Variational principles with generalized distances and applications to behavioral sciences

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Abstract. This paper has a two-fold focus on proving that the quasimetric and the weak τ-distance versions of the Ekeland variational principle are equivalent in the sense that one implies the other and on presenting the need of such extensions for possible applications in the formation and break of workers hiring and firing routines.

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1 Introduction

The Ekeland variational principle (EVP) is one of the most important results in nonlinear analysis; it allows us to study minimization problems in which the lower level set of a problem is not compact and thus the Bolzano-Weierstrass theorem might not be applied. Roughly speaking, starting at an approximate solution of a lower semicontinuous function, we can find a better approximate solution of the given function which is the unique solution of the perturbed function defined as a sum of the given function and a 'weighted' distance.

By now, a large number of equivalent versions and extensions of the EVP are known in the literature. On one hand, it has been proved to be equivalent to the Caristi fixed point theorem (CFPT) [9], the Takahashi minimal point theorem [47], the nonempty intersection theorem [32], the drop theorem [10], the petal theorem [34], the equilibrium theorem [32], the Krasnoselski-Zabrejko theorem on solvability of operator equations [51], Phelps' lemma [35], etc. On the other hand, the EVP has been extended in many directions in order to fit a new setting or a particular application such as vectorial and set-valued cost functions, quasimetrics, w-distances, τ-distances, etc.

In [20], Kada, Suzuki and Takahashi introduced a generalized distance, namely, w-distance, and sucessfully established extensions of both the CFPT and the EVP to w-distances which play the role of the metric in the original results. Several year later, Suzuki [44] generalized both w-distance and Tataru's distance [46] to τ-distance and proved that the Banach contraction principle holds with any τ-distance. Then, Lin and Du [27] established a version of the EVP for decreasingly-closed (known also as lower semicontinuous from above) functionals and τ-functions. Khanh and Quy [22, 23] further extended the EVP from τ-functions to a weak form of them.

To the best of our knowledge, a quasimetric version of the EVP was presented in [5, Corollary 3.3] for the class of quasimetric spaces in which the topology induced by a quasimetric is Hausdorff. It

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is worth mentioning that the Hausdorff requirement was missed in the formulations and proofs of quasimetric extensions of the EVP in [49] and some other publications.

In [2], Bao and Khanh proved that many generalizations and extensions of the EVP including Zhong's generalized version in [52] are equivalent to the original EVP. In the last decade, many new extensions of the EVP with w-distances, τ-distances, τ-functions, and weak τ-functions were added into the literature. Inspired by the title of the mentioned paper, "Are several recent generalizations of Ekeland's variational principle more general than the original principle?", we study the question: are newer versions of the EVP equivalent to the original EVP as well? This present paper provides a suprising yes answer.

This paper has a two-fold focus—on proving that the quasi-distance and the weak τ-distance versions of the EVP are equivalent in the sense that one implies the other and on presenting the need of such extensions for possible applications in the formation and break of worker's hiring and firing routines. The rest of the paper is organized as follows. Section 2 sets preliminaries on quasimetrics, w-distances, τ-functions, τ-distances and weak τ-distances. Section 3 gives behavioral motivations for generalized versions of the EVP. Section 4 presents two generalized versions of the EVP corresponding to quasimetrics and weak τ-distances. We also show that they are equivalent in the sense that one implies the other. Section 5 considers generalized versions of the CFPT. Section 6 focuses the attention on applications in Behavioral Sciences. The conclusion follows, giving possible extensions.

2 Preliminaries

Let us recall the definition of quasimetric spaces and the corresponding notions of convergence, closedness, limit, completeness, and Hausdorff separation in such spaces; cf. [4, 5, 6, 49].

Definition 2.1. (quasimetrics and metrics). *A functional $q : X \times X \to \mathbb{R}_+ := [0, +\infty)$ is said to be a* QUASIMETRIC *iff it satisfies*

(q1) $q(x, y) \geq 0$ *(nonnegativity);*

(q2) *if $x = y$, then $q(x, y) = 0$ (equality implies indistancy);*

(q3) $q(x, z) \leq q(x, y) + q(y, z)$ *(triangularity).*

If the quasimetric q enjoys the symmetry condition,

(q4) $q(x, y) = q(y, x)$ *(symmetry),*

then it is a METRIC.

Note that the validity of (q4) ensures the implication "indistancy implies equality". In other words, (q2) becomes that $q(x, y) = 0 \iff x = y$. We prefer to use d for a metric.

A quasimetric space X with a quasi-metric q will be denoted by (X, q) and a metric space X with a metric d will be denoted by (X, d). A quasimetric on the real numbers can be defined by setting

$$q(x, y) = x - y \text{ if } x \geq y, \text{ and } q(x, y) = 1 \text{ otherwise.} \tag{2.1}$$

The topological space underlying this quasimetric space is the *Sorgenfrey* line. This space describes the process of filing down a metal stick: it is easy to reduce its size, but it is difficult or impossible to grow it.

Below are some important concepts for quasimetrics which reduce to the known ones for metrics.

Definition 2.2. (left-sequential convergence). *For a quasimetric space (X, q), a sequence $\{x_n\}$ in X is said to be* LEFT-SEQUENTIALLY CONVERGENT *to a point $x_* \in X$, denoted by $x_n \to x_*$, iff the quasidistances $q(x_n, x_*)$ tend to zero as $n \to \infty$, i.e. $\lim_{n\to\infty} q(x_n, x_*) = 0$.*

Definition 2.3. (left-sequential closedness). *For a quasimetric space (X, q), a subset $\Omega \subset X$ is said to be* LEFT-SEQUENTIALLY CLOSED *iff for any sequence $\{x_n\} \subset X$ converging to $x_* \in X$, the limit x_* belongs to Ω.*

Definition 2.4. (left-sequential completeness). *For a quasimetric space (X, q), a sequence $\{x_n\} \subset X$ is said to be* LEFT-SEQUENTIAL CAUCHY *iff for each $k \in \mathbb{N}$ there exists $N_k \in \mathbb{N}$ such that*

$$q(x_n, x_m) < 1/k \ \text{ for all } \ m \geq n \geq N_k.$$

A quasimetric space is said to be LEFT-SEQUENTIALLY COMPLETE *iff each left-sequential Cauchy sequence is convergent.*

Definition 2.5. (left-sequentially Hausdorff separation). *X is said to be* HAUSDORFF *iff every left-sequentially convergent sequence has the unique limit, i.e.*

$$\text{if } \lim_{n\to\infty} q(x_n, x_*) = 0 \wedge \lim_{n\to\infty} q(x_n, y_*) = 0, \ \text{ then } \ x_* = y_*.$$

A quasimetric space which is not Hausdorff can be found in [5, page 3] or [28, Example 3.16]. For the sake of simplicity, we will not mention 'left-sequential' in the terminologies defined above.

Next, let us recall the definitions of several generalized distances used in the paper. w-distance was introduced by Kada et al. [20] and was used also by Park [33] and Lin and Du [27]. Then, another generalized distance was proposed by Tataru [46]. Next, τ-distance was defined by Suzuki [44], which is more general than both above-mentioned distances. Next, τ-function was presented by Lin and Du [27]. Finally, Khanh and Quy [23] considered a weak form of τ-function.

Definition 2.6. (w-distances [20]). *Let (X, d) be a metric space. Then, a function $p : X \times X \to \mathbb{R}_+$ is called a* w-DISTANCE *on X iff it satisfies*

(w1) $p(x, z) \leq p(x, y) + p(y, z)$ *(triangularity);*

(w2) *p is lower semicontinuous in its second variable (lower semicontinuity);*

(w3) $\forall \varepsilon > 0, \exists\, \delta > 0 : \ p(z, x) \leq \delta \ \wedge \ p(z, y) \leq \delta \implies d(x, y) \leq \varepsilon.$

Definition 2.7. (τ-distances [44]). *Let (X, d) be a metric space. A function $p : X \times X \to \mathbb{R}_+$ is called a* τ-DISTANCE *on X iff there is a function $\eta : X \times \mathbb{R}_+ \to \mathbb{R}_+$ such that the following conditions are, for $x, y, z \in X$ and $t \in \mathbb{R}_+$, satisfied*

($\tau 1'$) $p(x, z) \leq p(x, y) + p(y, z)$ *(triangle inequality);*

($\tau 2'$) *if $x_n \overset{d}{\to} x$ and $\lim_{n\to\infty} \sup\{\eta(z_n, p(z_n, x_m))|\ m \geq n\} = 0$ for some $\{z_n\} \subset X$, then $p(w, x) \leq \liminf_{n\to\infty} p(w, x_n)$ for all $w \in X$ (weak lower semicontinuity);*

($\tau 3'$) *if $\lim_{n\to\infty} \eta(x_n, z_n) = 0$ and $\lim_{n\to\infty} \sup\{p(x_n, y_m)|\ m \geq n\} = 0$, then $\lim_{n\to\infty} \eta(y_n, z_n) = 0$ (the uniqueness of (η, p)-convergence);*

($\tau 4'$) *$\lim_{n\to\infty} \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_{n\to\infty} \eta(z_n, p(z_n, y_n)) = 0$ imply that $\lim_{n\to\infty} d(x_n, y_n) = 0$ (the uniqueness of η-convergence);*

($\tau 5'$) *$\eta(x, 0) = 0$, $\eta(x, t) \geq t$ and $\eta(x,)$ is concave.*

Definition 2.8. (τ-functions [27]). *Let (X, d) be a metric space. A function $p : X \times X \to \mathbb{R}_+$ is called a* τ-FUNCTION *iff the following four conditions hold*

(τ1) *$p(x, z) \le p(x, y) + p(y, z)$ (triangle inequality);*

(τ2) *for all $x \in X$, $p(x, \cdot)$ is lower semicontinuous (lower semicontinuity);*

(τ3) *for all $\{x_n\}$, $\{y_n\}$ with $\lim_{n\to\infty} p(x_n, y_n) = 0$ and $\lim_{n\to\infty} \{\sup p(x_n, x_m) : m > n\} = 0$, one has $\lim_{n\to\infty} d(x_n, y_n) = 0$ (p-convergence implies d-convergence);*

(τ4) *$p(x, y) = 0$ and $p(x, z) = 0$ imply that $y = z$ (indistancy implies coincidence).*

Definition 2.9. (weak τ-functions [22, 23]). *Let (X, d) be a metric space. A function $p : X \times X \to \mathbb{R}_+$ is called a* WEAK τ-FUNCTION *iff it satisfies conditions* (τ1), (τ3), *and* (τ4).

It is known from [44, 27] that both τ-functions and τ-distances are w-distances, but the former two notions are incomparable.

3 Motivations for Generalized Versions of the EVP

In [41, 42], Soubeyran proposed a Variational Rationality (VR) approach which modelizes and unifies a huge list of stay/stability and change dynamical systems in Behavioral Sciences in different contexts in many disciplines; e.g. Psychology, Economics, Management Sciences, Decision Theory, Philosophy, Game Theory, Political Sciences, Artificial Intelligence, etc. He has shown how the original EVP, Theorem 4.1, and the use of quasidistances as costs to be able to change can be seen as a prototype which formalizes, in a crude but nice way, such dynamics as succession of worthwhile stays and changes which balance, each step, motivation- and resistance-to-change to finally end in some variational trap. Recently, Bao, Mordukhovich and Soubeyran further considered how set-valued versions of the EVP can be applied to the functioning of goal systems in psychology [4] and the capability theory of wellbeing [5]. In this section, we show how the introduction of w-distances in the EVP expands the range of applications in Behavioral Sciences; in particular, the formation and break of hiring and firing workers routines in Management Sciences.

The literature on routine's formation and break is enormous and represents a very important area of research. Our behavioral application considers a well known, more specific and concrete example, the case of hiring, firing and repeated employment routines in [15]. This example modelizes a very simple case of knowledge management within an organization.

A simple model. Consider a hierarchical firm where, each period, an entrepreneur (leader), can hire, keep again and fire numbers of employed workers in l different kinds of skilled and specialized works $x = (x^1, x^2, \ldots, x^l) \in X = \mathbb{R}^l_+$, where $x^j \ge 0$ is the number of employed workers of type $j \in J := \{1, 2, \ldots, l\}$, to produce a chosen quantity $Q(x)$ of a final good of a chosen quality $s(x)$; the endogenous quality $s(x)$ of this final good changes with the chosen profile of skilled workers $x \ge 0$.

The revenue of the entrepreneur is $\varphi[Q(x), s(x)]$. His operational cost $\rho(x)$ is the sum of costs to buy the non durable means used by each worker, and the wages paid to each employee worker. Then, if, in the past and current periods, the entrepeneur utilizes the profiles x and y of skilled workers, his past and current profits are $g(x) = \varphi\left[Q(x), s(x)\right] - \rho(x) \in R$ and $g(y) = \varphi\left[Q(y), s(y)\right] - \rho(y)$.

Let $\overline{g} = \sup\left\{g(y), y \in X\right\} < +\infty$ be the maximum profit the entrepreneur can expect. Then, given the choice of his past and current profiles of workers x and y, $f(x) = \overline{g} - g(x) \geq 0$ and $f(y) = \overline{g} - g(y) \geq 0$ represent his past and current unsatisfactions to do not have succeeded to reach his potential maximum profit $\overline{g} < +\infty$.

To be more precise, let us recall the Cobb Douglas production function in the O-Ring theory of the firm—the "O-Ring" terminology comes from the NASA Apollo program, where the quality of the rocket is zero if the quality of only one of its components is zero; see [25].

$$\varphi\left[Q(x), s(x)\right] = k^{\alpha}\left[\prod_{j=1}^{l} s^j x^j\right] L(x)B,$$

where $k > 0$ represents capital (machines), $\alpha > 0$ shows how a marginal increase in the use in capital will increase, more or less, the quality of the rocket, refering to the degree of concavity of the quality of the rocket with respect to the use of capital, $s^j \geq 0$ defines the skill (quality) of each type worker of type j (hence the quality of the component he produces), $L(x)$ is the numner of employed workers, and B is the output per worker. The endogenous quality of the final good is $s(x) = \Pi_{j=1}^{l} s^j x^j$. Obviously, $s(x) = 0$ if $s_j = 0$ for, at least, one $j \in J$.

Next, let us present main ingredients and important concepts in the VR approach; see [41, 42] for a general framework.

— n and $n+1$ stands for the past and current periods.

— $x_n = x$ and $x_{n+1} = y$ refer to a past and a current profile of employed workers.

— $x_n \curvearrowright x_{n+1}$ stands for a current move of the entrepreneur, i.e. he hires, keeps again or fires $x_{n+1}^j - x_n^j > 0$, $x_{n+1}^j - x_n^j = 0$, or $x_n^j - x_{n+1}^j > 0$ workers of type j.

— A move is called a **change** iff $x_{n+1}^j \neq x_n^j$ for some $j \in J$.

— A move is called a **stay** iff $x_{n+1}^j = x_n^j$ for all $j \in J$.

— The **advantage-to-change function** $A : X \times X \to \mathbb{R}$ is defined by $A(x, y) = g(y) - g(x)$ as the difference between the profit to use the profile of employed workers y and the profit to use x.

— The advantage-to-change $A(x_n, x_{n+1})$ from the old profile of employed workers x_n to a current profile x_{n+1} is the increase between the current profit $g(x_{n+1})$ to use the new profile of workers x_{n+1} and the profit $g(x_n)$ to use again the past profile of workers x_n.

—If we define $\overline{g} = \sup\left\{g(x) |\, x \in X\right\} < +\infty$ as the largest profit the entrepreneur can expect, then the function $f(x) := \overline{g} - g(x)$ measures the residual unsatisfaction to use the profile of employed workers x. Thus, the advantage-to-change from x_n to x_{n+1}

$$A(x_n, x_{n+1}) = f(x_n) - f(x_{n+1}) = [\overline{g} - g(x_n)] - [\overline{g} - g(x_{n+1})] = g(x_{n+1}) - g(x_n)$$

also refer to the difference between the residual unsatisfaction to use again the current profile of workers x_n and the residual unsatisfaction to use the new profile of workers x_{n+1}.

—The **inconvenience-to-change function** $I : X \times X \to \mathbb{R}$ is defined by $I(x, y) = C(x, y) - C(x, x) \geq 0$ as the difference between the cost-to-be-able-to-change $C(x, y) \geq 0$

and the cost-to-be-able-to-stay $C(x,x) \geq 0$. We will show later that, in this example, $C(x,y)$ is the sum of all the costs to be able to hire, keep again and fire workers.
— A move $x \curvearrowright y$ is **worthwhile** to the entrepreneur iff the advantage-to-change from x to y is bigger than the inconvenience-to-change up to a prior chosen degree of acceptability $\lambda > 0$, i.e. the increase in profit moving from using again the profile of workers x to using the new profile of workers y is more than the increase in costs to be able to change and to stay moving from using again the profile of workers x to using the new profile of workers y up to a the chosen degree of acceptability, i.e.

$$g(y) - g(x) \geq \lambda I(x,y) \text{ or } f(x) - f(y) \geq \lambda I(x,y).$$

—The multifunction $W : X \rightrightarrows X$ defined by

$$W_\lambda(x) := \{y \in X |\ A(x,y) \geq \lambda I(x,y)\}$$

is called a **worthwhile** mapping since for any $y \in W_\lambda(x)$, the move $x \curvearrowright y$ is worthwhile.
—A **worthwhile transition** $\{x_n\}$ is defined as a succession of worthwhile temporary stays and changes, i.e. $x_{n+1} \in W_\lambda(x_n),\ \forall\, n \in \mathbb{N}$.
—$x_* \in X$ is called an **aspiration point (strong or weak)** of a worthwhile transition $\{x_n\}$ iff $x_* \in W_\lambda(x_n),\ \forall\, n \in \mathbb{N}$ (the strong case) or $x_* \in W_\lambda(x_0)$ (the weak case).
—$x_* \in X$ is called a **worthwhile-to-stay trap** of a worthwhile transition $\{x_n\}$ iff

$$W_\lambda(x_*) = \{x_*\} \iff g(x) - g(x_*) = f(x_*) - f(x) < \lambda I(x_*, x),\ \forall\, x \neq x_*.$$

This means that being there, it is not worthwhile to move away.
—A **variational trap** is both worthwhile to reach and worthwhile to stay.

The main idea of a behavioral theory is to explain "why, where, how and when" agents perform actions and change, at each current period, along a path of stays and changes $\{x_0, x_1, \ldots, x_n, x_{n+1}, \ldots\ldots\}$: i) why the agent, first, has an incentive to take some steps away from his current position and, then, an incentive to stop changing one more step within this period, ii) when, starting from an initial position, a worthwhile transition converges to a variational trap, i.e. it approaches and ends in this trap.

Next, we will provide motivations for us to study extensions of the EVP into with quasimetrics, w-distances, etc. In the VR approach in [41, 42], costs-to-be-able-to-change verify, in the simplest prototype case, the following four assumptions:

(1) no change no cost $C(x,x) = 0,\ \forall\, x \in X$;

(2) it is not free to perform a move $C(x,y) > 0,\ \forall\, x \neq y$;

(3) a direct change costs less than any indirect change $C(x,y) \leq C(x,z) + C(z,y)$, $\forall\, x, y, z \in X$;

(4) the cost to perform a change cannot be equal to the cost to undo that change $C(x,y) \neq C(y,x),\ \forall\, x, y \in X$.

Mathematically, such a cost function is indeed a quasimetric in X.

The modelization of inconvenience-to-change in terms of a generalized distance. In the model described above, the inconvenience-to-change function can be defined in terms of hiring, keeping again, and firing costs. More presicely, to be able to hire one skilled worker of type j, ready to work, costs $c_H^j(t) \geq 0$. The variable t represents the last t worker of type j to be hired. Notice that this formulation considers that, in the current period, to hire one more worker can be more or less costly, depending on how many workers the entrepreneur have hired before this last one in this current period. These costs include search and training costs. To fire one worker of type j, costs $c_F^j(t) \geq 0$, where t represents the last t worker to be fired. These costs represent separation and compensation costs. To keep a worker, ready to work, one period more, costs $c_K^j(t) \geq 0$. These conservation costs include knowledge regeneration and motivation costs. Then, in the current period, costs to hire $y^j - x^j \geq 0$ (resp., costs to fire and costs to keep) workers of type j are defined by

$$C_H^j(x^j, y^j) = \int_{x^j}^{y^j} c_H^j(t)dt, \; C_F^j(x^j, y^j) = \int_{y^j}^{x^j} c_F^j(t)dt, \text{ and } C_K^j(x^j, x^j) = \int_0^{x^j} c_K^j(t)dt,$$

respectively. Then, costs to be able to change from using x^j workers of type j to y^j workers of type j are

$$C^j(x^j, y^j) := \begin{cases} C_K^j(x^j, x^j) + C_H^j(x^j, y^j) & \text{if } y^j \geq x^j, \\ C_K^j(y^j, y^j) + C_F^j(x^j, y^j) & \text{if } y^j \leq x^j, \end{cases}$$

for $j \in J = \{1, 2, \dots, l\}$.

Observe from the context that costs-to-be-able-to-stay $C^j(x^j, x^j) = C_K^j(x^j, x^j)$ **are strictly positive** since the cost to be able to keep x^j workers of type j does exist. Observe also that C^j does not enjoy the symmetricity property since, in general, we have the inequality

$$C^j(x^j, y^j) = C_K^j(x^j, x^j) + C_H^j(x^j, y^j) \neq C_K^j(y^j, y^j) + C_F^j(x^j, y^j) = C^j(y^j, x^j)$$

when $x^j \neq y^j$. Therefore, they are neither quasimetrics nor metrics.

Fix any $j \in J$. It is not difficult to check that $C^j(x^j, y^j)$ is a w**-distance** provided that the costs to hire, keep again, and fire functions c_H^j, c_K^j and c_F^j are continuous and satisfy

$$\inf_{a \in \mathbb{R}} \int_a^{a+r} c_H^j(t)dt > 0, \;\; \inf_{a \in \mathbb{R}} \int_a^{a+r} c_F^j(t)dt > 0, \;\; \text{and} \;\; \inf_{a \in \mathbb{R}} \int_a^{a+r} c_K^j(t)dt > 0.$$

The reader is referred to [20, Example 6] for details of a proof.

The inconvenience-to-change the working force type j from x^j to y^j, denoted by $I^j(x^j, y^j)$, is defined by

$$\begin{aligned} I^j(x^j, y^j) &= C^j(x^j, y^j) - C^j(x^j, x^j) \\ &= \begin{cases} C_H^j(x^j, y^j) + C_K^j(x^j, x^j) - C_K^j(x^j, x^j) = \int_{x^j}^{y^j} c_H^j(t)dt & \text{if } y^j \geq x^j \\ C_F^j(x^j, y^j) + C_K^j(y^j, y^j) - C_K^j(x^j, x^j) = \int_{y^j}^{x^j} \left[c_F^j - c_K^j\right](t)dt & \text{if } y^j \leq x^j. \end{cases} \end{aligned}$$

By Proposition 4.8 in this paper, the inconvenience-to-change the working force of type j functions, C^j, for $j \in J$ are quasimetrics.

General models. (a) To modelize a very complex concept of resistance-to-change including, among many different aspects, obstacles, barriers to change, difficulties to change and inertia, the VR approach uses as ingredients: costs-to-be-able-to-change, costs-to-be-able-to-stay, inconveniences-to-change and disutilities of these conveniences. Similarly to modelize the very complex concept of motivation-to-change (notice that, in Psychology, there exist more than 100 theories of motivation to change!), the VR approach uses as unifying ingredients: payoffs (gains to be increased and unsatisfactions to be decreased), advantages-to-change, and utility of these advantages.
—A **motivation-to-change function** is defined by $M(x,y) := U\left[A(x,y)\right]$ as the *pleasure* or *utility* of the advantage to change from x to y.
— A **resistance-to-change function** is defined by $R(x,y) := D\left[I(x,y)\right]$ as the *pain* or *disutility* of the inconvenience-to-change $I(x,y) = C(x,y) - C(x,x)$, which is the difference between the costs to be able to change from x to y and the costs to be able to stay at x.
In the simplest model, we assume that both the utility and the disutility are identity functions.

(b) Psychological inertia adds a fixed cost to accept to change rather than to stay denoted by $e(x) > 0$. This allows us to consider an resistance-to-change as a w-distance which can be strictly positive for some x and thus is not a quasimetric. This formulation means that a change $x \curvearrowright y$ is worthwhile if the motivation-to-change $M(x,y) = U\left[A(x,y)\right] := A(x,y)$ is higher than the inconvenience-to-change plus some fixed psychological cost to accept to change, i.e. $A(x,y) \geq I(x,y) + e(x)$. For a nice paper on different aspects of psychological inertia, see [16]. Let us show that if the inconvenience-to-change $I(x,y) \geq 0$ for all $x, y \in X$ is a w-distance and if the psychological inertia term $e(x)$ is positive, then the resistance-to-change $R(x,y) = I(x,y) + e(x)$ is also a w-distance such that $R(x,x) = e(x) > 0$ for some $x \in X$. It is sufficient to check the validity of three conditions (w1), (w2) and (w3) in Definition 2.6.

(w1) For all $x, y, z \in X$, $R(x,z) = I(x,z) + e(x) \leq I(x,y) + e(x) + I(y,z) + e(y) = R(x,y) + R(y,z)$ holds since $I(x,z) \leq I(x,y) + I(y,z)$.

(w2) For any $x \in X$, $R(x,\cdot) = I(x,\cdot) + e(x)$ is lower semicontinuous with respect to the second variable y because of that property of $I(x,\cdot)$.

(w3) For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $R(x,y) \leq \delta$ and $R(y,z) \leq \delta$, then $d(x,z) \leq \varepsilon$. This is true since $R(x,y) \geq I(x,y)$ and I satisfies (w3).

In summary, the model of the formation and break of hiring and firing workers routines described above ensures that it is essential to extend the EVP from metrics to generalized distances. The next section addresses this need.

4 EVPs with Generalized Distances

In this section, we establish a quasimetric and a weak τ-distance versions of the EVP and prove that they are indeed equivalent.

It is well recognized that the EVP is one of the most useful tools in nonlinear analysis and variational analysis; it allows us to study minimization problems in which the lower level set of the problem is not compact, i.e. when the Bolzano-Weierstrass theorem cannot be applied.

Theorem 4.1. (Ekeland's variational principle [14]). *Let (X,d) be a metric space, $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous, bounded below, and not identically equal to $+\infty$, $\varepsilon > 0$, and $x_0 \in \operatorname{dom}\varphi$ be an ε-minimal solution of φ, i.e.*

$$\varphi(x_0) \le \inf_{x\in X} \varphi(x) + \varepsilon.$$

Then, for each $\lambda > 0$, there exists some point $x_* \in \operatorname{dom}\varphi$ such that

(i) $\varphi(x_*) \le \varphi(x_0)$;

(ii) $d(x_0, x_*) \le \lambda$;

(iii) $\varphi(x) + (\varepsilon/\lambda)d(x_*, x) > \varphi(x_*),\ \forall\, x \in \operatorname{dom}\varphi \setminus \{x_*\}$.

Over the last four decades a great deal of effort has been done to look for another equivalent formulation of various types of the Ekeland variational principle as well as its generalizations in order to fit a new setting of some particular application. In [52], Zhong proved that the EVP is still valid for the class of generalized distances $p : X \times X \to \mathbb{R}\cup\{\infty\}$ defined by

$$p(x,y) := \frac{d(x,y)}{1 + h(d(x_0,x))}, \tag{4.2}$$

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function satisfying $\int_0^\infty \Big(1/(1+h(t))\Big)dt < \infty$. This class of generalized distances (4.2) has been enlarged to w-distances [20], Tataru's distances [46], τ-functions [27], and weak τ-functions [22, 23]; see also the bibiographies therein.

In [27], Lin and Du established a generalized EVP for decreasingly-closed (known also as lower semicontinuous from above) functionals and τ-functions defined on metric spaces. Then, in [22, 23], Khanh and Quy further extended it to weak τ-functions.

Recently, Bao, Mordukhovich and Soubeyran obtained extensions of the EVP for mappings acting from a quasimetric space to a vector space equipped with variable ordering structures in [4, 5, 6]. Below is a simplest version for a class of extended real-valued functions which is larger than that of lower semicontinuous ones.

Definition 4.2. (decreasing closedness). *Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a function. It is said to be* DECREASINGLY CLOSED *iff for any convergent sequence $\{x_k\} \subset \operatorname{dom}\varphi$, if the sequence of numbers $\{\varphi(x_k)\}$ is decreasing, then the limit x_* of the sequence $\{x_k\}$ belongs to* $\operatorname{dom}\varphi$.

Obviously, if φ is lower semicontinuous (known also as level-closed) on $\operatorname{dom}\varphi \setminus \operatorname{Max}(\varphi)$, then it is decreasingly closed, where $\operatorname{Max}(\varphi)$ is the collection of all the local maxima of φ.

Theorem 4.3. (a quasimetric version of the EVP [5, Corollary 3.3]). *Let (X, q) be a complete and Hausdorff quasimetric space, and let $\varphi : X \to \mathbb{R} \cup \{\infty\}$ be a decreasingly closed and bounded from below function on X with $\operatorname{dom}\varphi \neq \emptyset$. For any $\lambda > 0$ and $x_0 \in \operatorname{dom}\varphi$, there is $x_* \in \operatorname{dom}\varphi$ such that*

(i) $\varphi(x_*) + \lambda q(x_0, x_*) \leq \varphi(x_0)$;

(ii) $\varphi(x) + \lambda q(x_*, x) > \varphi(x_*), \quad \forall\, x \neq x_*$.

Furthermore, there is a sequence $\{x_n\}$ starting at x_0 and satisfying

$$x_{n+1} \in S_{\varphi,q,\lambda}(x_n),\ \forall\, n \in \mathbb{N} \ \ \textit{with} \ \ S_{\varphi,q,\lambda}(x) := \{y \in X |\ \varphi(y) + \lambda q(x, y) \leq \varphi(x)\}. \tag{4.3}$$

Obviously, conditions (i) *and* (ii) *are reduced to $x_* \in S_{\varphi,q,\lambda}(x_0)$ and $S_{\varphi,q,\lambda}(x_*) = \{x_*\}$, respectively.*

Technically, the decreasing closedness of the cost φ in Theorem 4.3 can be weaken to that of the set-valued mapping $S_{\varphi,q,\lambda}$.

Theorem 4.4. (an alternative quasimetric version of the EVP). *Let (X, q) be a complete and Hausdorff quasimetric space, and let $\varphi : X \to \mathbb{R} \cup \{\infty\}$ be a bounded from below function on X with $\operatorname{dom}\varphi \neq \emptyset$ which is not necessarily decreasingly closed. Given $\lambda > 0$. Assume that the mapping $S_{\varphi,q,\lambda} : X \rightrightarrows X$ defined by (4.3) enjoys the limiting monotonicity condition in the sense that for every Cauchy sequence $\{x_n\} \subset X$, if $x_{n+1} \in S_{\varphi,q,\lambda}(x_n)$, $\forall\, n \in \mathbb{N}$, then the limit x_* of the sequence satisfies $x_* \in S_{\varphi,q,\lambda}(x_n)$, $\forall\, x \in \mathbb{N}$. Then, for any $x_0 \in \operatorname{dom}\varphi$, there exists $x_* \in \operatorname{dom}\varphi$ such that conditions* (i) *and* (ii) *in Theorem 4.3 hold.*

Proof. Starting from $x_0 \in \operatorname{dom}\varphi$, we recursively construct a sequence $\{x_n\}$ by

$$\begin{cases} x_{n+1} \in S_{\varphi,q,\lambda}(x_n) \\ q(x_n, x_{n+1}) \geq \operatorname{radius}(x_{n+1}, S_{\varphi,q,\lambda}(x_n)) - 2^{-n} := \sup\limits_{y \in S_{\varphi,q,\lambda}(x_n)} q(x_{n+1}, y) - 2^{-n}. \end{cases} \tag{4.4}$$

It is easy to check that $x_n \in S_{\varphi,q,\lambda}(x_n)$ for all $n \in \mathbb{N}$ and that $S_{\varphi,q,\lambda}(x_m) \subset S_{\varphi,q,\lambda}(x_n)$ for all $m, n \in \mathbb{N}$ with $m > n$. Taking into account the definition of the mapping $S_{\varphi,q,\lambda}$ in (4.3) and the triangle inequality of the quasimetric q we get

$$\sum_{i=0}^{n} q(x_i, x_{i+1}) \leq \frac{1}{\lambda}\big(\varphi(x_0) - \varphi(x_n)\big) \text{ and thus } \sum_{i=0}^{\infty} q(x_i, x_{i+1}) \leq \frac{1}{\lambda}\big(\varphi(x_0) - \inf_{x \in X} \varphi(x)\big) < \infty.$$

The last inequality ensures that the sequence $\{x_n\}$ is Cauchy in the quasimetric space (X, q). Since the space is assumed to be complete and Hausdorff, it converges to a unique limit x_*. By the limiting monotonicity condition, $x_* \in S_{\varphi,q,\lambda}(x_n)$ for all $n \in \mathbb{N}$. We also have

$$S_{\varphi,q,\lambda}(x_*) \subset S_{\varphi,q,\lambda}(x_n),\ \forall\, n \in \mathbb{N}. \tag{4.5}$$

Obviously, (i) holds by (4.4) and (ii) is equivalent to $S_{\varphi,q,\lambda}(x_*) = \{x_*\}$. To complete the proof it is sufficient to prove that if $y_* \in S_{\varphi,q,\lambda}(x_*)$, then $y_* = x_*$. Indeed, by (4.4) and (4.5), the assumption $y_* \in S_{\varphi,q,\lambda}(x_*)$ produces

$$y_* \in S_{\varphi,q,\lambda}(x_n) \text{ and } q(x_n, y_*) \leq \text{radius}(x_n; S_{\varphi,q,\lambda}(x_n)) \leq q(x_n; x_{n+1}) + 2^{-n}, \ \forall\, n \in \mathbb{N}.$$

Therefore, $q(x_n, y_*) \to 0$ as $n \to \infty$ since $q(x_n, x_{n+1}) \to 0$. This means that y_* is a limit of the sequence $\{x_n\}$. Since (X, q) is Hausdorff and x_* is the limit of $\{x_n\}$, we have $y_* = x_*$. △

Proof. of Theorem 4.3. It is not difficult to check that if φ is decreasingly closed, then $S_{\varphi,q,\lambda}$ has the limiting monotonicity condition. Therefore, Theorem 4.4 $\Longrightarrow$ Theorem 4.3 and the proof is complete. The reader can find another proof in [5, Corollary 3.3]. △

Remark 4.5. (on directions of generalization). Since condition (i) with $\lambda = \varepsilon/\lambda$ implies both $\varphi(x_*) \leq \varphi(x_0)$ and $q(x_0, x_*) \leq \lambda$ provided that x_0 is ε-minimal to φ, Theorem 4.3 is an extension of Theorem 4.1 in two directions: (a) quasimetric spaces instead of metric spaces and (b) decreasingly closed functions instead of lower semicontinuous functions.

(a) The extension from the class of metric spaces to quasimetric spaces, i.e. from Theorem 4.1 to Theorem 4.3, is nontrivial. It allows us to apply this principle to applications in behavioral sciences in which the cost to change from a state to another one is not the same the cost to change it back (see the last section for an application). It is important to emphasize that in many publications on the Ekeland variational principle with quasimetric spaces the Hausdorff separation property of the topology induced from the quasimetric was missing; for example see, to our knowledge, the first extension of the EVP to quasi metric spaces in [49]. The topology induced by a metric unconditionally enjoys the Hausdorff property, but it does not hold for quasimetrics. The reader can find in [5] a counter example illustrating that the Ekeland variational principle does not hold true for a quasimetric space being not Hausdorff.

(b) The modification of the continuity assumption of the cost from the lower semicontinuity to the decreasing closedness is quite technical. That is that the proof in [14] is still valid for the latter continuity condition. The former one was used in order to apply the Cantor nonempty intersection result for closed sets. However, we can directly justify the existence of the intersection point with just the decreasing closedness of the functional φ. See, e.g. [4, 5, 6] and also [22, 23]. In addition, since the calculus rules of all the known subdifferentials including limiting, proximal and approximate ones work for lower semicontinuous functions only, if we need to formulate a condition in terms of subdifferentials of the cost function, we cannot use the decreasing closedness version.

In the next result, we work with a generalized distance similar to those defined by Suzuki, Lin-Du, and Khanh-Quy, but slightly different. In constrast to quasimetrics and metrics, when working with a w-distance, a τ-distance/function or a weak τ-distance in Section 2, the nonempty valuedness of the multifunction $S_{\varphi,p,\lambda}$ is no longer guaranteed.

Definition 4.6. (weak τ-distances and τ-functions in a quasimetric space). *Let (X, q) be a quasimetric space. A function $p : X \times X \to \mathbb{R}_+$ is called a* WEAK τ-FUNCTION *iff it satisfies conditions $(\tau 1)$ and $(\tau 3)$. It is called a* WEAK τ-DISTANCE *iff it satisfies conditions $(\tau 1)$, $(\tau 2)$, and $(\tau 3)$ in Definition 2.8 with q instead of d, i.e.*

$(\tau 1)$ *$p(x, z) \le p(x, y) + p(y, z)$ (triangle inequality);*

$(\tau 2)$ *for all $x \in X$, $p(x, \cdot)$ is lower semicontinuous (lower semicontinuity);*

$(\tau 3)$ *for all sequences $\{x_n\}, \{y_n\} \subset X$ with $\lim_{n\to\infty} p(x_n, y_n) = 0$ and $\lim_{n\to\infty} \sup_{m>n} p(x_n, x_m) = 0$, one has $\lim_{n\to\infty} q(x_n, y_n) = 0$ (p-convergence implies q-convergence).*

Theorem 4.7. (a weak τ-distance version of the EVP). *Let (X, q) and φ be as in Theorem 4.3 and let $p : X \times X \to \mathbb{R}_+$ be a weak τ-distance in Definition 4.6. For any $\lambda > 0$ and $x_0 \in \operatorname{dom} \varphi$ such that $S_{\varphi,p,\lambda}(x_0) \neq \emptyset$, there exists $x_* \in \operatorname{dom} \varphi$ such that*

(i′) *$x_* \in S_{\varphi,p,\lambda}(x_0)$, i.e. $\varphi(x_*) + \lambda p(x_0, x_*) \le \varphi(x_0)$;*

(ii′) *$S_{\varphi,p,\lambda}(x_*) \subset \{x_*\}$.*

In contrast to proofs of versions of the EVP with generalized distances; see, e.g. [27, 22, 23], we derive Theorem 4.7 from the quasimetric version of the EVP in Theorem 4.3. To do so, we need an auxiliary result.

Proposition 4.8. *Let (X, q) be a complete and Hausdorff quasimetric space and $p : X \times X \to \mathbb{R}_+$ be a weak τ-distance satisfying conditions $(\tau 1)$, $(\tau 2)$, and $(\tau 3)$ in Definition 4.6. Define a function $\overline{q} : X \times X \to \mathbb{R}_+$ by*

$$\overline{q}(x, y) := \begin{cases} p(x, y) & \text{if} \quad x \neq y, \\ 0 & \text{if} \quad x = y. \end{cases} \tag{4.6}$$

Then, $\overline{q}$ is a quasimetric on X and the space $(X, \overline{q})$ is complete and Hausdorff.

Proof. Obviously, $\overline{q}$ enjoys the nonnegativity and "equality implies indistancy" properties and thus it is a quasimetric on X. Of course, a sequence $\{x_n\}$ is Cauchy with respect to $\overline{q}$ means that $\lim_{n\to\infty} \{\sup \overline{q}(x_n, x_m) : m > n\} = 0$. We claim first that $(\tau 3)$ implies that every sequence $\{x_n\}$ satisfying $\lim_{n\to\infty} \{\sup \overline{q}(x_n, x_m) : m > n\} = 0$ is a Cauchy sequence in (X, q). Indeed, if the sequence $\{x_n\}$ is eventually constant, i.e. there is some integer $M > 0$ such that $x_{M+k} \equiv x_M$ for all $k \in \mathbb{N}$, then, by (q2), $q(x_{M+k}, x_M) = 0$ for all $k \in \mathbb{N}$, and thus x_M is a limit of the sequence $\{x_n\}$ and hence $\{x_n\}$ is Cauchy with respect to q. Otherwise, we may assume that $x_m \neq x_n$ for all $m \neq n$, by the definition of $\overline{q}$ we have $\overline{q}(x_n, x_m) = p(x_n, x_m)$ for all $m > n$. Then,

$$\lim_{n\to\infty} \{\sup \overline{q}(x_n, x_m) : m > n\} = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \{\sup p(x_n, x_m) : m > n\} = 0. \tag{4.7}$$

By Lemma 2.6 in [22], $\{x_n\}$ is Cauchy with respect to q, if q is a metric, but the proof of this lemma is still valid for the case where q is a quasimetric. Since (X, q) is complete, $\{x_n\}$

converges with respect to q. We have to show that it converges also with respect to $\overline{q}$, i.e. $\lim_{n\to\infty} \overline{q}(x_n, x) = 0$.

By condition (τ2) and the definition of $\overline{q}$, for any $n \in \mathbb{N}$, we have

$$\overline{q}(x_n, \overline{x}) \le p(x_n, \overline{x}) \le \liminf_{m\to\infty} p(x_n, x_{n+m}),$$

and thus $\overline{q}(x_n, \overline{x})$ tends to zero as $n \to \infty$ due to (4.7). This proves the completeness of the space $(X, \overline{q})$.

Next, we show that the Hausdorff property of the space (X, q) implies that of $(X, \overline{q})$. Suppose first, for the constant sequence $\{x_n\}$ with $x_n \equiv \overline{x}$ for all $n \in \mathbb{N}$, that there is $x_* \ne \overline{x}$ such that $\overline{q}(x_n, x_*) \equiv \overline{q}(\overline{x}, x_*) = 0$. Then, with $y_n \equiv x_*$, (τ3) and (4.7) imply that $q(\overline{x}, x_*) = 0$. Since (X, q) is Hausdorff, we have $\overline{x} = x_*$ contradicting to the imposed assumption that $\overline{x} \ne x_*$. The contradiction implies the unique limit of any constant sequence in $(X, \overline{q})$.

Assume now that there is a convergent (with respect to to $\bar{q}$) sequence $\{x_n\}$ with $x_m \ne x_n$ for all $m \ne n$ with two different limits $\overline{x}$ and x_*. Without loss of generality, we may assume also that $x_n \ne \overline{x}$ and $x_n \ne x_*$ for all $n \in \mathbb{N}$. By the definition of $\overline{q}$, we have $\overline{q}(x_n, \overline{x}) = p(x_n, \overline{x})$ and $\overline{q}(x_n, x_*) = p(x_n, x_*)$ for all $n \in \mathbb{N}$. By (τ 3) with the sequences $\{x_n\}$ and $\{y_n \equiv \overline{x}\}$, we have $\lim q(x_n, \overline{x}) = 0$. By ($\tau$3) with sequence $\{x_n\}$ and $\{y_n \equiv x_*\}$, we have $\lim q(x_n, x_*) = 0$. Since (X, q) is Hausdorff, we get $\overline{x} = x_*$ contradicting the imposed assumption $\overline{x} \ne x_*$. The contradiction implies the unique limit of any nonconstant sequence in $(X, \overline{q})$. The proof is complete. △

Proof. By Proposition 4.8, the space X equipped with the quasimetric $\overline{q}$ defined in (4.6) is a complete and Hausdorff quasimetric space. Employing Theorem 4.3 to the underlying quasimetric space $(X, \overline{q})$, for any $\lambda > 0$ and for each $x_0 \in \operatorname{dom}\varphi$, there exists $x_* \in \operatorname{dom}\varphi$ satisfying two relationships (i) and (ii) therein with $\overline{q}$, i.e.

(i) $\varphi(x_*) + \lambda \overline{q}(x_0, x_*) \le \varphi(x_0)$;

(ii) $\varphi(x) + \lambda p(x_*, x) = \varphi(x) + \lambda \overline{q}(x_*, x) > \varphi(x_*), \quad \forall\, x \ne x_*$.

Obviously, (ii) implies (ii′) in Theorem 4.7. To verify (i′) in the same theorem, we consider two cases:

Case 1: If $x_* \ne x_0$, then $\overline{q}(x_0, x_*) = p(x_0, x_*)$ by (4.6) and thus (i) reduces to (i′).

Case 2: If $x_* = x_0$, then (ii′) ensures that $S_{\varphi,p,\lambda}(x_0) \subset \{x_0\}$. This together with the imposed assumption of Theorem 4.7, $S_{\varphi,p,\lambda}(x_0) \ne \emptyset$, implies $S_{\varphi,p,\lambda}(x_0) = \{x_0\}$, i.e. $\varphi(x_*) + \lambda p(x_0, x_*) = \varphi(x_0) + \lambda p(x_0, x_0) \le f(x_0)$ clearly verifying (i′). We, furthermore, get $p(x_0, x_0) = 0$ in this case.

In summary, each case ensures the validity of (ii′). The proof is complete. △

Next, let us provide a weak τ-function version of the EVP for functionals which are not decreasingly closed.

Theorem 4.9. (a weak τ-function version of the EVP). *Let (X, q) and φ be as in Theorem 4.4 and let $p : X \times X \to \mathbb{R}_+$ be a weak τ-function in the sense of Definition 4.6. Assume that the mapping $S_{\varphi,p,\lambda}$ defined in (4.3) with p in replacement of q has* *the limiting monotonicity condition in the sense that for every Cauchy sequence* $\{x_n\}$ *in the space* (X, q)*, if* $x_{n+1} \in S_{\varphi,p,\lambda}(x_n)$, $\forall\, n \in \mathbb{N}$*, then the limit* x_* *of the sequence satisfies* $x_* \in S_{\varphi,p,\lambda}(x_n)$, $\forall\, x \in \mathbb{N}$*. For any* $\lambda > 0$ *and* $x_0 \in \operatorname{dom}\varphi$ *such that* $S_{\varphi,p,\lambda}(x_0) \neq \emptyset$*, there exists* $x_* \in \operatorname{dom}\varphi$ *such that conditions* (i′) *and* (ii′) *in Theorem 4.7 hold.*

Proof. First, we construct a sequence $\{x_n\}$. For any $n \in \mathbb{N}$, if $S_{\varphi,p,\lambda}(x_n) = \emptyset$, then STOP and set $x_* = x_n$; otherwise x_{n+i} is chosen such that

$$\begin{cases} x_{n+1} \in S_{\varphi,p,\lambda}(x_n) \\ p(x_n, x_{n+1}) \geq \operatorname{radius}(x_n, S_{\varphi,p,\lambda}(x_n)) - 2^{-n} := \sup\limits_{y \in S_{\varphi,p,\lambda}(x_n)} p(x_n, y) - 2^{-n}. \end{cases} \tag{4.8}$$

Obviously, if $S_{\varphi,p,\lambda}(x_*) = \emptyset$, x_* satisfies (ii′). In addition, we have $x_n \in S_{\varphi,p,\lambda}(x_{n-1}) \subset \cdots \subset S_{\varphi,p,\lambda}(x_0)$ which ensures the validity of (i′) and completes the proof of the theorem.

Next, we need to justify (i′) and (ii′) in the case where $S_{\varphi,p,\lambda}(x_n) \neq \emptyset$, $\forall\, n \in I\!N$. Proceed similarly the proof of Theorem 4.4. First, the sequence $\{x_n\}$ is Cauchy with respect to p. By ($\tau 3$), it is also Cauchy with respect to q and thus it has the unique limit $x_* \in X$ with repect to q, i.e. $q(x_n, x_*) \to 0$ as $n \to \infty$. By the limiting monotonicity condition, $x_* \in S_{\varphi,p,\lambda}(x_n)$, $\forall\, n \in \mathbb{N} \cup \{0\}$. Fix $n = 0$, we have (i′). Next, take an arbitrary element $y_* \in S_{\varphi,p,\lambda}(x_*)$ (if exists), we have $y_* \in S_{\varphi,p,\lambda}(x_n)$, $\forall\, n \in \mathbb{N}$. Taking into account now (4.8), we have estimates

$$p(x_n, y_*) \leq \sup_{y \in S_{\varphi,p,\lambda}(x_n)} p(x_n, y) \leq p(x_n, x_{n+1}) + 2^{-n},\ \forall\, n \in \mathbb{N}$$

clearly implying that $p(x_n, y_*) \to 0$ as $n \to \infty$. Property ($\tau 3$) applied to the sequence $\{x_n\}$ and the constant sequence $\{y_n\}$ with $y_n \equiv y_*$, $\forall\, n \in \mathbb{N}$ ensures that $q(x_n, y_*) \to 0$ as $n \to \infty$. Since (X, q) is Hausdorff and x_* is the limit of $\{x_n\}$, $y_* = x_*$ clearly veryfying that $S_{\varphi,p,\lambda}(x_*) \subset \{x_*\}$, i.e. (ii′) holds. The proof is complete. △

Theorem 4.10. *Theorem 4.3* $\Longleftrightarrow$ *Theorem 4.7.*

Proof. On one hand, the implication Theorem 4.7 $\Longrightarrow$ Theorem 4.3 holds since every quasimetric is a w-distance. On the other hand, the proof of Theorem 4.7 clearly justifies the validity of the inverse implication Theorem 4.3 $\Longrightarrow$ Theorem 4.7. Therefore, the equivalence is true. △

This result says that any known version of the EVP with either w-distances or τ-functions; in particular, the w-distances from Examples 1–8 in [20], is a consequence of the weak-τ-distance version in Theorem 4.7. It is easy to check that the relation between versions with the underlying metric spaces is similar/the-same as the presentation here for underlying quasimetric spaces. Therefore, such a result is not more general than the original EVP.

To conclude this section let us derive a version of Theorem 2.1 in [27] from the weak τ-distance version in Theorem 4.7.

Theorem 4.11. (a weak τ-distance version of [27, Theorem 2.1].) *Let (X, q), φ, and p as in Theorem 4.7, and let $\Lambda : \mathbb{R} \to \mathbb{R}_{++} := (0, \infty)$ be a* nondecreasing *function. Then, for any $x_0 \in \operatorname{dom}\varphi$ there exists $x_* \in \operatorname{dom}\varphi$ such that*

(ii$'''$) $p(x_*, x) > \Lambda(\varphi(x_*))\big(\varphi(x_*) - \varphi(x)\big)$, $\forall\, x \neq x_*$.

Indeed, Theorem 4.11 $\Longleftrightarrow$ Theorem 4.7.

Proof. Observe that if $S_{\varphi,p,\lambda_0} = \emptyset$ with $\lambda_0 = 1/\Lambda(\varphi(x_0)) > 0$, then $x_* = x_0$ satisfies (ii$'''$). Assume now that it is nonempty and get from Theorem 4.7 for φ, x_0 and λ_0 the following:

(i$'$) $\varphi(x_*) + \lambda_0 \overline{q}(x_0, x_*) \leq \varphi(x_0)$;

(ii$'$) $\varphi(x) + \lambda_0 p(x_*, x) = \varphi(x) + \lambda \overline{q}(x_*, x) > \varphi(x_*), \quad \forall\, x \neq x_*$.

Since $\overline{q}(x_0, x_*) \geq 0$, (i$'$) implies $\varphi(x_*) \leq \varphi(x_0)$. Then, the nondecreasing monotonicity of the function Λ gives $\Lambda(\varphi(x_*)) \leq \Lambda(\varphi(x_0))$. This together with (ii$'$) justifies (ii$'''$). Details are below:

$$
\begin{aligned}
(\mathrm{ii}') \quad &\Longleftrightarrow \quad \varphi(x) + \lambda_0 p(x_*, x) > \varphi(x_*), \quad \forall\, x \neq x_* \\
&\overset{\lambda_0 > 0}{\Longleftrightarrow} \quad p(x_*, x) > \frac{1}{\lambda_0}\Big(\varphi(x_*) - \varphi(x)\Big) = \Lambda(\varphi(x_0))\Big(\varphi(x_*) - \varphi(x)\Big), \quad \forall\, x \neq x_* \\
&\overset{\textit{monotonicity}}{\Longrightarrow} \quad p(x_*, x) > \Lambda(\varphi(x_*))\Big(\varphi(x_*) - \varphi(x)\Big), \quad \forall\, x \neq x_* \\
&\Longleftrightarrow \quad (\mathrm{ii}''').
\end{aligned}
$$

We have proved that Theorem 4.7 $\Longrightarrow$ Theorem 4.11. By taking $\Lambda(t) \equiv \lambda$ for all $t \in \mathbb{R}$, we also get the validity of the reverse of this implication. Therefore, the equivalence holds true. The proof is complete. △

Remark 4.12. (comparisons with known results). *When $q = d$ is a metric, Theorem 4.11 recaptures the result in [27, Theorem 2.1] and Theorem 4.9 improves the corresponding result in [22, 23, 24] for extended real-valued functions. It is important to emphasize that Khanh and Quy used weak τ-functions while we need only weak τ-distances and that every weak τ-function is a weak τ-distance, but not vise versa.*

Remark 4.13. (possible extensions). *For the sake of simplicity, we present in this paper versions of EVPs for scalar single-valued cost functions with quasimetrics and weak τ-distances while it is not difficult to further generalize our obtained results to other settings studied in recent publications on new developments of the EVP. Let us mention several important directions:*

- *vector-valued and set-valued cost mappings; see, e.g. [3, 17, 19, 21, 39]*
- *different kinds of optimal solutions in vector spaces with partial orders induced by ordering cones; see, e.g. [6, 18, 24]*

– *minimal and nondominated solutions in vector spaces with variable ordering structures; see, e.g. [4, 5, 6, 7] (Recall that in [50], Yu introduced the concept of variable ordering/domination structures and dominated solutions in the case where $D(z)$ is a convex cone for each $z \in Z$. For recent developments on vector optimization with variable ordering structures, the reader is referred to a recent Eichfelder's book [13].)*

– *vector-valued and set-valued distances; see, e.g. [6, 21, 22, 23]*

– *preference solutions in vector spaces equipped with general preorders, see, e.g. [5, 48]*

5 Caristi Fixed Point Theorems with Generalized Distances

In this section, we present extensions of the Caristi fixed point theorem (CFPT) to quasi-metrics and weak τ-distances which are equivalent to the Ekeland-type results obtained in the previous section.

First, let us present some developments on CFPT with w-distances and τ-distances.

Theorem 5.1. (Caristi's fixed point theorem [9]). *Let (X, d) be a complete metric space, $\varphi : X \to \mathbb{R}_+$ be a lower semicontinuous function, and $T : X \to X$ be a single-valued function such that*

$$d(x, T(x)) \leq \varphi(x) - \varphi(Tx) \text{ for all } x \in X.$$

Then, T has a fixed point.

In [20], Kada et al. introduced the concept of w-distance in order to prove a new fixed point theorem in which a w-distance plays the role of the metric of the underlying metric space, and then derived from it Subrahmanyam's fixed point theorem, Kannan's fixed point theorem, and Ȉirič's fixed point theorem as well as equivalent forms of fixed point theorems.

Theorem 5.2. (a w-distance version of the CFPT [20, Theorem 4]). *Let (X, d) be a complete metric space, $p : X \times X \to \mathbb{R}_+$ be a w-distance in Definition 2.6, and $T : X \to X$ be a function. Suppose that there exists $r \in [0, 1)$ such that*

$$p(Tx, T^2x) \leq rp(x, Tx), \ \forall\, x \in X \text{ and } \inf_{x \in X} \big(p(x, y) + p(x, Tx)\big) > 0, \ \forall\, y \in X \text{ with } y \neq Ty.$$

Then, there exists $x_* \in X$ such that $x_* = Tx_*$. Moreover, if $x_* = Tx_*$, then $p(x_*, x_*) = 0$.

In [26], Latif formulated the following extension of Theorem 5.2 to a set-valued mapping $T : X \rightrightarrows X$ and a parameter mapping $\Lambda : \mathbb{R}_+ \to \mathbb{R}_{++}$, where $\mathbb{R}_{++} := (0, \infty)$.

Theorem 5.3. (Latif's fixed point theorem [26, Theorem 2.3]). *Let X, p, and φ be as in Theorem 5.2 and let $T : X \rightrightarrows X$ be a set-valued mapping satisfying*

$$\forall\, x \in X, \ \exists\, y \in T(x)\colon p(x, y) \leq \Lambda\big(\varphi(x)\big)\big(\varphi(x) - \varphi(y)\big),$$

where $\Lambda : \mathbb{R} \to \mathbb{R}_{++}$ is a nondecreasing function. Then, T has a fixed point $x_* \in X$ satisfying $p(x_*, x_*) = 0$.

The above results were further generalized to τ-functions and weak τ-functions settings in, e.g. [27, Theorem 2.2] and [22, Theorem 3.3]. In this paper, we establish two new (quasimetric and weak τ-distance) versions of the CFPT which are directly derived from the corresponding version of the EVP. We also prove that they are equivalent.

Theorem 5.4. (a quasimetric version of the CFPT). *Let (X, q) be a complete and Hausdorff quasimetric space, $\varphi : X \to \mathbb{R} \cup \{\infty\}$ be proper, decreasingly closed, and bounded from below, $T : X \rightrightarrows X$ be a set-valued mapping. Assume that the set $\Xi := S_{\varphi,q,1}(x_0)$ is nonempty for some $x_0 \in \operatorname{dom} \varphi$ and the pair (T, Ξ) satisfies*

$$\forall\, x \in \Xi,\ \exists\, y \in T(x) :\ \varphi(y) + q(x, y) \le \varphi(x). \tag{5.9}$$

Then, T has a fixed point $x_* \in \Xi$, i.e. $x_* \in T(x_*)$ satisfying $p(x_*, x_*) = 0$.

Proof. By the quasimetric version of the Ekeland variational principle from Theorem 4.3, we can find some $x_* \in S_{\varphi,q,1}(x_0) = \Xi$ such that $S_{\varphi,q,1}(x_*) \subset \{x_*\}$, where $S_{\varphi,q,1}$ is defined by (4.3). We claim that this point is a fixed point of T. Arguing by contradiction, suppose that it is not true, i.e. $x_* \notin T(x_*)$. This together with condition (5.9) ensures the existence of $y \in T(x_*)$ with $y \ne x_*$ such that $\varphi(y) + q(x_*, y) \le \varphi(x_*)$, i.e.

$$x_* \ne y \in S_{\varphi,q,1}(x_*) \subset \{x_*\}.$$

This impossibility verifies that x_* is a fixed point of T. We also get from condition (5.9) that

$$\varphi(x_*) + q(x_*, x_*) \le \varphi(x_*)$$

and thus $q(x_*, x_*) = 0$ due to the nonnegativity property. The proof is complete. △

Theorem 5.5. (a weak τ-distance version of the CFPT). *Let (X, q) be a complete Hausdorff quasimetric space, $p : X \times X \to \mathbb{R}_+$ be a weak τ-distance in Definition 4.6, and $(\tau 3)$, $\varphi : X \to \mathbb{R} \cup \{\infty\}$ be proper, decreasingly closed, and bounded from below. Assume that the set $\Xi := S_{\varphi,q,1}(x_0)$ is nonempty for some $x_0 \in \operatorname{dom} \varphi$ and the pair (T, Ξ) satisfies*

$$\forall\, x \in \Xi,\ \exists\, y \in T(x) :\ \varphi(y) + p(x, y) \le \varphi(x).$$

Then, T has a fixed point $x_* \in X$ such that $p(x_*, x_*) = 0$.

Proof. By replacing, in the proof of Theorem 5.4, the quasimetric q and the quasimetric version of the EVP in Theorem 4.3 by the weak τ-distance p and the weak τ-distance version in Theorem 4.7, we have a proof for Theorem 5.5.

We learn from the proofs of Theorems 5.4 and 5.5 that Theorem 4.3 $\Longrightarrow$ Theorem 5.4 and Theorem 4.7 $\Longrightarrow$ Theorem 5.5. Next, we will show that these implications are reversible, i.e. they hold as equivalences.

Theorem 5.6. *Theorem 5.4 $\Longrightarrow$ Theorem 4.3.*

Proof. Assume that all the assumptions in Theorem 4.3 are fulfilled. Given $x_0 \in \operatorname{dom}\varphi$ and consider two cases:
Case 1: there is $x_* \in S_{\varphi,q,\lambda}(x_0)$ such that $S_{\varphi,q,\lambda}(x_0) = \{x_*\}$. Obviously, such an x_* satisfies both relationships (i) and (ii) in Theorem 4.3.
Case 2: for every $x \in S_{\varphi,q,\lambda}(x_0)$, there is $y \in S_{\varphi,q,\lambda}(x) \setminus \{x\}$. Observe that if q is a quasimetric in X, so is $q_\lambda := \lambda q$ and that $S_{\varphi,\lambda q,1}(x_0) = S_{\varphi,q,\lambda}(x_0)$. Set $\Xi := S_{\varphi,q_\lambda,1}(x_0)$ and construct a set-valued mapping $T : X \rightrightarrows X$ with

$$T(x) := \{y \in X|\ y \neq x \text{ and } \varphi(y) + q_\lambda(x,y) \leq \varphi(x)\} = S_{\varphi,q,\lambda}(x) \setminus \{x\}.$$

Obviously, $\Xi \subset \operatorname{dom} T$ and the pair (Ξ, T) satisfies condition (5.9) in the quasimetric space (X, q_λ). By Theorem 5.4, there is $x_* \in T(x_*)$ contradicting the structure of T.

In summary, case 2 never holds while case 1 always happens and proves the validity of Theorem 4.3. △

Theorem 5.7. *Theorem 5.5 $\Longrightarrow$ Theorem 4.7.*

Proof. Proceed similarly the proof of Theorem 5.6. △

As a result of the above consideration, four Theorems 4.3, 4.7, 5.4, and 5.5 are equivalent.

To conclude this section, we establish a version of the CFPT which is equivalent to the weak τ-distance version of the EVP in Theorem 4.11.

Theorem 5.8. (a weak τ-distance version of [26, Theorem 2.3]). *Let (X, q) be a complete Hausdorff quasimetric space, $p : X \times X \to \mathbb{R}_+$ be a weak τ-distance in Definition 4.6, $\varphi : X \to \mathbb{R} \cup \{\infty\}$ be proper, decreasingly closed, and bounded from below, and $T : X \rightrightarrows X$ be a set-valued mapping. Assume that there is $x_0 \in \operatorname{dom}\varphi$ such that the set*

$$\Xi := \{y \in X|\ p(x,y) \leq \Lambda\big(\varphi(x)\big)\big(\varphi(x) - \varphi(y)\big)\}$$

is nonempty and that the pair (T, Ξ) satisfies

$$\forall\, x \in X,\ \exists\, y \in T(x)\colon p(x,y) \leq \Lambda\big(\varphi(x)\big)\big(\varphi(x) - \varphi(y)\big),$$

where $\Lambda : \mathbb{R} \to \mathbb{R}_{++}$ is a nondecreasing function. Then, T has a fixed point $x_* \in X$ satisfying $p(x_*, x_*) = 0$. Indeed, Theorem 5.8 $\Longleftrightarrow$ Theorem 4.11 in the sense that one implies the other.

Proof. By using similar arguments of proving the equivalence between a version of the EVP and its coresponding version of the CFPT. △

We now have that Theorem 5.8 $\Longleftrightarrow$ Theorem 4.11 $\overset{Thm.\ 4.11}{\Longleftrightarrow}$ Theorem 4.7 $\overset{Fig.\ 1}{\Longleftrightarrow}$ Theorem 5.5. This implies that a weak τ distance version of [26, Theorem 2.3] is not more general than the original form without the monotonicity mapping Λ.

6 Applications

In this section, we describe the role of the EVP and the CFPT with generalized distances in the model of the formation and break of workers' hiring and firing routines. The reader is referred to [41, 42] for a unified framework in Behavioral Sciences; see also [4] for applications in Psychology, [5] for applications to capability theory of well-being, [8] for a numerical method in routine's formation with resistance to change, following worthwhile changes.

The minimization formulation of the EVP considers a complete and Hausdorff quasimetric space (X, q) with a weak τ-distance $p : X \times X \to \mathbb{R}_+$, a lower semicontinous, not identically $+\infty$, and bounded from below payoff-to-be-decreased function $f : X \to \mathbb{R} \cup \{+\infty\}$, and initial conditions $\lambda > 0$, and $x_0 \in \operatorname{dom} \varphi$ such that $f(x_0) - f(x) \geq \lambda p(x_0, x)$ for some x. Then, it shows that there exists $x_* \in X$ such that

(a) $f(x_0) - f(x_*) \geq \lambda p(x_0, x_*)$;

(b) $f(x_*) - f(x) < \lambda p(x_*, x)$ for all $x \neq x_*$;

(c) $p(x_0, x_*) \leq \frac{\varepsilon}{\lambda}$ provided that $f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon$ for some $\varepsilon > 0$.

Under the same setting, the maximization formulation of EVP considers a upper semicontinuous, not identically $-\infty$, and bounded from above payoff-to-be-increased function $g : X \to \mathbb{R}$, and initial conditions $\lambda > 0$, and $x_0 \in X$ such that $g(x) - g(x_0) \geq \lambda p(x_0, x)$ for some $x \in X$. Then, there exists $x_* \in X$ such that

(a′) $g(x_*) - g(x_0) \geq \lambda p(x_0, x_*)$;

(b′) $g(x) - g(x_*) < \lambda p(x_*, x)$ for all $x \neq x_*$;

(c′) $p(x_0, x_*) \leq \frac{\varepsilon}{\lambda}$ provided that $g(x_0) \geq \sup_{x \in X} g(x) - \varepsilon$ for some $\varepsilon > 0$.

Let us defined an advantage-to-change from x to y as a decrease in unsatisfaction $A(x, y) = f(x) - f(y) \geq 0$ or an increase in payoff (profit) $A(x, y) = g(y) - g(x) \geq 0$. Define also an resistance-to-change as a w-distance $R(x, y) = C(x, y) - C(x, x) + e(x) = p(x, y)$, where $e(x)$ can take positive values for some $x \in X$ and so is $R(x, x)$. Then, worthwhile-to-change sets are

$$W_\lambda(x) = \{y \in X | \; g(y) - g(x) \geq \lambda p(x, y)\} \text{ or } W_\lambda(x) = \{y \in X | \; f(x) - f(y) \geq \lambda p(x, y)\}.$$

Then, the EVPs say:

(a′) There exists an acceptable one step transition from an initial position to an end $x_* \in W_\lambda(x_0)$. This means that it is worthwhile to move directly from x_0 to x_*. The proof also shows that x_* is an aspiration point, i.e. $x_* \in W_\lambda(x_n)$ for all $n \in \mathbb{N}$. This means that it is worthwhile to reach x_* starting from each x_n for $n \in \mathbb{N}$.

(b′) The end is a stable position (a stationary trap): $W_\lambda(x_*) \subset \{x_*\}$. In other words, being at x_*, it is not worthwhile to move from x_* to any different action $x \neq x_*$.

(c′) The end can be reached in a feasible way: $C(x_0, x_*) = p(x_0, x_*) \leq \varepsilon/\lambda$. Then, if the agent cannot spend more than $\overline{C} > 0$ in terms of costs to move from x_0 to x_*, the agent must choose his/her acceptability ratio λ to satisfy $\varepsilon/\lambda \leq \overline{C}$. The conditions $0 \leq f(x_0) - \underline{f} < \varepsilon$ or $g(x_0) > \overline{g} - \varepsilon$ tell us that the gap between the initial and final unsatisfactions, or the gap between the maximun and initial profit is less than $\varepsilon > 0$, respectively, where

$\overline{g} := \sup_{x \in X} g(x)$ and $\underline{f} := \inf_{x \in X} f(x)$.

Then, given all these behavioral simplifications, the EVPs tell us that, starting from some initial position (action, or state, some being or having), there exists a worthwhile transition to a final variational trap x_* (both an aspiration point and a stable point).

How hiring and firing routines form and break.
Our model explains the formation of routines (a routinization process) in terms of the convergence of an organizational worthwhile stay and move dynamic to a variational trap, seen as a permanent routine. Our example shows how hiring and firing routines form, gradually, after a lot of repetitions, in response to a more and more similar stimuli (context, environment, cue). This modelizes fairly well the well known and concrete study of Feldman [15] on hiring and firing routines. Let us consider the main elements which favor such a routinization process to end in a routine.

(A) The repeated "habit-loop" of a routinization process. Following Duhigg, routines in [12] can be seen as a three-part "habit loop": a cue (stimuli), a behavior (action/response) and a reward (payoff). In our model, each current period $n+1$, the cue is the stimuli given by the past unsatisfaction to do not have succeeded to reach the optimum $\overline{g} - g(x_n) = f(x_n) > 0$, the behavior is the collective hiring, firing and repeated employment action x_{n+1} and the payoff is the difference between motivation and resistance to change

$$\Delta_\lambda(x_n, x_{n+1}) = M(x_n, x_{n+1}) - \lambda R(x_n, x_{n+1}) = A(x_n, x_{n+1}) - \lambda I(x_n, x_{n+1}).$$

(B) The psychological state of the entrepreneur and his rationality. In Psychology, this state describes how, each period, the entrepreneur self regulates his activity, i.e. how, each current period, he chooses his current goal (goal setting), how he tries to reach this goal (goal striving), and how he pursues or abandons this current goal (goal pursuit or goal disengagement). Using the machinery of the VR approach in [41, 42], the present paper supposes that the current proximal goal of the entrepreneur is, each current period $n+1$, to find a worthwhile change $x_{n+1} \in W_\lambda(x_n)$. Then, the agent is proactive, bounded and procedural rational. Each period, he tries to satisfy with not too much sacrifices, balancing motivation and resistance to change, desirability and feasibility issues. This generalizes in several directions the famous static satisficing approach by Simon [38], adding, in a dynamic context, motivation (desires) and resistance to change (sacrifices) to the analysis. Worthwhile to change conditions drive, implicitly, three main and famous psychological theories of the entrepreneur: i) in Economics, the Schumpeterian theory of the entrepreneur [37] shows how the entrepreneur, being resilient to the accumulation of obstacles to change have the energy to break resistances and the motivation to innovate, ii) in Psychology, Ajzen's theory of planned behavior [1] shows how intentions to act balance perceptions of personal attractiveness (desirability), social norms, and feasibility, iii) in Management Sciences, Shapero's theory of entrepreneurial [40] intentions balances perceptions of personal desirability, feasibility, and propensity to act.

C) The role of the environment/context. In Behavioral Sciences the stability of the context (the recurrence of a similar context/environment which acts as a tigger) is a necessary condition for the formation of routines. In this paper, the stability of the context is a condition for the convergence to a variational trap. This stability is defined by the regularity of the (VR) structure, i.e. the regularity (upper semicontinuity) of the payoff to be increased, in our model the current profit of the entrepreneur $g(.)$, and the lower semicontinuity in the second variable of the inconvenience-to-change function which is a w-distance.

D) The routinization process. The hiring/firing process becomes more and more similar to end in an employment routine where no worker is hired or fired. Then, the structure of competences remains the same and the firm stops to innovate. The process ends in a variational trap. In our (VR) model, the formation of a hiring and firing routine comes from the balance between motivation and resistance to change, when, at the end, resistance to change wins. Routines break in the opposite case, when the habit loop breaks because motivation to change wins.

Conclusions

— The main results of this paper are summarized in Figure 1.

— We formulate both weak τ-distance and quasi-distance versions of the Ekeland variational principle as well as the Caristi fixed point theorem and prove that they are equivalent in the sense that one can be derived from another.

— While they are equivalent, the generalized distance versions are essential for applications in the formation and break of workers hiring and firing routines.

— In [28], the authors introduced the notion of λ-spaces which is much weaker than cone metric spaces and then established some critical point theorems and Ekeland-type variational theorems in the setting of λ-spaces. In [29], a modification of the notion of a w-distance was presented to further extend some fixed point results for generalized contractive set-valued maps on complete preordered quasimetric spaces. In [36], new fixed point theorems under c-distance in ordered cone metric spaces were established. Our further research will examize whether they are equivalent to our results as well.

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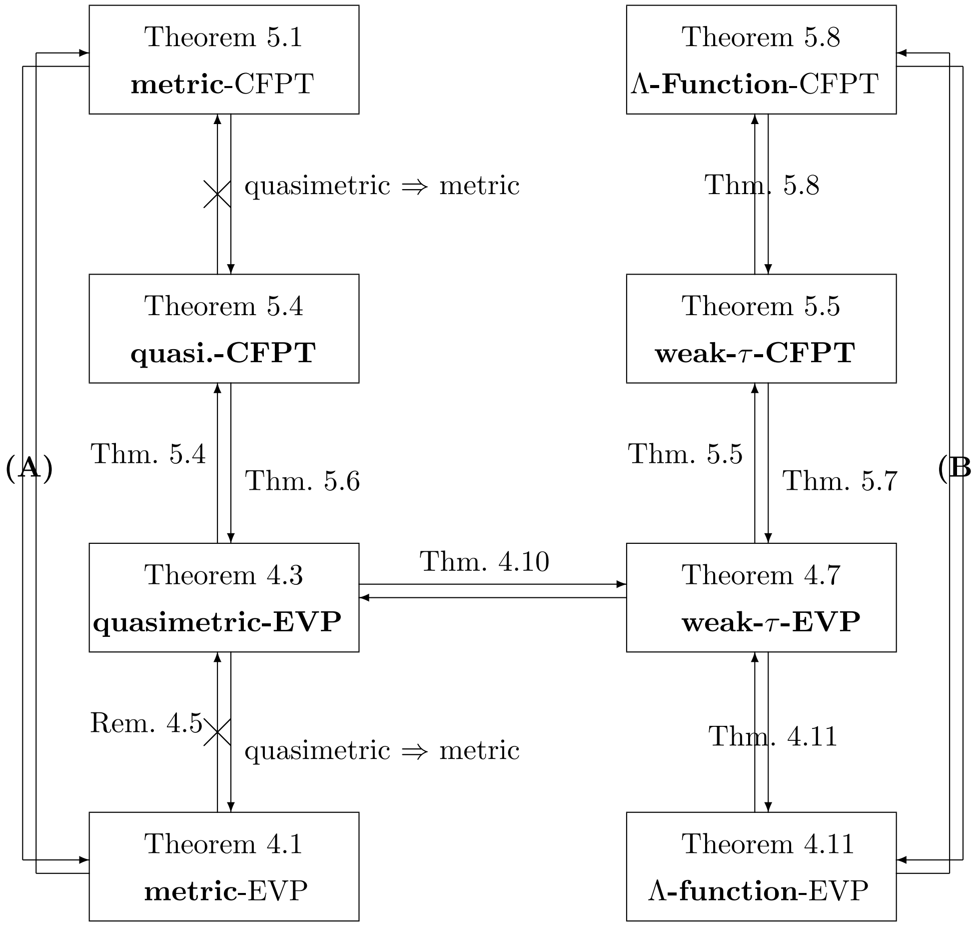

Figure 1: This diagram illustrates main results obtained in this paper. Equivalence (A) is well-known. Equivalence (B) was first established in [27] for w-distances in metric spaces.

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