

A Counterexample to “Threshold Boolean form for joint probabilistic constraints with random technology matrix”

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Abstract

Recently, in the paper “Threshold Boolean form for joint probabilistic constraints with random technology matrix” (Math. Program. 147:391–427, 2014), Kogan and Lejeune proposed a set of mixed-integer programming formulations for probabilistically constrained stochastic programs having random constraint matrix and finite support distribution. We show that the proposed formulations do not in general correctly model such problems. In particular, we characterize the structure of the feasible region defined by the proposed formulations, and provide an example of a probabilistically constrained stochastic program that has a feasible region that does not match this structure.

1 Introduction

Kogan and Lejeune [1] have recently proposed a set of mixed-integer programming formulations for probabilistically constrained stochastic programs. In this note, we provide an example that shows that the first of these formulations does not in general correctly model such problems. The additional formulations in [1] are equivalent to this first one, and so also do not model such problems.

As much as possible, we use the notation of [1]. A probabilistically constrained stochastic linear program is defined as:

$$\begin{aligned} \max \quad & q^T x \\ \text{s.t.} \quad & Ax \geq b \\ & \mathbb{P}(Tx \leq d) \geq p \\ & x \in \mathbb{R}_+^n. \end{aligned} \tag{PCLP}$$

Here, A is an $m \times n$ deterministic matrix, $b \in \mathbb{R}^m$, T is an $r \times n$ random matrix, $d \in \mathbb{R}^r$, and $p \in (0, 1)$. Define $J = \{1, \dots, n\}$ as the index set of the decision variables. It is assumed in [1]

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that $T_{ij} = s_{ij}\xi_j$, where s_{ij} is a positive number and ξ is an n -dimensional random vector, and that the random vector ξ has finite support given by $\{\omega^k : k \in \Omega\}$ ¹. In this note, we focus on the special case in which the constraints $Ax \geq b$ are not present and $r = 1$. Thus, T is a random n -dimensional vector and d is a scalar. We also consider the special case in which $s_{1j} = 1$ for all j , and so $T = \xi$.

For each $j \in J$, Kogan and Lejeune [1] define the cut-set associated with the random variable ξ as:

$$C_j = \{\omega_j^k : \mathbb{P}(\xi \leq \omega^k) \geq p, k \in \Omega\} = \{c_{j1}, \dots, c_{jn_j}\}$$

where $c_{j1} < c_{j2} < \dots < c_{jn_j}$ for each $j \in J$. Next, they consider all possible vectors obtained as ‘‘recombinations’’ of the cut-sets, i.e., $C_1 \times C_2 \times \dots \times C_n$, let $\{\omega^k : k \in \bar{\Omega}\}$ represent this set, and define

$$\bar{\Omega}^+ = \{k \in \bar{\Omega} : \mathbb{P}(\xi \leq \omega^k) \geq p\}, \quad \bar{\Omega}^- = \{k \in \bar{\Omega} : \mathbb{P}(\xi \leq \omega^k) < p\}.$$

A binary encoding β^k of each point in $k \in \bar{\Omega}$ is then defined as follows:

$$\beta_{jl}^k = \begin{cases} 1 & \text{if } \omega_j^k \geq c_{jl} \\ 0 & \text{otherwise,} \end{cases} \quad j \in J, l = 1, \dots, n_j.$$

Finally, the sets $\bar{\Omega}_B^-$ and $\bar{\Omega}_B^+$ are defined as the index sets of the boolean representations of the corresponding vectors in $\bar{\Omega}^-$ and $\bar{\Omega}^+$, respectively.²

We now present the mixed-integer bilinear programming formulation proposed in Theorem 28 of [1]. The theorem states that the problem (PCLP) can be reformulated as the following problem:

$$\max q^T x \tag{1a}$$

$$\text{s.t. } x \in \mathbb{R}_+^n \tag{1b}$$

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k \leq |J| - 1, \quad k \in \bar{\Omega}_B^- \tag{1c}$$

$$\sum_{l=1}^{n_j} \lambda_{jl} = 1, \quad j \in J \tag{1d}$$

$$\lambda_{jl} \in \{0, 1\}, \quad j \in J, l = 1, \dots, n_j \tag{1e}$$

$$\sum_{l=1}^{n_j} \lambda_{jl} c_{jl} = y_j, \quad j \in J \tag{1f}$$

$$\sum_{j \in J} y_j x_j \leq d \tag{1g}$$

$$c_{j1} \leq y_j \leq c_{jn_j}, \quad j \in J. \tag{1h}$$

¹In section 2.1 of [1], Ω is defined to be a vector in \mathbb{R}_+^n , but that definition is not consistent with its use in [1].

²There is some ambiguity in the definitions of these sets in [1], e.g., in Definition 10 they are defined as sets of boolean vectors, but in Theorem 11 and its later use, they are index sets. We follow the latter definition.

Here, we have specialized the formulation to our case of $r = 1$ and $s_{1j} = 1$ for all $j \in J$.

Let Y be the set of feasible solutions to (1), and let $X = \{x \in \mathbb{R}^n : \exists(\lambda, y) \text{ with } (x, \lambda, y) \in Y\}$ be the projection of Y onto the x variable space. Then, (1) is a valid formulation of (PCLP) if and only if X is the feasible region of (PCLP). We next describe the structure of X .

Lemma 1. *The set X has the form:*

$$X = \bigcup_{t=1}^T \{x \in \mathbb{R}_+^n : \bar{y}^t x \leq d\} \quad (2)$$

where $\{\bar{y}^t, t = 1, \dots, T\}$ is a finite set of vectors in \mathbb{R}^n .

Proof. Let $\bar{\lambda}^t, t = 1, \dots, T$ be the finite set of vectors that satisfy the constraints (1c) – (1e), and for each $t = 1, \dots, T$ define \bar{y}^t by $\bar{y}_j^t = \sum_{l=1}^{n_j} \bar{\lambda}_{jl}^t c_{jl}$, for $j \in J$. Then, for any $x \in X$, let (λ, y) be such that $(x, \lambda, y) \in Y$. Then, $\lambda = \bar{\lambda}^t$ for some t , and by (1f), $y = \bar{y}^t$. Thus, $\bar{y}^t x \leq d$, which proves the inclusion \subseteq . Conversely, suppose x satisfies $\bar{y}^t x \leq d$ for some $t \in \{1, \dots, T\}$. Then $(x, \bar{\lambda}^t, \bar{y}^t) \in Y$ and hence $x \in X$, proving the inclusion \supseteq . \square

We now provide an example of (PCLP) which has a feasible region that does not have the form specified in Lemma 1, which thus demonstrates that the formulation (1) does not in general correctly model (PCLP).

Table 1: Possible realizations of ξ in Example 1.

k	ω_1^k	ω_2^k	$\mathbb{P}(\xi = \omega^k)$	$F_1(\omega_1^k)$	$F_2(\omega_2^k)$	$F(\omega^k)$
1	1	2	1/3	1/3	1	1/3
2	2	1	1/3	1	1/3	1/3
3	2	2	1/3	1	1	1

Example 1. Let $n = 2$, $d = 3$, and let the three possible realizations of ξ be given in Table 1, along with the joint cumulative probability function F and the marginal cumulative distribution functions F_i , for $i = 1, 2$. Set $p = 2/3$ so that in order to be feasible to the probabilistic constraint a solution x must satisfy at least two of the following three inequalities:

$$x_1 + 2x_2 \leq 3, \quad 2x_1 + x_2 \leq 3, \quad 2x_1 + 2x_2 \leq 3.$$

Because the last inequality dominates the other two, the feasible region reduces to $F := \{x \in \mathbb{R}_+^2 : x_1 + 2x_2 \leq 3, 2x_1 + x_2 \leq 3\}$. This feasible region does not satisfy the form (2) given in Lemma 1. E.g., for a set X of the form (2), the set $\mathbb{R}_+^n \setminus X$ is convex. On the other hand, the set $\mathbb{R}_+^n \setminus F$ is not convex. In particular, the points $(2, 0)$ and $(0, 2)$ are not in F , but $(1, 1) \in F$.

We next derive the formulation (1) for this example. The sets of cut points are: $C_1 = \{\omega_j^k : F_j(\omega_j^k) \geq p, k \in \Omega\} = \{2\}$ and also $C_2 = \{2\}$. Thus, the only recombination is the vector $(2, 2)$, which satisfies $F((2, 2)) = 1 \geq p$, and so in this example, the set $\bar{\Omega}_B$ is empty and there are no constraints (1c). The constraints (1d) are simply $\lambda_{11} = 1$ and $\lambda_{21} = 1$. The constraint (1f) is:

$$x_1(2\lambda_{11}) + x_2(2\lambda_{21}) \leq 3.$$

Substituting in the only feasible λ solution we obtain:

$$X = \{x \in \mathbb{R}_+^2 : 2x_1 + 2x_2 \leq 3\}.$$

Thus, $(1, 1) \in F$ but $(1, 1) \notin X$. □

Formulation (1), including its generalization to multiple rows, does provide an *inner* approximation of the feasible region of (PCLP), which was shown in [1]. The error in the proof that it is an equivalent formulation lies in the reverse inclusion, that every feasible solution of (PCLP) has a representation in (1). In [1] Kogan and Lejeune show that any solution of (PCLP) *that is defined by* $k \in \bar{\Omega}_B^+$ is feasible for (1). This is indeed true, but there may be feasible solutions for (PCLP) that are not defined by any $k \in \bar{\Omega}_B^+$. In Example 1 the only vector in the set $\bar{\Omega}_B^+$ is $(2, 2)$. The solution $(1, 1)$ is feasible to (PCLP) but does not satisfy the constraint defined by this vector ($2x_1 + 2x_2 \leq 3$).

We conclude that (1) can be used as a heuristic for finding feasible solutions for (PCLP). Note, however, that the number of recombinations that have to be enumerated to construct the formulation (1) grows exponentially in n , so this heuristic does not scale well with the number of decision variables. For alternative heuristics that scale better with the number of decision variables, we refer the reader to [2–4]. For exact solution methods that can solve problems of the form (PCLP) with random constraint matrix and finite scenarios, we refer the reader to [5–8].

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