New Ranks for Even-Order Tensors and Their Applications in Low-Rank Tensor Optimization

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Abstract

In this paper, we propose three new tensor decompositions for even-order tensors corresponding respectively to the rank-one decompositions of some unfolded matrices. Consequently such new decompositions lead to three new notions of (even-order) tensor ranks, to be called the M-rank, the symmetric M-rank, and the strongly symmetric M-rank in this paper. We discuss the bounds between these new tensor ranks and the CP(CANDECOMP/PARAFAC)-rank and the symmetric CP-rank of an even-order tensor. In particular, we show: (1) these newly defined ranks actually coincide with each other if the even-order tensor in question is super-symmetric; (2) the CP-rank and symmetric CP-rank for a fourth-order tensor can be both lower and upper bounded (up to a constant factor) by the corresponding M-rank. Since the M-rank is much easier to compute than the CP-rank, we can replace the CP-rank by the M-rank in the low-CP-rank tensor recovery model. Numerical results on both synthetic data and real data from colored video completion and decomposition problems show that the M-rank is indeed an effective and easy computable approximation of the CP-rank in the context of low-rank tensor recovery.

Keywords: Matrix Unfolding, Tensor Decomposition, CP-Rank, Tensor Completion, Tensor Robust PCA.

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1 Introduction

Tensor data have appeared frequently in modern applications such as computer vision [40], diffusion magnetic resonance imaging [12, 3, 34], quantum entanglement problem [15], spectral hypergraph theory [16] and higher-order Markov chains [27]. Tensor-based multi-dimensional data analysis has shown that tensor models can take full advantage of the multi-dimensionality structures of the data, and generate more useful information. A common observation for huge-scale data analysis is that the data exhibits a low-dimensionality property, or its most representative part lies in a low-dimensional subspace. Due to this low-dimensionality property of the tensor data, it becomes imperative to understand (and compute) the rank of a tensor, which, however, is notoriously known to be a thorny issue. The most commonly used definition of tensor rank is the so-called *CP-rank*, where "C" stands for CANDECOMP while "P" corresponds to PARAFAC and these are two alternate names for the same tensor decomposition. As we shall see later, being a most natural notion of tensor rank, the CP-rank is unfortunately very difficult to compute numerically. Nonetheless, let us formally introduce the CP-rank below.

Definition 1.1 *Given a d-th order tensor* $\mathcal{F} \in \mathbb{C}^{n_1\times n_2\times\cdots\times n_d}$ *in complex domain, its CP-rank (denoted as* $\mathrm{rank}_{CP}(\mathcal{F})$*) is the smallest integer* r *exhibiting the following decomposition*

$$\mathcal{F} = \sum_{i=1}^{r} a^{1,i} \otimes a^{2,i} \otimes \cdots \otimes a^{d,i}, \tag{1}$$

where $a^{k,i} \in \mathbb{C}^{n_i}$ *for* $k = 1,\ldots,d$ *and* $i = 1,\ldots,r$*. Similarly, for a real-valued tensor* $\mathcal{F} \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_d}$*, its CP-rank in the real domain (denoted as* $\mathrm{rank}^{\mathbb{R}}_{CP}(\mathcal{F})$*) is the smallest integer* r *such that there exists a real-valued decomposition* (1) *with* $a^{k,i} \in \mathbb{R}^{n_i}$ *for* $k = 1,\ldots,d$ *and* $i = 1,\ldots,r$*.*

An extreme case is $r = 1$, where $\mathcal{F}$ is called a rank-1 tensor in this case. For a given tensor, finding its best rank-1 approximation, also known as finding the largest eigenvalue of a given tensor, has been studied in [28, 33, 9, 21, 19]. It should be noted that the CP-rank of a real-valued tensor can be different over $\mathbb{R}$ and $\mathbb{C}$; i.e., it may hold that $\mathrm{rank}_{CP}(\mathcal{F}) < \mathrm{rank}^{\mathbb{R}}_{CP}(\mathcal{F})$ for a real-valued tensor $\mathcal{F}$. As an example, a real-valued $2 \times 2 \times 2$ tensor $\mathcal{F}$ is given in [24] and it can be shown that $\mathrm{rank}^{\mathbb{R}}_{CP}(\mathcal{F}) = 3$ while $\mathrm{rank}_{CP}(\mathcal{F}) = 2$. In this paper, we shall focus on the notion of CP-rank in the complex domain and discuss two low-CP-rank tensor recovery problems: tensor completion and robust tensor recovery.

Low-CP-rank tensor recovery problem seeks to recover a low-CP-rank tensor based on limited observations. This problem can be formulated as

$$\min_{\mathcal{X}\in\mathbb{C}^{n_1\times n_2\cdots\times n_d}} \ \mathrm{rank}_{CP}(\mathcal{X}), \quad \text{s.t. } \boldsymbol{L}(\mathcal{X}) = b, \tag{2}$$

where $\boldsymbol{L}: \mathbb{C}^{n_1 \times n_2 \cdots \times n_d} \to \mathbb{C}^p$ is a linear mapping and $b \in \mathbb{C}^p$ denotes the observation to $\mathcal{X}$ under $\boldsymbol{L}$. Low-CP-rank tensor completion is a special case of (2) where the linear mapping in the constraint picks certain entries of the tensor. The low-CP-rank tensor completion can be formulated as

$$\min_{\mathcal{X} \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_d}} \operatorname{rank}_{CP}(\mathcal{X}), \quad \text{s.t. } P_\Omega(\mathcal{X}) = P_\Omega(\mathcal{X}_0), \tag{3}$$

where $\mathcal{X}_0 \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_d}$ is a given tensor, Ω is an index set and

$$[P_\Omega(\mathcal{X})]_{i_1, i_2, \ldots, i_d} = \begin{cases} \mathcal{X}_{i_1, i_2, \ldots, i_d}, & \text{if } (i_1, i_2, \ldots, i_d) \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

In practice, the underlying low-CP-rank tensor data $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_d}$ may be heavily corrupted by a sparse noise tensor $\mathcal{Y}$. To identify and remove the noise, one can solve the following robust tensor recovery problem:

$$\min_{\mathcal{Y}, \mathcal{Z} \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_d}} \operatorname{rank}_{CP}(\mathcal{Y}) + \lambda \|\mathcal{Z}\|_0, \quad \text{s.t. } \mathcal{Y} + \mathcal{Z} = \mathcal{F}, \tag{4}$$

where $\lambda > 0$ is a weighting parameter and $\|\mathcal{Z}\|_0$ denotes the number of nonzero entries of $\mathcal{Z}$.

Solving (2) and (4), however, are very difficult in practice. In fact, determining the CP-rank of a given tensor is already proved to be NP-hard in general [14]. Even worse than an ordinary NP-hard problem, computing the CP-rank for small size instances remains a difficult task. For example, a particular $9 \times 9 \times 9$ tensor is given in [25] and its CP-rank is only known to be in between 18 and 23 to this date. This makes the above low-CP-rank tensor recovery problems (2) and (4) appear to be hopeless. One way out of this dismay is to approximate the CP-rank by some more reasonable objects. Since computing the rank of a matrix is easy, a classical way for tackling tensor optimization problem is to unfold the tensor into certain matrix and then resort to some well established solution methods for matrix-rank optimization. A typical matrix unfolding technique is the so-called mode-n matricization [20]. Specifically, for a given tensor $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_d}$, we denote its mode-$n$ matricization by $F(n)$, which is obtained by arranging the n-th index of $\mathcal{F}$ as the column index of $F(n)$ and merging other indices of $\mathcal{F}$ as the row index of $F(n)$. The Tucker rank (or the mode-n-rank) of $\mathcal{F}$ is defined as the vector $(\operatorname{rank}(F(1)), \ldots, \operatorname{rank}(F(d)))$. For simplicity the average-rank of the different mold unfoldings is often used and denoted as

$$\operatorname{rank}_n(\mathcal{F}) := \frac{1}{d} \sum_{j=1}^{d} \operatorname{rank}(F(j)).$$

As such, $\operatorname{rank}_n(\mathcal{F})$ is much easier to compute than the CP-rank, since it is just the average of d matrix ranks. Therefore, the following low-n-rank minimization model has been proposed for tensor recovery [29, 11]:

$$\min_{\mathcal{X} \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_d}} \operatorname{rank}_n(\mathcal{X}), \quad \text{s.t. } \boldsymbol{L}(\mathcal{X}) = b, \tag{5}$$

where $\boldsymbol{L}$ and b are the same as the ones in (2). Since minimizing the rank function is still difficult, it was suggested in [29] and [11] to convexify the matrix rank function by the nuclear norm, which has become a common practice due to the seminal works on matrix rank minimization (see, e.g., [10, 35, 6, 7]). That is, the following convex optimization problem is solved instead of (5):

$$\min_{\mathcal{X}\in\mathbb{C}^{n_1\times n_2\cdots\times n_d}} \ \frac{1}{d}\sum_{j=1}^{d}\|X(j)\|_*, \quad \text{s.t. } \boldsymbol{L}(\mathcal{X})=b, \tag{6}$$

where the nuclear norm $\|X(j)\|_*$ is defined as the sum of singular values of matrix $X(j)$. Some efficient algorithms such as alternating direction method of multipliers and Douglas-Rachford operator splitting methods were proposed in [29] and [11] to solve (6).

However, to our best knowledge, the relationship between the CP-rank and mode-n-rank of a tensor is still unclear so far. It is easy to see (from similar argument as in Theorem 3.3) that the mode-n-rank is a lower bound for the CP-rank. But whether it can also lead to an upper bound (up to a constant factor) of the CP-rank is not known. In other words, the low-n-rank minimization (5) can only be regarded as minimizing a lower bound of the CP-rank. Consequently, the CP-rank of $\mathcal{X}^*$ obtained by solving (5) (or its convex relaxation (6)) may be still high, even though it is already of low mode-n-rank. Convexifying the robust tensor recovery problem (4) was also studied by Tomioka et al. [38] and Goldfarb and Qin [13]. Specifically, they used mode-n-rank to replace the CP-rank of $\mathcal{Y}$ and $\|\mathcal{Z}\|_1$ to replace $\|\mathcal{Z}\|_0$ in the objective of (4). However, it is again not guaranteed that the resulting solution $\mathcal{X}$ has a low CP-rank, as the model considered is based on minimizing a different tensor rank. Other works on this topic include [39, 36, 32, 22, 23]. Specifically, [39] analyzed the statistical performance of the convex relaxation of the low-n-rank minimization model. [36] compared the performance of the convex relaxation of the low-n-rank minimization model and the low-rank matrix completion model on applications in spectral image reconstruction. Mu et al. [32] proposed a square unfolding technique, which was also considered by Jiang et al. in [19] for tensor PCA, in tensor completion problem, and they further showed that this square unfolding is a better way to recover a low-rank tensor compared with the unfolding techniques used in low-n-rank minimization. [22] proposed a Riemannian manifold optimization algorithm for finding a local optimum of the Tucker rank constrained optimization problem. [23] studied some adaptive sampling algorithms for low-rank tensor completion.

Note that in the definition of CP-rank, $\mathcal{F}\in\mathbb{C}^{n_1\times n_2\times\cdots\times n_d}$ is decomposed to the sum of rank-1 tensors. If $\mathcal{F}$ is a super-symmetric tensor, i.e., its component is invariant under any permutation of indices, then a natural extension is to decompose $\mathcal{F}$ into the sum of symmetric rank-1 tensors, and this leads to the definition of symmetric CP-rank (see, e.g., [8]).

Definition 1.2 *Given a d-th order n-dimensional super-symmetric complex-valued tensor $\mathcal{F}$, its* *symmetric CP-rank (denoted by* $\mathrm{rank}_{SCP}(\mathcal{F})$*) is the smallest integer* r *such that*

$$\mathcal{F} = \sum_{i=1}^{r} \underbrace{a^i \otimes \cdots \otimes a^i}_{d},$$

with $a^i \in \mathbb{C}^n, i = 1, \ldots, r$*.*

It is obvious that $\mathrm{rank}_{CP}(\mathcal{F}) \leq \mathrm{rank}_{SCP}(\mathcal{F})$ for any given super-symmetric tensor $\mathcal{F}$. In the matrix case, i.e., when $d = 2$, it is known that the rank and symmetric rank are identical. However, in the higher order case, whether the CP-rank equals the symmetric CP-rank is still unknown, and this has become an interesting and challenging open problem (see Comon et al. [8]). There has been some recent progress towards solving this problem. In particular, Zhang et al. [41] recently proved $\mathrm{rank}_{CP}(\mathcal{F}) = \mathrm{rank}_{SCP}(\mathcal{F})$ for any d-th order super-symmetric tensor with $\mathrm{rank}_{CP}(\mathcal{F}) \leq d$.

At this point, it is important to remark that the CP-rank stems from the idea of decomposing a general tensor into a sum of simpler – viz. *rank-one* in this context – tensors. The nature of the "simpler components" in the sum, however, can be made flexible and inclusive. In fact, in many cases it does not have to be a rank-one tensor as in the CP-decomposition. In particular, for a $2d$-th order tensor, the "simple tensor" being decomposed into could be the outer product of two tensors with lower degree, which is d in this paper, and we call this new decomposition to be *M-decomposition*. It is easy to see (in our later discussion) that after square unfolding each term in the M-decomposition is actually a rank-one matrix. Consequently, the minimum number of such simple terms deserves to be called a *rank* too, or indeed the *M-rank* in our context, to be differentiated from other existing notions of tensor ranks. By imposing further symmetry on the "simple tensor" that composes the M-decomposition, the notion of symmetric M-decomposition (symmetric M-rank), and strongly symmetric M-decomposition (strongly symmetric M-rank) naturally follow. We will introduce the formal definitions later.

To summarize, the main contributions of this paper lie in several folds. We first introduce several new notions of tensor decomposition for the even order tensors, followed by the new notions of tensor M-rank, symmetric M-rank and strongly M-rank respectively. Second, we prove the equivalence of these three rank definitions for even-order super-symmetric tensors. Third, we establish the connection between these ranks and the CP-rank and symmetric CP-rank. Basically, we show that for a fourth-order tensor, both CP-rank and symmetric CP-rank can be lower and upper bounded (up to a constant factor) by the M-rank. Finally, we apply the newly developed M-rank to obtain tractable approximations of the tensor completion and robust tensor recovery problems, and test their performances on some video processing problems.

Notation. We use $\mathbb{C}^n$ to denote the n-dimensional complex-valued vector space. We use calligraphic letter to denote a tensor, i.e. $\mathcal{A} = (\mathcal{A}_{i_1 i_2 \cdots i_d})_{n_1 \times n_2 \times \cdots \times n_d}$. $\mathbb{C}^{n_1 \times n_2 \times \cdots \times n_d}$ denotes the space of d-th order $n_1 \times n_2 \times \cdots \times n_d$ dimensional complex-valued tensor. $\pi(i_1, i_2, \cdots, i_d)$ denotes a permutation of indices $(i_1, i_2, \cdots, i_d)$. We use $\mathcal{A}_\pi$ to denote the tensor obtained by permuting the indices of $\mathcal{A}$ according to permutation π. Formally speaking, a tensor $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_d}$ is called super-symmetric, if $n_1 = n_2 = \ldots = n_d$ and $\mathcal{F} = \mathcal{F}_\pi$ for any permutation π. The space where $\underbrace{n \times n \times \cdots \times n}_{d}$ super-symmetric tensors reside is denoted by $\mathbb{S}^{n^d}$. We use $\otimes$ to denote the outer product of two tensors; that is, for $\mathcal{A}_1 \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_d}$ and $\mathcal{A}_2 \in \mathbf{R}^{n_{d+1} \times n_{d+2} \times \cdots \times n_{d+\ell}}$, $\mathcal{A}_1 \otimes \mathcal{A}_2 \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_{d+\ell}}$ and

$$(\mathcal{A}_1 \otimes \mathcal{A}_2)_{i_1 i_2 \cdots i_{d+\ell}} = (\mathcal{A}_1)_{i_1 i_2 \cdots i_d} (\mathcal{A}_2)_{i_{d+1} \cdots i_{d+\ell}}.$$

2 M-rank, symmetric M-rank and strongly symmetric M-rank

In this section, we shall introduce the M-decomposition (correspondingly M-rank), the symmetric M-decomposition (correspondingly symmetric M-rank), and the strongly symmetric M-decomposition (correspondingly strongly symmetric M-rank) for tensors, which will be used to provide lower and upper bounds for the CP-rank and the symmetric CP-rank.

2.1 M-rank of even-order tensor

Let us formally define the M-decomposition as follows:

Definition 2.1 *For an even-order tensor $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_{2d}}$, the M-decomposition is to find some tensors $\mathcal{A}^i \in \mathbb{C}^{n_1 \times \cdots \times n_d}$, $\mathcal{B}^i \in \mathbb{C}^{n_{d+1} \times \cdots \times n_{2d}}$ with $i = 1, \ldots, r$ such that*

$$\mathcal{F} = \sum_{i=1}^{r} \mathcal{A}^i \otimes \mathcal{B}^i. \tag{7}$$

The motivation to study such decomposition is based on the following novel matricization technique called square unfolding that has been considered in [19] and [32].

Definition 2.2 *The square unfolding of an even-order tensor $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_{2d}}$ (denoted by $\boldsymbol{M}(\mathcal{F}) \in \mathbb{C}^{(n_1 \cdots n_d) \times (n_{d+1} \cdots n_{2d})}$) is a matrix that is defined as*

$$\boldsymbol{M}(\mathcal{F})_{k\ell} := \mathcal{F}_{i_1 \cdots i_{2d}},$$

where

$$k = \sum_{j=2}^{d} (i_j - 1) \prod_{q=1}^{j-1} n_q + i_1,\ 1 \le i_j \le n_j,\ 1 \le j \le d,$$

$$\ell = \sum_{j=d+2}^{2d} (i_j - 1) \prod_{q=d+1}^{j-1} n_q + i_{d+1},\ 1 \le i_j \le n_j,\ d+1 \le j \le 2d.$$

In Definition 2.2, the square unfolding merges d indices of $\mathcal{F}$ as the row index of $\boldsymbol{M}(\mathcal{F})$, and merges the other d indices of $\mathcal{F}$ as the column index of $\boldsymbol{M}(\mathcal{F})$. In this pattern of unfolding, we can see that the M-decomposition (7) can be rewritten as

$$\boldsymbol{M}(\mathcal{F}) = \sum_{i=1}^{r} \mathbf{a}^i (\mathbf{b}^i)^\top,$$

where $a^i = \boldsymbol{V}(\mathcal{A}^i)$, $b^i = \boldsymbol{V}(\mathcal{B}^i)$ for $i = 1, \dots, r$, and $\boldsymbol{V}(\cdot)$ is the vectorization operator. Specifically, for a given tensor $\mathcal{F} \in C^{(n_1 \times n_2 \cdots \times n_d)}$, $\boldsymbol{V}(\mathcal{F})_k := \mathcal{F}_{i_1 \cdots i_d}$ with

$$k = \sum_{j=2}^{d} (i_j - 1) \prod_{q=1}^{j-1} n_q + i_1, 1 \le i_j \le n_j, 1 \le j \le d.$$

Therefore, the M-decomposition is exactly the rank-one decomposition of the matrix $\boldsymbol{M}(\mathcal{F})$. Unless $\mathcal{F}$ is super-symmetric, there are different ways to unfold the tensor $\mathcal{F}$ by permuting the $2d$ indices. Taking this into account, the M-rank of even-order tensor is defined as follows.

Definition 2.3 *Given an even-order tensor $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \cdots \times n_{2d}}$, its M-rank (denoted by $\operatorname{rank}_M(\mathcal{F})$) is the smallest rank of all the possible square unfolding matrices, i.e.*

$$\operatorname{rank}_M(\mathcal{F}) = \min_{\pi \in \Pi(1,\dots,2d)} \operatorname{rank}\left(\boldsymbol{M}(\mathcal{F}_\pi)\right), \tag{8}$$

where $\Pi(1, \dots, 2d)$ denotes the set of all possible permutations of indices $(1, \dots, 2d)$. In other words, $\operatorname{rank}_M(\mathcal{F})$ is the smallest integer r such that

$$\mathcal{F}_\pi = \sum_{i=1}^{r} \mathcal{A}^i \otimes \mathcal{B}^i, \tag{9}$$

holds for some permutation $\pi \in \Pi(1, \dots, 2d)$, $\mathcal{A}^i \in \mathbb{C}^{n_{i_1} \times \cdots \times n_{i_d}}$, $\mathcal{B}^i \in \mathbb{C}^{n_{i_{d+1}} \times \cdots \times n_{i_{2d}}}$ with $(i_1, \cdots, i_{2d}) = \pi(1, \cdots, 2d)$, $i = 1, \dots, r$.

2.2 Symmetric M-rank and strongly symmetric M-rank of even-order super-symmetric tensor

As we mentioned earlier, the decomposition (7) is essentially based on the matrix rank-one decomposition of the matricized tensor. In the matrix case, it is clear that there are different ways to decompose a symmetric matrix; for instance,

$$ab^\top + ba^\top = \frac{1}{2}(a+b)(a+b)^\top - \frac{1}{2}(a-b)(a-b)^\top.$$

In other words, a given symmetric matrix may be decomposed as a sum of *symmetric* rank-one terms, as well as a sum of *non-symmetric* rank-one terms, however yielding the same rank: the minimum number of respective decomposed terms. A natural question arises when dealing with tensors: Does the same property holds for the super-symmetric tensors? The tensor M-decomposition is in fact subtler: the decomposed terms can be restricted to symmetric products, and moreover they can also be further restricted to be super-symmetric themselves.

Specifically, we can define symmetric M-rank and strongly symmetric M-rank for even-order super-symmetric tensor as follows.

Definition 2.4 *For an even-order super-symmetric tensor $\mathcal{F} \in \mathbb{S}^{n^{2d}}$, its symmetric M-decomposition is defined to be*

$$\mathcal{F} = \sum_{i=1}^{r} \mathcal{B}^i \otimes \mathcal{B}^i, \quad \mathcal{B}^i \in \mathbb{C}^{n^d}, i = 1, \ldots, r. \tag{10}$$

The matrix-rank of $\mathcal{F}$ (denoted by $\mathrm{rank}_{SM}(\mathcal{F})$) is the rank of the symmetric matrix $\boldsymbol{M}(\mathcal{F})$, i.e., $\mathrm{rank}_{SM}(\mathcal{F}) = \mathrm{rank}\left(\boldsymbol{M}(\mathcal{F})\right) = \mathrm{rank}_M\left(\mathcal{F}\right)$; or equivalently $\mathrm{rank}_{SM}(\mathcal{F})$ is the smallest integer r such that (10) holds.

In a similar vein, the strongly symmetric M-decomposition is

$$\mathcal{F} = \sum_{i=1}^{r} \mathcal{A}^i \otimes \mathcal{A}^i, \quad \mathcal{A}^i \in \mathbb{S}^{n^d}, i = 1, \ldots, r, \tag{11}$$

and the strongly symmetric M-rank of $\mathcal{F}$ (denoted by $\mathrm{rank}_{SSM}(\mathcal{F})$) is defined as the smallest integer r such that (11) holds.

The fact that the M-rank and symmetric M-rank of an even-order super-symmetric tensor are always equal follows from the similar property for the symmetric matrices. (Note however the M-decompositions may be different). Interestingly, we can show that $\mathrm{rank}_{SM}(\mathcal{F}) = \mathrm{rank}_{SSM}(\mathcal{F})$ for any even-order super-symmetric tensor $\mathcal{F}$, which appears to be a new property of the super-symmetric even-order tensors.

2.3 Equivalence of symmetric M-rank and strongly symmetric M-rank

To show the equivalence of the symmetric M-rank and the strongly symmetric M-rank, we need to introduce the concept of partial symmetric and some lemmas first.

Definition 2.5 *We say a tensor $\mathcal{F} \in \mathbb{C}^{n^d}$ partial symmetric with respect to indices $\{1, \ldots, m\}$, $m < d$, if*

$$\mathcal{F}_{i_1,\ldots,i_m,i_{m+1},\ldots,i_d} = \mathcal{F}_{\pi(i_1,\ldots,i_m),i_{m+1},\ldots,i_d}, \quad \forall \pi \in \Pi(1, \ldots, m).$$

We use $\pi_{i,j} \in \Pi(1,\cdots,d)$ to denote the specific permutation that exchanges the i-th and the j-th indices and keeps other indices unchanged.

The following result holds directly from Definition 2.5.

Lemma 2.1 *Suppose tensor $\mathcal{F} \in \mathbb{C}^{n^d}$ is partial symmetric with respect to indices $\{1,\ldots,m\}$, $m < d$. Then the tensor*

$$\mathcal{F} + \sum_{j=1}^{m} \mathcal{F}_{\pi_{j,m+1}}$$

is partial symmetric with respect to indices $\{1,\ldots,m+1\}$. Moreover, it is easy to verify that for $\ell \le k \le m$,

$$\begin{aligned}\left(\sum_{j=1}^{k}\left(\mathcal{F} - \mathcal{F}_{\pi_{j,m+1}}\right)\right)_{\pi_{\ell,m+1}} &= k\cdot\mathcal{F}_{\pi_{\ell,m+1}} - \sum_{j\neq\ell}\mathcal{F}_{\pi_{j,m+1}} - \mathcal{F}\\ &= -k\left(\mathcal{F} - \mathcal{F}_{\pi_{\ell,m+1}}\right) + \sum_{j\neq\ell}\left(\mathcal{F} - \mathcal{F}_{\pi_{j,m+1}}\right).\end{aligned} \tag{12}$$

We are now ready to prove the following key lemma.

Lemma 2.2 *Suppose $\mathcal{F} \in \mathbb{S}^{n^{2d}}$ and*

$$\mathcal{F} = \sum_{i=1}^{r} \mathcal{B}^i \otimes \mathcal{B}^i, \text{ where } \mathcal{B}^i \in \mathbb{C}^{n^d} \text{ is partial symmetric with respect to } \{1,\ldots,m\}, m < d.$$

Then there exist tensors $\mathcal{A}^i \in \mathbb{C}^{n^d}$, which is partial symmetric with respect to $\{1,\ldots,m+1\}$, for $i = 1,\ldots,r$, such that

$$\mathcal{F} = \sum_{i=1}^{r} \mathcal{A}^i \otimes \mathcal{A}^i.$$

Proof. Define $\mathcal{A}^i = \frac{1}{m+1}\left(\mathcal{B}^i + \sum\limits_{j=1}^{m} \mathcal{B}^i_{\pi_{j,m+1}}\right)$. From Lemma 2.1 we know that $\mathcal{A}^i$ is partial symmetric with respect to $\{1,\ldots,m+1\}$, for $i = 1,\ldots,r$. It is easy to show that

$$\mathcal{B}^i = \mathcal{A}^i + \sum_{j=1}^{m} \mathcal{C}^i_j, \text{ with } \quad \mathcal{C}^i_j = \frac{1}{m+1}\left(\mathcal{B}^i - \mathcal{B}^i_{\pi_{j,m+1}}\right).$$

Because $\mathcal{F}$ is super-symmetric, we have $\mathcal{F} = \mathcal{F}_{\pi_{d+1,d+m+1}} = \mathcal{F}_{\pi_{1,m+1}} = (\mathcal{F}_{\pi_{1,m+1}})_{\pi_{d+1,d+m+1}}$, which implies

$$\mathcal{F} = \sum_{i=1}^{r} \mathcal{B}^i \otimes \mathcal{B}^i = \sum_{i=1}^{r} \mathcal{B}^i \otimes \mathcal{B}^i_{\pi_{1,m+1}} = \sum_{i=1}^{r} \mathcal{B}^i_{\pi_{1,m+1}} \otimes \mathcal{B}^i = \sum_{i=1}^{r} \mathcal{B}^i_{\pi_{1,m+1}} \otimes \mathcal{B}^i_{\pi_{1,m+1}}. \tag{13}$$

By using (12), we have

$$\mathcal{B}^i_{\pi_{1,m+1}} = \left(\mathcal{A}^i + \sum_{j=1}^{m} \mathcal{C}^i_j\right)_{\pi_{1,m+1}} = \mathcal{A}^i + \sum_{j=2}^{m} \mathcal{C}^i_j - m \cdot \mathcal{C}^i_1. \tag{14}$$

Combining (14) and (13) yields

$$\begin{aligned}
\mathcal{F} &= \sum_{i=1}^{r} \mathcal{B}^i \otimes \mathcal{B}^i = \sum_{i=1}^{r} \left(\mathcal{A}^i + \sum_{j=1}^{m} \mathcal{C}^i_j\right) \otimes \left(\mathcal{A}^i + \sum_{j=1}^{m} \mathcal{C}^i_j\right) && (15)\\
&= \sum_{i=1}^{r} \mathcal{B}^i \otimes \mathcal{B}^i_{\pi_{1,m+1}} = \sum_{i=1}^{r} \left(\mathcal{A}^i + \sum_{j=1}^{m} \mathcal{C}^i_j\right) \otimes \left(\mathcal{A}^i + \sum_{j=2}^{m} \mathcal{C}^i_j - m \cdot \mathcal{C}^i_1\right) && (16)\\
&= \sum_{i=1}^{r} \mathcal{B}^i_{\pi_{1,m+1}} \otimes \mathcal{B}^i = \sum_{i=1}^{r} \left(\mathcal{A}^i + \sum_{j=2}^{m} \mathcal{C}^i_j - m \cdot \mathcal{C}^i_1\right) \otimes \left(\mathcal{A}^i + \sum_{j=1}^{m} \mathcal{C}^i_j\right) && (17)\\
&= \sum_{i=1}^{r} \mathcal{B}^i_{\pi_{1,m+1}} \otimes \mathcal{B}^i_{\pi_{1,m+1}} = \sum_{i=1}^{r} \left(\mathcal{A}^i + \sum_{j=2}^{m} \mathcal{C}^i_j - m \cdot \mathcal{C}^i_1\right) \otimes \left(\mathcal{A}^i + \sum_{j=2}^{m} \mathcal{C}^i_j - m \cdot \mathcal{C}^i_1\right). && (18)
\end{aligned}$$

It is easy to check that

$$\frac{(18) + m \times (17) + m \times (16) + m^2 \times (15)}{(1+m)^2} \implies \mathcal{F} = \sum_{i=1}^{r} \left(\mathcal{A}^i + \sum_{j=2}^{m} \mathcal{C}^i_j\right) \otimes \left(\mathcal{A}^i + \sum_{j=2}^{m} \mathcal{C}^i_j\right).$$

Then we repeat this procedure. That is, since $\mathcal{F} \in \mathbb{S}^{n^{2d}}$, we have $\mathcal{F} = \mathcal{F}_{\pi_{d+2,d+m+1}} = \mathcal{F}_{\pi_{2,m+1}} = (\mathcal{F}_{\pi_{2,m+1}})_{\pi_{d+2,d+m+1}}$. By letting $\mathcal{B}^i = \mathcal{A}^i + \sum\limits_{j=2}^{d} \mathcal{C}^i_j$, we can apply the same procedure as above to obtain $\mathcal{F} = \sum_{i=1}^{r} \left(\mathcal{A}^i + \sum\limits_{j=3}^{m} \mathcal{C}^i_j\right) \otimes \left(\mathcal{A}^i + \sum\limits_{j=3}^{m} \mathcal{C}^i_j\right)$. We just repeat this procedure until $\mathcal{F} = \sum_{i=1}^{r} \mathcal{A}^i \otimes \mathcal{A}^i$ and this completes the proof. □

Now we are ready to present the equivalence of symmetric M-rank and strongly symmetric M-rank.

Theorem 2.3 *For an even-order super-symmetric tensor $\mathcal{F} \in \mathbb{S}^{n^{2d}}$, its M-rank, symmetric M-rank and strongly symmetric M-rank are the same, i.e.* $\mathrm{rank}_M(\mathcal{F}) = \mathrm{rank}_{SM}(\mathcal{F}) = \mathrm{rank}_{SSM}(\mathcal{F})$.

Proof. That $\mathrm{rank}_M(\mathcal{F}) = \mathrm{rank}_{SM}(\mathcal{F})$ follows directly from the definition of symmetric M-rank. We thus only need to prove $\mathrm{rank}_{SM}(\mathcal{F}) = \mathrm{rank}_{SSM}(\mathcal{F})$. Suppose $\mathrm{rank}_{SM}(\mathcal{F}) = r$, which means there exist $\mathcal{B}^i \in \mathbb{C}^{n^d}, i = 1, \ldots, r$, such that $\mathcal{F} = \sum\limits_{i=1}^{r} \mathcal{B}^i \otimes \mathcal{B}^i$. By applying Lemma 2.2 at most d times, we can find super-symmetric tensors $\mathcal{A}^i \in \mathbb{S}^{n^d}, i = 1, \ldots, r$ such that $\mathcal{F} = \sum_{i=1}^{r} \mathcal{A}^i \otimes \mathcal{A}^i$. Hence, we have $\text{rank}_{SSM}(\mathcal{F}) \leq r = \text{rank}_{SM}(\mathcal{F})$. On the other hand, it is obvious that $\text{rank}_{SM}(\mathcal{F}) \leq \text{rank}_{SSM}(\mathcal{F})$. Combining these two inequalities yields $\text{rank}_{SM}(\mathcal{F}) = \text{rank}_{SSM}(\mathcal{F})$. □

3 Bounding CP-rank for even-order tensor using M-rank

In this section, for even-order tensor, we establish the equivalence between the symmetric CP-rank and the M-rank under the rank-one assumption. Then we particularly focus on the fourth-order tensors. This is because many multi-dimensional data from real practice are in fact fourth-order tensors. For example, the colored video completion and decomposition problems considered in [11, 13, 29] can be formulated as low-CP-rank fourth-order tensor recovery problems. We also show that the CP-rank and symmetric CP-rank for fourth-order tensor can be both lower and upper bounded (up to a constant factor) by the corresponding M-rank.

3.1 Rank-one equivalence for super-symmetric even-order tensor

We first consider a super-symmetric even-order tensor $\mathcal{F}$, and recall that we have shown a rank-one equivalence between $\mathcal{F}$ and its square unfolding in [19] if $\mathcal{F}$ is real-valued and the decomposition is performed in the real domain. Actually a similar result can be established when $\mathcal{F} \in \mathbb{S}^{n^{2d}}$. To see this, we first present the following lemma.

Lemma 3.1 *If a d-th order tensor $\mathcal{A} = a^1 \otimes a^2 \otimes \cdots \otimes a^d \in \mathbb{S}^{n^d}$ is super-symmetric, then we have $a^i = \pm a^1$ for $i = 2, \ldots, d$ and $\mathcal{A} = \underbrace{b \otimes b \otimes \cdots \otimes b}_{d}$ for some $b \in \mathbb{C}^n$.*

Proof. Since $\mathcal{A}$ is super-symmetric, construct $\mathcal{T} = \bar{\mathcal{A}} \otimes \mathcal{A}$ and it is easy to show that

$$\mathcal{T}_{i_1 \ldots i_d i_{d+1} \ldots i_{2d}} = \mathcal{T}_{j_1 \ldots j_d j_{d+1} \ldots j_{2d}}, \;\forall\, (j_1 \ldots j_d) \in \Pi(i_1 \ldots i_d),\; (j_{d+1} \ldots j_{2d}) \in \Pi(i_{d+1} \ldots i_{2d}),$$

and

$$\mathcal{T}_{i_1 \ldots i_d i_{d+1} \ldots i_{2d}} = \overline{\mathcal{T}_{i_{d+1} \ldots i_{2d} i_1 \ldots i_d}}, \;\forall 1 \leq i_1 \leq \cdots \leq i_d \leq n,\; 1 \leq i_{d+1} \leq \cdots \leq i_{2d} \leq n.$$

Therefore, $\mathcal{T}$ belongs to the so-called conjugate partial symmetric tensor introduced in [18]. Moreover, from Theorem 6.5 in [18], we know that

$$\max_{\|x\|=1} \mathcal{T}(\underbrace{\overline{x}, \ldots, \overline{x}}_{d}, \underbrace{x, \ldots, x}_{d}) = \max_{\|x^i\|=1,\, i=1,\ldots,d} \mathcal{T}(\overline{x^1}, \ldots, \overline{x^d}, x^1, \ldots, x^d) = \|a^1\|^2 \cdot \|a^2\|^2 \cdot \cdots \cdot \|a^d\|^2.$$

So there must exist an $\hat{x}$ with $\|\hat{x}\| = 1$ such that $|(a^i)^\top \hat{x}| = \|a^i\|$ for all i, which implies that $a^i = \pm a^1$ for $i = 2, \dots, d$, and $\mathcal{A} = \lambda \underbrace{a^1 \otimes a^1 \otimes \cdots \otimes a^1}_{d}$ for some $\lambda = \pm 1$. Finally by taking $b = \sqrt[d]{\lambda} a^1$, the conclusion follows. $\square$

The rank-one equivalence is established in the following theorem.

Theorem 3.2 *Suppose $\mathcal{F} \in \mathbb{S}^{n^{2d}}$ and we have*

$$\operatorname{rank}_M(\mathcal{F}) = 1 \Longleftrightarrow \operatorname{rank}_{SCP}(\mathcal{F}) = 1.$$

Proof. Suppose $\operatorname{rank}_{SCP}(\mathcal{F}) = 1$ and $\mathcal{F} = \underbrace{x \otimes \cdots \otimes x}_{2d}$ for some $x \in \mathbb{C}^n$. By constructing $\mathcal{A} = \underbrace{x \otimes \cdots \otimes x}_{d}$, we have $\mathcal{F} = \mathcal{A} \otimes \mathcal{A}$ with $\mathcal{A} \in \mathbb{S}^{n^d}$. Thus, $\operatorname{rank}_M(\mathcal{F}) = 1$.

To prove the other direction, suppose that we have $\mathcal{F} \in \mathbb{S}^{n^{2d}}$ and its M-rank is one, i.e. $\mathcal{F} = \mathcal{A} \otimes \mathcal{A}$ for some $\mathcal{A} \in \mathbb{S}^{n^d}$. By similar arguments as in Lemma 2.1 and Proposition 2.3 in [19], one has that the Tucker rank of $\mathcal{A}$ is $(1, 1, \cdots, 1)$ and consequently the asymmetric CP rank of $\mathcal{A}$ is one. This fact together with Lemma 3.1 implies that the symmetric CP rank of $\mathcal{A}$ is one as well, i.e., $\mathcal{A} = \underbrace{b \otimes \cdots \otimes b}_{d}$ for some $b \in \mathbb{C}^n$. It follows from $\mathcal{F} = \mathcal{A} \otimes \mathcal{A} = \underbrace{b \otimes \cdots \otimes b}_{2d}$ that $\operatorname{rank}_{SCP}(\mathcal{F}) = 1$. $\square$

3.2 Bound for asymmetric fourth-order tensor

For an asymmetric fourth-order tensor, the relation between its CP-rank and the corresponding M-rank is summarized in the following result.

Theorem 3.3 *Suppose $\mathcal{F} \in \mathbb{C}^{n_1 \times n_2 \times n_3 \times n_4}$ with $n_1 \le n_2 \le n_3 \le n_4$. Then it holds that*

$$\operatorname{rank}_M(\mathcal{F}) \le \operatorname{rank}_{CP}(\mathcal{F}) \le n_1 n_3 \cdot \operatorname{rank}_M(\mathcal{F}).$$

Proof. Suppose $\operatorname{rank}_{CP}(\mathcal{F}) = r$, i.e.

$$\mathcal{F} = \sum_{i=1}^{r} a^{1,i} \otimes a^{2,i} \otimes a^{3,i} \otimes a^{4,i} \quad \text{with} \quad a^{k,i} \in \mathbb{C}^{n_i} \text{ for } k = 1, \dots, 4 \text{ and } i = 1, \dots, r.$$

By letting $A^i = a^{1,i} \otimes a^{2,i}$ and $B^i = a^{3,i} \otimes a^{4,i}$, we get $\mathcal{F} = \sum_{i=1}^r A^i \otimes B^i$. Thus $\operatorname{rank}_M(\mathcal{F}) \le r = \operatorname{rank}_{CP}(\mathcal{F})$.

On the other hand, suppose that the M-rank of $\mathcal{F}$ is r_M. Then there exists some permutation $(j_1, j_2, j_3, j_4) = \pi(1, 2, 3, 4)$ such that

$$\mathcal{F} = \sum_{i=1}^{r_M} A^i \otimes B^i \text{ with matrices } A^i \in \mathbb{C}^{n_{j_1} \times n_{j_2}}, B^i \in \mathbb{C}^{n_{j_3} \times n_{j_4}} \text{ for } i = 1, \ldots, r_M.$$

Then it follows that $\operatorname{rank}(A^i) \leq \ell_1$ and $\operatorname{rank}(B^i) \leq \ell_2$ for all $i = 1, \ldots, r_M$, where $\ell_1 := \min\{n_{j_1}, n_{j_2}\}$ and $\ell_2 := \min\{n_{j_3}, n_{j_4}\}$. That is to say, matrices A^i and B^i admit some rank-one decompositions with at most ℓ_1 and ℓ_2 terms, respectively. Consequently, $\mathcal{F}$ can be decomposed as the sum of at most $r_M \ell_1 \ell_2$ rank-one tensors, or equivalently $\operatorname{rank}_{CP}(\mathcal{F}) \leq \min\{n_{j_1}, n_{j_2}\} \cdot \min\{n_{j_3}, n_{j_4}\} \cdot \operatorname{rank}_M(\mathcal{F}) \leq n_1 n_3 \cdot \operatorname{rank}_M(\mathcal{F})$, and this completes the proof. □

3.3 Bound for super-symmetric fourth-order tensor

Theorem 3.2 essentially says that M-rank and symmetric CP-rank are the same for the rank-one case. This equivalence, however, does not hold if the rank is not equal to one. In this subsection, we show that the symmetric CP-rank of $\mathcal{F} \in \mathbb{S}^{n^{2d}}$ can be both lower and upper bounded (up to a constant factor) by the corresponding M-rank.

Before we provide lower and upper bounds for symmetric CP-rank, we need the following technical results.

Lemma 3.4 *Suppose* $\sum\limits_{i=1}^{r} A^i \otimes A^i = \mathcal{F} \in \mathbb{S}^{n^4}$ *with* $A^i = \sum\limits_{j_i=1}^{m_i} a^{j_i} \otimes a^{j_i}$ *and* $a^{j_i} \in \mathbb{C}^n$ *for* $i = 1, \ldots, r$, $j_i = 1, \ldots, m_i$, $m_i \leq n$. *Then it holds that*

$$\begin{aligned} \mathcal{F} \;=\; & \sum_{i=1}^{r} \sum_{j_i=1}^{m_i} a^{j_i} \otimes a^{j_i} \otimes a^{j_i} \otimes a^{j_i} + \\ & \sum_{i=1}^{r} \sum_{j_i \neq k_i} \tfrac{1}{3} \left(a^{j_i} \otimes a^{j_i} \otimes a^{k_i} \otimes a^{k_i} + a^{j_i} \otimes a^{k_i} \otimes a^{j_i} \otimes a^{k_i} + a^{j_i} \otimes a^{k_i} \otimes a^{k_i} \otimes a^{j_i} \right). \end{aligned}$$

Proof. Since $\mathcal{F}$ is super-symmetric, we have $\mathcal{F}_{ijk\ell} = \mathcal{F}_{ikj\ell} = \mathcal{F}_{i\ell kj}$. Consequently,

$$\begin{aligned}
\mathcal{F} &= \sum_{i=1}^{r} A^i \otimes A^i \\
&= \sum_{i=1}^{r} \left(\sum_{j_i=1}^{m_i} a^{j_i} \otimes a^{j_i} \right) \otimes \left(\sum_{j_i=1}^{m_i} a^{j_i} \otimes a^{j_i} \right) \\
&= \sum_{i=1}^{r} \left(\sum_{j_i=1}^{m_i} a^{j_i} \otimes a^{j_i} \otimes a^{j_i} \otimes a^{j_i} + \sum_{j_i \neq k_i} a^{j_i} \otimes a^{j_i} \otimes a^{k_i} \otimes a^{k_i} \right) \qquad (19) \\
&= \sum_{i=1}^{r} \left(\sum_{j_i=1}^{m_i} a^{j_i} \otimes a^{j_i} \otimes a^{j_i} \otimes a^{j_i} + \sum_{j_i \neq k_i} a^{j_i} \otimes a^{k_i} \otimes a^{j_i} \otimes a^{k_i} \right) \qquad (20) \\
&= \sum_{i=1}^{r} \left(\sum_{j_i=1}^{m_i} a^{j_i} \otimes a^{j_i} \otimes a^{j_i} \otimes a^{j_i} + \sum_{j_i \neq k_i} a^{j_i} \otimes a^{k_i} \otimes a^{k_i} \otimes a^{j_i} \right) \qquad (21)
\end{aligned}$$

Then the conclusion follows by taking the average of (19), (20) and (21). ◻

Lemma 3.5 *Suppose $a_1, \cdots, a_m$ are m vectors and $\xi_1, \cdots, \xi_m$ are i.i.d. symmetric Bernoulli random variables, i.e.,*

$$\xi_i = \begin{cases} -1, & \text{with probability } 1/2 \\ -1, & \text{with probability } 1/2 \end{cases} \qquad i = 1, \ldots, m.$$

Then it holds that

$$\begin{aligned}
&\mathsf{E}\left[\left(\sum_{j=1}^{m} \xi_j a^j \right) \otimes \left(\sum_{j=1}^{m} \xi_j a^j \right) \otimes \left(\sum_{j=1}^{m} \xi_j a^j \right) \otimes \left(\sum_{j=1}^{m} \xi_j a^j \right) \right] \\
= &\sum_{j=1}^{m} a^j \otimes a^j \otimes a^j \otimes a^j + \sum_{i \neq j} \left(a^i \otimes a^i \otimes a^j \otimes a^j + a^i \otimes a^j \otimes a^i \otimes a^j + a^i \otimes a^j \otimes a^j \otimes a^i \right) \quad (22)
\end{aligned}$$

Proof. The expectation can be rewritten as

$$\begin{aligned}
&\mathsf{E}\left[\left(\sum_{j=1}^{m} \xi_j a^j \right) \otimes \left(\sum_{j=1}^{m} \xi_j a^j \right) \otimes \left(\sum_{j=1}^{m} \xi_j a^j \right) \otimes \left(\sum_{j=1}^{m} \xi_j a^j \right) \right] \\
= &\sum_{i,j,k,\ell} \mathsf{E}\left[\xi_i a^i \otimes \xi_j a^j \otimes \xi_k a^k \otimes \xi_\ell a^\ell \right] \\
= &\sum_{i,j,k,\ell} \mathsf{E}\left[\xi_i \xi_j \xi_k \xi_\ell \right] a^i \otimes a^j \otimes a^k \otimes a^\ell. \qquad (23)
\end{aligned}$$

Since $\xi_1, \cdots, \xi_m$ are i.i.d. with $\mathsf{E}[\xi_i] = 0$, $i = 1, \ldots, m$, it follows that

$$\mathsf{E}\left[\xi_i \xi_j \xi_k \xi_\ell\right] = \begin{cases} 1, & \text{if } \{i,j,k,\ell\} = \{u,u,v,v\}, \text{ or } \{u,v,u,v\}, \text{ or } \{u,v,v,u\}, \text{ for some } u,v; \\ 0, & \text{otherwise} . \end{cases} \tag{24}$$

Therefore,

$$\begin{aligned} & \sum_{i,j,k,\ell} \mathsf{E}\left[\xi_i \xi_j \xi_k \xi_\ell\right] a^i \otimes a^j \otimes a^k \otimes a^\ell \\ = & \sum_{j=1}^m a^j \otimes a^j \otimes a^j \otimes a^j + \sum_{i \neq j} \left(a^i \otimes a^i \otimes a^j \otimes a^j + a^i \otimes a^j \otimes a^i \otimes a^j + a^i \otimes a^j \otimes a^j \otimes a^i \right), \end{aligned}$$

which combining with (23) yields (22). □

Now we are ready to present the main result in this subsection.

Theorem 3.6 *For any given $\mathcal{F} \in \mathbb{S}^{n^4}$, it holds that*

$$\operatorname{rank}_M(\mathcal{F}) \le \operatorname{rank}_{SCP}(\mathcal{F}) \le (n + 4n^2) \operatorname{rank}_M(\mathcal{F}).$$

Proof. We prove $\operatorname{rank}_M(\mathcal{F}) \le \operatorname{rank}_{SCP}(\mathcal{F})$ first. Suppose $\operatorname{rank}_{SCP}(\mathcal{F}) = r$, i.e.,

$$\mathcal{F} = \sum_{i=1}^r a^i \otimes a^i \otimes a^i \otimes a^i \text{ with } a^i \in \mathbb{C}^n \text{ for } i = 1, \ldots, r.$$

By letting $A^i = a^i \otimes a^i$, we get $\mathcal{F} = \sum_{i=1}^r A^i \otimes A^i$ with $A^i \in \mathbb{S}^{n^2}$. Thus $\operatorname{rank}_M(\mathcal{F}) \le r = \operatorname{rank}_{SCP}(\mathcal{F})$.

We now prove $\operatorname{rank}_{SCP}(\mathcal{F}) \le (n + 4n^2) \operatorname{rank}_M(\mathcal{F})$. Suppose that $\operatorname{rank}_M(\mathcal{F}) = r$, then from (2.3) it holds that $\mathcal{F} = \sum\limits_{i=1}^r A^i \otimes A^i$ with $A^i \in \mathbb{S}^{n^2}$. Assume $A^i = \sum\limits_{j=1}^{m_i} a^{i_j} \otimes a^{i_j}$, $m_i \le n$, $i = 1, \ldots, r$. Let $\xi_{1_1}, \cdots, \xi_{1_{m_1}}, \xi_{2_1}, \cdots, \xi_{r_{m_r}}$ be i.i.d. symmetric Bernoulli random variables. Then by combining Lemmas 3.4 and 3.5, we have that

$$\begin{aligned} \mathcal{F} \;=\; & \sum_{i=1}^r A^i \otimes A^i = \frac{2}{3} \sum_{i=1}^r \sum_{j=1}^{m_i} a^{i_j} \otimes a^{i_j} \otimes a^{i_j} \otimes a^{i_j} + \\ & \frac{1}{3} \sum_{i=1}^r \mathsf{E}\left[\left(\sum_{j=1}^{m_i} \xi_{i_j} a^{i_j} \right) \otimes \left(\sum_{j=1}^{m_i} \xi_{i_j} a^{i_j} \right) \otimes \left(\sum_{j=1}^{m_i} \xi_{i_j} a^{i_j} \right) \otimes \left(\sum_{j=1}^{m_i} \xi_{i_j} a^{i_j} \right) \right]. \end{aligned} \tag{25}$$

Assume that $\eta_{i_1}, \ldots, \eta_{i_{m_i}}$ are 4-wise independent (see, e.g., [1, 17]) and identical symmetric Bernoulli random variables for all $i = 1, \ldots, r$. Then we have that

$$\begin{aligned} & \mathsf{E}\left[\left(\sum_{j=1}^{m_i} \xi_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \xi_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \xi_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \xi_{i_j} a^{i_j}\right)\right] \\ = \; & \mathsf{E}\left[\left(\sum_{j=1}^{m_i} \eta_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \eta_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \eta_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \eta_{i_j} a^{i_j}\right)\right], \text{ for } i = 1, \ldots, r. \quad (26) \end{aligned}$$

According to Proposition 6.5 in [1], we know that the right hand side of (26) can be written as the sum of at most $4m_i^2$ symmetric rank-one tensors. By combining (25) and (26) we get that

$$\begin{aligned} \mathcal{F} \; = \; & \frac{2}{3} \sum_{i=1}^{r} \sum_{j=1}^{m_i} a^{i_j} \otimes a^{i_j} \otimes a^{i_j} \otimes a^{i_j} + \\ & \frac{1}{3} \sum_{i=1}^{r} \mathsf{E}\left[\left(\sum_{j=1}^{m_i} \eta_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \eta_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \eta_{i_j} a^{i_j}\right) \otimes \left(\sum_{j=1}^{m_i} \eta_{i_j} a^{i_j}\right)\right], \end{aligned}$$

which indicates that

$$\text{rank}_{SCP}(\mathcal{F}) \le \sum_{i=1}^{r} m_i + \sum_{i=1}^{r} 4m_i^2 \le rn + 4rn^2 = (n + 4n^2)\text{rank}_M(\mathcal{F}).$$

□

4 Low-M-rank tensor recovery

In this section, we consider the low-CP-rank tensor recovery problems (2) and (4). We focus on the fourth-order tensor case. We denote $X = \boldsymbol{M}(\mathcal{X})$, $Y = \boldsymbol{M}(\mathcal{Y})$ and $F = \boldsymbol{M}(\mathcal{F})$. According to Theorems 3.3 and 3.6, we know that by multiplying a constant factor, the M-rank can provide an upper bound for the CP-rank and the symmetric CP-rank, and so dose $\text{rank}(X)$ since $\text{rank}_M(\mathcal{X}) \le \text{rank}(X)$. Thus we can replace the CP-rank in the objective of (2) and (4) by $\text{rank}(X)$, and hopefully by minimizing $\text{rank}(X)$, $\text{rank}_{CP}(\mathcal{X})$ will be small as well. In other words, rather than solving (2) and (4), we consider solving the following two matrix problems

$$\min_{X} \; \text{rank}(X), \quad \text{s.t.,} \quad \bar{\boldsymbol{L}}(X) = b, \tag{27}$$

and

$$\min_{Y,Z} \; \text{rank}(Y) + \lambda \|Z\|_0, \quad \text{s.t.,} \quad Y + Z = F, \tag{28}$$

where $\bar{\boldsymbol{L}}$ is a linear mapping such that $\bar{\boldsymbol{L}}(X) = \boldsymbol{L}(\mathcal{X})$.

It is now very natural to consider the convex relaxations of the two matrix problems (27) and (28); i.e., we replace the rank function by the nuclear norm and replace the cardinality function by the ℓ_1 norm. This results in the following two convex relaxations for (27) and (28):

$$\min_X \ \|X\|_*, \quad \text{s.t.,} \quad \bar{\boldsymbol{L}}(X) = b, \tag{29}$$

and

$$\min_{Y,Z} \ \|Y\|_* + \lambda\|Z\|_1, \quad \text{s.t.,} \quad Y + Z = F. \tag{30}$$

Note that all the variables in (27) and (28) are complex-valued. Thus, the ℓ_1 norm is defined as $\|Z\|_1 := \sum_{ij} \sqrt{(\text{Re }(Z_{ij}))^2 + (\text{Im }(Z_{ij}))^2}$. Although (27) and (28) are complex-valued problems, they can still be solved by the methods in [5, 31, 30, 2, 37] with minor modifications. We omit the details for succinctness.

When the tensors are super-symmetric, we can impose the super-symmetric constraint and get the following formulation:

$$\begin{array}{ll} \min\limits_X & \|X\|_* \\ \text{s.t.} & \bar{\boldsymbol{L}}(Y) = b, \ \boldsymbol{M}^{-1}(X) \in \mathbb{S}^{n^4}. \end{array}$$

Note that the above problem is equivalent to

$$\begin{array}{ll} \min\limits_X & \|X\|_* \\ \text{s.t.} & \boldsymbol{M}^{-1}(Y) \in \mathbb{S}^{n^4}, \\ & \bar{\boldsymbol{L}}(Y) = b, \quad X = Y, \end{array} \tag{31}$$

which can be efficiently solved by the standard alternating direction method of multipliers (see the survey paper [4] for more details).

5 Numerical results

In this section, we perform numerical experiments on low-CP-rank tensor recovery problems using the formulations (29) and (30) on both synthetic data and real data from colored video processing.

5.1 Synthetic data

5.1.1 Low-rank tensor completion problems

In this subsection we use the FPCA algorithm proposed in [31] to solve (29) for complex-valued fourth-order tensor completion. The testing examples are generated as follows. We generate random complex-valued tensors $\mathcal{X}_0$ of sizes $(20 \times 20 \times 30 \times 30)$ and $(30 \times 30 \times 30 \times 30)$ with CP-rank no more than 10, and tensors of sizes $(30 \times 30 \times 40 \times 40)$ and $(40 \times 40 \times 40 \times 40)$ with CP-rank no more than 18. Thus the sizes of the resulting unfolding matrices in (29) are 400×900, 900×900, 900×1600, 1600×1600 respectively. Under each setting, we randomly generate 10 instances. For each instance, we randomly select 30% of the entries as the observed ones for tensor completion. We report the relative error, the CPU time (in seconds) and the M-rank of the recovered tensor $\mathcal{X}^*$ in Table 1, where the relative error is defined as

$$RelErr := \frac{\|\mathcal{X}^* - \mathcal{X}_0\|_F}{\|\mathcal{X}_0\|_F}.$$

From Table 1 we can see that with only 30% of entries observed, FPCA can recover the tensor very well by solving (29). It is noted that the relative error is usually at the order of 10^{-5} or 10^{-6}. We also note that the M-rank of the recovered tensor $\mathcal{X}^*$ is *always* equal to the CP-rank of the original tensor $\mathcal{X}_0$, which indicates that the M-rank is a good approximation of the CP-rank as alluded to in Theorem 3.3.

As an observation in the computational experiments, we notice that the mode-n-rank can be strictly less than the M-rank in practice. This combined with the lower bound results implies that the M-rank is superior to the mode-n-rank in terms of approximating the CP-rank at least under some circumstances. A numerical example leading to such observation can be constructed as follows. We randomly generate a $(10 \times 10 \times 10 \times 10)$ tensor with CP-rank equal to 12. Then the mode-n-rank is at most 10, because it is the average of ranks of four 10×1000 matrices. However, we observe that the M-rank of this tensor is typically 12, which is strictly larger than 10 and is equal to the CP-rank.

In another set of tests, we aim to observe the relationship between the M-rank and the symmetric CP-rank via solving the super-symmetric tensor completion problem (31). To this end, 40 complex-valued super-symmetric tensors with varying dimensions are randomly generated. In particular, the symmetric CP-rank is no more than 8 if the tensor size is $(10 \times 10 \times 10 \times 10)$ or $(15 \times 15 \times 15 \times 15)$, and the symmetric CP-rank is no more than 15 when the tensor size is $(20 \times 20 \times 20 \times 20)$ or $(25 \times 25 \times 25 \times 25)$. For each generated tensor, we randomly remove 60% of the components and solve the tensor completion problem (31). The results are summarized in Table 2, from which we see that the tensor is nicely recovered (usually with the relative error at the order of 10^{-6}). Moreover, the M-rank and the symmetric CP-rank listed in the table are *always* identical, implying that the M-rank remains a good approximation of the symmetric CP-rank. We also note that solving problem (31) is much more time consuming than solving (29), this is due to the super-symmetric constraint, which is essentially equivalent to $O(n^4)$ linear constraints and is costly to deal with.

5.1.2 Robust tensor recovery problem

For the robust tensor recovery problem (4), we choose λ as $1/\sqrt{n_1 n_2}$ in (30), and apply the alternating linearization method in [2] to solve (30). We randomly generate complex-valued tensors $\mathcal{Y}_0$ of sizes $(20\times20\times20\times20)$ and $(20\times20\times30\times30)$ with CP-rank no more than 10, and tensors of sizes $(30\times30\times30\times30)$ and $(30\times30\times40\times40)$ with CP-rank no more than 15. In our test, another random complex-valued fourth-order tensor $\mathcal{Z}_0$ is generated with cardinality equal to $0.05\cdot n_1n_2n_3n_4$. Therefore, $\mathcal{F}=\mathcal{Y}_0+\mathcal{Z}_0$ is the observed data in (4). For each setting, we randomly generate 10 instances, and in Table 3 we report the relative error, the M-rank of $\mathcal{Y}^*$ and the cardinality of $\mathcal{Z}^*$ after solving (30). Notice that the relative error in this case is defined as

$$RelErr := \frac{\|\mathcal{Y}^*+\mathcal{Z}^*-\mathcal{F}\|_F}{\|\mathcal{F}\|_F}.$$

From Table 3 we see that the relative error is always at the order of 10^{-7} for the problems we tested, and the cardinality of the recovered tensor $\mathcal{Z}^*$ remains unchanged compared to the original sparse part. Moreover, we see that the M-rank of the recovered tensor $\mathcal{Y}^*$ is *always* equal to the CP-rank of the original tensor $\mathcal{Y}_0$, which is consistent with the observations in the low-rank tensor completion problems.

5.2 Colored video completion and decomposition

In this subsection, we apply the matrix completion and matrix robust PCA models (27) and (28) for the purpose of colored video completion and decomposition, which can be formulated as the fourth-order tensor recovery problems (2) and (4) respectively. A colored video file consists of n_4 frames, and each frame is an image stored in the RGB format as a $n_1\times n_2\times 3$ array. As a result, filling in the missing entries of the colored video and decomposing the video to static background and moving foreground can be regarded as low-rank tensor completion (2) and robust tensor recovery (4), respectively.

In our experiment for tensor completion, we chose 50 frames from a video taken in a lobby, which was introduced by Li et al. in [26]. Each frame in this video is a colored image with size $128\times160\times3$. The images in the first row of Figure 1 are three frames of the video. Basically we chose the 50 frames such that they only contain static background, and thus the $128\times160\times3\times50$ fourth-order tensor formed by them are expected to have low rank, because the background is almost the same in each frame. We then randomly remove 80% of the entries from the video, and the images in the second row of Figure 1 are the frames after the removal. We then apply the FPCA proposed in [31] to solve (29) with the square unfolding matrix having the size 20480×150, to complete the missing entries in the target tensor. The images in the third row of Figure 1 are the frames recovered by

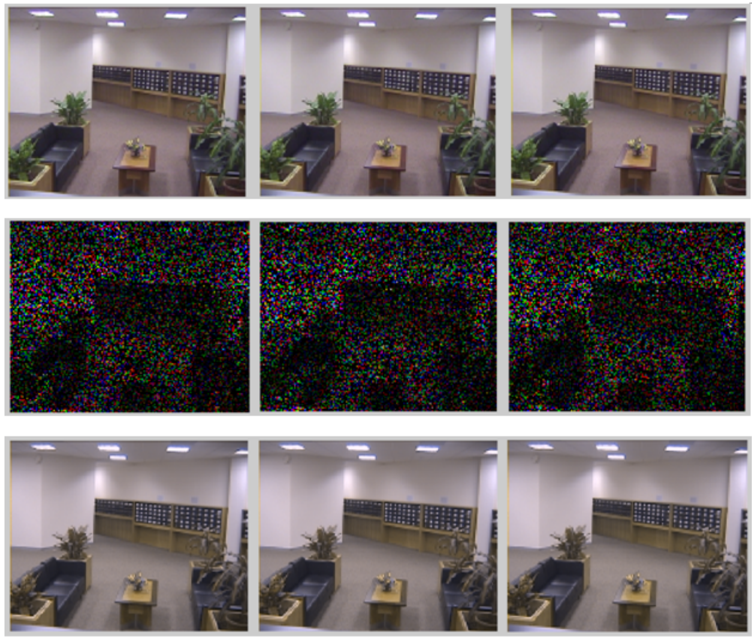

Figure 1: Results for video completion. The first row: frames of the original video; the second row: frames with 80% missing entries; the third row: recovered images using tensor completion.

FPCA for solving (29). We can see that we are able to recover the video very well even though 80% entries are missing.

In our experiment for robust tensor recovery, we chose another 50 frames from the same video in [26]. These frames were chosen such that the frames contain some moving foregrounds. The task in robust tensor recovery is to decompose the given tensor into two parts: a low-rank tensor corresponding to the static background, and a sparse tensor corresponding to the moving foreground. Note that the tensor corresponding to the moving foreground is sparse because the foreground usually only occupies a small portion of the frame. Thus this decomposition can be found by solving the robust tensor recovery problem (4). Here we again apply the alternating liearization method proposed in [2] to solve (28) for the task of robust tensor recovery, where λ in (30) is chosen as $1/\sqrt{n_1 n_2}$ and n_1, n_2 are the first two dimensions of the fourth-order tensor. The decomposition results are shown in Figure 2. The images in the first row of Figure 2 are frames of the original video. The images in the second and third rows of Figure 2 are the corresponding static background and moving foreground, respectively. We can see that our approach very effectively decomposes the video, which is a fourth-order tensor.

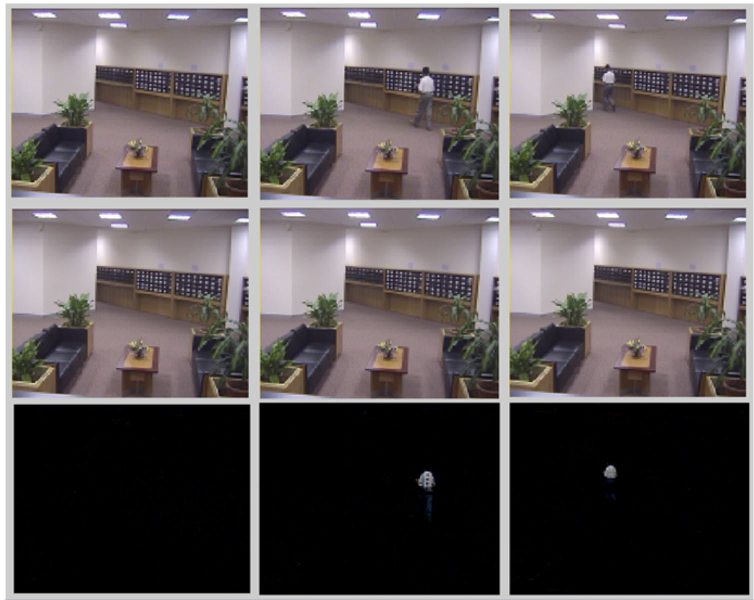

Figure 2: Robust Video Recovery. The first row are the 3 frames of the original video sequence. The second row are recovered background. The last row are recovered foreground.

6 Conclusion

In this paper, we proposed some new notions of tensor decomposition for the even order tensors, which yield some new rank definitions for tensors, namely, the M-rank, the symmetric M-rank and the strongly symmetric M-rank. We showed that these three definitions are equivalent if the tensor under consideration is even-order and super-symmetric. We then showed that the CP-rank and symmetric CP-rank of a given fourth-order tensor can be both lower and upper bounded (up to a constant factor) by the corresponding M-rank. This provided a theoretical foundation for using the M-rank to replace the CP-rank in low-CP-rank tensor recovery problems. This is encouraging since the M-rank is much easier to compute than the CP-rank. We then solved the low-M-rank tensor recovery problems using some existing methods for matrix completion and matrix robust PCA. The results showed that our method can recover the tensors very well, confirming that the M-rank is a good approximation to substitute the CP-rank in such applications.

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Inst.	$\text{rank}_{CP}(\mathcal{X}_0)$	$RelErr$	CPU	$\text{rank}_M(\mathcal{X}^*)$
Dimension $20 \times 20 \times 30 \times 30$				
1	10	3.36e-05	11.42	10
2	10	3.10e-05	10.95	10
3	10	3.46e-05	11.00	10
4	10	3.47e-05	11.03	10
5	10	2.82e-05	10.76	10
6	10	3.28e-05	10.78	10
7	10	3.83e-05	10.95	10
8	10	3.21e-05	11.17	10
9	10	2.92e-05	10.97	10
10	10	3.34e-05	11.34	10
Dimension $30 \times 30 \times 30 \times 30$				
1	10	1.67e-05	33.88	10
2	10	1.93e-05	34.96	10
3	10	1.92e-05	33.35	10
4	10	1.80e-05	33.31	10
5	10	1.69e-05	33.31	10
6	10	1.74e-05	32.64	10
7	10	1.74e-05	33.77	10
8	10	1.73e-05	31.79	10
9	10	1.15e-05	33.07	10
10	10	1.81e-05	32.67	10
Dimension $30 \times 30 \times 40 \times 40$				
1	18	1.30e-05	59.00	18
2	18	1.58e-05	63.34	18
3	18	1.48e-05	59.98	18
4	18	1.53e-05	63.70	18
5	18	1.46e-05	55.36	18
6	18	1.50e-05	52.92	18
7	18	1.41e-05	55.83	18
8	18	1.51e-05	55.01	18
9	18	1.48e-05	54.62	18
10	18	1.47e-05	53.84	18
Dimension $40 \times 40 \times 40 \times 40$				
1	18	9.11e-06	119.79	18
2	18	1.05e-05	116.39	18
3	18	1.02e-05	119.90	18
4	18	8.68e-06	130.78	18
5	18	8.69e-06	145.50	18
6	18	1.02e-05	159.95	18
7	18	9.99e-06	152.21	18
8	18	8.80e-06	151.66	18
9	18	1.09e-05	145.05	18
10	18	9.94e-06	144.86	18

Table 1: Numerical results for low-rank tensor completion by solving (29)

Inst.	$\text{rank}_{SCP}(\mathcal{X}_0)$	$RelErr$	CPU	$\text{rank}_M(\mathcal{X}^*)$
Dimension $10 \times 10 \times 10 \times 10$				
1	8	2.55e-006	5.96	8
2	8	1.25e-006	2.26	8
3	8	7.16e-006	2.17	8
4	8	7.96e-006	4.21	8
5	8	1.19e-006	3.03	8
6	8	2.84e-006	1.33	8
7	8	8.83e-006	3.71	8
8	8	1.15e-005	4.04	8
9	8	7.31e-006	5.38	8
10	8	2.31e-006	4.62	8
Dimension $15 \times 15 \times 15 \times 15$				
1	8	1.44e-006	54.66	8
2	8	2.86e-006	39.25	8
3	8	8.95e-006	51.68	8
4	8	2.67e-006	31.39	8
5	8	3.55e-006	35.19	8
6	8	1.30e-006	18.19	8
7	8	5.16e-006	57.16	8
8	8	6.15e-006	84.32	8
9	8	2.60e-006	22.50	8
10	8	7.11e-007	26.41	8
Dimension $20 \times 20 \times 20 \times 20$				
1	15	3.16e-006	390.88	15
2	15	8.70e-006	416.35	15
3	15	8.51e-006	523.54	15
4	15	4.87e-006	353.58	15
5	15	3.41e-006	417.79	15
6	15	5.43e-006	356.06	15
7	15	5.79e-006	597.83	15
8	15	8.08e-006	304.83	15
9	15	6.63e-006	623.36	15
10	15	2.88e-006	357.90	15
Dimension $25 \times 25 \times 25 \times 25$				
1	15	1.99e-006	949.98	15
2	15	1.04e-006	1046.49	15
3	15	1.77e-006	1350.69	15
4	15	4.21e-006	1552.74	15
5	15	3.34e-006	1109.26	15
6	15	2.71e-006	1163.05	15
7	15	4.99e-006	1409.80	15
8	15	2.17e-006	832.45	15
9	15	1.87e-006	981.00	15
10	15	4.98e-006	1463.07	15

Table 2: Numerical results for low-rank super-symmetric tensor completion by solving (31)

Inst.	$\text{rank}_{CP}(\mathcal{Y}_0)$	$\|\mathcal{Z}_0\|_0$	$RelErr$	CPU	$\text{rank}_M(\mathcal{Y}^*)$	$\|\mathcal{Z}^*\|$
Dimension $20 \times 20 \times 20 \times 20$						
1	10	8000	6.80e-007	28.26	10	8000
2	10	8000	7.60e-007	28.26	10	8000
3	10	8000	8.74e-007	27.75	10	8000
4	10	8000	6.99e-007	28.52	10	8000
5	10	8000	9.86e-007	31.25	10	8000
6	10	8000	9.71e-007	36.24	10	8000
7	10	8000	7.27e-007	33.94	10	8000
8	10	8000	6.66e-007	32.66	10	8000
9	10	8000	7.02e-007	31.99	10	8000
10	10	8000	9.54e-007	30.93	10	8000
Dimension $20 \times 20 \times 30 \times 30$						
1	10	18000	8.56e-007	56.16	10	18000
2	10	18000	7.78e-007	57.30	10	18000
3	10	18000	7.29e-007	63.92	10	18000
4	10	18000	7.83e-007	53.90	10	18000
5	10	18000	7.62e-007	52.81	10	18000
6	10	18000	9.17e-007	53.66	10	18000
7	10	18000	8.81e-007	56.03	10	18000
8	10	18000	9.55e-007	52.88	10	18000
9	10	18000	7.62e-007	53.93	10	18000
10	10	18000	8.13e-007	54.55	10	18000
Dimension $30 \times 30 \times 30 \times 30$						
1	15	40500	7.93e-007	453.66	15	40500
2	15	40500	7.96e-007	430.11	15	40500
3	15	40500	7.61e-007	437.26	15	40500
4	15	40500	7.87e-007	433.16	15	40500
5	15	40500	7.58e-007	428.68	15	40500
6	15	40500	8.29e-007	455.13	15	40500
7	15	40500	8.65e-007	421.81	15	40500
8	15	40500	7.21e-007	420.99	15	40500
9	15	40500	7.76e-007	438.62	15	40500
10	15	40500	7.59e-007	444.84	15	40500
Dimension $30 \times 30 \times 40 \times 40$						
1	15	72000	8.32e-007	636.56	15	72000
2	15	72000	8.70e-007	629.76	15	72000
3	15	72000	8.75e-007	621.89	15	72000
4	15	72000	6.88e-007	644.74	15	72000
5	15	72000	8.91e-007	617.52	15	72000
6	15	72000	9.02e-007	620.96	15	72000
7	15	72000	8.29e-007	671.75	15	72000
8	15	72000	9.23e-007	629.99	15	72000
9	15	72000	8.10e-007	639.81	15	72000
10	15	72000	8.46e-007	629.77	15	72000

Table 3: Numerical results for robust tensor recovery by solving (30)