquad \square$

Proposition 3.3. *If $\Delta y \neq 0$, then*

$$(3.5) \qquad\qquad \bar{b}^{\mathrm{T}} \Delta y / \|\Delta y\| \geq \bar{b}^{\mathrm{T}} u / \|u\|, \quad \forall \; 0 \neq u \in \text{Null}(B^{\mathrm{T}}).$$

Proof. For P defined by (3.1), it holds that

$$(3.6) \qquad\qquad Pu = u \neq 0, \quad \forall \; 0 \neq u \in \text{Null}(B^{\mathrm{T}})$$

It is known from Cauchy inequality that

$$\|P\bar{b}\| \|Pu\| \geq (P\bar{b})^{\mathrm{T}} (Pu),$$

Multiplying the preceding inequality by $1/\|Pu\|$ renders

$$\|P\bar{b}\| \geq (P\bar{b})^{\mathrm{T}} (Pu) / \|Pu\|,$$

i.e.,

$$(P\bar{b})^{\mathrm{T}} (P\bar{b}) / \|P\bar{b}\| \geq (P\bar{b})^{\mathrm{T}} (Pu) / \|Pu\|,$$

combining which, $P^2 = P$, $P^{\mathrm{T}} = P$, (3.6) and $\Delta y = P\bar{b}$ leads to (3.5). $\qquad \square$

The preceding Proposition says that the nonzero Δy forms the most acute angle with the objective gradient $\bar{b}$ of the local dual program, over the null of B^{T}. As the *steepest uphill*, it could be viewed as the best choice for the search direction in y-space.

4. Updating Dual Solution

Assume now that $\Delta y \neq 0$, and search vectors, defined by (3.2) and (3.3), are available.

Then, a new dual feasible solution can be determined via the following line search scheme:

$$(4.1) \qquad\qquad\qquad \hat{y} \;=\; \bar{y} + \beta \Delta y,$$
$$(4.2) \qquad\qquad\qquad \hat{z}_N \;=\; \bar{z}_N + \beta \Delta z_N,$$

where the largest possible stepsize β, maintaining $\hat{z}_\Gamma \geq 0$ and $\hat{z}_\Pi \leq 0$, and corresponding entering index q are defined as follows: where

$$(4.3) \qquad \beta = -\bar{z}_q/\Delta z_q = \min\left\{ -\bar{z}_j/\Delta z_j \;\middle|\; \begin{array}{ll} \Delta z_j < 0, & j \in \Gamma \\ \Delta z_j > 0, & j \in \Pi \end{array} \right\}$$

If some components of $\bar{z}_N$ vanishes, the dual face solution $(\bar{y}, \bar{z}_N)$ is said to be *dual degenerate*, as is a case in which the stepsize β would vanish, yielding a solution just the same as the old itself.

Using the preceding notation, we have the following Lemma.

Lemma 4.1. *Assume $\Delta y \neq 0$.*
(i) If $\Delta z_\Gamma \geq 0$ and $\Delta z_\Pi \leq 0$, problem (2.2) is upper unbounded; else,
(ii) $(\hat{y}, \hat{z}_N)$ is a boundary point of the feasible region, with the dual objective value not decreasing, or even strictly increasing if $(\bar{y}, \bar{z}_N)$ is nondegenerate.

Proof. (i) In this case, it is seen that $(\bar{y}, \bar{z}_N) \in D(y, z_N) \; \forall \beta > 0$. According to Proposition 3.2, moreover, $\Delta y \neq 0$ implies

$$(4.4) \qquad \bar{b}^T \Delta y > 0.$$

Since the new objective value is

$$(4.5) \qquad \bar{b}^T \hat{y} = \bar{b}^T \bar{y} + \beta \bar{b}^T \Delta y,$$

therefore, it follows that the objective value tends to $+\infty$, as β tends to $+\infty$.

(ii) It is noted that the new point $(\bar{y}, \bar{z}_N)$ is well-defined in this case. By $(\bar{y}, \bar{z}_N) \in D(y, z_N)$, (4.3) and Proposition 3.1, it is known that $(\hat{y}, \hat{z}_N) \in D(y, z_N)$. Furthermore, by (4.5), the associated objective value does not decrease, and strictly increases in case of nondegeneracy ($\beta > 0$). In addition, it is clear that $\hat{z}_q = 0$, as indicates that the new iterate is on the boundary. □

4.1. **Dual face contraction.** Assume that $\Delta y \neq 0$, and $\Delta z_\Gamma \ngeq 0$ or $\Delta z_\Pi \nleq 0$. After updating the feasible dual solution by (4.1) and (4.2), we update (B, N) by moving q from N to B accordingly. Denote the resulting face matrix by $\breve{B} \in \mathcal{R}^{m \times (k+1)}$.

Proposition 4.2. *If* rank $B = k$, *then* rank $\breve{B} = k + 1$.

Proof. It is clear that $k \leq \text{rank } \breve{B} \leq k + 1$. Assume rank $\breve{B} \neq k + 1$. Then, there is a vector $u \neq 0$ such that $a_q = Bu$. Since $B^{\mathrm{T}} \Delta y = 0$ (Proposition 3.1), it holds that

$$a_q^{\mathrm{T}} \Delta y = u^{\mathrm{T}} B^{\mathrm{T}} \Delta y = 0.$$

But, (4.3) implies that $a_q^{\mathrm{T}} \Delta y \neq 0$, as is a contradiction. Therefore rank $\breve{B} = k + 1$. □

Consequently, the dimension of the dual face decreased by one. So, the dual face is contracted in this case. In addition, $\bar{b}$ must be updated accordingly, as q is dropped from Γ or Π. However, it is too time consuming to compute new $\bar{b}$ from scratch by (2.6). According to (2.9), fortunately, it is possible instead to bypass this difficulty via

$$\bar{b} = \bar{b} + \bar{x}_q a_q$$

It is noticeable that if $\Delta y \neq 0$, the solution $\bar{x}_B$ to the second system in (3.2) is only a least squares solution to $Bx_B = \bar{b}$. If $\Delta y = 0$, on the other hand, it is an

exact solution to $Bx_B = \bar{b}$, and hence $\bar{x}_B$ with $\bar{x}_N$ gives a primal solution matching up the current dual feasible solution, as will be addressed in the next section.

5. Optimality Test

Assume that $\Delta y = 0$. In this case, the associated search direction vectors are useless, and no further progress can be made on the current dual face, as the following Lemma reveals.

Lemma 5.1. *If $\Delta y = 0$, then $D(y, z_N)$ is a dual level face; and vice versa if there is $(\bar{y}, \bar{z}_N) \in D(y, z_N)$ such that $\bar{z}_\Gamma > 0$ and $\bar{z}_\Pi < 0$.*

Proof. Assume that $\bar{x}_B$ is the solution to $B^{\mathrm{T}} B x_B = B^{\mathrm{T}} \bar{b}$. From $\Delta y = 0$ and the first expression of (3.2), it follows that

$$\bar{b} = B\bar{x}_B.$$

For all $(y, z_N) \in D(y, z_N))$, transposing the two sides of the preceding equality and postmultiplying the result by y gives

$$\bar{b}^{\mathrm{T}} y = \bar{x}_B^{\mathrm{T}} B^{\mathrm{T}} y.$$

In addition, it is known from (2.7) that

$$B^{\mathrm{T}} y = c_B.$$

Combining the preceding two expressions leads to

$$\bar{b}^{\mathrm{T}} y = \bar{x}_B^{\mathrm{T}} c_B.$$

Therefore, $D(y, z_N)$ is a level face, as the dual objective value is constant over it,

Note that $\bar{z}_\Gamma > 0$ and $\bar{z}_\Pi < 0$ together imply nondegeneracy of $(\bar{y}, \bar{z}_N)$. To show the last half of the Lemma, suppose that $D(y, z_N)$ is a level face but $\Delta y \neq 0$. If $\Delta z_N \geq 0$, it is known by Lemma 4.1(i) that the objective value is upper unbounded over $D(y, z_N)$, as contradicts that it is a level face; if, otherwise, $\Delta z_N \not\geq 0$, then, under the nondegeneracy it follows from Lemma 4.1(ii) that $(\hat{y}, \hat{z}_N)$, defined by (4.1) and (4.2), belongs to $D(y, z_N)$, with a strictly increased objective value. This contradicts that $D(y, z_N)$ is a level face, and therefore Δy must vanish. $\square$

We are now able to give the optimality condition as follows.

Theorem 5.2. *Assume $\Delta y = 0$. If $l_B \leq \bar{x}_B \leq u_B$, then $D(y, z_N)$ is a dual optimal face, and $\bar{x}$ and $(\bar{y}, \bar{z}_N)$ are a pair of primal and dual optimal solutions.*

Proof. From (3.2) and $\Delta y = 0$, it follows that $\bar{b} = B\bar{x}_B$. Since $\bar{b} = b - \bar{x}_N$ and $\bar{x}_N$ are on bounds, furthermore, it is known under $l_B \leq \bar{x}_B \leq u_B$ that $\bar{x}$ is a feasible solution to (2.1). On the other hand, it is verified that the corresponding $(\bar{y}, \bar{v}\bar{w})$ is a feasible solution to (2.2), and exhibits slackness complementarity with $\bar{x}$. Therefore they are a pair of primal and dual optimal solutions. By Lemma 5.1, in addition, $D(y, z_N)$ is a dual level face; moreover, it is a dual optimal face, since $D(y, z_N)$ includes the optimal solution. $\square$

5.1. **Dual face expansion.** Assume that Δy vanishes and $D(y, z_N)$ is hence a level dual face, but the optimality condition is not satisfied. Then the dual face has to be expanded to make further progress.

Define the face index set

$$B = \{j_1, \cdots, j_k\}$$

and introduce "bound-violation" quantities:

(5.1) $$\rho_i = \begin{cases} l_{j_i} - \bar{x}_{j_i}, & \text{if } \bar{x}_{j_i} < l_{j_i}, \\ u_{j_i} - \bar{x}_{j_i}, & \text{if } \bar{x}_{j_i} > u_{j_i}, \\ 0, & \text{if } l_{j_i} \leq \bar{x}_{j_i} \leq u_{j_i}, \end{cases} \qquad i = 1, \cdots, k.$$

Determine leaving index j_s such that

(5.2) $$s \in \arg\max\{|\rho_i| \mid i = 1, \cdots, k\}.$$

If $\rho_s = 0$, hence $l_B \leq \bar{x}_B \leq u_B$ holds, the optimality is achieved (Theorem 5.2); otherwise, achievement of optimality can not be declared.

Assume now that $\rho_s \neq 0$. If $\rho_s > 0$, implying that $\bar{x}_p$ violates the lower bound, j_s is moved to Γ with Π remaining unchanged; if, else, $\rho_s < 0$, j_s is moved to Π with Γ unchanged. Accordingly, B is updated by

(5.3) $$\hat{B} = B \backslash \{j_s\}.$$

It is clear that the new face matrix is of full column rank $k - 1$; therefore, the dimension of the dual face increases by one.

Accordingly, the leaving component of $\bar{x}$ is updated by

$$\bar{x}_{j_s} := \bar{x}_{j_s} + \rho_s,$$

which will be fixed on the lower or upper bound, and $\bar{b}$ is updated by

$$\bar{b} := \bar{b} - \bar{x}_{j_s} a_{j_s}.$$

6. Dual face algorithm

The solution process is described as follows. In each iteration, firstly compute the search vectors. To do so, solve the $k \times k$ system

(6.1) $$B^{\mathrm{T}} B x_B = B^{\mathrm{T}} \bar{b}$$

for $\bar{x}_B$ and then compute search direction Δy in y-space by

(6.2) $$\Delta y = \bar{b} - B \bar{x}_B.$$

There are following two cases arising:

(i) $\Delta y \neq 0$. Calculate Δz_N by (3.3), so that it is possible to update the dual solution and determine the leaving index by (4.1) and (4.2). Then, carry out the according dual face contraction, and go to the next iteration. Under nondegeneracy, the new dual feasible solution corresponds to a strictly higher objective value.

(ii) $\Delta y = 0$. Reached now is a level dual face, and hence no further progress can be made on it. In this case, $(\bar{y}, \bar{z})$ and $\bar{x}$, where $\bar{x}_B$ solves $B x_B = \bar{b}$, are a pair of dual and primal feasible solutions. If $l_B \leq \bar{x}_B \leq u_B$, they are dual and primal optimal solutions, together with the dual face optimal. Otherwise, associated dual face expansion is conducted before going to the next iteration.

A level face corresponds to a deficient basis. In fact, it is seen from (6.2) that vanishing Δy implies that the range of B includes $\bar{b}$; and vice versa. So, an increment in the dual objective value would be achieved only when B is not a deficient

basis. In this sense, the dual face method may be regarded as a reasonable extension of the deficient-basis method [6]-[11]. If $k = m$, in particular, no labor on Δy is needed at all, because $\Delta y = 0$ is predictable, as is a case corresponding to the conventional basis.

The overall steps can be summarized into the following algorithm.

Algorithm 1.(Generalized dual face algorithm) Initial $B, \Pi, \Gamma,\ 1 \leq k \leq m$; dual feasible solution $(\bar{y}, \bar{z}_N)$; $\bar{b}$. This algorithm solves the pair of problems (2.2) and (2.1).

1. Solve system $B^{\mathrm{T}} B \bar{x}_B = B^{\mathrm{T}} \bar{b}$.
2. Compute $\Delta y = \bar{b} - B \bar{x}_B$.
3. Go to step 12 if $\Delta y = 0$.
4. Compute $\Delta z_N = -N^T \Delta y$.
5. Stop if $\Delta z_\Gamma \geq 0$ and $\Delta_\Pi \leq 0$.
6. Determine index q and stepsize β by (4.3).
7. If $\beta \neq 0$, update $\bar{y} = \bar{y} + \beta \Delta y, \bar{z}_N = \bar{z}_N + \beta \Delta z_N$.
8. Update $\bar{b} = \bar{b} + \bar{x}_q a_q$ and $B = B \cup \{q\}$; if $q \in \Gamma$, $\Gamma = \Gamma \backslash \{q\}$; else, $\Pi = \Pi \backslash \{q\}$.
9. Set $k = k + 1$.
10. Go to step 1 if $k \neq m$.
11. Solve system $B \bar{x}_B = \bar{b}$.
12. Compute ρ_i by (5.1), and determine s by (5.2).
13. Stop if $\rho_s = 0$.
14. Update $x_{j_s} = \bar{x}_{j_s} + \rho_s$.
15. Update $\bar{b} = \bar{b} - \bar{x}_{j_s} a_{j_s}$ and $B = B \backslash \{j_s\}$; if $\rho_s > 0$, $\Gamma = \Gamma \cup \{j_s\}$; else, $\Pi = \Pi \cup \{j_s\}$.
16. Set $k = k - 1$, and go to step 1.

Theorem 6.1. *Assume dual nondegeneracy throughout the solution process, Algorithm 1 terminates either at*

(i) step 5, detecting upper unboundedness of (2.2); or at

(ii) step 13, generating a dual optimal face together with a pair of primal and dual optimal solutions.

Proof. Under the nondegeneracy assumption, the proof of finiteness of the Algorithm is the same as that for the simplex method. The meanings of its exits are derived from Lemma 4.1 and Theorem 5.2. □

To be an initial dual face matrix, a submatrix $B \in \mathcal{R}^{m \times k}, 1 \leq k \leq m$ with small k seems to be preferable, though any of full column rank is eligible. The advantage of such doing is that the subsequent systems are small, and quickly to solve. To this end, one may start even with $k = 0$ by setting $B = 0$, $N = A$ and $P = I$ initially. Then, replacing the first formula in (3.2) by $\Delta y = \bar{b}$ enables subsequent steps to go smoothly.

An issue left is how to handle the linear system involved in step 1, which constitutes the core part of computations. To this end, one may apply the Cholesky factorization $B^{\mathrm{T}} B = L L^T$ and solve two triangular $k \times k$ systems in succession, i.e.

$$Lu = B^{\mathrm{T}} \bar{b}, \qquad L^{\mathrm{T}} x_B = u,$$

where the Cholesky factor L is updated in each iteration. Alternatively, one may directly use

$$\bar{x}_B = (B^{\mathrm{T}} B)^{-1} B^{\mathrm{T}} \bar{b},$$

and update the inverse $(B^\mathrm{T}B)^{-1}$ in each iteration (see [12] for more details).

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