On the Adaptivity Gap in Two-Stage Robust Linear Optimization under Uncertain Constraints

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Abstract

In this paper, we study the performance of static solutions in two-stage adjustable robust packing linear optimization problem with uncertain constraint coefficients. Such problems arise in many important applications such as revenue management and resource allocation problems where demand requests have uncertain resource requirements. The goal is to find a two-stage solution that maximizes the worst case objective value over all possible realizations of the second-stage constraints from a given uncertainty set. We consider the case where the uncertainty set is *column-wise* and *constraint-wise* (any constraint describing the set involve entries of only a single column or a single row). This is a fairly general class of uncertainty sets to model constraint coefficient uncertainty. We show that the two-stage adjustable robust problem is $\Omega(\log n)$-hard to approximate. On the positive side, we show that a static solution is an $O\big(\log n \cdot \min(\log \Gamma, \log(m+n))\big)$-approximation for the two-stage adjustable robust problem where m and n denote the numbers of rows and columns of the constraint matrix and Γ is the maximum possible ratio of upper bounds of the uncertain constraint coefficients. Therefore, for constant Γ, surprisingly the performance bound for static solutions matches the hardness of approximation for the adjustable problem. Furthermore, in general the static solution provides nearly the best efficient approximation for the two-stage adjustable robust problem.

1 Introduction

In most real world applications, problem parameters are uncertain at the optimization phase. Stochastic and robust optimization are two different paradigms that have been studied to address this parameter uncertainty. In a stochastic optimization approach, uncertainty is modeled using probability distributions and the goal is to optimize an expected objective. This has been extensively studied and we refer the reader to several textbooks including Kall and Wallace [19], Prekopa [20], Shapiro [21], Shapiro et al. [22] for a comprehensive review of stochastic optimization. However, this approach suffers from the "curse of dimensionality" and is intractable even if an approximate solution is desired (see Shapiro and Nemirovski [23]). In a robust optimization approach, the uncertainty is modeled using a deterministic *uncertainty set* and the goal is to optimize over the worst case uncertainty realization (see Ben-Tal and Nemirovski [3, 4, 5], El Ghaoui

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and Lebret [14], Bertsimas and Sim [11, 12], Goldfarb and Iyengar [17]). This is a computationally tractable approach for a large class of problems if we want to compute a static solution feasible for all scenarios. We refer the reader to the recent book by Ben-Tal et al. [2] and the references therein for a comprehensive review. However, computing an optimal dynamic (or adjustable) solution is hard in general even in the robust optimization approach. Feige et al. [15] show that it is hard to approximate the two-stage robust fractional set covering problem with uncertain right-hand-side within a factor better than $\Omega(\log m/\log\log m)$.

In this paper, we consider the following two-stage (adjustable) robust packing linear optimization problem Π_{AR} under uncertain constraint coefficients.

$$\begin{aligned} z_{\mathsf{AR}} = \max\ & \boldsymbol{c}^T\boldsymbol{x} + \min_{\boldsymbol{B}\in\mathcal{U}} \max_{\boldsymbol{y}(\boldsymbol{B})} \boldsymbol{d}^T\boldsymbol{y}(\boldsymbol{B}) \\ & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y}(\boldsymbol{B}) \ \leq\ \boldsymbol{h} \\ & \boldsymbol{x} \ \in\ \mathbb{R}^{n_1}_+ \\ & \boldsymbol{y}(\boldsymbol{B}) \ \in\ \mathbb{R}^{n_2}_+, \end{aligned} \tag{1.1}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n_1}, \boldsymbol{c} \in \mathbb{R}^{n_1}_+, \boldsymbol{d} \in \mathbb{R}^{n_2}_+, \boldsymbol{h} \in \mathbb{R}^m$. The second-stage constraint matrix $\boldsymbol{B} \in \mathbb{R}^{m\times n_2}_+$ is uncertain and belongs to a full-dimensional compact convex uncertainty set $\mathcal{U} \subseteq \mathbb{R}^{m\times n_2}_+$ (non-negative orthant). The decision maker selects the first-stage solution $\boldsymbol{x}$ and for each second-stage matrix $\boldsymbol{B}$, recourse decision $\boldsymbol{y}(\boldsymbol{B})$ such that the worst case objective value is maximized (the term *adjustable* emphasizes the fact that we can select a recourse decision after the uncertain constraint matrix is known). We would like to emphasize that the objective coefficients $\boldsymbol{c}, \boldsymbol{d}$, constraint coefficients $\boldsymbol{B}$, and the decision variables $\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{B})$ are all non-negative. Also, the uncertainty set $\mathcal{U}$ of second-stage constraint matrices is contained in the non-negative orthant. We can assume without loss of generality that $n_1 = n_2 = n$ and $\mathcal{U}$ is *down-monotone* [8], i.e., $\boldsymbol{B} \in \mathcal{U}$ and $\boldsymbol{0} \leq \hat{\boldsymbol{B}} \leq \boldsymbol{B}$ implies that $\hat{\boldsymbol{B}} \in \mathcal{U}$. When $m = 1$, the above problem reduces to a fractional knapsack problem with uncertain item sizes. The stochastic version of this knapsack problem has been extensively studied in the literature (for instance, see Dean et al. [13], Goel and Indyk [16], Goyal and Ravi [18]).

The above model is fairly general and captures many important applications. For instance, in the resource allocation problem considered in [26], m corresponds to the number of resources with capacities $\boldsymbol{h}$. The linear constraints correspond to capacity constraints on the resources, the first-stage matrix $\boldsymbol{A}$ denotes the resource requirements of known first-stage demands and $\boldsymbol{B}$ denotes the uncertain resource requirements for future demands. In the framework of (1.1), we want to compute first-stage (fractional) allocation decisions $\boldsymbol{x}$ such that the worst case total revenue over all possible future demand arrivals from $\mathcal{U}$ is maximized.

As another example, consider a multi-server scheduling problem as in [10] where jobs arrive with uncertain processing times and we need to make the scheduling decisions to maximize the utility. The first-stage matrix $\boldsymbol{A}$ denotes the known processing time of first-stage jobs, $\boldsymbol{h}$ denotes the available timespan and $\boldsymbol{B}$ represents the time requirements of unknown arriving jobs. If we employ a pathwise enumeration for the uncertain time requirement, such stochastic project scheduling problem can be modeled as two-stage packing linear programming problems with uncertain constraint coefficients as in (1.1).

As mentioned earlier, computing an optimal two-stage adjustable robust solution is intractable in general. This motivates us to consider approximate solution approaches. In particular, we consider the corresponding static robust optimization problem Π_{Rob}, which can be formulated as follows.

$$
\begin{aligned}
z_{\mathsf{Rob}} = \max\ & \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y} \\
& \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} \ \leq\ \boldsymbol{h},\ \forall \boldsymbol{B} \in \mathcal{U} \\
& \boldsymbol{x} \ \in\ \mathbb{R}^{n_1}_{+} \\
& \boldsymbol{y} \ \in\ \mathbb{R}^{n_2}_{+},
\end{aligned}
\tag{1.2}
$$

where the "second-stage" variable $\boldsymbol{y}$ is static and independent of the realization of $\boldsymbol{B}$. Therefore, both $\boldsymbol{x}$ and $\boldsymbol{y}$ are chosen before $\boldsymbol{B}$ is known and are feasible for all realizations of $\boldsymbol{B}$. An optimal solution (1.2) can be computed efficiently (Bertsimas et al. [6], Ben-Tal et al. [2]). In fact, if $\mathcal{U}$ is a polyhedron, we can compute the optimal static robust solution by solving a single linear program (Soyster [24], Ben-Tal and Nemirovski [4]). However, since there is no recourse, a static solution is believed to be highly conservative.

The performance of static solution has been studied in the literature. Bertsimas and Goyal [7], Bertsimas et al. [9] study the performance of static solution for two-stage and multi-stage adjustable robust linear covering problems under right-hand-side uncertainty, and relate it to the *symmetry* of the uncertainty set. Ben-Tal and Nemirovski [4] show that the static solution is optimal for adjustable robust problem with uncertain constraint coefficients if the uncertainty set of matrices is *constraint-wise*, i.e., a Cartesian product of row sets. Bertsimas et al. [8] provide a tight approximation bound on the performance of static robust solution for two-stage adjustable robust problems under constraint uncertainty. In particular, the approximation bound in [8] is related to a measure of non-convexity of a transformation of the uncertainty set. However, the authors show that for the following family of uncertainty sets of non-negative diagonal matrices with an upper bound on the ℓ_1-norm of the diagonal vector

$$
\mathcal{U} = \left\{ \boldsymbol{B} \in \mathbb{R}^{m\times m}_{+} \ \middle|\ B_{ij} = 0,\ \forall i \neq j,\ \sum_{i=1}^{m} B_{ii} \leq 1 \right\},
$$

the measure of non-convexity is m. Moreover, it is not necessarily tractable to compute the measure of non-convexity for an arbitrary convex compact set. We would like to note that such (diagonal) uncertainty sets do not arise naturally in practice. For instance, consider the resource allocation problem where the uncertainty set $\mathcal{U}$ represents the set of uncertain resource requirement matrices. A constraint on the diagonal relates requirements of different resources across different demands, which is not a naturally arising relation. This motivates us to study the special class of *column-wise* and *constraint-wise* sets. In particular,

$$
\mathcal{U} = \{\boldsymbol{B} \in \mathbb{R}^{m\times n}_{+} \mid \boldsymbol{B}\boldsymbol{e}_j \in C_j,\ j \in [n], \boldsymbol{B}^T\boldsymbol{e}_i \in R_i, i \in [m]\},
$$

where $C_j \subseteq \mathbb{R}^m_+$ for all $j \in [n]$ and $R_i \subseteq \mathbb{R}^n_+$ for all $i \in [m]$ are compact, convex and down-monotone sets. We assume that the sets $C_j,\ j \in [n]$ and $R_i,\ i \in [m]$ are such that linear optimization problems over $\mathcal{U}$ can be solved in time polynomial in the encoding length of $\mathcal{U}$. We refer to the above uncertainty set as a column-wise and constraint-wise set since the constraints describing the uncertainty set $\mathcal{U}$ involve entries of only a single column or a single row of the matrix. In the resource allocation problem, this would imply that we can have a constraint on the resource requirements of a particular resource for different demands, and a constraint on resource requirements of different resources for any particular demand.

1.1 Our Contributions.

Our main contributions are as follows.

Hardness of Approximation. We show that the two-stage adjustable robust problem Π_{AR} (1.1) is $\Omega(\log n)$ hard to approximate even for the case of column-wise uncertainty sets. In other words, there is no polynomial time algorithm that computes an adjustable two-stage solution with worst case objective value within a factor better than $\Omega(\log n)$ of the optimal. Our proof is based on an approximation preserving reduction from the set cover problem [25]. In particular, we show that any instance of set cover problem can be reduced to an instance of the two-stage adjustable robust problem with column-wise sets where each column set is a simplex. For the more general case where the uncertainty set $\mathcal{U}$ and objective coefficients $\boldsymbol{d}$ are not constrained to be in the non-negative orthant, we show that the two-stage adjustable robust problem is $\Omega(2^{\log^{1-\epsilon} m})$-hard to approximate for any constant $0 < \epsilon < 1$ by a reduction from the Label-Cover-Problem [1]. The hardness of approximation results motivate us to find good approximations for the two-stage adjustable robust problem.

Adaptivity Gap: Performance of static solutions. We show that a static solution provides an $O\big(\log n \cdot \min(\log \Gamma, \log(m+n))\big)$-approximation for the two-stage adjustable robust problem for the case of column-wise and constraint-wise uncertainty sets where Γ is the maximum possible ratio of the upper bounds of different matrix entries in the uncertainty set (See Section 4 for details). Therefore, if Γ is a constant, a static solution gives a $O(\log n)$-approximation for the adjustable robust problem for column-wise and constraint-wise uncertainty sets; thereby, matching the hardness of approximation. This is quite surprising as it shows the static solution is the best possible efficient approximation for the adjustable robust problem in this case. We would like to note that the two-stage adjustable robust optimization problem is $\Omega(\log n)$-hard even for the case when Γ is a constant. Furthermore, when Γ is large, we show that a static solution gives a $O(\log n \cdot \log(m+n))$-approximation for the adjustable robust problem. Therefore, the static solution provides a nearly optimal approximation for the two-stage adjustable robust problem for column-wise and constraint-wise uncertainty sets in general.

We first consider the case when the uncertainty set is *column-wise*, i.e., each column $j \in [n]$ of the uncertain matrix $\boldsymbol{B}$ belongs to a compact convex set $\mathcal{U}_j \subseteq \mathbb{R}^m_+$ unrelated to other columns

$$\mathcal{U} = \{[\boldsymbol{b}_1\ \boldsymbol{b}_2 \ldots \boldsymbol{b}_n] \mid \boldsymbol{b}_j \in \mathcal{U}_j, j \in [n]\}, \tag{1.3}$$

and prove a bound of $O\big(\log n \cdot \min(\log \Gamma, \log(m+n))\big)$ on the adaptivity gap for the adjustable robust problem Our analysis is based on the structural properties of the optimal adjustable and static robust solutions. In particular, we first show that the worst adaptivity gap is achieved when each column is a simplex. This is based on the property of the optimal static robust solution that it depends only on the hypercube containing the given uncertainty set $\mathcal{U}$ (Soyster [24]). We formalize this in Theorems 3.1 and 3.2. Furthermore, for the simplex column-wise uncertainty sets, we relate the adjustable robust problem to an appropriate set cover problem and relate the static robust problem to the corresponding LP relaxation in order to obtain the bound on the adaptivity gap.

We extend the analysis to the case when $\mathcal{U}$ is a column-wise and constraint-wise uncertainty set and prove a similar bound on the performance of static solutions. In particular, we show that if a static solution provides an α-approximation for the adjustable robust problem with column-wise uncertainty sets, then a static solution is an α-approximation for the case of column-wise and constraint-wise uncertainty sets. Moreover, we also extend our result to the case where the second-stage objective coefficients are also uncertain and show that the same bound holds when the uncertainty in the objective coefficients does not depend on the column-wise and constraint-wise constraint coefficient uncertainty sets.

Our results confirm the power of static robust solutions for the two-stage adjustable robust problem. In particular, its performance nearly matches the hardness of approximation factor for the adjustable robust problem, which indicates that it is nearly the best approximation possible for the problem. In addition, we would like to note that our approximation bound only compares the optimal objective values of the adjustable and static robust problems. The performance of the static robust solution policy can potentially be better: if $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is an optimal static robust solution, we only implement the first-stage solution $\boldsymbol{x}^*$ and compute the recourse solution after observing the realization of the uncertain matrix $\boldsymbol{B}$. Therefore, the objective value of the recourse solution can potentially be better than that of $\boldsymbol{y}^*$.

Outline. In Section 2, we present the hardness of approximation for the two-stage adjustable robust problems. In Section 3, we present the one-stage reformulation of the adjustable robust problem and the corresponding problem. In Sections 4 and 5, we present the bounds on the adaptivity gap for column-wise uncertainty sets. We extend the analysis to the general case of column-wise and constraint-wise uncertainty sets in Section 6. In Section 7, we compare our result with the measure of non-convexity bound in Bertsimas et al. [8] and extend our bound to the case where the objective coefficients are also uncertain in Section 8.

2 Hardness of Approximation.

In this section, we show that the two-stage adjustable robust problem Π_{AR} is $\Omega(\log n)$-hard to approximate even for column-wise uncertainty sets (1.3). In other words, there is no polynomial time algorithm that guarantees an approximation within a factor of $\Omega(\log n)$ of the optimal two-stage adjustable robust solution. We achieve this via an approximation preserving reduction from the set cover problem , which is $\Omega(\log n)$-hard to approximate [25]. In particular, we have the following theorem.

Theorem 2.1. *The two-stage adjustable robust problem,* Π_{AR} *as defined in* (1.1) *is* $\Omega(\log n)$*-hard to approximate for column-wise uncertainty sets.*

Proof. Consider an instance $\mathcal{I}$ of the set cover problem with ground set of elements $S = \{1, \dots, n\}$ and a family of subsets $\mathcal{S}_1, \dots, \mathcal{S}_m \subseteq S$. The goal is to find minimum cardinality collection C of subsets $\mathcal{S}_i, i \in [m]$ that covers all $j \in [n]$. We construct an instance $\mathcal{I}'$ of the two-stage adjustable robust problem Π_{AR} (1.1) with a column-wise uncertainty set $\mathcal{U}$ as follows.

$$\begin{aligned}
&\boldsymbol{c} = \boldsymbol{0}, \boldsymbol{A} = \boldsymbol{0}, \;\; h_i = 1, \; \forall i \in [m], \;\; d_j = 1, \; \forall j \in [n] \\
&\mathcal{U}_j = \left\{ \boldsymbol{b} \in [0,1]^m \;\middle|\; \sum_{i=1}^m b_i \le 1, \; b_i = 0, \; \forall i \text{ s.t. } j \notin \mathcal{S}_i \right\} \\
&\mathcal{U} = \left\{ [\boldsymbol{b}_1 \; \boldsymbol{b}_2 \; \dots \; \boldsymbol{b}_n] \mid \boldsymbol{b}_j \in \mathcal{U}_j \right\}.
\end{aligned}$$

Note that there is a row corresponding to each subset $\mathcal{S}_i$ and a column corresponding to each element j. Moreover, $\mathcal{U}$ is a column-wise uncertainty set. Let $\boldsymbol{e}$ denote the vector of all ones (of appropriate dimension) and $\boldsymbol{e}_j$ be a standard unit vector where the j^{th} is one and all other components are zero. Now,

$$\begin{aligned}
z_{\mathsf{AR}} &= \min_{\boldsymbol{b}_j \in \mathcal{U}_j, j \in [n]} \max_{\boldsymbol{y} \in \mathbb{R}^n_+} \left\{ \boldsymbol{e}^T \boldsymbol{y} \;\middle|\; \sum_{j=1}^n y_j \boldsymbol{b}_j \le \boldsymbol{e} \right\} \\
&= \min_{\boldsymbol{b}_j \in \mathcal{U}_j, j \in [n]} \min_{\boldsymbol{v} \in \mathbb{R}^m_+} \{ \boldsymbol{e}^T \boldsymbol{v} \mid \boldsymbol{b}_j^T \boldsymbol{v} \ge 1, \forall j \in [n] \},
\end{aligned}$$

where the second reformulation follows from taking the dual of the inner maximization problem in the original formulation. Suppose $\hat{\boldsymbol{v}}, \hat{\boldsymbol{b}}_j$ for all $j \in [n]$ is a feasible solution for instance $\mathcal{I}'$. Then, we can compute a solution for instance $\mathcal{I}$ with cost at most $\boldsymbol{e}^T\hat{\boldsymbol{v}}$. To prove this, we show that we can construct an integral solution $\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{b}}_j$ for all $j \in [n]$ such that

$$\boldsymbol{e}^T\tilde{\boldsymbol{v}} \leq \boldsymbol{e}^T\hat{\boldsymbol{v}}.$$

Note that $\hat{\boldsymbol{b}}_j$ may not necessarily be integral. For each $j \in [n]$, consider a basic optimal solution $\tilde{\boldsymbol{b}}_j$ where

$$\tilde{\boldsymbol{b}}_j \in \arg\max\{\boldsymbol{b}^T\hat{\boldsymbol{v}} \mid \boldsymbol{b} \in \mathcal{U}_j\}.$$

Therefore, $\boldsymbol{b}_j$ is a vertex of $\mathcal{U}_j$ for any $j \in [n]$, which implies $\tilde{\boldsymbol{b}}_j = \boldsymbol{e}_{i_j}$ for some $i_j \in \mathcal{S}_j$. Also,

$$\tilde{\boldsymbol{b}}_j^T\hat{\boldsymbol{v}} \geq \hat{\boldsymbol{b}}_j^T\hat{\boldsymbol{v}} \geq 1, \ \forall j \in [n].$$

Now, let

$$\tilde{\boldsymbol{v}} \in \arg\min\{\boldsymbol{e}^T\boldsymbol{v} \mid \tilde{\boldsymbol{b}}_j^T\boldsymbol{v} \geq 1, \forall j \in [n], \boldsymbol{v} \geq \boldsymbol{0}\}.$$

Clearly, $\boldsymbol{e}^T\tilde{\boldsymbol{v}} \leq \boldsymbol{e}^T\hat{\boldsymbol{v}}$. Also, for all $j \in [n]$, since $\tilde{\boldsymbol{b}}_j = \boldsymbol{e}_{i_j}$ for some $i_j \in \mathcal{S}_j$,

$$\tilde{\boldsymbol{b}}_j^T\tilde{\boldsymbol{v}} \geq 1 \implies \tilde{v}_{i_j} = 1, \ \forall j \in [n].$$

Therefore, $\tilde{\boldsymbol{v}} \in \{0,1\}^m$. Let

$$C = \{\mathcal{S}_i \mid \tilde{v}_i = 1\}.$$

Clearly, C covers all the element $j \in [n]$ and $|C| = \boldsymbol{e}^T\tilde{\boldsymbol{v}} \leq \boldsymbol{e}^T\hat{\boldsymbol{v}}$.

Conversely, consider set cover $C \subseteq \{\mathcal{S}_i, i \in [m]\}$ of instance $\mathcal{I}$. For any $j \in [n]$, there exists $i_j \in [m]$ such that $j \in \mathcal{S}_{i_j}$ and $\mathcal{S}_{i_j} \in C$. Now, we can construct a feasible solution $\bar{\boldsymbol{v}}, \bar{\boldsymbol{b}}_j$ for all $j \in [n]$ for z_{AR} as follows.

$$\bar{\boldsymbol{b}}_j = \boldsymbol{e}_{i_j}, \ \forall j \in [n]$$

$$\bar{v}_i = \begin{cases} 1 & \text{if } \mathcal{S}_i \in C \\ 0 & \text{otherwise} \end{cases}, \ \forall i \in [m].$$

It is easy to observe that $\bar{\boldsymbol{b}}_j^T\bar{\boldsymbol{v}} \geq 1$ for all $j \in [n]$ and $\boldsymbol{e}^T\bar{\boldsymbol{v}} = |C|$. □

2.1 General Two-stage Adjustable Robust Problem.

If the uncertainty set $\mathcal{U}$ of second-stage constraint matrices and the objective coefficients $\boldsymbol{d}$ are not constrained to be in the non-negative orthant in Π_{AR}, we can prove a stronger hardness of approximation result. In particular, consider the following general problem $\Pi_{\mathsf{AR}}^{\mathsf{Gen}}$:

$$\begin{aligned} z_{\mathsf{AR}}^{\mathsf{Gen}} = \ & \max \ \boldsymbol{c}^T\boldsymbol{x} + \min_{\boldsymbol{B}\in\mathcal{U}} \max_{\boldsymbol{y}(\boldsymbol{B})} \boldsymbol{d}^T\boldsymbol{y}(\boldsymbol{B}) \\ & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y}(\boldsymbol{B}) \ \leq \ \boldsymbol{h} \\ & \boldsymbol{y}(\boldsymbol{B}) \geq \boldsymbol{0}, \end{aligned} \tag{2.1}$$

where $\mathcal{U} \subseteq \mathbb{R}^{m\times n}$ is a convex compact column-wise set, $\boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^n$ and $\boldsymbol{A} \in \mathbb{R}^{m\times n}$. We show that it is $\Omega(2^{\log^{1-\epsilon} m})$-hard to approximate for any constant $0 < \epsilon < 1$.

Theorem 2.2. *The adjustable robust problem* $\Pi_{\mathsf{AR}}^{\mathsf{Gen}}$ (2.1) *is* $\Omega(2^{\log^{1-\epsilon} m})$*-hard to approximate for any constant* $0 < \epsilon < 1$.

We prove this by an approximation preserving reduction from the Label-Cover-Problem [1]. The proof is presented in Appendix A.

3 Adjustable Robust Problem: Separation Problem.

Before proving the adaptivity gap for the general column-wise and constraint-wise uncertainty sets, we first consider the case where the uncertainty set $\mathcal{U}$ is column-wise. Recall that $\mathcal{U}$ being column-wise implies that

$$\mathcal{U} = \{[\boldsymbol{b}_1\ \boldsymbol{b}_2 \ldots \boldsymbol{b}_n] \mid \boldsymbol{b}_j \in \mathcal{U}_j, j \in [n]\},$$

where $\mathcal{U}_j \subseteq \mathbb{R}^m_+$ is a compact, convex, down-monotone set for all $j \in [n]$.

3.1 The Separation Problem.

In this section, we consider the separation problem for the two-stage adjustable robust problem and a reformulation of the one-stage static robust problem introduced by Soyster [24]. In particular, we have the following epigraph reformulation of Π_{AR}.

$$\begin{aligned} z_{\mathsf{AR}} = \max\ & \boldsymbol{c}^T\boldsymbol{x} + z \\ & z \ \leq\ \boldsymbol{d}^T\boldsymbol{y}(\boldsymbol{B}),\ \forall \boldsymbol{B} \in \mathcal{U} \\ & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y}(\boldsymbol{B}) \ \leq\ \boldsymbol{h},\ \forall \boldsymbol{B} \in \mathcal{U} \\ & \boldsymbol{x}, \boldsymbol{y}(\boldsymbol{B}) \ \geq\ \boldsymbol{0}. \end{aligned}$$

Consider the following separation problem.

Separation problem: Given $\boldsymbol{x} \geq \boldsymbol{0}, z$, decide whether

$$\min_{\boldsymbol{B}\in\mathcal{U}} \max_{\boldsymbol{y}\geq\boldsymbol{0}} \{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}\} \ \geq\ z, \tag{3.1}$$

or give a violating hyperplane by exhibiting $\boldsymbol{B} \in \mathcal{U}$ such that

$$\max_{\boldsymbol{y}\geq\boldsymbol{0}} \{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}\} < z.$$

We can show that a γ-approximate algorithm for the separation problem (3.1) implies a γ-approximate algorithm for the two-stage adjustable robust problem (See Appendix B). Moreover, we can assume without loss of generality that $\boldsymbol{h} - \boldsymbol{A}\boldsymbol{x} > \boldsymbol{0}$ (See Appendix C). Therefore, we can rescale $\mathcal{U}$ by $\hat{\mathcal{U}} = [\mathsf{diag}(\boldsymbol{h} - \boldsymbol{A}\boldsymbol{x})]^{-1}\mathcal{U}$ so that the right-hand-side $(\boldsymbol{h} - \boldsymbol{A}\boldsymbol{x})$ is $\boldsymbol{e}$. Note that $\hat{\mathcal{U}}$ is also a convex, compact, down-monotone and column-wise set. Therefore, we can assume without loss of generality that the right-hand-side is $\boldsymbol{e}$. In addition, we can interpret the separation problem as the following one-stage adjustable robust problem.

$$\begin{aligned} z^I_{\mathsf{AR}} &= \min_{\boldsymbol{B}\in\mathcal{U}} \max_{\boldsymbol{y}\geq\boldsymbol{0}} \{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{e}\} \\ &= \min\{\boldsymbol{e}^T\boldsymbol{v} \mid \boldsymbol{B}^T\boldsymbol{v} \geq \boldsymbol{d}, \boldsymbol{B} \in \mathcal{U}, \boldsymbol{v} \geq \boldsymbol{0}\}, \end{aligned} \tag{3.2}$$

where the second reformulation follows by taking the dual of the inner maximization problem. On the other hand, the corresponding one-stage static robust problem can be defined as follows.

$$z^I_{\mathsf{Rob}} = \max_{\boldsymbol{y}\geq\boldsymbol{0}} \{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{e}, \forall \boldsymbol{B} \in \mathcal{U}\}.$$

We can reformulate z^I_{Rob} as a compact LP using the following result of Soyster [24].

Theorem 3.1 (Soyster [24])**.** *Suppose $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n}$ is a compact, convex, and column-wise uncertainty set. Let $\hat{\boldsymbol{B}} \in \mathbb{R}^{m\times n}$ be such that*

$$\hat{B}_{ij} = \max\{B_{ij} \mid \boldsymbol{B} \in \mathcal{U}\},\ \forall i \in [m], j \in [n]. \tag{3.3}$$

Then,

$$\max_{\boldsymbol{y}\geq \boldsymbol{0}}\{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{e}, \forall \boldsymbol{B} \in \mathcal{U}\} = \max\{\boldsymbol{d}^T\boldsymbol{y} \mid \hat{\boldsymbol{B}}\boldsymbol{y} \leq \boldsymbol{e}, \boldsymbol{y} \geq \boldsymbol{0}\}. \tag{3.4}$$

For the sake of completeness, we provide the proof of Theorem 3.1 in Appendix D. Therefore, we can reformulate z^I_{Rob} as follows.

$$z^I_{\mathsf{Rob}} = \min\{\boldsymbol{e}^T\boldsymbol{v} \mid \hat{\boldsymbol{B}}^T\boldsymbol{v} \geq \boldsymbol{d}, \boldsymbol{v} \geq \boldsymbol{0}\}, \tag{3.5}$$

where $\hat{\boldsymbol{B}}$ is as defined in (3.3).

3.2 Worst Case Instances for Adaptivity Gap.

In this section, we show that the adaptivity gap is worst on column-wise uncertainty set when each column set is a simplex. In particular, we prove the following theorem.

Theorem 3.2. *Given an arbitrary convex, compact, down-monotone and column-wise uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n}$ with $\mathcal{U} = \mathcal{U}_1 \times \ldots \times \mathcal{U}_n$, let $\hat{\boldsymbol{B}}$ be defined as in (3.3). For each $j \in [n]$, let*

$$\hat{\mathcal{U}}_j = \left\{\boldsymbol{b} \in \mathbb{R}_+^m \;\middle|\; \sum_{i=1}^m \frac{1}{\hat{B}_{ij}} b_i \leq 1, b_i = 0, \forall i : \hat{B}_{ij} = 0\right\},\ \forall j \in [n].$$

and

$$\hat{\mathcal{U}} = \left\{[\boldsymbol{b}_1\ \boldsymbol{b}_2\ \ldots\ \boldsymbol{b}_n] \;\middle|\; \boldsymbol{b}_j \in \hat{\mathcal{U}}_j,\ \forall j \in [n]\right\}.$$

Let $z_{\mathsf{AR}}(\mathcal{U})$ $(z_{\mathsf{AR}}(\hat{\mathcal{U}})$ respectively) and $z_{\mathsf{Rob}}(\mathcal{U})$ $(z_{\mathsf{Rob}}(\hat{\mathcal{U}})$ respectively) be the optimal values of the two-stage adjustable robust problem and the static robust problem over uncertainty set $\mathcal{U}$ ($\hat{\mathcal{U}}$ respectively). Then,

$$z_{\mathsf{AR}}(\hat{\mathcal{U}}) \geq z_{\mathsf{AR}}(\mathcal{U}) \text{ and } z_{\mathsf{Rob}}(\hat{\mathcal{U}}) = z_{\mathsf{Rob}}(\mathcal{U}).$$

Proof. Given arbitrary $\boldsymbol{b} \in \hat{\mathcal{U}}_j, j \in [n]$, $\boldsymbol{b}$ is a convex combination of $\hat{B}_{ij}\boldsymbol{e}_i, i \in [m]$, which further implies that $\boldsymbol{b} \in \mathcal{U}_j$. Therefore, $\boldsymbol{B} \in \hat{\mathcal{U}}$ implies that $\boldsymbol{B} \in \mathcal{U}$ and we have $\hat{\mathcal{U}} \subseteq \mathcal{U}$. Therefore, any $\boldsymbol{x}$ that is feasible for $\Pi_{\mathsf{AR}}(\mathcal{U})$ is feasible for $\Pi_{\mathsf{AR}}(\hat{\mathcal{U}})$, and we have $z_{\mathsf{AR}}(\hat{\mathcal{U}}) \geq z_{\mathsf{AR}}(\mathcal{U})$.

Since $\hat{\mathcal{U}} \subseteq \mathcal{U}$, any feasible solution for $\Pi_{\mathsf{Rob}}(\mathcal{U})$ is also feasible for $\Pi_{\mathsf{Rob}}(\hat{\mathcal{U}})$. Therefore, $z_{\mathsf{Rob}}(\hat{\mathcal{U}}) \geq z_{\mathsf{Rob}}(\mathcal{U})$. Conversely, let $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ be the optimal solution of $\Pi_{\mathsf{Rob}}(\hat{\mathcal{U}})$. Noting that $(\hat{\boldsymbol{x}}, \boldsymbol{0})$ is a feasible solution for $\Pi_{\mathsf{Rob}}(\mathcal{U})$, we have

$$\begin{aligned} z_{\mathsf{Rob}}(\mathcal{U}) &\geq \boldsymbol{c}^T\hat{\boldsymbol{x}} + \max\{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{h} - \boldsymbol{A}\hat{\boldsymbol{x}}, \forall \boldsymbol{B} \in \mathcal{U}\} \\ &= \boldsymbol{c}^T\hat{\boldsymbol{x}} + \max\{\boldsymbol{d}^T\boldsymbol{y} \mid \hat{\boldsymbol{B}}\boldsymbol{y} \leq \boldsymbol{h} - \boldsymbol{A}\hat{\boldsymbol{x}}\}, \end{aligned}$$

where the last equality follows from Theorem 3.1. Furthermore,

$$\begin{aligned} z_{\mathsf{Rob}}(\hat{\mathcal{U}}) &= \boldsymbol{c}^T\hat{\boldsymbol{x}} + \max\{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{h} - \boldsymbol{A}\hat{\boldsymbol{x}}, \forall \boldsymbol{B} \in \hat{\mathcal{U}}\} \\ &= \boldsymbol{c}^T\hat{\boldsymbol{x}} + \max\{\boldsymbol{d}^T\boldsymbol{y} \mid \hat{\boldsymbol{B}}\boldsymbol{y} \leq \boldsymbol{h} - \boldsymbol{A}\hat{\boldsymbol{x}}\}, \end{aligned}$$

where the last equality follows from Theorem 3.1 and the fact that $\mathcal{U}$ and $\hat{\mathcal{U}}$ have the same $\hat{\boldsymbol{B}}$. Therefore, $z_{\mathsf{Rob}}(\mathcal{U}) = z_{\mathsf{Rob}}(\hat{\mathcal{U}})$. □

The above theorem shows that the for column-wise uncertainty sets, the gap between the optimal values of Π_{AR} and Π_{Rob} for a column-wise set is largest when each column set is a simplex. Therefore, to provide the tight bound on the performance of static solutions, we can assume without loss of generality that the column-wise, convex compact uncertainty $\mathcal{U}$ is a Cartesian product of simplices. The worst known instance of Π_{AR} with a column-wise uncertainty set has an adaptivity gap of $\Theta(\log n)$. We present the family of instances below.

Family of Adaptivity Gap Examples. Consider the following instance (I^{LB}) of Π_{AR}:

$$\boldsymbol{A} = \boldsymbol{0}, \boldsymbol{c} = \boldsymbol{0}, \boldsymbol{d} = \boldsymbol{e}, \boldsymbol{h} = \boldsymbol{e}, \mathcal{U} = \{[\boldsymbol{b}_1\ \boldsymbol{b}_2 \ldots \boldsymbol{b}_n] \mid \boldsymbol{b}_j \in \mathcal{U}_j, j \in [n]\}, \tag{I^{LB}}$$

where

$$\begin{aligned}
\mathcal{U}_1 &= \left\{\boldsymbol{b} \in \mathbb{R}^n_+ \mid 1 \cdot b_1 + 2 \cdot b_2 + \ldots + (n-1) \cdot b_{n-1} + n \cdot b_n \leq 1\right\}, \\
\mathcal{U}_2 &= \left\{\boldsymbol{b} \in \mathbb{R}^n_+ \mid n \cdot b_1 + 1 \cdot b_2 + \ldots + (n-2) \cdot b_{n-1} + (n-1) \cdot b_n \leq 1\right\}, \\
&\ \ \vdots \\
\mathcal{U}_n &= \left\{\boldsymbol{b} \in \mathbb{R}^n_+ \mid 2 \cdot b_1 + 3 \cdot b_2 + \ldots + n \cdot b_{n-1} + 1 \cdot b_n \leq 1\right\}.
\end{aligned}$$

Therefore,

$$\mathcal{U}_j = \left\{ \boldsymbol{b} \in \mathbb{R}^n_+ \;\middle|\; \sum_{i=1}^{n} [(n+i-j+1) \bmod n] \cdot b_i \leq 1 \right\}, \forall j \in [n]$$

where mod is the standard remainder operation and let $(0 \bmod n) = n$. We have the following lemma.

Lemma 3.3. *Let z_{AR} be the optimal objective value of the instance (I^{LB}) of Π_{AR} and z_{Rob} be the optimal objective value of the corresponding static robust problem. Then,*

$$z_{\mathsf{AR}} = \Theta(\log n) \cdot z_{\mathsf{Rob}}.$$

We provide the proof in Appendix E.

4 $O(\log n \cdot \log \Gamma)$ Adaptivity Gap for Column-wise Uncertainty Sets

In this section, we first consider the case of column-wise uncertainty sets and show that a static solution gives a $O(\log n \cdot \log \Gamma)$-approximation for the two-stage adjustable robust problem where Γ is defined as follows.

$$\begin{aligned}
\beta_{\max} &= \max\{\hat{B}_{ij} \mid i \in [m], j \in [n]\} \\
\beta_{\min} &= \min\{\hat{B}_{ij} \mid i \in [m], j \in [n], \hat{B}_{ij} \neq 0\} \\
\Gamma &= 2 \cdot \frac{\beta_{\max}}{\beta_{\min}},
\end{aligned} \tag{4.1}$$

From Theorem 3.2, the worst case adaptivity gap for two-stage adjustable robust problem with column-wise uncertainty sets is achieved when $\mathcal{U}$ is a Cartesian product of simplices. Therefore, to provide a bound on the performance of static solutions, we assume that $\mathcal{U}$ is a Cartesian product of simplices.

4.1 One-stage Problems

We first compare the one-stage adjustable and static robust problems. Recall,

$$z_{\mathsf{AR}}^I = \min\{\boldsymbol{e}^T\boldsymbol{v} \mid \boldsymbol{B}^T\boldsymbol{v} \geq \boldsymbol{d}, \boldsymbol{B} \in \mathcal{U}, \boldsymbol{v} \geq \boldsymbol{0}\}$$
$$z_{\mathsf{Rob}}^I = \min\{\boldsymbol{e}^T\boldsymbol{v} \mid \hat{\boldsymbol{B}}^T\boldsymbol{v} \geq \boldsymbol{d}, \boldsymbol{v} \geq \boldsymbol{0}\}.$$

Let where $\hat{\boldsymbol{B}}$ is defined as in (3.3). The following lemma compares the objective values of one-stage adjustable robust problem z_{AR}^I and static robust problem z_{Rob}^I.

Theorem 4.1. *Given $\boldsymbol{d} \in \mathbb{R}_+^n$ and a convex, compact and down-monotone uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n}$ that is column-wise with simplex column uncertainty sets $\mathcal{U}_1, \ldots, \mathcal{U}_n$. Let z_{AR}^I be as defined in* (3.2)*, and z_{Rob}^I be as defined in* (3.5)*. Then*

$$z_{\mathsf{AR}}^I \leq O(\log\Gamma \log n) \cdot z_{\mathsf{Rob}}^I.$$

Our proof exploits the structural properties of the optimal solutions for the adjustable robust and static robust problems. In particular, we relate the one-stage adjustable robust problem to an integer set cover problem and relate the static robust problem to the dual of the corresponding LP relaxation. As earlier, by appropriate rescaling of $\mathcal{U}$, we can assume that the cost $\boldsymbol{d}$ is $\boldsymbol{e}$. We can write the one-stage adjustable robust problem as

$$z_{\mathsf{AR}}^I = \min\{\boldsymbol{e}^T\boldsymbol{v} \mid \boldsymbol{v}^T\boldsymbol{b}^j \geq 1, \boldsymbol{b}^j \in \mathcal{U}_j, \forall j \in [n], \boldsymbol{v} \geq \boldsymbol{0}\}. \tag{4.2}$$

and the corresponding static robust problem:

$$\begin{aligned} z_{\mathsf{Rob}}^I &= \max\left\{\sum_{j=1}^n y_j \,\middle|\, \sum_{j=1}^n \beta_i^j y_j \leq 1,\ \forall i \in [m], \boldsymbol{y} \geq \boldsymbol{0}\right\} &(4.3)\\ &= \min\{\boldsymbol{e}^T\boldsymbol{v} \mid \boldsymbol{v}^T\boldsymbol{\beta}^j \geq 1, \forall j \in [n], \boldsymbol{v} \geq \boldsymbol{0}\}, &(4.4)\end{aligned}$$

where

$$\beta_i^j = \hat{B}_{ij},\ \forall i \in [m], j \in [n]. \tag{4.5}$$

We first show that there exists an "integral" optimal solution for the one-stage adjustable robust problem (4.2).

Lemma 4.2. *Consider the one-stage adjustable robust problem* (4.2) *where the uncertainty set $\mathcal{U}$ is a Cartesian product of simplices $\mathcal{U}_j$, $j \in [n]$. Let $\boldsymbol{\beta}^j, j \in [n]$ be defined as in* (4.5)*. Then, there exists an optimal solution $(\bar{\boldsymbol{v}}, \bar{\boldsymbol{b}}^j, j \in [n])$ for* (4.2) *such that*

$$\bar{\boldsymbol{b}}^j = \beta_{i_j}^j \boldsymbol{e}_{i_j} \text{ for some } i_j \in [m],\ \forall j \in [n]$$
$$\bar{v}_i \in \left\{0, 1/\beta_i^j \,\middle|\, j \in [n]\right\},\ \forall i \in [m].$$

Proof. Suppose this is not the case. Let $(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{b}}^j)$ be an optimal solution for (3.2). For all $j \in [n]$, let $\bar{\boldsymbol{b}}^j$ be an extreme point optimal for

$$\max\{\tilde{\boldsymbol{v}}^T\boldsymbol{x} \mid \boldsymbol{x} \in \mathcal{U}_j\}.$$

Since $\mathcal{U}_j$ is a down-monotone simplex, $\bar{\boldsymbol{b}}^j = \beta^j_{i_j} \boldsymbol{e}_{i_j}$ for some $i_j \in [m]$. Note that $\tilde{\boldsymbol{v}}^T \bar{\boldsymbol{b}}^j \geq 1$. Therefore, $(\tilde{\boldsymbol{v}}, \bar{\boldsymbol{b}}^j, j \in [n])$ is also an optimal solution for (3.2). Now, we can reformulate the separation problem as follows.

$$z^I_{\mathsf{AR}} = \min\{\boldsymbol{e}^T \boldsymbol{v} \mid \boldsymbol{v}^T \bar{\boldsymbol{b}}^j \geq 1, \forall j \in [n]\},$$

where only $\boldsymbol{v}$ is the decision variable. Let $\bar{\boldsymbol{v}}$ be an extreme point optimal of the above LP. Then for all $j \in [n]$,

$$\bar{v}_{i_j} \bar{\boldsymbol{b}}^j_{i_j} = v_{i_j} \beta^j_{i_j} \geq 1,$$

as $\bar{\boldsymbol{b}}^j = \beta^j_{i_j} \boldsymbol{e}_{i_j}$. Therefore, we have

$$\bar{v}_i \in \left\{0, 1/\beta^j_i \;\middle|\; j \in [n]\right\}, \ \forall i \in [m]$$

at optimality. □

Therefore, we can reformulate the one-stage adjustable robust problem (3.2) as

$$z^I_{\mathsf{AR}} = \min\left\{\sum_{i=1}^m v_i \;\middle|\; \forall j \in [n], \exists i_j \in [m] \text{ s.t. } v_{i_j} \beta^j_{i_j} \geq 1, \boldsymbol{v} \geq \boldsymbol{0}\right\}. \tag{4.6}$$

A 0-1 formulation of z^I_{AR}. We can approximately formulate (4.6) as a 0-1 integer program. From Lemma 4.2, we know that there is an optimal solution $(\boldsymbol{v}, \boldsymbol{b}^j, j \in [n])$ for (4.6) such that

$$v_i \in \left\{0, 1/\beta^j_i \;\middle|\; j \in [n]\right\}, \ \forall i \in [m].$$

Therefore, if $v_i \neq 0$, then

$$\frac{1}{\beta_{\max}} \leq v_i \leq \frac{1}{\beta_{\min}}.$$

To formulate an approximate 0-1 program, we consider discrete values of v_i in multiples of 2 starting from $1/\beta_{\max}$. Denote $T = \lceil \log \Gamma \rceil$ and $\mathcal{T} = \{0, \ldots, T\}$. We consider

$$v_i \in \{0\} \cup \left\{\frac{2^t}{\beta_{\max}} \;\middle|\; t \in \mathcal{T}\right\}.$$

For any $i \in [m]$, $t \in \mathcal{T}$, let C_{it} denote the set of columns $j \in [n]$ that can be covered by setting $v_i = 2^t/\beta_{\max}$, i.e.,

$$C_{it} = \left\{j \in [n] \;\middle|\; \frac{2^t}{\beta_{\max}} \cdot \beta^j_i \geq 1\right\}.$$

Also, for all $i \in [m]$, $t \in \mathcal{T}$, let

$$x_{it} = \begin{cases} 1, \ \text{if } v_i = \dfrac{2^t}{\beta_{\max}}, \\ 0, \ \text{otherwise}, \end{cases}$$

$$c_t = \frac{2^t}{\beta_{\max}}.$$

Consider the following 0-1 integer program.

$$z^{\mathsf{mod}}_{\mathsf{AR}} = \min\left\{\sum_{i=1}^m \sum_{t=0}^T c_t x_{it} \;\middle|\; \sum_{i=1}^m \sum_{t \in \mathcal{T}:\, j \in C_{it}} x_{it} \geq 1, \ \forall j \in [n], \ x_{it} \in \{0,1\}\right\}. \tag{4.7}$$

In the following lemma, we show that the above integer program approximates z^I_{AR} within a constant factor.

Lemma 4.3. *The IP problem in (4.7) is feasible and provides a near-optimal solution for the one-stage adjustable robust problem* z^I_{AR} *(4.6). In particular, we have*

$$\frac{1}{2} z^{\mathsf{mod}}_{\mathsf{AR}} \le z^I_{\mathsf{AR}} \le z^{\mathsf{mod}}_{\mathsf{AR}}.$$

Proof. Consider an optimal solution $\boldsymbol{v}^*$ for z^I_{AR} (4.6). Note that for all $i \in [m]$, $t \in \mathcal{T}$, let

$$\bar{x}_{it} = \begin{cases} 1, \text{ if } \frac{c_t}{2} < v^*_i \le c_t, \\ 0, \text{ otherwise.} \end{cases}$$

For any $j \in [n]$, there exists $i \in [m],\ t \in \mathcal{T}$ such that

$$v^*_i \beta^j_i \ge 1.$$

Then, $\bar{\boldsymbol{x}}$ is a feasible solution to the IP problem (4.7) and

$$z^{\mathsf{mod}}_{\mathsf{AR}} \le \sum_{i=1}^{m} \sum_{t=0}^{T} c_t \bar{x}_{it} \le 2\boldsymbol{e}^T \boldsymbol{v}^* = 2 \cdot z^I_{\mathsf{AR}}.$$

Conversely, suppose $x^*_{it},\ i \in [m],\ t \in \mathcal{T}$ is an optimal solution for (4.7). We construct a feasible solution $\tilde{\boldsymbol{v}}$ for (4.6) as follows:

$$\tilde{v}_i = \sum_{t \in \mathcal{T}} c_t \cdot x_{it},\ \forall i \in [m].$$

For each $j \in [n]$, there exists $i \in [m]$ and $t \in \mathcal{T}$ such that $j \in C_{it}$ and $x^*_{it} = 1$. Therefore,

$$v_i \ge c_t = \frac{2^t}{\beta_{\max}},$$

and

$$v_i \beta^j_i \ge \frac{2^t}{\beta_{\max}} \cdot \beta^j_i \ge 1,$$

since $j \in C_{it}$. Therefore, $\tilde{\boldsymbol{v}}$ is a feasible solution for the one-stage adjustable robust problem (4.6) and

$$z^I_{\mathsf{AR}} \le \boldsymbol{e}^T \tilde{\boldsymbol{v}} \le \sum_{i=1}^{m} \sum_{t=0}^{T} c_t x^*_{it} = z^{\mathsf{mod}}_{\mathsf{AR}}.$$

□

Note that (4.7) as a 0-1 formulation for the set cover instance problem on ground set of elements $\{1, \ldots, n\}$ and family of subsets C_{it} for all $i \in [m],\ t \in \mathcal{T}$ where C_{it} has cost c_t. We can formulate the LP relaxation of (4.7) as follows.

$$z_{\mathsf{LP}} = \min \left\{ \sum_{i=1}^{m} \sum_{t=0}^{T} c_t x_{it} \;\middle|\; \sum_{i=1}^{m} \sum_{t \in \mathcal{T}:\, j \in C_{it}} x_{it} \ge 1,\ \forall j \in [n],\ x_{it} \ge 0 \right\}. \tag{4.8}$$

From [25], we know that the LP relaxation (4.8) is an $O(\log n)$-approximation for (4.7), i.e.,

$$z^{\mathsf{mod}}_{\mathsf{AR}} \le O(\log n) \cdot z_{\mathsf{LP}}.$$

Consider the dual of (4.8).

$$z_{\mathsf{LP}} \quad = \quad \max \left\{ \sum_{j=1}^{n} y_j \;\middle|\; \sum_{j \in C_{it}} y_j \leq c_t, \ \forall i \in [m], t \in \mathcal{T}, \ y_j \geq 0, \forall j \in [n] \right\} \tag{4.9}$$

We relate the dual of (4.8) to the one-stage static robust problem (3.5) to obtain the desired bound on the adaptivity gap.

Proof of Theorem 4.1. From Lemma 4.3, it is sufficient to show that

$$z_{\mathsf{LP}} \leq O(\log \Gamma) \cdot z^I_{\mathsf{Rob}}.$$

Let $\boldsymbol{y}^*$ by an optimal solution of (4.9). We show that we can construct a feasible solution for (4.3) by scaling $\boldsymbol{y}^*$ by a factor of $O(\log \Gamma)$. For each $i \in [m]$, we have

$$\sum_{j : \beta_i^j \geq \frac{\beta_{\max}}{2^t}} \frac{\beta_{\max}}{2^t} y_j^* \leq 1, \ \forall t \in \mathcal{T}.$$

Sum over all $t \in \mathcal{T}$, we have

$$\sum_{t=0}^{T} \sum_{j : \beta_i^j \geq \frac{\beta_{\max}}{2^t}} \frac{\beta_{\max}}{2^t} y_j^* \leq T + 1, \forall i \in [m].$$

Switching the summation, we have

$$\sum_{j=1}^{n} \sum_{t \in \mathcal{T} : \frac{\beta_{\max}}{2^t} \leq \beta_i^j} \frac{\beta_{\max}}{2^t} y_j^* \leq T + 1 \leq \log \Gamma + 2, \forall i \in [m]$$

Note that

$$\frac{\beta_{\max}}{2^T} \leq \beta_{\min} \leq \beta_i^j \leq \beta_{\max},$$

which implies

$$\frac{1}{2} \beta_i^j \leq \sum_{t \in \mathcal{T} : \frac{\beta_{\max}}{2^t} \leq \beta_i^j} \frac{\beta_{\max}}{2^t} \leq 2 \beta_i^j.$$

Therefore,

$$\hat{y}_j = \frac{1}{2(\log \Gamma + 2)} y_j^*, \ \forall j \in [n]$$

is a feasible solution to the maximization formulation of z^I_{Rob} (4.3) and

$$z_{\mathsf{LP}} = \boldsymbol{e}^T \boldsymbol{y}^* = O(\log \Gamma) \cdot \boldsymbol{e}^T \hat{\boldsymbol{y}} \leq O(\log \Gamma) \cdot z^I_{\mathsf{Rob}},$$

which completes the proof.

4.2 $O(\log n \cdot \log \Gamma)$ Bound on Adaptivity Gap

Based on the result in Theorem 4.1, we show that a static solution gives an $O(\log n \cdot \log \Gamma)$-approximation for the two-stage adjustable robust problem (1.1) for column-wise uncertainty sets. In particular, we prove the following theorem.

Theorem 4.4. *Let z_{AR} be the objective value of an optimal fully-adjustable solution for Π_{AR} (1.1), and z_{Rob} be the optimal objective value of the corresponding static robust problem Π_{Rob} (1.2). If $\mathcal{U}$ is a column-wise uncertainty set, then,*

$$z_{AR} \leq O(\log n \cdot \log \Gamma) \cdot z_{Rob}.$$

Proof. Let $(\boldsymbol{x}^*, \boldsymbol{y}^*(\boldsymbol{B}), \boldsymbol{B} \in \mathcal{U})$ be an optimal fully-adjustable solution to Π_{AR}. Then,

$$z_{\mathsf{AR}} = \boldsymbol{c}^T\boldsymbol{x}^* + \min_{\boldsymbol{B}\in\mathcal{U}} \max_{y(B)\geq\boldsymbol{0}} \{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y}(\boldsymbol{B}) \leq \boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}^*\}.$$

From Appendix C, we can assume without loss of generality that $(\boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}^*) > \boldsymbol{0}$. Let

$$\mathcal{U}^* = [\mathsf{diag}(\boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}^*)]^{-1}\mathcal{U}.$$

Then,

$$z_{\mathsf{AR}} = \boldsymbol{c}^T\boldsymbol{x}^* + \min_{\boldsymbol{B}\in\mathcal{U}^*} \max_{y(B)\geq\boldsymbol{0}} \{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y}(\boldsymbol{B}) \leq \boldsymbol{e}\}.$$

By writing the dual of the inner maximization problem, we have

$$z_{\mathsf{AR}} = \boldsymbol{c}^T\boldsymbol{x}^* + \min_{\boldsymbol{B},\boldsymbol{\mu}}\{\boldsymbol{e}^T\boldsymbol{\mu} \mid \boldsymbol{B}^T\boldsymbol{\mu} \geq \boldsymbol{d}, \boldsymbol{B} \in \mathcal{U}^*, \boldsymbol{\mu} \geq \boldsymbol{0}\}.$$

On the other hand, since $(\boldsymbol{x}^*, \boldsymbol{0})$ is a feasible solution of Π_{Rob}, we have

$$\begin{aligned} z_{\mathsf{Rob}} &\geq \boldsymbol{c}^T\boldsymbol{x}^* + \max_{\boldsymbol{y}\geq\boldsymbol{0}}\{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}^*, \forall \boldsymbol{B} \in \mathcal{U}\} \\ &= \boldsymbol{c}^T\boldsymbol{x}^* + \max_{\boldsymbol{y}\geq\boldsymbol{0}}\{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{e}, \forall \boldsymbol{B} \in \mathcal{U}^*\}. \end{aligned}$$

Let $\hat{B}$ be defined as in (3.3). For $\mathcal{U}^*$, from Theorem 3.1, we have

$$\begin{aligned} z_{\mathsf{Rob}} &\geq \boldsymbol{c}^T\boldsymbol{x}^* + \max\{\boldsymbol{d}^T\boldsymbol{y} \mid \hat{\boldsymbol{B}}\boldsymbol{y} \leq \boldsymbol{e}, \boldsymbol{y} \geq \boldsymbol{0}\} \\ &= \boldsymbol{c}^T\boldsymbol{x}^* + \min_{\boldsymbol{v}\geq\boldsymbol{0}}\{\boldsymbol{e}^T\boldsymbol{v} \mid \hat{\boldsymbol{B}}^T\boldsymbol{v} \geq \boldsymbol{d}\}. \end{aligned}$$

Note that $\mathcal{U}^*$ is compact, convex, down-monotone and column-wise. Therefore, from Theorem 4.1, we have

$$\begin{aligned} z_{\mathsf{AR}} &= \boldsymbol{c}^T\boldsymbol{x}^* + \min_{\boldsymbol{B},\boldsymbol{\mu}}\{\boldsymbol{e}^T\boldsymbol{\mu} \mid \boldsymbol{B}^T\boldsymbol{\mu} \geq \boldsymbol{d}, \boldsymbol{B} \in \mathcal{U}^*, \boldsymbol{\mu} \geq \boldsymbol{0}\} \\ &\leq \boldsymbol{c}^T\boldsymbol{x}^* + O(\log \Gamma \log n) \cdot \min_{\boldsymbol{v}\geq\boldsymbol{0}}\{\boldsymbol{e}^T\boldsymbol{v} \mid \hat{\boldsymbol{B}}^T\boldsymbol{v} \geq \boldsymbol{d}\} \\ &\leq O(\log \Gamma \log n) \cdot \left(\boldsymbol{c}^T\boldsymbol{x}^* + \min_{\boldsymbol{v}\geq\boldsymbol{0}}\{\boldsymbol{e}^T\boldsymbol{v} \mid \hat{\boldsymbol{B}}^T\boldsymbol{v} \geq \boldsymbol{d}\}\right) \\ &\leq O(\log n \cdot \log \Gamma) \cdot z_{\mathsf{Rob}} \end{aligned}$$

where the second last inequality follows as $\boldsymbol{c}, \boldsymbol{x}^* \geq \boldsymbol{0}$. ☐

We would like to note that if the ratio between the largest and smallest entries of $\hat{\boldsymbol{B}}$ is constant, then static solution provides an $O(\log n)$-approximation for the two-stage adjustable robust problem. The two-stage adjustable robust problem is hard to approximate within a factor better than $O(\log n)$ even when the ratio is one. Therefore, quite surprisingly, the performance of the static solution matches the hardness of approximation in this case. Furthermore, in the following section, we show that even when the ratio is large, the static solution still provides a near-optimal approximation for the adjustable robust problem.

5 $O(\log n \cdot \log(m+n))$ Bound on Adaptivity Gap

In this section, we show that a static solution provides an $O(\log n \cdot \log(m+n))$-approximation for the two-stage adjustable robust problem Π_{AR} (1.1) with column-wise uncertainty sets. Note that this bound on adaptivity gap is uniform across instances and does not depend on Γ. In particular, we have the following theorem.

Theorem 5.1. *Let z_{AR} be the objective value of an optimal fully-adjustable solution for Π_{AR} (1.1), and z_{Rob} be the optimal objective value of the corresponding static robust problem Π_{Rob} (1.2). If $\mathcal{U}$ is a column-wise uncertainty set, then,*

$$z_{AR} \leq O(\log(m+n)\log n) \cdot z_{Rob}.$$

To prove Theorem 5.1, it is sufficient to prove the approximation bound for corresponding one-stage problems since we can extend the bound to the two-stage problem using arguments as in Theorem 4.4.

Theorem 5.2. *Let z^I_{AR} be as defined in (4.6), and z^I_{Rob} be as defined in (3.5). If the uncertainty set $\mathcal{U}$ is column-wise, then*

$$z^I_{\mathsf{AR}} \leq O(\log(m+n)\log n) \cdot z^I_{\mathsf{Rob}}.$$

Our proof is similar in the spirit of Theorem 4.1. If $\log\Gamma = O(\log(m+n))$, the result follows. However, if Γ is large, we need to handle extreme values of $\hat{B}_{ij}$ differently in order to avoid the dependence on Γ. Let $\boldsymbol{v}^*$ be an optimal solution for the one-stage adjustable robust problem (4.6) and $\theta = ||\boldsymbol{v}^*||_\infty$. Let

$$J_1 = \left\{ j \in [n] \;\middle|\; \text{there exists } i \in [m] \text{ s.t. } \beta^j_i \geq \frac{2m}{\theta} \right\}.$$

We show that we can delete the columns in J_1 from z^I_{AR} (4.6) (corresponding to the large values of $\hat{B}_{ij}$) such that the modified problem is only within a constant factor of z^I_{AR}. As before, we consider only discrete values of v_i for all $i \in [m]$. Let $T = \lceil \max\{\log m, \log n\} \rceil$ and $\mathcal{T} = \{-T, \ldots, T\}$. For all $i \in [m]$, we consider

$$v_i \in \{0\} \cup \left\{ \frac{\theta}{2^t} \;\middle|\; t \in \mathcal{T} \right\}.$$

Also, for all $i \in [m]$, $t \in \mathcal{T}$, let C_{it} denote the set of columns in $[n] \setminus J_1$ that can be covered by setting $v_i = \theta/2^t$, i.e.,

$$\begin{aligned} C_{it} &= \left\{ j \in [n] \setminus J_1 \;\middle|\; \beta^j_i \geq \frac{2^t}{\theta} \right\}, \text{and} \\ c_t &= \frac{\theta}{2^t}. \end{aligned}$$

Consider the following 0-1 formulation for the modified one-stage problem.

$$z_{\mathsf{AR}}^{\mathsf{mod}} = \min \left\{ \sum_{i=1}^{m} \sum_{t=-T}^{T} c_t x_{it} \;\middle|\; \sum_{i=1}^{m} \sum_{t \in \mathcal{T}: j \in C_{it}} x_{it} \geq 1,\ \forall j \in [n] \setminus J_1,\ x_{it} \in \{0,1\} \right\}. \tag{5.1}$$

We have the following lemma.

Lemma 5.3. *The IP problem in* (5.1) *is feasible and provides a near-optimal solution for the one-stage adjustable robust problem* z_{AR}^{I} (4.6)*. In particular, we have*

$$\frac{1}{2} z_{\mathsf{AR}}^{\mathsf{mod}} \leq z_{\mathsf{AR}}^{I} \leq 2 z_{\mathsf{AR}}^{\mathsf{mod}}.$$

Proof. Consider an optimal solution $\boldsymbol{v}^*$ for (4.6). We construct a feasible solution for (5.1) as follows. Now, for all $i \in [m]$, $t \in \mathcal{T}$, let

$$\bar{x}_{it} = \begin{cases} 1, \text{ if } \dfrac{c_t}{2} < v_i^* \leq c_t \\ 0, \text{ otherwise.} \end{cases}$$

Since $\boldsymbol{v}^*$ is feasible, $\bar{\boldsymbol{x}}$ is a feasible solution to the set cover problem (5.1) and

$$z_{\mathsf{AR}}^{\mathsf{mod}} \leq \sum_{i=1}^{m} \sum_{t=-T}^{T} c_t \bar{x}_{it} \leq 2 \boldsymbol{e}^T \boldsymbol{v}^* = 2 z_{\mathsf{AR}}^{I}.$$

Conversely, consider an optimal solution $\boldsymbol{x}^*$ for the set cover problem (5.1). We construct a feasible solution $\tilde{\boldsymbol{v}}$ for (4.6) as follows. For all $i \in [m]$,

$$\tilde{v}_i = \frac{\theta}{2m} + \sum_{t \in \mathcal{T}} c_t x_{it}^*.$$

Note that we add $\theta/2m$ to each v_i in order to handle the constraints for columns in J_1 that are not considered in (5.1). For each $j \in J_1$, there exists $i \in [m]$ such that $\beta_i^j \geq 2m/\theta$ and $v_i \beta_i^j \geq 1$. For all $j \in [n] \setminus J_1$, there exists $i \in [m]$ and $t \in \{-T, \ldots, T\}$ such that $j \in C_{it}$ and $x_{it}^* = 1$. Therefore, $v_i \geq c_t$ which implies that $v_i \cdot \beta_i^j \geq 1$. Therefore, $\tilde{\boldsymbol{v}}$ is a feasible solution for the one-stage adjustable robust problem z_{AR}^{I} (4.6). Moreover, we have

$$z_{\mathsf{AR}}^{I} \leq \boldsymbol{e}^T \tilde{\boldsymbol{v}} \leq \left(\frac{\theta}{2} + z_{\mathsf{AR}}^{\mathsf{mod}} \right) \leq \frac{z_{\mathsf{AR}}^{I}}{2} + z_{\mathsf{AR}}^{\mathsf{mod}} \Rightarrow z_{\mathsf{AR}}^{I} \leq 2 \cdot z_{\mathsf{AR}}^{\mathsf{mod}},$$

which completes the proof. □

We can formulate the LP relaxation of set cover problem in (5.1) as follows.

$$z_{\mathsf{LP}} = \min \left\{ \sum_{i=1}^{m} \sum_{t=-T}^{T} c_{it} x_{it} \;\middle|\; \sum_{i=1}^{m} \sum_{t \in \mathcal{T}: \frac{2^t}{\theta} \leq \beta_i^j} x_{it} \geq 1,\ \forall j \in [n] \setminus J_1,\ x_{it} \geq 0 \right\}. \tag{5.2}$$

We have

$$z_{\mathsf{AR}}^{\mathsf{mod}} \leq O(\log n) \cdot z_{\mathsf{LP}}.$$

Consider the dual of (5.2).

$$z_{\mathsf{LP}} \quad = \quad \max\left\{ \sum_{j \notin J_1} y_j \;\middle|\; \sum_{j \in C_{it}} y_j \leq c_t,\ \forall i \in [m], t \in \mathcal{T},\ y_j \geq 0, \forall j \in [n] \setminus J_1 \right\} \tag{5.3}$$

We will construct a feasible solution for the one-stage static robust problem (4.3) from (5.3).

Proof of Theorem 5.2. From Lemma 5.3, it is sufficient to show that

$$z_{\mathsf{LP}} \leq O(\log(m+n)) \cdot z^I_{\mathsf{Rob}}.$$

Let $\boldsymbol{y}^*$ by an optimal solution of (5.3). We construct a feasible solution for (4.3) by scaling $\boldsymbol{y}^*$ by a factor of $O(\log(m+n))$. For $t = 0$, we have

$$\sum_{j \notin J_1 : \beta_i^j \geq \frac{1}{\theta}} \frac{1}{\theta} y_j^* \leq 1, \forall i \in [m].$$

Let $\boldsymbol{v}^*$ be an optimal solution for (4.6). From Lemma 4.2, for each $j \in [n]$, there exist $i \in [m]$ such that

$$\beta_i^j v_i^* \geq 1 \Rightarrow \beta_i^j \geq \frac{1}{v_i^*} \geq \frac{1}{\theta}.$$

Therefore, for each $j \in [n] \setminus J_1$, we have $y_j^* \leq \theta$. Since $\boldsymbol{y}^*$ is an optimal solution of (5.3), we have

$$\sum_{j \notin J_1 : \beta_i^j \geq \frac{2^t}{\theta}} \frac{2^t}{\theta} y_j^* \leq 1,\ \forall t \in \mathcal{T}.$$

Sum over all $t \in \mathcal{T}$, we have

$$\sum_{t \in \mathcal{T}} \sum_{j \notin J_1 : \beta_i^j \geq \frac{2^t}{\theta}} \frac{2^t}{\theta} y_j^* \leq 2T + 1, \forall i \in [m].$$

Switching the summation, we have

$$\sum_{j \notin J_1} \sum_{t \in \mathcal{T} : \frac{2^t}{\theta} \leq \beta_i^j} \frac{2^t}{\theta} y_j^* \leq 2T + 1, \forall i \in [m]$$

Note that if $\beta_i^j \geq 1/n\theta$ and $j \notin J_1$, then

$$\frac{1}{2} \beta_i^j \leq \sum_{t : \frac{2^t}{\theta} \leq \beta_i^j} \frac{2^t}{\theta} \leq 2 \beta_i^j.$$

Let

$$\hat{y}_j = \begin{cases} \dfrac{1}{4T+3} y_j^*, & \text{if } j \in [n] \setminus J_1 \\ 0, & j \in J_1 \end{cases}$$

For any $i \in [m]$, we have

$$\sum_{j=1}^{n} \beta_i^j \hat{y}_j = \sum_{j \in J_1} \beta_i^j \hat{y}_j + \frac{1}{4T+3} \left(\sum_{j \notin J_1 : \beta_i^j < 1/n\theta} \beta_i^j y_j^* + \sum_{j \notin J_1 : \beta_i^j \geq 1/n\theta} \beta_i^j y_j^* \right)$$
$$\leq 0 + \frac{1}{4T+3} \left(1 + 2 \sum_{j=1}^{n} \sum_{t : \frac{2^t}{\theta} \beta_i^j} \frac{2^t}{\theta} y_j^* \right)$$
$$\leq 1$$

Therefore, $\hat{\boldsymbol{y}}$ is a feasible solution to the dual of z_{Rob}^I (4.3). Note that $T = O(\log(m+n))$. Therefore, we have

$$z_{\mathsf{LP}} = \boldsymbol{e}^T \boldsymbol{y}^* = O(\log(m+n)) \cdot \boldsymbol{e}^T \hat{\boldsymbol{y}} \leq O(\log(m+n)) \cdot z_{\mathsf{Rob}}^I,$$

which completes the proof.

From Theorems 4.4 and 5.1, we have the following corollary.

Corollary 5.4. Let z_{AR} be the objective value of an optimal fully-adjustable solution for Π_{AR} (1.1), and z_{Rob} be the optimal objective value of the corresponding static robust problem Π_{Rob} (1.2). If $\mathcal{U}$ is a column-wise uncertainty set, then,

$$z_{AR} \leq O(\log n \cdot \min(\log \Gamma, \log(m+n))) \cdot z_{Rob}.$$

6 Column-wise and Constraint-wise Uncertainty Sets.

In this section, we consider the general case where the uncertainty set is the intersection of column-wise and constraint-wise sets. Recall that a column-wise and constraint-wise uncertainty set $\mathcal{U}$ implies that

$$\mathcal{U} = \left\{ \boldsymbol{B} \in \mathbb{R}_+^{m \times n} \mid \boldsymbol{B}\boldsymbol{e}_j \in C_j, \ \forall j \in [n], \ \boldsymbol{B}^T \boldsymbol{e}_i \in R_i, \ \forall i \in [m] \right\}, \tag{6.1}$$

where $C_j \subseteq \mathbb{R}_+^m$ for all $j \in [n]$ and $R_i \subseteq \mathbb{R}_+^n$ for all $i \in [m]$ are compact, convex and down-monotone sets. We refer to the above uncertainty set as a column-wise and constraint-wise set since the constraints on the uncertainty set $\mathcal{U}$ are either over the columns or the rows of the matrix. As mentioned previously, we assume that optimization problems with linear objective over $\mathcal{U}$ can be solved in polynomial time in the encoding length of $\mathcal{U}$.

We show that a static solution provides an $O(\log n \cdot \min(\log \Gamma, \log(m+n))))$-approximation for the two-stage adjustable robust problem Π_{AR} for the above column-wise and constraint-wise uncertainty set where Γ is defined in (4.1). In particular, we have the following theorem.

Theorem 6.1. *Consider a convex, compact and down-monotone uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^{m \times n}$ that is column-wise and constraint-wise as in* (6.1). *Let $z_{\mathsf{AR}}(\mathcal{U})$ and $z_{\mathsf{Rob}}(\mathcal{U})$ be the optimal values of the two-stage adjustable robust problem $\Pi_{\mathsf{AR}}(\mathcal{U})$* (1.1) *and the static robust problem $\Pi_{\mathsf{Rob}}(\mathcal{U})$* (1.2) *over uncertainty set $\mathcal{U}$, respectively. Then,*

$$z_{\mathsf{AR}}(\mathcal{U}) \leq O(\log n \cdot \min(\log \Gamma, \log(m+n))) \cdot z_{\mathsf{Rob}}(\mathcal{U}).$$

Our proof is based on a transformation of the static robust problem into a equivalent formulation over a constraint-wise uncertainty set. In particular, we construct the constraint-wise uncertainty set as follows. For each $i \in [m]$, let

$$\tilde{\mathcal{R}}_i = \{ \boldsymbol{B}^T \boldsymbol{e}_i \mid \boldsymbol{B} \in \mathcal{U} \}, \tag{6.2}$$

i.e., $\tilde{\mathcal{R}}_i$ is the projection of the uncertainty set $\mathcal{U}$ for the i^{th} row. Let

$$\tilde{\mathcal{U}} = \tilde{\mathcal{R}}_1 \times \tilde{\mathcal{R}}_2 \times \ldots \times \tilde{\mathcal{R}}_m, \tag{6.3}$$

i.e., a Cartesian product of $\tilde{\mathcal{R}}_i, i \in [m]$. Note that for any $\boldsymbol{B} \in \tilde{\mathcal{U}}$, the constraints corresponding to row-sets $R_1, \ldots, R_m$ are satisfied. However, the constraints corresponding to column-sets $C_1, \ldots, C_n$ may not be satisfied. We have the following lemma.

Lemma 6.2. *Given a convex, compact and down-monotone uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n}$ that is column-wise and constraint-wise and any $\boldsymbol{\mu} \in [0,1]^m$ such that $\boldsymbol{e}^T\boldsymbol{\mu} = 1$, let $\tilde{\mathcal{U}}$ be defined as* (6.3)*. Then, for any $\boldsymbol{B} \in \tilde{\mathcal{U}}$, we have*

$$\mathsf{diag}(\boldsymbol{\mu})\boldsymbol{B} \in \mathcal{U}.$$

Proof. Noting that $\boldsymbol{B}^T\boldsymbol{e}_i \in \tilde{\mathcal{R}}_i$ and $\mathsf{diag}(\boldsymbol{e}_i)\boldsymbol{B}$ has the i^{th} row as $\boldsymbol{B}^T\boldsymbol{e}_i$ and other rows as $\boldsymbol{0}$, we have $\mathsf{diag}(\boldsymbol{e}_i)\boldsymbol{B} \in \mathcal{U}$ since $\mathcal{U}$ is down-monotone. Moreover, $\boldsymbol{\mu}$ is convex multiplier,

$$\mathsf{diag}(\boldsymbol{\mu})\boldsymbol{B} = \sum_{i=1}^{m} \mu_i \mathsf{diag}(\boldsymbol{e}_i)\boldsymbol{B}$$

and $\mathcal{U}$ is convex, we have $\mathsf{diag}(\boldsymbol{\mu})\boldsymbol{B} \in \mathcal{U}$. □

In the following lemma, we show that the static robust problem has the same optimal objective value for uncertainty sets $\mathcal{U}$ and $\tilde{\mathcal{U}}$.

Lemma 6.3. *Given a convex, compact and down-monotone uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n}$ that is column-wise and constraint-wise, let $\tilde{\mathcal{U}}$ be defined as in* (6.3)*. Let $z_{\mathsf{Rob}}(\mathcal{U})$ and $z_{\mathsf{Rob}}(\tilde{\mathcal{U}})$ be the optimal values of the static adjustable robust problem Π_{Rob}* (1.2) *over uncertainty set $\mathcal{U}$ and $\tilde{\mathcal{U}}$, respectively. Then*

$$z_{\mathsf{Rob}}(\mathcal{U}) = z_{\mathsf{Rob}}(\tilde{\mathcal{U}}).$$

Proof. For any $\boldsymbol{B} \in \mathcal{U}$, we have $\boldsymbol{B}^T\boldsymbol{e}_i \in \tilde{\mathcal{R}}_i$ for all $i \in [m]$, which implies that $\boldsymbol{B} \in \tilde{\mathcal{U}}$ since $\tilde{\mathcal{U}}$ is constraint-wise. Therefore, $\mathcal{U} \subseteq \tilde{\mathcal{U}}$ and any solution that is feasible for $\Pi_{\mathsf{Rob}}(\tilde{\mathcal{U}})$ must be feasible for $\Pi_{\mathsf{Rob}}(\mathcal{U})$. Therefore,

$$z_{\mathsf{Rob}}(\tilde{\mathcal{U}}) \leq z_{\mathsf{Rob}}(\mathcal{U}).$$

Conversely, suppose $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ is an optimal solution for $\Pi_{\mathsf{Rob}}(\mathcal{U})$. We show that it is feasible for $\Pi_{\mathsf{Rob}}(\tilde{\mathcal{U}})$. For the sake of contradiction, assume that there exists a $\tilde{\boldsymbol{B}} \in \tilde{\mathcal{U}}$ such that

$$(\tilde{\boldsymbol{B}}\hat{\boldsymbol{y}})_i > h_i - (\boldsymbol{A}\hat{\boldsymbol{x}})_i \text{ for some } i \in [m] \Rightarrow (\mathsf{diag}(\boldsymbol{e}_i)\tilde{\boldsymbol{B}}\hat{\boldsymbol{y}})_i > h_i - (\boldsymbol{A}\hat{\boldsymbol{x}})_i.$$

However, from Lemma 6.2, $\mathsf{diag}(\boldsymbol{e}_i)\tilde{\boldsymbol{B}} \in \mathcal{U}$, which contradicts the assumption that $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ is feasible for $\Pi_{\mathsf{Rob}}(\mathcal{U})$. Therefore, $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ is feasible for $\Pi_{\mathsf{Rob}}(\tilde{\mathcal{U}})$ and $z_{\mathsf{Rob}}(\mathcal{U}) \leq z_{\mathsf{Rob}}(\tilde{\mathcal{U}})$. □

From Ben-Tal and Nemirovski [4] and Bertsimas et al. [8], we know that

$$z_{\mathsf{Rob}}(\tilde{\mathcal{U}}) = z_{\mathsf{AR}}(\tilde{\mathcal{U}}),$$

since $\tilde{\mathcal{U}}$ is a constraint-wise uncertainty set and a static solution is optimal for the adjustable robust problem. Therefore, to prove Theorem 6.1, it is now sufficient to show

$$z_{\mathsf{AR}}(\mathcal{U}) \leq O(\log n \cdot \min(\log \Gamma, \log(m+n))) \cdot z_{\mathsf{AR}}(\tilde{\mathcal{U}}).$$

Proof of Theorem 6.1 Let $(\boldsymbol{x}^*, \boldsymbol{y}^*(\boldsymbol{B}), \boldsymbol{B} \in \mathcal{U})$ be an optimal fully-adjustable solution to $\Pi_{\mathsf{AR}}(\mathcal{U})$. Therefore,

$$z_{\mathsf{AR}}(\mathcal{U}) = \boldsymbol{c}^T\boldsymbol{x}^* + \min_{\boldsymbol{B}\in\mathcal{U}} \max\{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \le \boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}^*,\ \boldsymbol{y} \ge \boldsymbol{0}\}.$$

As discussed in Appendix C, we can assume without loss of generality $(\boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}^*) > \boldsymbol{0}$. Therefore, we can rescale $\mathcal{U}$ and $\tilde{\mathcal{U}}$ as

$$\mathcal{S} = [\mathsf{diag}(\boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}^*)]^{-1}\mathcal{U}, \ \text{ and } \tilde{\mathcal{S}} = [\mathsf{diag}(\boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}^*)]^{-1}\tilde{\mathcal{U}}.$$

Note that $\tilde{\mathcal{S}}$ is the Cartesian product of the row projections of $\mathcal{S}$. For any $\mathcal{H} \subseteq \mathbb{R}_+^{m\times n}$, let

$$z^I_{\mathsf{AR}}(\mathcal{H}) = \min\{\boldsymbol{e}^T\boldsymbol{v} \mid \boldsymbol{B}^T\boldsymbol{v} \ge \boldsymbol{d}, \boldsymbol{B} \in \mathcal{H}, \boldsymbol{v} \ge \boldsymbol{0}\}.$$

Now,

$$\begin{aligned} z_{\mathsf{AR}}(\mathcal{U}) &= \boldsymbol{c}^T\boldsymbol{x}^* + \min_{\boldsymbol{B}\in\mathcal{S}} \max\{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \le \boldsymbol{e},\ \boldsymbol{y} \ge \boldsymbol{0}\} \\ &= \boldsymbol{c}^T\boldsymbol{x}^* + \min\{\boldsymbol{e}^T\boldsymbol{v} \mid \boldsymbol{B}^T\boldsymbol{v} \ge \boldsymbol{d}, \boldsymbol{B} \in \mathcal{S},\ \boldsymbol{v} \ge \boldsymbol{0}\} \\ &= \boldsymbol{c}^T\boldsymbol{x}^* + z^I_{\mathsf{AR}}(\mathcal{S}), \end{aligned}$$

where the second equation follows by taking the dual of the inner maximization problem. Also,

$$\begin{aligned} z_{\mathsf{AR}}(\tilde{\mathcal{U}}) &\ge \boldsymbol{c}^T\boldsymbol{x}^* + \min_{\boldsymbol{B}\in\tilde{\mathcal{U}}} \max\{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \le \boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}^*,\ \boldsymbol{y} \ge \boldsymbol{0}\} \\ &= \boldsymbol{c}^T\boldsymbol{x}^* + z^I_{\mathsf{AR}}(\tilde{\mathcal{S}}). \end{aligned}$$

Therefore, to complete the proof, it is sufficient to show that

$$z^I_{\mathsf{AR}}(\mathcal{S}) \le O(\log n \cdot \min(\log \Gamma, \log(m+n))) \cdot z^I_{\mathsf{AR}}(\tilde{\mathcal{S}}). \tag{6.4}$$

Let $\tilde{\boldsymbol{B}} \in \tilde{\mathcal{S}}$ be the minimizer of $z^I_{\mathsf{AR}}(\tilde{\mathcal{S}})$. We construct a simplex column-wise uncertainty set, $\mathcal{H} \subseteq \mathbb{R}_+^{m\times n}$ where each simplex column set, $H_j \subseteq \mathbb{R}_+^m,\ j \in [n]$ is defined from $\tilde{\boldsymbol{B}}$ as follows.

$$H_j = \mathsf{conv}\left(\{\boldsymbol{0}\} \bigcup \left\{\tilde{B}_{ij}\boldsymbol{e}_i \mid i = 1, \ldots, m\right\}\right).$$

and

$$\mathcal{H} = \{[\boldsymbol{b}_1 \cdots \boldsymbol{b}_n] \mid \boldsymbol{b}_j \in H_j, \forall j \in [n]\}.$$

We would like to note that $\mathcal{H} \subseteq \mathcal{S}$: For any $\boldsymbol{b} \in H_j, j \in [n]$, we have $\boldsymbol{b} \le \mathsf{diag}(\boldsymbol{\mu})\tilde{\boldsymbol{B}}\boldsymbol{e}_j$ for some convex multiplier $\boldsymbol{\mu}$. From Lemma 6.2, $\mathsf{diag}(\boldsymbol{\mu})\tilde{\boldsymbol{B}} \in \mathcal{S}$, which indicates that $H_j \subseteq [\mathsf{diag}(\boldsymbol{h} - \boldsymbol{A}\boldsymbol{x})]^{-1}C_j$. Moreover, $\tilde{\boldsymbol{B}}$ satisfies the row constraints of $\mathcal{S}$ and $\boldsymbol{e}_i^T\boldsymbol{B} \le \boldsymbol{e}_i^T\tilde{\boldsymbol{B}}$ for any $\boldsymbol{B} \in \mathcal{H}, i \in [m]$. Therefore, $\mathcal{H} \subseteq \mathcal{S}$ and

$$z^I_{\mathsf{AR}}(\mathcal{S}) \le z^I_{\mathsf{AR}}(\mathcal{H}) \le O(\log n \cdot \min(\log \Gamma, \log(m+n))) \cdot z^I_{\mathsf{Rob}}(\mathcal{H}) \tag{6.5}$$

where the second inequality follows from Theorems 4.1 and 5.2. Note that $\tilde{\boldsymbol{B}}$ is the entry-wise maximum matrix over $\mathcal{H}$ as defined in (3.3). Therefore,

$$z^I_{\mathsf{Rob}}(\mathcal{H}) = \min\{\boldsymbol{e}^T\boldsymbol{v} \mid \tilde{\boldsymbol{B}}^T\boldsymbol{v} \ge \boldsymbol{d}\} = z^I_{\mathsf{AR}}(\tilde{\mathcal{S}}),$$

where the first equality follows from Theorem 3.1 and the second equality follows from the fact that $\tilde{\boldsymbol{B}} \in \tilde{\mathcal{S}}$ is a minimizer for $z^I_{\mathsf{AR}}(\tilde{\mathcal{S}})$. Therefore, from (6.5), we have $z^I_{\mathsf{AR}}(\mathcal{S}) \le O(\log n \cdot \min(\log \Gamma, \log(m+n))) \cdot z^I_{\mathsf{AR}}(\tilde{\mathcal{S}})$.

7 Comparison with Measure of Non-convexity Bound

In this section, we compare our bound with the measure of non-convexity bound introduced by Bertsimas et al. [8]. We show that our bound provides an upper bound on the measure of non-convexity for column-wise and constraint-wise uncertainty sets. In [8], the authors introduce the following transformation of uncertainty set $\mathcal{U} \in \mathbb{R}_+^{m\times n}$ for general right-hand-side vector $\boldsymbol{h} > \boldsymbol{0}$:

$$T(\mathcal{U}, \boldsymbol{h}) = \left\{ \boldsymbol{B}^T \boldsymbol{\mu} \;\middle|\; \boldsymbol{h}^T \boldsymbol{\mu} = 1,\ \boldsymbol{B} \in \mathcal{U},\ \boldsymbol{\mu} \geq \boldsymbol{0} \right\}. \tag{7.1}$$

and define a measure of non-convexity κ for general compact set $\mathcal{S}$ as follows:

$$\kappa(\mathcal{S}) = \min \left\{ \alpha \mid \mathsf{conv}(\mathcal{S}) \subseteq \alpha \mathcal{S} \right\}. \tag{7.2}$$

The authors prove the following tight bound for the adaptivity gap:

$$z_{\mathsf{AR}} \leq \max\{\kappa(T(\mathcal{U}, \boldsymbol{h})) \mid \boldsymbol{h} > \boldsymbol{0}\} \cdot z_{\mathsf{Rob}}.$$

However, the measure of non-convexity is not necessarily efficiently computable in general. Moreover, it can be as large as m. In fact, Bertsimas et al. [8] show that the measure of non-convexity of $T(\mathcal{U}, \boldsymbol{e})$ for the following uncertainty set $\mathcal{U}$ is m:

$$\mathcal{U} = \left\{ \boldsymbol{B} \in [0,1]^{m\times m} \;\middle|\; B_{ij} = 0, \forall i \neq j, \sum_{i=1}^m B_{ii} \leq 1 \right\}.$$

We show that for general column-wise and constraint-wise uncertainty sets, our analysis provides an upper bound on the measure of non-convexity. Specifically, we have the following theorem.

Theorem 7.1. *Given a convex, compact and down-monotone uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^{m\times n}$ that is column-wise and constraint-wise as in* (6.1) *and $\boldsymbol{h} > \boldsymbol{0}$, let $T(\mathcal{U}, \boldsymbol{h})$ and $\kappa(T(\mathcal{U}, \boldsymbol{h}))$ be defined as in* (7.1) *and* (7.2)*, respectively. Then,*

$$\kappa(T(\mathcal{U}, \boldsymbol{h})) \leq O(\log n \cdot \min(\log \Gamma, \log(m+n))).$$

Proof. Let $\alpha = \log n \cdot \min(\log \Gamma, \log(m+n))$. Let $\tilde{\mathcal{R}}_i, i \in [m]$ be defined as in (6.2). From Bertsimas et al. [8], we have

$$\mathsf{conv}(T(\mathcal{U}, \boldsymbol{h})) = \mathsf{conv}\left(\bigcup_{i=1}^m \frac{1}{h_i} \cdot \tilde{\mathcal{R}}_i \right).$$

Given any $\boldsymbol{d} \in \mathsf{conv}(T(\mathcal{U}, \boldsymbol{h}))$, we have

$$\boldsymbol{d} = \sum_{i=1}^m \frac{\lambda_i}{h_i} \tilde{\boldsymbol{b}}_i$$

where $\tilde{\boldsymbol{b}}_i \in \tilde{\mathcal{R}}_i, i \in [m]$, $\boldsymbol{\lambda} \geq \boldsymbol{0}$ and $\boldsymbol{e}^T \boldsymbol{\lambda} = 1$. For all $i \in [m]$, let $\boldsymbol{B}_i = \boldsymbol{e}_i \tilde{\boldsymbol{b}}_i^T$. Since $\mathcal{U}$ is down-monotone, $\boldsymbol{B}_i \in \mathcal{U}$. Let

$$\tilde{\boldsymbol{B}} = [\mathsf{diag}(\boldsymbol{h})]^{-1} \sum_{i=1}^m \boldsymbol{B}_i.$$

Therefore, $\tilde{\boldsymbol{B}}^T \boldsymbol{\lambda} = \boldsymbol{d}$. We construct a simplex column-wise uncertainty set $\mathcal{H} \subseteq \mathbb{R}_+^{m\times n}$ using $\tilde{\boldsymbol{B}}$ similar to the proof of Theorem 6.1. Let

$$\mathcal{H} = \{[\boldsymbol{b}_1 \cdots \boldsymbol{b}_n] \mid \boldsymbol{b}_j \in H_j, \forall j \in [n]\}$$

where

$$H_j = \mathsf{conv}\left(\{\mathbf{0}\} \bigcup \left\{ \tilde{B}_{ij}\boldsymbol{e}_i \;\middle|\; i = 1, \ldots, m \right\}\right)$$

for all $j \in [n]$. Note that $H_j \subseteq [\mathsf{diag}(\boldsymbol{h})]^{-1} C_j$, which implies that $\mathcal{H} \subseteq [\mathsf{diag}(\boldsymbol{h})]^{-1}\mathcal{U}$. From Theorem 3.1, we know that

$$z^I_{\mathsf{Rob}}(\mathcal{H}) = \min\{\boldsymbol{e}^T\boldsymbol{v} \mid \tilde{\boldsymbol{B}}^T\boldsymbol{v} \geq \boldsymbol{d}, \boldsymbol{v} \geq \mathbf{0}\},$$

and $\boldsymbol{\lambda}$ is a feasible solution for $z_{\mathsf{Rob}}(\mathcal{H})$. Therefore, $z^I_{\mathsf{Rob}}(\mathcal{H}) \leq \boldsymbol{e}^T\boldsymbol{\lambda} = 1$. Furthermore,

$$z^I_{\mathsf{AR}}([\mathsf{diag}(\boldsymbol{h})]^{-1}\mathcal{U}) \leq z^I_{\mathsf{AR}}(\mathcal{H}) \leq O(\alpha) \cdot z^I_{\mathsf{Rob}}(\mathcal{H}) \leq O(\alpha),$$

where the first inequality follows as $\mathcal{H} \subseteq [\mathsf{diag}(\boldsymbol{h})]^{-1}\mathcal{U}$ and the second inequality follows from Theorems 4.1 and 5.2. Therefore, there exists $(\boldsymbol{v}^*, \boldsymbol{B}^*)$ such that

$$(\boldsymbol{B}^*)^T\boldsymbol{v}^* \geq \boldsymbol{d},\ \boldsymbol{B}^* \in [\mathsf{diag}(\boldsymbol{h})]^{-1}\mathcal{U},\ \text{and } \boldsymbol{e}^T\boldsymbol{v}^* \leq O(\alpha).$$

Now, let

$$\boldsymbol{Q} = \mathsf{diag}(\boldsymbol{h})\boldsymbol{B}^* \text{ and } \boldsymbol{\mu} = \frac{1}{\boldsymbol{e}^T\boldsymbol{v}^*}[\mathsf{diag}(\boldsymbol{h})]^{-1}\boldsymbol{v}^*.$$

Then, $\boldsymbol{Q} \in \mathcal{U}$ and $\boldsymbol{h}^T\boldsymbol{\mu} = 1$, which implies that $\boldsymbol{Q}^T\boldsymbol{\mu} \in T(\mathcal{U}, \boldsymbol{h})$. Note that

$$\boldsymbol{Q}^T\boldsymbol{\mu} = \frac{1}{\boldsymbol{e}^T\boldsymbol{v}^*}(\boldsymbol{B}^*)^T\boldsymbol{v}^* \geq \frac{1}{O(\alpha)}\boldsymbol{d}.$$

Since $\mathcal{U}$ is down-monotone, so is $T(\mathcal{U}, \boldsymbol{h})$. Therefore, for $\boldsymbol{d} \in \mathsf{conv}(T(\mathcal{U}, \boldsymbol{h}))$, we have

$$\frac{1}{O(\alpha)}\boldsymbol{d} \in T(\mathcal{U}, \boldsymbol{h}),$$

which implies that $\kappa(T(\mathcal{U}, \boldsymbol{h})) \leq O(\log n \cdot \min(\log \Gamma, \log(m+n)))$. □

8 Adaptivity Gap under Constraint and Objective Uncertainty.

In this section, we show that our result can be generalized to the case where both constraint and objective coefficients are uncertain. In particular, we consider the following two-stage adjustable robust problem $\Pi^{(B,d)}_{\mathsf{AR}}$.

$$\begin{aligned} z^{(B,d)}_{\mathsf{AR}} = \max\ & \boldsymbol{c}^T\boldsymbol{x} + \min_{(\boldsymbol{B},\boldsymbol{d})\in\mathcal{U}}\ \max_{\boldsymbol{y}(\boldsymbol{B},\boldsymbol{d})} \boldsymbol{d}^T\boldsymbol{y}(\boldsymbol{B},\boldsymbol{d}) \\ & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y}(\boldsymbol{B},\boldsymbol{d}) \ \leq\ \boldsymbol{h} \\ & \boldsymbol{x} \ \in\ \mathbb{R}^n_+,\ \boldsymbol{y}(\boldsymbol{B},\boldsymbol{d}) \ \in\ \mathbb{R}^n_+ \end{aligned} \tag{8.1}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{c} \in \mathbb{R}^n_+$, $\boldsymbol{h} \in \mathbb{R}^m_+$, and $(\boldsymbol{B}, \boldsymbol{d})$ are uncertain second-stage constraint matrix and objective that belong to a convex compact uncertainty set $\mathcal{U} \subseteq \mathbb{R}^{m\times n}_+ \times \mathbb{R}^n_+$. We consider the case where the uncertainty in constraint matrix $\boldsymbol{B}$ is column-wise and constraint-wise and does not depend on the uncertainty in objective coefficients $\boldsymbol{d}$. Therefore,

$$\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d,$$

where $\mathcal{U}^B \subseteq \mathbb{R}_+^{m\times n}$ is a convex compact uncertainty set of constraint matrices that is column-wise and constraint-wise, and $\mathcal{U}^d \subseteq \mathbb{R}_+^n$ is a convex compact uncertainty set of the second-stage objective. As previous sections, we can assume without loss of generality that $\mathcal{U}^B$ is down-monotone.

We formulate the corresponding static robust problem $\Pi_{\mathsf{Rob}}^{(B,d)}$ as follows.

$$z_{\mathsf{Rob}}^{(B,d)} = \max_{\boldsymbol{x},\boldsymbol{y}} \min_{\boldsymbol{d}\in\mathcal{U}^d} \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y}$$
$$\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} \le \boldsymbol{h},\ \forall \boldsymbol{B}\in\mathcal{U}^B \tag{8.2}$$
$$\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}_+^n.$$

We prove the following theorem.

Theorem 8.1. *Let $z_{\mathsf{AR}}^{(B,d)}$ be the optimal objective value of $\Pi_{\mathsf{AR}}^{(B,d)}$ in (8.1) defined over the uncertainty $\mathcal{U} = \mathcal{U}^B \times \mathcal{U}^d$, where $\mathcal{U}^B \subseteq \mathbb{R}_+^{m\times n}$ is a convex compact uncertainty set of constraint matrices that is column-wise and constraint-wise, and $\mathcal{U}^d \subseteq \mathbb{R}_+^n$ is a convex compact uncertainty set of the second-stage objective. Let $z_{\mathsf{Rob}}^{(B,d)}$ be the optimal objective value of $\Pi_{\mathsf{Rob}}^{(B,d)}$ in (8.2). Then,*

$$z_{\mathsf{AR}}^{(B,d)} \le O(\log n \cdot \min(\log\Gamma, \log(m+n))) \cdot z_{\mathsf{Rob}}^{(B,d)}.$$

Proof. In Bertsimas et al. [8], the authors prove that

$$z_{\mathsf{AR}}^{(B,d)} \le \max\{\kappa(T(\mathcal{U},\boldsymbol{h})) \mid \boldsymbol{h} > \boldsymbol{0}\} \cdot z_{\mathsf{Rob}}^{(B,d)}.$$

From Theorem 7.1, we have

$$\max\{\kappa(T(\mathcal{U},\boldsymbol{h})) \mid \boldsymbol{h} > \boldsymbol{0}\} \le O(\log n \cdot \min(\log\Gamma, \log(m+n))),$$

which completes the proof. □

9 Conclusion.

In this paper, we study the adaptivity gap in two-stage adjustable robust linear optimization problem under column-wise and constraint-wise uncertainty sets. We show that in this case, the adjustable problem is $\Omega(\log n)$-hard to approximate. In fact, for a more general case where the uncertainty set $\mathcal{U}$ and objective coefficients $\boldsymbol{d}$ are not constrained in the non-negative orthant, we show that the adjustable robust problem is $\Omega(2^{\log^{1-\epsilon} m})$-hard to approximate for any constant $0 < \epsilon < 1$. We show that a static solution is an $O(\log(m+n)\log n)$-approximation for the adjustable robust problem when the uncertainty set is column-wise and constraint-wise, which indicates that static solution provides an near-optimal approximation. Our result confirms the power of static solution. Our analysis provides a uniform upper bound on the measure of non-convexity bound given in Bertsimas et al [8] for column-wise and constraint-wise uncertainty sets. Moreover, our bound can be extended to the case where the objective coefficients are also uncertain and the uncertainty is unrelated to the column-wise and constraint-wise constraint uncertainty set.

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A Proof of Theorem 2.2.

In this section, we show that the general two-stage adjustable robust problem $\Pi_{\mathsf{AR}}^{\mathsf{Gen}}$ (2.1) is $\Omega(2^{\log^{1-\epsilon} m})$-hard to approximate for any constant $0 < \epsilon < 1$. We prove this by an approximation preserving reduction from the Label-Cover-Problem. The reduction is similar in spirit to the reduction from the set cover problem to the two-stage adjustable robust problem.

Label-Cover-Problem: We are given a finite set V ($|V| = m$), a family of subset $\{\mathcal{V}_1, \ldots, \mathcal{V}_K\}$ of V and graph $G = (V, E)$. Let H be a supergraph with vertices $\{\mathcal{V}_1, \ldots, \mathcal{V}_K\}$ and edges F where $(\mathcal{V}_i, \mathcal{V}_j) \in F$ if there exists $(k, l) \in E$ such that $k \in \mathcal{V}_i, l \in \mathcal{V}_j$. The goal is to find the smallest cardinality set $C \subseteq V$ such that F is covered, i.e., for each $(\mathcal{V}_i, \mathcal{V}_j) \in F$, there exists $k \in \mathcal{V}_i \cap C, l \in \mathcal{V}_j \cap C$ such that $(k, l) \in E$.

The label cover problem is $\Omega(2^{\log^{1-\epsilon} m})$-hard to approximate for any constant $0 < \epsilon < 1$, i.e., there is no polynomial time approximation algorithm that give an $O(2^{\log^{1-\epsilon} m})$-approximation for any constant $0 < \epsilon < 1$ unless $\mathbf{NP} \subseteq \mathbf{DTIME}(m^{\text{polylog}(m)})$ [1].

Proof of Theorem 2.2 Consider an instance $\mathcal{I}$ of Label-Cover-Problem with ground elements V ($|V| = m$), graph $G = (V, E)$, a family of subset of V: $(\mathcal{V}_1, \ldots, \mathcal{V}_K)$ and a supergraph $H = (\{\mathcal{V}_1, \ldots, \mathcal{V}_K\}, F)$ where $|F| = n$. We construct the following instance $\mathcal{I}'$ of the general adjustable robust problem $\Pi_{\mathsf{AR}}^{\mathsf{Gen}}$ (2.1):

$$\boldsymbol{A} = \boldsymbol{0},\ \boldsymbol{c} = \boldsymbol{0},\ \boldsymbol{d} = \begin{pmatrix} \boldsymbol{e} \\ -\boldsymbol{e} \end{pmatrix} \in \mathbb{R}^{n+m},\ \boldsymbol{h} = \boldsymbol{e} \in \mathbb{R}^m,\ \mathcal{U} = \{[\boldsymbol{B}\ \ -\boldsymbol{I}_m] \mid \boldsymbol{B} \in \mathcal{U}_F\}$$

where $d_1 = d_2 = \ldots = d_n = 1$, $\boldsymbol{I}_m$ is the m-dimensional identity matrix and each column set of $\mathcal{U}_F \subseteq \mathbb{R}_+^{m \times n}$ corresponds to an edge $(\mathcal{V}_i, \mathcal{V}_j) \in F$ with

$$\mathcal{U}_{(\mathcal{V}_i,\mathcal{V}_j)} = \mathsf{conv}\left(\{\boldsymbol{0}\} \bigcup \left\{ \frac{1}{2}(\boldsymbol{e}_k + \boldsymbol{e}_l) \ \middle|\ (k, l) \in E, k \in \mathcal{V}_i, l \in \mathcal{V}_j \right\}\right) \subseteq \mathbb{R}_+^m.$$

Therefore, $\mathcal{U}$ is column-wise with column sets $\mathcal{U}_{(\mathcal{V}_i,\mathcal{V}_j)}, \forall (\mathcal{V}_i, \mathcal{V}_j) \in F$ and $\mathcal{U}_j, j \in [m]$ where $\mathcal{U}_j = \{-\boldsymbol{e}_j\}$, i.e., there is no uncertainty in $\mathcal{U}_j$. The instance $\mathcal{I}'$ of $\Pi_{\mathsf{AR}}^{\mathsf{Gen}}$ can be formulated as

$$\begin{aligned} z_{\mathsf{AR}}^{\mathsf{Gen}} &= \min_{\boldsymbol{B} \in \mathcal{U}_F} \max_{\boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{z} \geq \boldsymbol{0}} \{\boldsymbol{e}^T \boldsymbol{y} - \boldsymbol{e}^T \boldsymbol{z} \mid \boldsymbol{B}\boldsymbol{y} - \boldsymbol{z} \leq \boldsymbol{e}, \boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{z} \geq \boldsymbol{0}\} \\ &= \min_{\boldsymbol{b}_{(\mathcal{V}_i,\mathcal{V}_j)} \in \mathcal{U}_{(\mathcal{V}_i,\mathcal{V}_j)}} \max_{\boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{z} \geq \boldsymbol{0}} \left\{ \boldsymbol{e}^T \boldsymbol{y} - \boldsymbol{e}^T \boldsymbol{z} \ \middle|\ \sum_{(\mathcal{V}_i,\mathcal{V}_j) \in F} y_{(\mathcal{V}_i,\mathcal{V}_j)} \boldsymbol{b}_{(\mathcal{V}_i,\mathcal{V}_j)} - \boldsymbol{z} \leq \boldsymbol{e}, \boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{z} \geq \boldsymbol{0} \right\}. \end{aligned}$$

Suppose $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}, \hat{\boldsymbol{b}}_{(\mathcal{V}_i,\mathcal{V}_j)}, (\mathcal{V}_i, \mathcal{V}_j) \in F)$ is a feasible solution for instance $\mathcal{I}'$. Then, we can compute a label cover of instance $\mathcal{I}$ with cardinality at most $\boldsymbol{e}^T \hat{\boldsymbol{y}} - \boldsymbol{e}^T \hat{\boldsymbol{z}}$. From strong duality, there exists an optimal solution $\hat{\boldsymbol{\mu}}$ for

$$\min\{\boldsymbol{e}^T\boldsymbol{\mu} \mid \hat{\boldsymbol{b}}^T_{(\mathcal{V}_i,\mathcal{V}_j)}\boldsymbol{\mu} \geq 1, \forall(\mathcal{V}_i,\mathcal{V}_j) \in F, \mu \in [0,1]^m\}$$

and $\boldsymbol{e}^T\hat{\boldsymbol{\mu}} = \boldsymbol{e}^T\hat{\boldsymbol{y}} - \boldsymbol{e}^T\hat{\boldsymbol{z}}$. For each $(\mathcal{V}_i,\mathcal{V}_j) \in F$, consider a basic optimal solution $(\tilde{\boldsymbol{b}}_{(\mathcal{V}_i,\mathcal{V}_j)}, (\mathcal{V}_i,\mathcal{V}_j) \in F)$ where

$$\tilde{\boldsymbol{b}}_{(\mathcal{V}_i,\mathcal{V}_j)} \in \arg\max\{\boldsymbol{b}^T\hat{\boldsymbol{\mu}} \mid \boldsymbol{b} \in \mathcal{U}_{(\mathcal{V}_i,\mathcal{V}_j)}\}.$$

Therefore, $\tilde{\boldsymbol{b}}_{(\mathcal{V}_i,\mathcal{V}_j)}$ is a vertex of $\mathcal{U}_{(\mathcal{V}_i,\mathcal{V}_j)}$ for each $(\mathcal{V}_i,\mathcal{V}_j) \in F$, which implies that $\tilde{\boldsymbol{b}}_{(\mathcal{V}_i,\mathcal{V}_j)} = \frac{1}{2}(\boldsymbol{e}_{k_i} + \boldsymbol{e}_{l_j})$ for some $(k_i,l_j) \in E$ and $k_i \in \mathcal{V}_i, l_j \in \mathcal{V}_j$. Also, $\tilde{\boldsymbol{b}}^T_{(\mathcal{V}_i,\mathcal{V}_j)}\hat{\boldsymbol{\mu}} \geq 1, \forall(\mathcal{V}_i,\mathcal{V}_j) \in F$. Now, let $\tilde{\boldsymbol{\mu}}$ the optimal solution of the following LP:

$$\min\{\boldsymbol{e}^T\boldsymbol{\mu} \mid \tilde{\boldsymbol{b}}^T_{(\mathcal{V}_i,\mathcal{V}_j)}\boldsymbol{\mu} \geq 1, \forall(\mathcal{V}_i,\mathcal{V}_j) \in F, \boldsymbol{0} \leq \boldsymbol{\mu} \leq \boldsymbol{e}\}.$$

Clearly, $\boldsymbol{e}^T\tilde{\boldsymbol{\mu}} \leq \boldsymbol{e}^T\hat{\boldsymbol{\mu}}$. Also, since $\tilde{\boldsymbol{b}}_{(\mathcal{V}_i,\mathcal{V}_j)} = \frac{1}{2}(\boldsymbol{e}_{k_i} + \boldsymbol{e}_{l_j})$ and $\tilde{\boldsymbol{b}}^T_{(\mathcal{V}_i,\mathcal{V}_j)}\tilde{\boldsymbol{\mu}} \geq 1$, $\tilde{\mu}_{k_i} = \tilde{\mu}_{l_j} = 1$. Therefore, $\tilde{\boldsymbol{\mu}} \in \{0,1\}^m$. Let

$$C = \{j \mid \tilde{\mu}_j = 1\}.$$

Clearly, C is a valid label cover for F and $|C| = \boldsymbol{e}^T\tilde{\boldsymbol{\mu}} \leq \boldsymbol{e}^T\hat{\boldsymbol{\mu}} = \boldsymbol{e}^T\hat{\boldsymbol{y}} - \boldsymbol{e}^T\hat{\boldsymbol{z}}$.

Conversely, given a label cover C of instance $\mathcal{I}$, for any $j \in [m]$, let $\bar{\mu}_j = 1$ if $j \in C$ and zero otherwise. This implies that $\boldsymbol{e}^T\bar{\boldsymbol{\mu}} = |C|$. For any $(\mathcal{V}_i,\mathcal{V}_j) \in F$, let $\bar{\boldsymbol{b}}_{(\mathcal{V}_i,\mathcal{V}_j)} = \frac{1}{2}(\boldsymbol{e}_{k_i} + \boldsymbol{e}_{l_j})$ where $k_i \in \mathcal{V}_i \cap C, l_j \in \mathcal{V}_j \cap C$ such that $(k_i,l_j) \in E$. Then, let $\boldsymbol{\mu}'$ be an optimal solution for the following LP

$$\min\{\boldsymbol{e}^T\boldsymbol{\mu} \mid \bar{\boldsymbol{b}}^T_{(\mathcal{V}_i,\mathcal{V}_j)}\boldsymbol{\mu} \geq 1, \forall(\mathcal{V}_i,\mathcal{V}_j) \in F, \boldsymbol{0} \leq \boldsymbol{\mu} \leq \boldsymbol{e}\}.$$

Then, $\boldsymbol{e}^T\boldsymbol{\mu}' \leq \boldsymbol{e}^T\bar{\boldsymbol{\mu}}$ as $\bar{\boldsymbol{\mu}}$ is feasible for the above LP. From strong duality, there exists $\bar{\boldsymbol{y}} \in \mathbb{R}^n_+$ and $\bar{\boldsymbol{z}} \in \mathbb{R}^m_+$ such that $(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{b}}_{(\mathcal{V}_i,\mathcal{V}_j)}, (\mathcal{V}_i,\mathcal{V}_j) \in F)$ is a feasible solution for instance $\mathcal{I}'$ of $\Pi^{\mathsf{Gen}}_{\mathsf{AR}}$ with cost $\boldsymbol{e}^T\bar{\boldsymbol{y}} - \boldsymbol{e}^T\bar{\boldsymbol{z}} = \boldsymbol{e}^T\boldsymbol{\mu}' \leq \boldsymbol{e}^T\bar{\boldsymbol{\mu}} = |C|$.

B Approximate Separation to Optimization.

For any $\boldsymbol{x} \in \mathbb{R}^n_+$, let

$$Q^*(\boldsymbol{x}) = \min_{\boldsymbol{B}\in\mathcal{U}} \max_{\boldsymbol{y}\geq\boldsymbol{0}}\{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}\}.$$

We show that if we can approximate the separation problem, we can also approximate Π_{AR}. Let $\mathcal{A}$ be a γ-approximate algorithm for the separation problem (3.1), i.e., $\mathcal{A}$ computes a γ-approximation algorithm for the min-max problem in (3.1). For any $\boldsymbol{x} \in \mathbb{R}^n_+$, let $\boldsymbol{B}^{\mathcal{A}}(\boldsymbol{x})$ denote the matrix returned by $\mathcal{A}$ and let

$$Q^{\mathcal{A}}(\boldsymbol{x}) = \max_{\boldsymbol{y}\geq\boldsymbol{0}}\{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}^{\mathcal{A}}(\boldsymbol{x})\boldsymbol{y} \leq \boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}\}.$$

Therefore, the approximate separation based on Algorihm $\mathcal{A}$ is as follows: for any $(\boldsymbol{x}, z)$, return feasible if $Q^{\mathcal{A}}(\boldsymbol{x}) \geq z$. Otherwise give a violating hyperplane corresponding to $\boldsymbol{B}^{\mathcal{A}}(\boldsymbol{x})$. Now, we prove the following theorem.

Theorem B.1. *Suppose we have an Algorithm $\mathcal{A}$ that is a γ-approximation for the separation problem* (3.1)*. Then we can compute a γ-approximation for the two-stage adjustable robust problem* Π_{AR} (1.1)*.*

Proof. Since $\mathcal{A}$ is a γ-approximation to the min-max problem in (3.1), for any $\boldsymbol{x} \in \mathbb{R}^n_+$,

$$Q^*(\boldsymbol{x}) \leq Q^{\mathcal{A}}(\boldsymbol{x}) \leq \gamma \cdot Q^*(\boldsymbol{x}).$$

Let $(\boldsymbol{x}^*, z^*)$ be an optimal solution for Π_{AR} and let

$$\mathsf{OPT} = \boldsymbol{c}^T \boldsymbol{x}^* + z^*.$$

Consider the optimization algorithm based on the approximate separation algorithm $\mathcal{A}$ and suppose it returns the solution $(\hat{\boldsymbol{x}}, \hat{z})$. Note that $(\boldsymbol{x}^*, z^*)$ is feasible according to the approximate separation algorithm $\mathcal{A}$ as $Q^{\mathcal{A}}(\boldsymbol{x}^*) \geq Q^*(\boldsymbol{x}^*) = z^*$. Therefore,

$$\boldsymbol{c}^T \hat{\boldsymbol{x}} + \hat{z} \geq \boldsymbol{c}^T \boldsymbol{x}^* + z^*. \tag{B.1}$$

Note that $\hat{z}$ is an approximation for the worst case second-stage objective value when the first stage solution is $\hat{\boldsymbol{x}}$. The true objective value for the first stage solution $\hat{\boldsymbol{x}}$ is given by

$$\begin{aligned} \boldsymbol{c}^T \hat{\boldsymbol{x}} + Q^*(\hat{\boldsymbol{x}}) &\geq \boldsymbol{c}^T \hat{\boldsymbol{x}} + \frac{1}{\gamma} Q^{\mathcal{A}}(\hat{\boldsymbol{x}}) \\ &\geq \boldsymbol{c}^T \hat{\boldsymbol{x}} + \frac{1}{\gamma} \hat{z} \qquad \text{(B.2)} \\ &\geq \frac{1}{\gamma}(\boldsymbol{c}^T \hat{\boldsymbol{x}} + \hat{z}) \\ &\geq \frac{1}{\gamma} \mathsf{OPT}, \end{aligned}$$

where the first inequality follows as $\mathcal{A}$ is a γ-approximation and $Q^{\mathcal{A}}(\hat{\boldsymbol{x}}) \leq \gamma \cdot Q^*(\hat{\boldsymbol{x}})$. Inequality (B.2) follows as $(\hat{\boldsymbol{x}}, \hat{z})$ is feasible according to $\mathcal{A}$ and therefore, $\hat{z} \leq Q^{\mathcal{A}}(\hat{\boldsymbol{x}})$ and the last inequality follows from (B.1). Therefore, the optimization problem based on algorithm $\mathcal{A}$ computes a γ-approximation for Π_{AR}. □

C Transformation of the Adjustable Robust Problem.

Let $\boldsymbol{x}^*$ be the optimal first-stage solution for Π_{AR}, i.e.,

$$z_{\mathsf{AR}} = \boldsymbol{c}^T \boldsymbol{x}^* + \min_{\boldsymbol{B}} \max_{\boldsymbol{y}} \{\boldsymbol{d}^T \boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}^*, \boldsymbol{B} \in \mathcal{U}, \boldsymbol{y} \geq \boldsymbol{0}\}.$$

Note that $(\boldsymbol{x}^*, \boldsymbol{0})$ is a feasible solution for Π_{Rob}. We have

$$z_{\mathsf{Rob}} \geq \boldsymbol{c}^T \boldsymbol{x}^* + \max_{\boldsymbol{y} \geq \boldsymbol{0}} \{\boldsymbol{d}^T \boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}^*, \forall \boldsymbol{B} \in \mathcal{U}\}.$$

Since $\boldsymbol{c}$ and $\boldsymbol{x}^*$ are both non-negative, to prove Theorem 5.1, it suffice to show

$$\min_{\boldsymbol{B} \in \mathcal{U}} \max_{\boldsymbol{y} \geq \boldsymbol{0}} \{\boldsymbol{d}^T \boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}^*\} \leq O(\log(m+n)\log n) \cdot \max_{\boldsymbol{y} \geq \boldsymbol{0}} \{\boldsymbol{d}^T \boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}^*, \forall \boldsymbol{B} \in \mathcal{U}\}.$$

In this section, we show that we can assume without loss of generality that $(\boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}^*) > \boldsymbol{0}$, as otherwise the static solution is optimal for the two-stage adjustable robust problem Π_{AR} (1.1), i.e., $z_{\mathsf{AR}} = z_{\mathsf{Rob}}$: Note that $(\boldsymbol{h} - \boldsymbol{A}\boldsymbol{x}^*) \geq \boldsymbol{0}$, since otherwise the inner problem becomes infeasible. Now, suppose that $(\boldsymbol{h}-\boldsymbol{A}\boldsymbol{x})_i = 0$ for some $i \in [m]$. Since $\mathcal{U}$ is a full-dimensional convex set, there exist $\boldsymbol{B}^* \in \mathcal{U}$ such that $B^*_{ij} > 0$ for all $j \in [n]$. Therefore,

$$\min_{\boldsymbol{B}\in\mathcal{U}} \max_{\boldsymbol{y}\geq\boldsymbol{0}}\{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{h}-\boldsymbol{A}\boldsymbol{x}\} \leq \max_{\boldsymbol{y}\geq\boldsymbol{0}}\{\boldsymbol{d}^T\boldsymbol{y} \mid \boldsymbol{B}^*\boldsymbol{y} \leq \boldsymbol{h}-\boldsymbol{A}\boldsymbol{x}\} = 0,$$

which implies that $z_{\mathsf{AR}} = \boldsymbol{c}^T\boldsymbol{x}^*$ since $\boldsymbol{d}, \boldsymbol{y}$ are non-negative. On the other hand, $(\boldsymbol{x}^*, \boldsymbol{0})$ is a feasible solution for Π_{Rob}. Therefore,

$$z_{\mathsf{Rob}} \geq \boldsymbol{c}^T\boldsymbol{x}^* = z_{\mathsf{AR}}.$$

However, suppose $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is an optimal solution for Π_{Rob}, then $\boldsymbol{x} = \bar{\boldsymbol{x}}, \boldsymbol{y}(\boldsymbol{B}) = \bar{\boldsymbol{y}}$ for all $\boldsymbol{B} \in \mathcal{U}$ is feasible for Π_{AR}. Therefore, $z_{\mathsf{AR}} \geq z_{\mathsf{Rob}}$.

D Proof of Theorem 3.1

Let $\boldsymbol{y}^*$ be such that $\hat{\boldsymbol{B}}\boldsymbol{y}^* \leq \boldsymbol{h}$. For any $\boldsymbol{B} \in \mathcal{U}$, we have $\boldsymbol{B} \leq \hat{\boldsymbol{B}}$ component-wise by construction. Note that $\boldsymbol{y}^* \geq \boldsymbol{0}$, this implies $\boldsymbol{B}\boldsymbol{y}^* \leq \hat{\boldsymbol{B}}\boldsymbol{y}^* \leq \boldsymbol{h}$ for all $\boldsymbol{B} \in \mathcal{U}$.

Conversely, suppose $\tilde{\boldsymbol{y}}$ satisfies $\boldsymbol{B}\tilde{\boldsymbol{y}} \leq \boldsymbol{h}$ for all $\boldsymbol{B} \in \mathcal{U}$. For each $i \in [m]$, note that $\mathsf{diag}(\boldsymbol{e}_i)\hat{\boldsymbol{B}} \in \mathcal{U}$ by construction. Therefore, $\boldsymbol{e}_i^T\hat{\boldsymbol{B}}\tilde{\boldsymbol{y}} \leq h_i$ for all $i \in [m]$, which implies that $\hat{\boldsymbol{B}}\tilde{\boldsymbol{y}} \leq \boldsymbol{h}$.

E Proof of Lemma 3.3.

Let

$$\hat{B}_{ij} = \frac{1}{(n+i-j+1) \bmod m}.$$

From Theorem 3.1, Π_{Rob} is equivalent to

$$z_{\mathsf{Rob}} = \max\{\boldsymbol{e}^T\boldsymbol{y} \mid \hat{\boldsymbol{B}}\boldsymbol{y} \leq \boldsymbol{e}, \boldsymbol{y} \geq \boldsymbol{0}\}.$$

The dual problem is

$$z_{\mathsf{Rob}} = \min\{\boldsymbol{e}^T\boldsymbol{z} \mid \hat{\boldsymbol{B}}^T\boldsymbol{z} \geq \boldsymbol{e}, \boldsymbol{z} \geq \boldsymbol{0}\}.$$

Let

$$s = \sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n).$$

It is easy to observe that $\frac{1}{s}\boldsymbol{e}$ is a feasible solution for both the primal and the dual formulations of z_{Rob}. Moreover, they have the same objective value. Therefore,

$$z_{\mathsf{Rob}} = \frac{n}{s}.$$

On the other hand, for each $j \in [n]$, denote

$$\mathcal{U}_j = \left\{ \boldsymbol{b} \in \mathbb{R}^n_+ \;\middle|\; \sum_{i=1}^{n} [(n+i-j+1) \bmod n] \cdot b_i \leq 1 \right\}.$$

By writing the dual of the inner maximization problem of Π_{AR}, we have

$$\begin{aligned} z_{\mathsf{AR}} &= \min\{\boldsymbol{e}^T\boldsymbol{\alpha} \mid \boldsymbol{B}^T\boldsymbol{\alpha} \geq \boldsymbol{e}, \boldsymbol{\alpha} \geq \boldsymbol{0}, \boldsymbol{B} \in \mathcal{U}\} \\ &= \min\{\lambda \mid \lambda\boldsymbol{B}^T\boldsymbol{\mu} \geq \boldsymbol{e}, \boldsymbol{e}^T\boldsymbol{\mu} = 1, \boldsymbol{\mu} \geq \boldsymbol{0}, \boldsymbol{B} \in \mathcal{U}\} \\ &= \min\left\{ \frac{1}{\theta} \;\middle|\; \boldsymbol{b}_j^T\boldsymbol{\mu} \geq \theta, \boldsymbol{b}_j \in \mathcal{U}_j, \boldsymbol{e}^T\boldsymbol{\mu} = 1, \boldsymbol{\mu} \geq \boldsymbol{0} \right\}. \end{aligned}$$

Therefore, we just need to solve

$$\frac{1}{z_{\mathsf{AR}}} = \max\{\theta \mid \boldsymbol{b}_j^T \boldsymbol{\mu} \geq \theta, \boldsymbol{b}_j \in \mathcal{U}_j, \boldsymbol{e}^T \boldsymbol{\mu} = 1, \boldsymbol{\mu} \geq \boldsymbol{0}\} \tag{E.1}$$

Suppose $(\hat{\theta}, \hat{\boldsymbol{\mu}}, \hat{\boldsymbol{b}}_j, j \in [m])$ is an optimal solution for (E.1). For each $j \in [n]$, consider a basic optimal solution $\tilde{\boldsymbol{b}}_j$ of the following LP:

$$\tilde{\boldsymbol{b}}_j \in \arg\max\{\boldsymbol{b}^T \hat{\boldsymbol{\mu}} \mid \boldsymbol{b} \in \mathcal{U}_j\}.$$

Therefore, $\tilde{\boldsymbol{b}}_j$ is a vertex of $\mathcal{U}_j$, which implies that $\tilde{\boldsymbol{b}}_j = \hat{B}_{i_j j} \boldsymbol{e}_{i_j}$ for some $i_j \in [n]$ and $\tilde{\boldsymbol{b}}_j^T \hat{\boldsymbol{\mu}} \geq \hat{\theta}$. For each $i \in [n]$, let $\mathcal{S}_i = \{j \mid i_j = i\}$. We have $\sum_{i=1}^n |\mathcal{S}_i| = n$. For each $i \in [n]$ such that $\mathcal{S}_i \neq \emptyset$, $\hat{B}_{ij}$ can only take values in $\{1, 1/2, \ldots, 1/n\}$ for $j \in \mathcal{S}_i$. Moreover, $\hat{B}_{ij} \neq \hat{B}_{ik}$ for $j \neq k$. Therefore, there exists $l_i \in \mathcal{S}_i$ such that

$$\hat{B}_{il_i} \leq \frac{1}{|\mathcal{S}_i|}, \text{ and } \tilde{\boldsymbol{b}}_{l_i}^T \hat{\boldsymbol{\mu}} = \hat{B}_{il_i} \hat{\mu}_i \geq \hat{\theta}.$$

We have

$$1 = \sum_{i:\mathcal{S}_i \neq \emptyset} \hat{\mu}_i \geq \sum_{i:\mathcal{S}_i \neq \emptyset} \frac{\hat{\theta}}{\hat{B}_{il_i}} \geq \sum_{i:\mathcal{S}_i \neq \emptyset} \hat{\theta} |\mathcal{S}_i| = \hat{\theta} n.$$

Therefore, $\hat{\theta} \leq \frac{1}{n}$, which implies that $z_{\mathsf{AR}} \geq n$.

On the other hand, it is easy to observe that $z_{\mathsf{AR}} \leq n$: $\boldsymbol{b}_j = \boldsymbol{e}_j$, $\boldsymbol{\mu} = 1/n \cdot \boldsymbol{e}$ and $\theta = 1/n$ is a feasible solution for (E.1). Therefore,

$$z_{\mathsf{AR}} = n = \sum_{i=1}^n \frac{1}{i} \cdot z_{\mathsf{Rob}} = \Theta(\log n) \cdot z_{\mathsf{Rob}},$$

which completes the proof.