Quadratic Cone Cutting Surfaces for Quadratic Programs with On-Off Constraints

Hyemin Jeon, Jeff Linderoth
Department of Industrial and Systems Engineering
University of Wisconsin-Madison
jeon5@wisc.edu, linderoth@wisc.edu

Andrew Miller
UPS
foresomenteneikona@gmail.com

January 15, 2015

Abstract

We study the convex hull of a set arising as a relaxation of difficult convex mixed integer quadratic programs (MIQP). We characterize the extreme points of our set and the extreme points of its continuous relaxation. We derive four quadratic cutting surfaces that improve the strength of the continuous relaxation. Each of the cutting surfaces is second-order-cone representable. Via a shooting experiment, we provide empirical evidence as to the importance of each inequality type in improving the relaxation. Computational results that employ the new cutting surfaces to strengthen the relaxation for MIQPs arising from portfolio optimization applications are promising.

1 Introduction

This paper is concerned with a mixed-integer nonlinear set of the form

$$X = \{(x, z, v) \in \mathbb{R}^n_+ \times \mathbb{B}^n \times \mathbb{R} \mid v \geq x^T Q x, x_j \leq z_j \forall j \in N\}, \quad (1)$$

where $N = \{1, 2, \ldots, n\}$, and the matrix Q is positive semidefinite. The set X appears as a substructure in many practical mixed integer quadratic programs (MIQPs), including those arising from portfolio management [1], model selection [5], sparse digital filter design [18], and linear-quadratic optimal control with setup costs [9]. The aim of this study is to obtain a good approximation of $\text{conv}(X)$, which can help with the solution process of the optimization problems that have X as a relaxation.

In the case that $Q = \text{diag}(q) \geq 0$, Stubbs [17] characterized $\text{conv}(X)$ with the following extended formulation:

$$\text{conv}(X) = \text{proj}_{x,z,v}\{(x, z, v, t) \in \mathbb{R}^n_+ \times [0, 1]^n \times \mathbb{R} \times \mathbb{R}^n_+ \mid v \geq \sum_{j \in N} q_j t_j,$$
$$t_j z_j \geq x_j^2 \ \forall j \in N, x_j \leq z_j \ \forall j \in N\}. \quad (2)$$

The description of $\mathrm{conv}(X)$ in (2) is obtained by applying a transformation known as the *perspective reformulation* [11], which is itself a specialization of a more general theorem on the convex hull of a disjunctive set found in the work of Ceria and Soares [3]. The *perspective cuts* of Frangioni and Gentile [6] can be derived as linearizations of the nonlinear inequalities appearing in the description of $\mathrm{conv}(X)$.

Where applicable, the perspective reformulation has been shown to help tremendously in solving practical MIQP problems. The interested reader is invited to look at the works [8, 12, 2] for a host of applications on which the perspective reformulation has improved performance. Unfortunately, direct application of the perspective reformulation to the set X is not possible, as x^TQx is not in general a separable function of the decision variables x. The focus of this work is to attempt to extend the perspective reformulation to the non-separable case.

Since $Q \succeq 0$, it has a Cholesky factorization $Q = MM^T$ for some lower triangular matrix M. We could thus induce separability through an affine transformation, arriving at the set

$$Y \stackrel{\text{def}}{=} \Big\{(x, y, z, v) \in \mathbb{R}^n_+ \times \mathbb{R}^n \times \mathbb{B}^n \times \mathbb{R} \mid v \geq y^T y, y = M^T x, x \leq z\Big\}. \tag{3}$$

The set Y is a reformulation of X, since $X = \mathrm{proj}_{x,z,v}(Y)$. If we project out the x variables from Y, we arrive at the set

$$\mathrm{proj}_{y,z,v}(Y) = \Big\{(y, z, v) \in \mathbb{R}^n \times \mathbb{B}^n \times \mathbb{R} \mid v \geq y^T y, 0 \leq M^{-T} y \leq z\Big\}, \tag{4}$$

where the quadratic function is separable in y, but whose interaction between continuous and binary variables ($0 \leq M^{-T}y \leq z$) becomes more complicated than simple "on-off" relationships between x and z in the set X.

Frangioni and Gentile [6] suggest a clever matrix-splitting approach for applying the perspective reformulation to X. Specifically, the matrix Q may be decomposed into $Q = R + D$, for some matrix $D = \mathrm{diag}(d) \geq 0$ such that $R = Q - D \succeq 0$. With this transformation, the set X can be reformulated as

$$X = \mathrm{proj}_{x,z,v} \Big\{(x, z, v, t) \in \mathbb{R}^n_+ \times \mathbb{B}^n \times \mathbb{R} \times \mathbb{R}^n_+ \mid v \geq x^T Rx + e^T t \quad t_j \geq d_j x_j^2 \ \forall j \in N, x \leq z\Big\}. \tag{5}$$

The continuous relaxation of the set (5) can be strengthened via the perspective reformulation to

$$\mathcal{P}(X) \stackrel{\text{def}}{=} \mathrm{proj}_{x,z,v} \Big\{(x, z, v, t) \in \mathbb{R}^n_+ \times [0, 1]^n \times \mathbb{R} \times \mathbb{R}^n_+ \mid v \geq x^T Rx + e^T t \quad z_j t_j \geq d_j x_j^2, \ \forall j \in N, x \leq z\Big\}. \tag{6}$$

Note that if $R = 0$, then $\mathcal{P}(X) = \mathrm{conv}(X)$.

The set $\mathcal{P}(X)$ depends on the extracted diagonal vector d. Zheng et al. [19] give a semidefinite program (SDP) that will find the vector d that leads to "strongest" (in terms of bound improvement) perspective relaxation (6). Dong and Linderoth [4] give a different proof of the same result, study the set X in a lifted space that introduces variables for all quadratic terms $x_i x_j$, and use SDP relaxations for strengthening the relaxations of X. In this work, we focus on reformulations that are not lifted to the "full" matrix-space, and we strengthen the relaxations using only second-order-cone representable constraints.

We propose to combine the separability inducing-Cholesky factorization (4) with the matrix-splitting approach (5). Specifically, we consider a more generalized decomposition of Q into $Q = R + B$, for some

2×2 block diagonal matrix $B \succeq 0$ such that $R = Q - B \succeq 0$. This has the effect of extracting more of the nonlinear function $x^T Q x$ into separable, low-dimensional components. Our focus will then be on obtaining strong relaxations for the extracted low-dimensional components. To ease the exposition, we assume that n is even. Splitting $Q = R + B$ and adding variables to capture the extracted portion of the quadratic function gives the extended formulation

$$X = \text{proj}_{x,z,v} \Big\{ (x, z, v, s) \in \mathbb{R}^n_+ \times \mathbb{B}^n \times \mathbb{R} \times \mathbb{R}^{(n/2)} \mid v \geq x^T R x + e^T s \\ x \leq z, s_j \geq B_{jj} x_j^2 + 2B_{j,j+1} x_j x_{j+1} + B_{j+1,j+1} x_{j+1}^2 \; \forall j = 1, 3, 5, \ldots, n-1 \Big\}. \quad (7)$$

If the constraint $v \geq x^T R x + e^T s$ is removed from X in (7), then the set decomposes into blocks consisting of one s variable, two x variables and two z variables. Thus, to strengthen the relaxation of the reformulation (7) of X we should understand the interactions between the on-off constraints $0 \leq x_j \leq z_j, 0 \leq x_{j+1} \leq z_{j+1}$ and the epigraph of a convex quadratic function

$$s_j \geq B_{jj} x_j^2 + 2B_{j,j+1} x_j x_{j+1} + B_{j+1,j+1} x_{j+1}^2.$$

To understand this set, we find it convenient to make the quadratic function separable using the Cholesky factorization of $B = LL^T$ and to perform the substitution $x = L^{-T} y$ (like in Equation (4)). Performing these transformations and focusing on one block yields a 6-dimensional set S to study, where the nonlinearity is confined to two simple inequalities

$$
\begin{aligned}
S \stackrel{\text{def}}{=} \Big\{ (y, t, z) \in \mathbb{R}^2 \times \mathbb{R}^2_+ \times \mathbb{B}^2 \mid\ & t_1 \geq y_1^2 && \text{(8a)} \\
& t_2 \geq y_2^2 && \text{(8b)} \\
& 0 \leq a_{11} y_1 + a_{12} y_2 \leq z_1 && \text{(8c)} \\
& 0 \leq \qquad\quad a_{22} y_2 \leq z_2 \Big\}, && \text{(8d)}
\end{aligned}
$$

and

$$L^{-T} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$$

is an upper-triangular matrix. The linear constraints $0 \leq L^{-T} y \leq z$ imply bounds on the variables $y_1 \in [l_1, u_1]$ and $y_2 \in [l_2, u_2]$, with

$$
\begin{aligned}
l_1 &\stackrel{\text{def}}{=} \min\{0, -\frac{a_{12}}{a_{11}} \frac{1}{a_{22}}\} \\
u_1 &\stackrel{\text{def}}{=} \max\{\frac{1}{a_{11}}, \frac{1}{a_{11}} - \frac{a_{12}}{a_{11}} \frac{1}{a_{22}}\} \\
l_2 &\stackrel{\text{def}}{=} 0 \\
u_2 &\stackrel{\text{def}}{=} \frac{1}{a_{22}}.
\end{aligned}
$$

For ease of notation, we typically ignore the dependence of the set S on the parameters a_{11}, a_{12}, and a_{22}. However, if we must explicitly write the dependence, we refer to the set S as $S(a_{11}, a_{12}, a_{22})$.

The remainder of this paper is focused on studying the set S. In Section 2, we characterize the extreme points of $\text{conv}(S)$ and the extreme points of its natural continuous relaxation. We derive four nonlinear inequalities that are valid for $\mathrm{conv}(S)$, but cut off all extreme points of the continuous relaxation. All of the inequalities we derive for $\mathrm{conv}(S)$ are second-order cone representable. In Section 3, we motivate the importance of our new inequalities in two ways. We perform analysis to show which extreme points of the continuous relaxation are cut-off by each of our new inequalities, and we show at which extreme points of $\mathrm{conv}(S)$ our new inequalities are tight. Section 4 describes a number of computational experiments designed to test the utility of the new inequalities on portfolio optimization instances.

2 Properties of the Set S

The set S is the union of 4 convex sets wherein (z_1, z_2) take the values $(0,0), (0,1), (1,0)$, and $(1,1)$. By adapting a result of Ceria and Soares [3], it is possible to write an extended formulation of $\mathrm{clconv}(S)$.

Proposition 2.1 (Ceria and Soares [3]). *The point $p = (y_1, y_2, t_1, t_2, z_1, z_2) \in \mathrm{clconv}(S)$ if and only if there exist scalars*

$$t_{001}, t_{002}, y_{011}, y_{012}, t_{011} t_{012}, y_{101}, t_{101}, t_{102}, y_{111}, y_{112}, t_{111}, t_{112}, \lambda_{00}, \lambda_{01}, \lambda_{10}, \lambda_{11}$$

such that the following system is feasible:

$$\begin{aligned}
y_{011} + y_{101} + y_{111} &= y_1 \\
y_{012} + y_{112} &= y_2 \\
t_{001} + t_{011} + t_{101} + t_{111} &= t_1 \\
t_{002} + t_{012} + t_{102} + t_{112} &= t_2 \\
\lambda_{10} + \lambda_{11} &= z_1 \\
\lambda_{01} + \lambda_{11} &= z_2 \\
\lambda_{00} + \lambda_{01} + \lambda_{10} + \lambda_{11} &= 1 \\
\lambda_{00}, \lambda_{01}, \lambda_{10}, \lambda_{11} &\geq 0 \\
\hline
t_{001} &\geq 0 \\
t_{002} &\geq 0
\end{aligned}
\qquad
\begin{aligned}
a_{11} y_{011} + a_{12} y_{012} &= 0 \\
0 \leq a_{22} y_{012} &\leq \lambda_{01} \\
\lambda_{01} t_{011} &\geq y_{011}^2 \\
\lambda_{01} t_{012} &\geq y_{012}^2 \\
\hline
0 \leq a_{11} y_{101} &\leq \lambda_{10} \\
\lambda_{10} t_{101} &\geq y_{101}^2 \\
t_{102} &\geq 0 \\
\hline
0 \leq a_{11} y_{111} + a_{12} y_{112} &\leq \lambda_{11} \\
0 \leq a_{22} y_{112} &\leq \lambda_{11} \\
\lambda_{11} t_{111} &\geq y_{111}^2 \\
\lambda_{11} t_{112} &\geq y_{112}^2
\end{aligned}$$

Using Proposition 2.1, one can write an extended formulation for $\mathrm{conv}(S)$ using 23 variables. Our focus is on convexifying S in the original space of variables, as ultimately we wish to use inequalities generated from this study to solve instances that may contain *many* low-dimensional sets S. In this case, adding 17 extra variables in order to convexify S may not be computationally attractive.

The convex sets whose union is S are all defined by the same set of quadratic inequalities and only differ in the polyhedral feasible region for y_1, y_2. We define $P_{z_1 z_2}$ as the following polyhedron for each element of the disjunction:

$$P_{z_1 z_2} \stackrel{\text{def}}{=} \Big\{ (y_1, y_2) \in \mathbb{R}^2 \mid 0 \leq a_{11} y_1 + a_{12} y_2 \leq z_1, \\ 0 \leq \qquad a_{22} y_2 \leq z_2 \Big\},$$

for $(z_1, z_2) \in \{(0,0), (0,1), (1,0), (1,1)\}$.

The section begins by characterizing the extreme points of $\text{conv}(S)$ and the extreme points of the continuous relaxation of S. We then derive four quadratic cone inequalities that are valid for $\text{conv}(S)$.

2.1 Extreme Point Characterizations

We denote the continuous relaxation of S by

$$\begin{aligned}\mathcal{R}(S) \stackrel{\text{def}}{=} \Big\{(y,t,z) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \mid t_1 \geq y_1^2, t_2 \geq y_2^2, \\ 0 \leq a_{11}y_1 + a_{12}y_2 \leq z_1, \\ 0 \leq \qquad a_{22}y_2 \leq z_2, \\ 0 \leq z_1 \leq 1,\ 0 \leq z_2 \leq 1 \quad \Big\}.\end{aligned}$$

Lemma 2.1 establishes that all extreme points for $\mathcal{R}(S)$ have the property that $t_1 = y_1^2, t_2 = y_2^2$.

Lemma 2.1. *If* $p = (\hat{y}_1, \hat{y}_2, \hat{t}_1, \hat{t}_2, \hat{z}_1, \hat{z}_2) \in \text{ext}(\mathcal{R}(S))$*, then* $\hat{t}_1 = \hat{y}_1^2, \hat{t}_2 = \hat{y}_2^2$.

Proof. If $p^* = (y_1^*, y_2^*, t_1^*, t_2^*, z_1^*, z_2^*)$ is an optimal solution to the optimization problem $\min_{p \in \mathcal{R}(S)}\{c^T p\}$ with $c > 0$, then it must be that $t_1^* = (y_1^*)^2, t_2^* = (y_2^*)^2$. If not, a solution with strictly smaller objective function value would exist. □

Lemma 2.1 provides some insight into the nature of the extreme points of $\mathcal{R}(S)$. Theorem 2.1 completely characterizes the extreme points by establishing that extreme points $p = (y_1, y_2, t_1, t_2, z_1, z_2)$ of $\mathcal{R}(S)$ must have one of the following four forms, parameterized by $\alpha, \beta \in \mathbb{R}$, where (α, β) lie in the polytope P_{11}:

$$(\ \alpha,\ \beta,\ \alpha^2,\ \beta^2,\ \quad 1,\ \quad 1\ ) \tag{9}$$

$$(\ \alpha,\ \beta,\ \alpha^2,\ \beta^2,\ \quad 1,\ a_{22}\beta\ ) \tag{10}$$

$$(\ \alpha,\ \beta,\ \alpha^2,\ \beta^2,\ a_{11}\alpha + a_{12}\beta,\ \quad 1\ ) \tag{11}$$

$$(\ \alpha,\ \beta,\ \alpha^2,\ \beta^2,\ a_{11}\alpha + a_{12}\beta,\ a_{22}\beta\ ) \tag{12}$$

Theorem 2.1.

$$\text{ext}(\mathcal{R}(S)) = \{(9), (10), (11), (12) \mid (\alpha, \beta) \in P_{11}\}.$$

Proof. To establish that $(\text{ext}(\mathcal{R}(S)) \subseteq \{(9), (10), (11), (12) \mid (\alpha, \beta) \in P_{11}\})$, suppose that $p = (\hat{y}_1, \hat{y}_2, \hat{t}_1, \hat{t}_2, \hat{z}_1, \hat{z}_2)$ is one of the forms (9) - (12), but not an extreme point. Then there must exist 2 points distinct from p in $\mathcal{R}(S)$ which we denote by

$$\begin{aligned} u =& (y_{1u}, y_{2u}, t_{1u}, t_{2u}, z_{1u}, z_{2u}), \\ v =& (y_{1v}, y_{2v}, t_{1v}, t_{2v}, z_{1v}, z_{2v}) \end{aligned}$$

such that $(1 - \lambda)u + \lambda v = p$ for some $\lambda \in (0, 1)$.

This relationship and the definition of $\mathcal{R}(S)$ indicate

$$\begin{aligned}\hat{y}_1 &= \alpha = (1-\lambda)y_{1u} + \lambda y_{1v} \\ \hat{t}_1 &= \alpha^2 = (1-\lambda)t_{1u} + \lambda t_{1v} \\ &t_{1u} \geq y_{1u}^2,\ t_{1v} \geq y_{1v}^2,\end{aligned}$$

and this system of equations and inequalities implies that

$$\begin{aligned}\hat{t}_1 = \alpha^2 =&(1-\lambda)t_{1u} + \lambda t_{1v} \\ \geq&(1-\lambda)y_{1u}^2 + \lambda y_{1v}^2 \\ \geq&((1-\lambda)y_{1u} + \lambda y_{1v})^2 \\ =&\hat{y}_1^2.\end{aligned}$$

Due to the strict convexity of function $f(y) = y^2$, the last inequality holds at equality only if $y_{1u} = y_{1v}$. It follows that $y_{1u} = y_{1v} = \alpha,\ t_{1u} = t_{1v} = \alpha^2$. The same argument applies to the system of equations and inequalities involving $\hat{y}_2, y_{2u}, y_{2v}, \hat{t}_2, t_{2u}, t_{2v}$ and establishes that $y_{2u} = y_{2v} = \beta,\ t_{2u} = t_{2v} = \beta^2$. Then the definition of $\mathcal{R}(S)$ implies that

$$\begin{aligned}0 \leq a_{11}\alpha + a_{12}\beta &\leq \hat{z}_1 \leq 1, \\ 0 \leq \qquad a_{22}\beta &\leq \hat{z}_2 \leq 1,\end{aligned}$$

and thus for a point p of the form (9),(10),(11),or (12), any solution to the equation $(1-\lambda)u + \lambda v = p,\ \lambda \in (0,1)$ satisfies $z_{1u} = z_{1v} = \hat{z}_1,\ z_{2u} = z_{2v} = \hat{z}_2$. We have established that $p = u = v$, which contradicts the assumption that p is not an extreme point. Therefore any point in $\mathcal{R}(S)$ of the given form is an extreme point.

To establish that $(\mathrm{ext}(\mathcal{R}(S)) \supseteq \{(9),(10),(11),(12) \mid (\alpha,\beta) \in P_{11}\})$, suppose there exists an extreme point $p = (\hat{y}_1, \hat{y}_2, \hat{t}_1, \hat{t}_2, \hat{z}_1, \hat{z}_2)$ that cannot be written in the form of (9),(10),(11), or (12). From Lemma 2.1, we know p can be written as $(\alpha, \beta, \alpha^2, \beta^2, \hat{z}_1, \hat{z}_2)$. Since $p \in \mathcal{R}(S)$, it satisfies

$$\begin{aligned}0 \leq& a_{11}\alpha + a_{12}\beta \leq \hat{z}_1 \leq 1, \\ 0 \leq& \qquad a_{22}\beta \leq \hat{z}_2 \leq 1.\end{aligned}$$

As we assumed $\hat{z}_1 \notin \{a_{11}\alpha + a_{12}\beta, 1\}$, $a_{11}\alpha + a_{12}\beta < \hat{z}_1 < 1$. Then we can take two points in $\mathcal{R}(S)$

$$\begin{aligned}u =&(\alpha, \beta, \alpha^2, \beta^2, a_{11}\alpha + a_{12}\beta, \hat{z}_2), \\ v =&(\alpha, \beta, \alpha^2, \beta^2, 1, \hat{z}_2)\end{aligned}$$

that satisfy $p = (1-\lambda)u + \lambda v$ where $\lambda = \frac{\hat{z}_1 - (a_{11}\alpha + a_{12}\beta)}{1 - (a_{11}\alpha + a_{12}\beta)} \in (0,1)$. This contradicts the assumption that p is extreme, and thus we conclude that $\hat{z}_1 \in \{a_{11}\alpha + a_{12}\beta, 1\}$. A similar argument shows that $\hat{z}_2 \in \{a_{22}\beta, 1\}$, proving that any extreme point can be written in the forms (9),(10),(11), or (12). □

Using similar simple arguments, in Theorem 2.2, we also characterize the extreme points $p = (y_1, y_2, t_1, t_2, z_1, z_2)$ of $\mathrm{conv}(S)$ to have one of the following forms, parameterized by two real numbers (α, β) lying in polyhedra $P_{z_1 z_2}$:

$$(\alpha, \beta, \alpha^2, \beta^2, 1, 1) \tag{13}$$
$$(\alpha, \beta, \alpha^2, \beta^2, 1, 0) \tag{14}$$
$$(\alpha, \beta, \alpha^2, \beta^2, 0, 1) \tag{15}$$
$$(\alpha, \beta, \alpha^2, \beta^2, 0, 0) \tag{16}$$

Theorem 2.2.

$$\begin{aligned}\text{ext}(\text{conv}(\mathcal{R}(S))) =& \{(13) \mid (\alpha, \beta) \in P_{11}\} \cup \{(14) \mid (\alpha, \beta) \in P_{10}\} \\ & \cup \{(15) \mid (\alpha, \beta) \in P_{01}\} \cup \{(16) \mid (\alpha, \beta) \in P_{00}\} \stackrel{\text{def}}{=} \mathcal{E}.\end{aligned}$$

Proof. $(\text{ext}(\text{conv}(\mathcal{R}(S))) \subseteq \mathcal{E})$: Suppose for a contradiction that there exists a point $p = (\hat{y}_1, \hat{y}_2, \hat{t}_1, \hat{t}_2, \hat{z}_1, \hat{z}_2)$ in the given set, but p is not an extreme point. Then there exist two distinct points in $\text{conv}(S)$ which we denote by

$$\begin{aligned} u =& (y_{1u}, y_{2u}, t_{1u}, t_{2u}, z_{1u}, z_{2u}), \\ v =& (y_{1v}, y_{2v}, t_{1v}, t_{2v}, z_{1v}, z_{2v}) \end{aligned}$$

such that $(1 - \lambda)u + \lambda v = p$ for some $\lambda \in (0, 1)$. Since $(\hat{z}_1, \hat{z}_2) \in \mathbb{B}^2$ and $z_{iu}, z_{iv} \in [0, 1], \ i = 1, 2$, any solution to this equation satisfies $z_{iu} = z_{iv} = \hat{z}_i, \ i = 1, 2$. Applying the same argument from the proof of Theorem 2.1, we can deduce that $y_{iu} = y_{iv} = \hat{y}_i, \ t_{iu} = t_{iv} = \hat{t}_i = \hat{y}_i^2, \ i = 1, 2$. This results in $u = v = p$, which contradicts the assumption that p is not extreme.

$(\text{ext}(\text{conv}(\mathcal{R}(S))) \supseteq \mathcal{E})$: Suppose for contradiction that there exists a point $p \in \text{ext}(\text{conv}(S))$, $p = (\hat{y}_1, \hat{y}_2, \hat{t}_1, \hat{t}_2, \hat{z}_1, \hat{z}_2)$ that cannot be written in the forms (13)-(16). First note that $(\hat{z}_1, \hat{z}_2) \in \mathbb{B}^2$ is a necessary condition for p to be extreme. If $(\hat{z}_1, \hat{z}_2) \notin \mathbb{B}^2$, then $p \in \text{conv}(S) \backslash S$ which means that p is written as a convex combination of two or more points in S and by definition not an extreme point. The condition $(\hat{y}_1, \hat{y}_2) \in P_{\hat{z}_1 \hat{z}_2}$ follows from the feasibility requirement. Therefore, that p is not of one of the forms (13) - (16) means $\hat{t}_i > \hat{y}_i^2$ for $i = 1$ or 2. Let $\hat{t}_i = \hat{y}_i^2 + \epsilon_i$ for $\epsilon_i > 0, \ i = 1, 2$ and take

$$\begin{aligned} u =& (\hat{y}_1, \hat{y}_2, \hat{y}_1^2 + 2\epsilon_1, \hat{y}_2^2 + 2\epsilon_2, \hat{z}_1, \hat{z}_2), \\ v =& (\hat{y}_1, \hat{y}_2, \hat{y}_1^2, \hat{y}_2^2, \hat{z}_1, \hat{z}_2). \end{aligned}$$

Then $p = 0.5u + 0.5v$, so p is not extreme, contradicting the assumption.

□

2.2 Valid Inequalities for S

In this subsection, we describe four valid nonlinear inequalities for the set S. Throughout our discussion, we will assume that $a_{12} \neq 0$, since if $a_{12} = 0$, the set S can be further decomposed into smaller blocks. The first valid inequality is a result of direct application of perspective reformulation to the variables t_2, y_2 and z_2.

Proposition 2.2. *The inequality*

$$z_2 t_2 \geq y_2^2. \qquad \text{(Ineq 1)}$$

is valid for S.

Proof. If $z_2 = 1$, then (Ineq 1) is the same as (8a) in the description of S. If $z_2 = 0$, then (8d) implies that $y_2 = 0$, so (Ineq 1) is valid in this case as well. □

Note that (Ineq 1) is in the form of a rotated second order cone, since $z_2, t_2 \geq 0$.

The second valid inequality is obtained by fixing the variable $z_1 = 0$, substituting variable y_1 with $-(a_{12}/a_{11})y_2$, applying the perspective reformulation on t_1, y_2 and z_2 and lifting in order to make the inequality valid when $z_1 = 1$.

Proposition 2.3. *The inequality*

$$\kappa z_1 z_2 + z_2 t_1 \geq \frac{a_{12}^2}{a_{11}^2} y_2^2 \qquad \text{(Ineq 2)}$$

where

$$\kappa = \begin{cases} 0 & \textit{if } a_{12} < 0, \\ \left(\frac{a_{12}}{a_{11}a_{22}}\right)^2 & \textit{if } 0 \leq a_{12} \leq a_{22}, \\ \left(\frac{1}{a_{11}}\right)^2 \left(\frac{2a_{12}}{a_{22}} - 1\right) & \textit{if } a_{22} \leq a_{12}. \end{cases}$$

is valid for S.

Proof. When $z_2 = 0$, (8d) implies that $y_2 = 0$, so (Ineq 2) is valid in this case. If $z_2 = 1$, and $z_1 = 0$, then (8c) implies that $y_2 = -a_{12}y_1/a_{11}$, so (Ineq 2) reduces to the inequality (8c) in the description of S in this case. Thus, we need only prove that (8d) is valid when $z_1 = z_2 = 1$. For (8d) to be valid in this case, we must have $\kappa \geq (a_{12}/a_{11})^2 y_2^2 - t_1 \ \forall (y_1, y_2, t_1, t_2, z_1, z_2) \in S$. Thus the minimum lifting coefficient κ is attained by solving

$$\kappa = \max\{(a_{12}/a_{11})^2 y_2^2 - y_1^2 \mid 0 \leq a_{11}y_1 + a_{12}y_2 \leq 1, 0 \leq a_{22}y_2 \leq 1\}. \qquad (17)$$

Note that in the objective of the lifting problem (17), y_1^2 replaces t_1 as $t_1 = y_1^2$ at an optimal solution. We can calculate κ by giving a closed-form for the optimal solution y_1^*, y_2^* to (17) by exploiting separability. To that end, define

$$\begin{aligned} y_1^*(y_2) &= \operatorname{argmax}_{y_1}\{(a_{12}/a_{11})^2 y_2^2 - y_1^2 \mid -a_{12}y_2 \leq a_{11}y_1 \leq 1 - a_{12}y_2\} \\ &= \operatorname{argmin}_{y_1}\{|y_1| \mid -a_{12}y_2 \leq a_{11}y_1 \leq 1 - a_{12}y_2\}, \end{aligned}$$

and note that $y_2 \geq 0 \ \forall y_2 \in S$. The solution to this 1-variable optimization problem can be characterized as

$$y_1^*(y_2) = \begin{cases} \frac{-a_{12}y_2}{a_{11}} & \text{if } a_{12} < 0 \\ 0 & \text{if } a_{12} > 0,\ 1 - a_{12}y_2 \geq 0 \\ \frac{1-a_{12}y_2}{a_{11}} & \text{if } a_{12} > 0,\ 1 - a_{12}y_2 < 0. \end{cases}$$

The lifting problem (17) now becomes an optimization problem over only y_2.

$$\kappa = \max_{y_2} \Big\{ (a_{12}/a_{11})^2 y_2^2 - (y_1^*(y_2))^2 \mid 0 \le a_{11} y_1^*(y_2) + a_{12} y_2 \le 1, 0 \le a_{22} y_2 \le 1 \Big\}. \tag{18}$$

If $a_{12} < 0$, then substituting $y_1^*(y_2) = -a_{12}y_2/a_{11}$ into (18) gives

$$\kappa = \max_{y_2} \{0 \mid 0 \le a_{22}y_2 \le 1\} = 0.$$

If $a_{12} > 0$, then we must consider two subcases depending on the value of $y_1^*(y_2)$. Specifically, in the case $y_2 \le 1/a_{12}$, we substitute $y_1^*(y_2) = 0$ into (18), and in the case $y_2 \le 1/a_{12}$, we substitute $y_1^*(y_2) = (1 - a_{12}y_2)/a_{11}$ into (18). Performing these substitutions and simplifying the expressions yields

$$\begin{aligned} \kappa = \max \Big\{ & \max_{y_2 \le 1/a_{12}} \{(a_{12}/a_{11})^2 y_2^2 \mid 0 \le a_{12} y_2 \le 1, 0 \le a_{22} y_2 \le 1\}, \\ & \max_{y_2 > 1/a_{12}} \{(1/a_{11})^2 (2a_{12}y_2 - 1) \mid 0 \le a_{22}y_2 \le 1\} \Big\}. \end{aligned} \tag{19}$$

We can solve each of the two optimization problems in the definition of κ in closed form. If $a_{12} \le a_{22}$, the first maximization problem in (19) has optimal value $(a_{12}/(a_{11}a_{22}))^2$, and the second optimization problem is infeasible. Thus $\kappa = (a_{12}/(a_{11}a_{22}))^2$ when $a_{12} \le a_{22}$. If $a_{12} \ge a_{22}$, the first maximization problem in (19) has optimal value $1/a_{11}^2$, and the second maximization problem in (19) has optimal value $(1/a_{11}^2)(2a_{12}/a_{22} - 1)$. Since $(2a_{12}/a_{22} - 1) \ge 1$ when $a_{12} \ge a_{22}$, we know that $\kappa = (1/a_{11}^2)(2a_{12}/a_{22} - 1)$ when $a_{12} \ge a_{22}$. Having established the proper value of κ in all cases, the proof is complete. □

Note that $\kappa \ge 0$, so (Ineq 2) can be written in the form of a rotated second order cone constraint:

$$(\kappa z_1 + t_1) z_2 \ge ((a_{12}/a_{11}) y_2)^2.$$

A lifting procedure similar to that used for (Ineq 2) can be used to derive the following inequality.

Proposition 2.4. *The inequality*

$$\nu z_1 z_2 + a_{11}^2 z_1 t_1 \ge (a_{11} y_1 + a_{12} y_2)^2, \tag{Ineq 3}$$

where

$$\nu = \begin{cases} 0 & \textit{if } a_{12} < 0 \\ 1 & \textit{if } a_{12} > 0, a_{22} < a_{12} \\ -(\frac{a_{12}}{a_{22}})^2 + \frac{2a_{12}}{a_{22}} & \textit{if } 0 \le a_{12} \le a_{22} \end{cases}$$

is valid for S.

Proof. If $z_1 = 0$, then the inequality reduces to $0 \ge (a_{11}y_1 + a_{12}y_2)^2$, which is implied by (8c) in the description of S. If $z_1 = 1, z_2 = 0$, then the inequality is $a_{11}^2 t_1 \ge (a_{11}y_1)^2$, which is implied by (8a) in the description of S. Thus, we need only prove validity for the case $z_1 = z_2 = 1$. In this case, the inequality will be valid as long as

$$\begin{aligned} \nu &= \max\{(a_{11}y_1 + a_{12}y_2)^2 - a_{11}^2 t_1 \mid 0 \le a_{11}y_1 + a_{12}y_2 \le 1, 0 \le a_{22}y_2 \le 1, t_1 \ge y_1^2\} \\ &= \max\{(a_{11}y_1 + a_{12}y_2)^2 - a_{11}^2 y_1^2 \mid 0 \le a_{11}y_1 + a_{12}y_2 \le 1, 0 \le a_{22}y_2 \le 1\}. \end{aligned}$$

Defining $Y = a_{11}y_1 + a_{12}y_2$, we can rewrite this as

$$\nu = \max\{Y^2 - a_{11}^2 y_1^2 \mid 0 \le Y \le 1, 0 \le \frac{a_{22}}{a_{12}}Y - \frac{a_{11}a_{22}}{a_{12}}y_1 \le 1\}. \tag{20}$$

We again can exploit separability to derive a closed-form solution to this problem. Let

$$\begin{aligned} y_1^*(Y) &:= \text{argmax}_{y_1}\{Y^2 - a_{11}^2 y_1^2 \mid -\frac{a_{22}}{a_{12}}Y \le \frac{-a_{11}a_{22}}{a_{12}}y_1 \le 1 - \frac{a_{22}}{a_{12}}Y\} \\ &= \text{argmin}_{y_1}\{|y_1| \mid -\frac{a_{22}}{a_{12}}Y \le \frac{-a_{11}a_{22}}{a_{12}}y_1 \le 1 - \frac{a_{22}}{a_{12}}Y\}. \end{aligned}$$

The definition of ν in (20) can be replaced by a 1-dimensional optimization problem

$$\nu = \max_Y\{Y^2 - a_{11}^2 (y_1^*(Y))^2 \mid 0 \le Y \le 1, 0 \le \frac{a_{22}}{a_{12}}Y - \frac{a_{11}a_{22}}{a_{12}}y_1^*(Y) \le 1\}. \tag{21}$$

If $a_{12} \le 0$, then

$$y_1^*(Y) = \text{argmin}_{y_1}\{|y_1| \mid Y/a_{11} \le y_1 \le Y/a_{11} - a_{12}/(a_{11}a_{22})\} = Y/a_{11},$$

since $Y/a_{11} \ge 0$. Substituting this into (21), we get that

$$\nu = \max_Y\{0 \mid 0 \le Y \le 1\} = 0.$$

If $a_{12} > 0$, then

$$\begin{aligned} y_1^*(Y) &= \text{argmin}_{y_1}\{|y_1| \mid Y/a_{11} \ge y_1 \ge Y/a_{11} - a_{12}/(a_{11}a_{22})\} \\ &= \begin{cases} 0 & \text{if } Y \le a_{12}/a_{22} \\ Y/a_{11} - a_{12}/(a_{11}a_{22}) & \text{if } Y \ge a_{12}/a_{22}. \end{cases} \end{aligned}$$

$$\begin{aligned} \nu = \max\Big\{ &\max_{Y \le a_{12}/a_{22}} \{Y^2 \mid 0 \le Y \le 1, 0 \le (a_{22}/a_{12})Y \le 1\}, \\ &\max_{Y > a_{12}/a_{22}} \{2a_{12}Y/a_{22} - (a_{12}/a_{22})^2 \mid 0 \le Y \le 1\}\Big\} \end{aligned} \tag{22}$$

If $a_{12} \ge a_{22}$, the first maximization problem in (22) has optimal value 1, and the second optimization problem is infeasible. So $\nu = 1$ in the case that $a_{12} > 0, a_{12} \ge a_{22}$. If $a_{12} \le a_{22}$, the first maximization problem in (22) has optimal value $(a_{12}/a_{22})^2$, and the second maximization problem in (22) has optimal value $2a_{12}/a_{22} - (a_{12}/a_{22})^2$. Since

$$2a_{12}/a_{22} - (a_{12}/a_{22})^2 - (a_{12}/a_{22})^2 = \frac{2a_{12}}{a_{22}}\left(1 - \frac{a_{12}}{a_{22}}\right) \ge 0$$

when $a_{22} > a_{12}$, we can conclude that $\nu = 2a_{12}/a_{22} - (a_{12}/a_{22})^2$ in this case. Having established the value of ν in all cases, the proof is complete.

□

Since if $0 \le a_{12} \le a_{22}$, we have that $\nu = (a_{12}/a_{22})(2 - a_{12}/a_{22}) \ge 0$, (Ineq 3) can be written as the rotated second order cone constraint

$$(\nu z_2 + a_{11}^2 t_1)z_1 \ge (a_{11}y_1 + a_{12}y_2)^2.$$

The final valid inequality is equivalent to applying the perspective reformulation to the variables t_1, y_1 and $(z_1 + z_2)$ which is applicable because $z_1 + z_2 = 0$ implies $y_1 = 0$.

Proposition 2.5. *The inequality*

$$t_1(z_1 + z_2) \geq y_1^2. \tag{Ineq 4}$$

is valid for S.

Proof. We verify the validity for each value that z_1, z_2 take.

1. $(z_1 = 0,\ z_2 = 0)$
   (Ineq 4) simply reduces to $0 \geq 0$.

2. $(z_1 = 0,\ z_2 = 1$ or $z_1 = 1,\ z_2 = 0)$
   (Ineq 4) is equivalent to $t_1 \geq y_1^2$, which is valid for S.

3. $(z_1 = 1,\ z_2 = 1)$
   (Ineq 4) is equivalent to $2t_1 \geq y_1^2$, which is dominated by (8a) in the description of S.

□

Note that the feasible region for (Ineq 4) is convex since it is clearly in rotated second-order cone form.

Adding the valid second-order cone constraints (Ineq 1)-(Ineq 4) to the continuous relaxation of S results in a strengthened relaxation of S, which we denote as

$$\begin{aligned}\mathcal{T}(S) \stackrel{\text{def}}{=} \Big\{(y,t,z) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \mid\ & t_1 \geq y_1^2, t_2 \geq y_2^2, \\ & z_2 t_2 \geq y_2^2 \\ & \kappa z_1 z_2 + z_2 t_1 \geq \frac{a_{12}^2}{a_{11}^2} y_2^2 \\ & \nu z_1 z_2 + a_{11}^2 z_1 t_1 \geq (a_{11}y_1 + a_{12}y_2)^2 \\ & t_1(z_1 + z_2) \geq y_1^2 \\ & 0 \leq a_{11}y_1 + a_{12}y_2 \leq z_1, \\ & 0 \leq \qquad\quad a_{22}y_2 \leq z_2, \\ & 0 \leq z_1 \leq 1,\ 0 \leq z_2 \leq 1 \quad \Big\}.\end{aligned}$$

3 Measuring the Impact of Conic Inequalities

In mixed integer linear programming, it is known that the convex hull of a mixed integer linear set is a polyhedron [16]. Theorem 2.2 demonstrated that $\text{conv}(S)$ is not a polyhedron, so we have no nice way of demonstrating that the inequalities we derived are necessary in the description of $\text{conv}(S)$, like demonstrating facets of a polyhedron. The theory of mixed integer nonlinear sets is not fully-developed, but Kılınç-Karzan [14] has done some work to extend the notion of the necessity of inequalities in the description of mixed-integer nonlinear sets. In this section, we take a different approach to providing evidence of the usefulness of the conic inequalities (Ineq 1)-(Ineq 4). First, we provide a short empirical experiment to demonstrate that adding the inequalities to the continuous relaxation $\mathcal{R}(S)$ does *not* provide a complete description of $\text{conv}(S)$, i.e. $\mathcal{T}(S) \neq \text{conv}(S)$. The next two subsections provide evidence that the inequalities (Ineq 1)-(Ineq 4) may be useful by characterizing which fractional extreme points of

Instance Classes	$\mathcal{R}(S)$	$\mathcal{T}(S)$
A ($a_{12} < 0$)	0.064	0.084
B ($0 \leq a_{12} \leq a_{22}$)	0.052	0.14
C ($a_{22} < a_{12}$)	0.048	0.092

Table 1: Probability of having fractional extreme points

the continuous relaxation are cut off by each of the inequalities and by characterizing at which points of $\text{conv}(S)$ each of the inequalities is tight. The section also contains results of a "shooting experiment," wherein we estimate the size of each surface of $\mathcal{T}(S)$.

3.1 Proportion of Fractional Extreme Points

A natural question to ask is whether the conic inequalities (Ineq 1)-(Ineq 4) are the only nontrivial inequalities necessary to describe $\text{conv}(S)$: Does $\mathcal{T}(S) = \text{conv}(S)$? We answer this question in the negative by estimating the probability that $\mathcal{T}(S)$ has fractional extreme points through computational experiments. Note that by Theorem 2.2, if $\mathcal{T}(S) = \text{conv}(S)$, then all extreme points of $\mathcal{T}(S)$ would be integral, so the associated probability would be zero.

The extreme points of $\mathcal{T}(S)$ were obtained by solving

$$\min_{(y,t,z)\in\mathcal{T}(S)} \alpha^T(y,t,z) \tag{23}$$

where $\alpha \in \mathbb{R}^6$. The coefficients $\alpha_1, \alpha_2, \alpha_5, \alpha_6$ were drawn from a uniform distribution in $[-1, 1]$, and the coefficients for the t variables α_3, α_4 were drawn from a uniform distribution in $[0, 1]$ since they need to be non-negative to ensure that (23) is not unbounded.

Thirty random instances were generated, broken into three classes based on the relative magnitudes of the coefficients defining S. For each instance 50 random linear objectives α were used. For comparison, the same experiment was repeated using the continuous relaxation $\mathcal{R}(S)$ instead of $\mathcal{T}(S)$ in (23). Table 1 summarizes the proportion of fractional extreme points out of all extreme points obtained by solving (23) in the different cases.

Since the probabilities associated with $\mathcal{T}(S)$ are not zero, we can be sure that $\mathcal{T}(S) \neq \text{conv}(S)$. Note that it appears more likely to have fractional extreme points using the relaxation $\mathcal{T}(S)$ than $\mathcal{R}(S)$. This implies that the valid inequalities (Ineq 1)-(Ineq 4) from the definition of $\mathcal{T}(S)$ cut off extreme points of $\mathcal{R}(S)$ while generating different fractional extreme points.

3.2 Extreme points cut off by the inequalities

We investigate the usefulness of the inequalities by examining which of the points of $\text{ext}(\mathcal{R}(S))$ are cut off by each of the inequalities introduced in Section 2.2. Table 2 demonstrates whether the fractional extreme points in each of the families (10), (11), and (12) are cut off by inequalities (Ineq 1) - (Ineq 4), or the condition under which they are cut off. In the table, it is assumed that $0 < a_{11}\alpha + a_{12}\beta < 1$ and $0 < a_{22}\beta < 1$ so that points represented by (10) - (12) are fractional. The entries of the table mean either the points are always cut off (A), never cut off (X), or are cut off under some condition given in the equations referenced.

Inequality		Extreme Points		
		(10)	(11)	(12)
(Ineq 1)		A	X	A
(Ineq 2)	$a_{12} < 0$	(24)	X	(24)
	$0 < a_{12} \leq a_{22}$	X	(25)	(26)
	$a_{12} > a_{22}$	(27)	(28)	(29)
(Ineq 3)	$a_{12} < 0$	X	(30)	(30)
	$0 < a_{12} \leq a_{22}$	(31)	X	(32)
	$a_{12} > a_{22}$	(33)	(34)	(35)
(Ineq 4)		X	X	(36)

$$a_{22}\alpha^2 - \left(\frac{a_{12}}{a_{11}}\right)^2 \beta < 0 \tag{24}$$

$$\left(\frac{a_{12}}{a_{11}a_{22}}\right)^2 (a_{11}\alpha + a_{12}\beta) + \alpha^2 - \left(\frac{a_{12}}{a_{11}}\right)^2 \beta^2 < 0 \tag{25}$$

$$\left(\frac{a_{12}}{a_{11}a_{22}}\right)^2 (a_{11}\alpha + a_{12}\beta) + \alpha^2 - \frac{a_{12}^2}{a_{11}^2 a_{22}}\beta^2 < 0 \tag{26}$$

$$\left(\frac{2a_{12} - a_{22}}{a_{11}^2 a_{22}}\right) + \alpha^2 - \left(\frac{a_{12}}{a_{11}a_{22}}\right)^2 \beta < 0 \tag{27}$$

$$\left(\frac{2a_{12} - a_{22}}{a_{11}^2 a_{22}}\right) (a_{11}\alpha + a_{12}\beta) + \alpha^2 - \left(\frac{a_{12}}{a_{11}}\right)^2 \beta^2 < 0 \tag{28}$$

$$\left(\frac{2a_{12} - a_{22}}{a_{11}^2 a_{22}}\right) (a_{11}\alpha + a_{12}\beta) + \alpha^2 - \frac{a_{12}^2}{a_{11}^2 a_{22}}\beta < 0 \tag{29}$$

$$a_{11}^2\alpha^2 - a_{11}\alpha - a_{12}\beta < 0 \tag{30}$$

$$a_{22}\beta + a_{11}^2\alpha^2 - (a_{11}\alpha + a_{12}\beta)^2 < 0 \tag{31}$$

$$a_{11}^2\alpha^2 - a_{11}\alpha - a_{12}\beta + a_{22}\beta < 0 \tag{32}$$

$$\left(\frac{2a_{12}a_{22} - a_{12}^2}{a_{22}}\right) \beta + a_{11}^2\alpha^2 - (a_{11}\alpha + a_{12}\beta)^2 < 0 \tag{33}$$

$$\left(\frac{2a_{12}a_{22} - a_{12}^2}{a_{22}^2}\right) + a_{11}^2\alpha^2 - (a_{11}\alpha + a_{12}\beta) < 0 \tag{34}$$

$$\left(\frac{2a_{12}a_{22} - a_{12}^2}{a_{22}}\right) \beta + a_{11}^2\alpha^2 - (a_{11}\alpha + a_{12}\beta) < 0 \tag{35}$$

$$a_{11}\alpha + a_{12}\beta + a_{22}\beta < 1 \tag{36}$$

Table 2: Extreme points of $\mathcal{R}(S)$ cut off by inequalities

The analysis demonstrates that all inequalities are able to eliminate some fractional extreme points of the continuous relaxation $\mathcal{R}(S)$. The simple perspective inequality always cuts off extreme points that have variable z_2 fractional. Of course, intersecting the set of points feasible for one of the inequalities (Ineq 1)-(Ineq 4) with $\mathcal{R}(S)$ may result in new fractional extreme points, which also is suggested by the earlier computational experiment summarized in Table 1.

3.3 Tightness of inequalities

Another property of inequalities that may point toward their usefulness in strengthening the continuous relaxation of a mixed integer nonlinear set S is whether or not the inequality is tight at extreme points of the convex hull of S. In Table 3 we examine for each class of extreme points of $\text{conv}(S)$ whether or not each inequality (Ineq 1) - (Ineq 4) holds with equality at that point. In the table, the notation *A* denotes that the inequality is always tight for all extreme points of the associated type. The notation *X* denotes that the inequality is never tight for that class of extreme points. If the inequality is tight for some of the extreme points in the class, then the conditions under which the extreme point is tight is given by the equation number listed in the table.

Inequality		Extreme Points			
		(13)	(14)	(15)	(16)
(Ineq 1)		A	A	A	A
(Ineq 2)	$a_{12} < 0$	A	A	(37)	(38)
	$0 < a_{12} \leq a_{22}$	A	A	(37)	(39)
	$a_{12} > a_{22}$	A	A	(37)	(40)
(Ineq 3)	$a_{12} < 0$	A	A	A	(37)
	$0 < a_{12} \leq a_{22}$	A	A	A	(41)
	$a_{12} > a_{22}$	A	A	A	(42)
(Ineq 4)		A	A	A	X

$$\beta = 0 \tag{37}$$

$$a_{11}\alpha + a_{12}\beta = 0 \text{ or } a_{11}\alpha - a_{12}\beta = 0 \tag{38}$$

$$\alpha = 0 \text{ and } \beta = \frac{1}{a_{22}} \tag{39}$$

$$\frac{1}{a_{11}^2}\left(\frac{2a_{12}}{a_{22}} - 1\right) + \alpha^2 = \left(\frac{a_{12}}{a_{11}}\right)^2 \tag{40}$$

$$-\left(\frac{a_{12}}{a_{22}}\right)^2 + \frac{2a_{12}}{a_{22}} + a_{11}^2\alpha^2 = (a_{11}\alpha + a_{12}\beta)^2 \tag{41}$$

$$\alpha = 0 \text{ and } \beta = \frac{1}{a_{12}} \tag{42}$$

Table 3: Tightness of inequalities at extreme points of $\text{conv}(S)$

3.4 Shooting Experiment

One way of measuring the importance of a facet of a polyhedron is by a "shooting experiment." To our knowledge, it was first proposed by Kuhn [15] as a means to identify new facets of the TSP polytope. Random rays from a fixed point in the interior of the polytope are extended until they intersect a facet of the polyhedron, and the facet hit by the ray is recorded. Later Gomory et al. [10] reported a similar experiment conducted on the cyclic group polyhedra where the random rays originate from outside of the polyhedra and the number of rays intersected for each facet is recorded. The facets hit by the highest percentage of rays are considered the most important. Hunsaker [13] found that the facets whose "hit percentage" is large correlate strongly with empirical usefulness of the inequalities. We take Kuhn's approach and use a fixed point in the interior of $\text{conv}(S)$ as the origin of the shooting. We would like to understand the importance of each of the surfaces in our relaxation $\mathcal{T}(S)$ by estimating the likelihood of hitting that surface first if traversing a random direction from that fixed point.

A shooting with ray $r \in \mathbb{R}^6$ originating from a point $\hat{p} = (\hat{y}_1, \hat{y}_2, \hat{t}_1, \hat{t}_2, \hat{z}_1, \hat{z}_2)$ corresponds to solving the following conic program:

$$\begin{aligned} \max_{y,t,z,d} \quad & d \\ \text{s.t.} \quad & (y,t,z) = (\hat{p} + rd) \\ & (y,t,z) \in \mathcal{T}(S) \\ & d \in \mathbb{R}_+. \end{aligned} \tag{43}$$

By selecting $\hat{p}$ as a convex combination of the extreme points of $\text{conv}(S)$ (characterized in Theorem (2.2)), we can be sure that $\hat{p} \in \text{conv}(S)$. In an attempt to approximate the centroid of $\text{conv}(S)$, we consider all possible combinations of the lower and upper bounds on variables y_1, y_2 as the values of α, β in each of the 4 cases (13) - (16) to obtain the following 9 points:

$$\begin{aligned} p_1 &= (0,0,0,0,1,1) \\ p_2 &= (1/a_{11}, 0, (1/a_{11})^2, 0, 1, 1) \\ p_3 &= (-a_{12}/a_{11}a_{22}, 1/a_{22}, (a_{12}/a_{11}a_{22})^2, (1/a_{22})^2, 1, 1) \\ p_4 &= (1/a_{11} - a_{12}/a_{11}a_{22}, 1/a_{22}, (1/a_{11} - a_{12}/a_{11}a_{22})^2, (1/a_{22})^2, 1, 1) \\ \hline p_5 &= (0,0,0,0,0,1) \\ p_6 &= (-a_{12}/a_{11}a_{22}, 1/a_{22}, (a_{12}/a_{11}a_{22})^2, (1/a_{22})^2, 0, 1) \\ \hline p_7 &= (0,0,0,0,1,0) \\ p_8 &= (1/a_{11}, 0, (1/a_{11})^2, 0, 1, 0) \\ \hline p_9 &= (0,0,0,0,0,0). \end{aligned}$$

As the starting point for our shooting experiment we use

$$\hat{p} = \frac{1}{9} \sum_{j=1}^{9} p_j.$$

We choose the ray $r \in \mathbb{R}^6$ to be uniformly distributed over an appropriate hemisphere in $\mathbb{R}^6$. Since $\text{conv}(S)$ is unbounded in the directions $(0,0,1,0,0,0)$ and $(0,0,0,1,0,0)$, we are not interested in rays

r that have positive components in the t directions ($r_3 > 0$ or $r_4 > 0$). We generate the direction by first generating a vector r from the multivariate normal distribution $r \sim \mathcal{N}(0, I)$, thus ensuring that r is uniformly distributed on the unit sphere in $\mathbb{R}^6$. If $r_3 > 0$ or $r_4 > 0$, we simply flip the sign for that component of the vector. For each random instance of S, the shooting problem (43) was solved 1,000 times using different directions r chosen in this manner.

Table (4) summarizes the result of the shooting experiment for 10 different sets $S(a_{11}, a_{12}, a_{22})$, characterized by the values of the coefficients (a_{11}, a_{12}, a_{22}). Entries in the table denote the number of times out of 1,000 the inequalities (Ineq 1) - (Ineq 4), and the original inequalities in the description of $\mathcal{R}(S)$ are tight at the optimal solution of shooting problem. For numerical purposes, we define the inequality $f(x) \leq 0$ to be *tight* at $\hat{x}$ if $\|f(\hat{x})\| < 10^{-6}$.

(a_{11}, a_{12}, a_{22})	(Ineq 1)	(Ineq 2)	(Ineq 3)	(Ineq 4)	$\mathcal{R}(S)$
$(5, -9, 6)$	841	2	7	0	156
$(9, -9, 4)$	565	38	0	0	408
$(7, 5, 5)$	623	0	0	315	390
$(8, 2, 9)$	687	0	0	5	347
$(3, 5, 1)$	254	1	0	0	745
$(5, 4, 1)$	172	0	2	3	827
$(5, -9, 3)$	668	10	0	0	325
$(3, 3, 9)$	954	0	0	0	62
$(1, 2, 8)$	963	0	0	0	53
$(8, 2, 6)$	452	0	0	21	564

Table 4: Shooting Experiment

The results of the experiment show convincingly that the most important nonlinear inequalities are the perspective inequalities (Ineq 1). However, each of the inequalities is sometimes useful in strengthening the set $\mathcal{R}(S)$.

4 Computations

In this section, we demonstrate that the new conic inequalities strengthen the relaxation of X on small instances of a portfolio optimization problem. In order to employ the inequalities, we first must discuss strategies for extracting diagonal or block-diagonal components from the original matrix Q.

4.1 Matrix Decomposition

As described in Section 1, the diagonal matrix splitting approach decomposes the original matrix into $Q = R + D$, for some matrix $D = \text{diag}(d) \geq 0$ such that $R = Q - D \succeq 0$. Frangioni and Gentile [6] suggest using $D = \lambda_n I$, where $\lambda_n > 0$ is the smallest eigenvalue of Q. In subsequent work, they show how more of the separable structure of Q can be extracted into D by solving a semidefinite program (SDP) [7]. Since the perspective reformulation and tightening will be performed on the diagonal elements of the extracted matrix D, a reasonable metric for the extracted matrix D seems to be to maximize $\sum_{j \in N} d_j = \text{tr}\, D$. This results in the SDP

$$\max_{d\geq 0}\{\sum_{j\in N} d_j \mid Q - \sum_{j\in N} d_j(e_j e_j^T) \succeq 0\}. \tag{44}$$

In our approach, we suggest a different decomposition $Q = R + B$ where $B \succeq 0$ is a 2×2 block diagonal matrix such that $R = Q - B \succeq 0$. Following the intuition of Frangioni and Gentile [7], we can construct a SDP that extracts a large 2×2 block diagonal matrix $B \in \mathbb{R}^{n\times n}$. To help construct the blocks, let $O = \{j \in N \mid j \text{ is odd}\}$ be the set of odd elements of N. The SDP that will maximize the amount extracted from the 2×2 block diagonal is

$$\begin{aligned}
\max \quad & r\sum_{j\in N}\mu_j + s\sum_{j\in O}\lambda_j, && (45)\\
\text{s.t.} \quad & Q - \sum_{j\in N}\mu_j e_j e_j^T - \sum_{j\in O}\lambda_j(e_j e_{j+1}^T + e_{j+1}e_j^T) \succeq 0, &&\\
& \mu_j e_j e_j^T + \mu_{j+1}e_{j+1}e_{j+1}^T + \lambda_j(e_j e_{j+1}^T + e_{j+1}e_j^T) \succeq 0 \;\forall j \in O, &&
\end{aligned}$$

where μ_j and λ_j represent the diagonal and off-diagonal elements of a symmetric 2×2 block-diagonal matrix B, respectively. The scalar parameters r and s allow flexibility in the extraction. Larger values of s put larger weight on the off-diagonal elements. We use the values $r = 5, s = 2$ in all of our computational experiments.

4.1.1 Minimum Variance Portfolio Problem

To demonstrate the potential of the block diagonal extraction and the strong conic inequalities introduced in Section 2, we report the results of an experiment computing root node bounds for small instances of a minimum variance portfolio problem. The problem can be stated mathematically as

$$\begin{aligned}
z^* := \min \quad & x^T Q x && (46)\\
\text{s.t.} \quad & e^T x = 1, &&\\
& e^T z \leq K, &&\\
& \alpha^T x \geq \rho, &&\\
& 0 \leq x_j \leq z_j \; j \in N, &&\\
& x \in \mathbb{R}^n, z \in \mathbb{B}^n. &&
\end{aligned}$$

We created instances of (46) randomly in the following manner. The parameters α_j were drawn from uniform distributions in the range $[-0.02, 0.5]$, and the parameter ρ from a uniform distribution in the range $[0, 0.02]$. A random, positive semidefinite matrix Q is constructed as

$$Q = A^T A + \text{diag}(\zeta) + \sum_{j=1}^{n/2}\gamma_j(e_{2j-1} + e_{2j})(e_{2j-1} + e_{2j})^T$$

where $A \in \mathbb{R}^{n\times n}, \zeta \in \mathbb{R}^n, \gamma \in \mathbb{R}^{n/2}$. Each element of A is drawn uniformly at random from the range $[-5, 5]$, each ζ_j is a uniform random variable in the range $[0, \bar{\zeta}]$ and γ_j is a uniform random variable in the range $[0, \bar{\gamma}]$. The third addend in Q is a 2×2 block-diagonal matrix whose j^{th} block has all 4 entries equal to $\gamma_j, j = 1, 2, \cdots, n/2$. By increasing $\bar{\zeta}$, the resulting matrix Q becomes more diagonally dominant. By increasing $\bar{\gamma}$, we can make Q more "block"-diagonally dominant. We would expect the block extraction technique with the conic inequalities (Ineq 1)-(Ineq 4) to perform best if $\bar{\gamma}$ is large. Note that by construction we have $Q \succeq 0$. We created three families of random instances. In the first family, $\bar{\zeta} = \bar{\gamma} = 0$; in the second family, $\bar{\zeta} = 100, \bar{\gamma} = 0$; and in the third family $\bar{\zeta} = 0, \bar{\gamma} = 100$.

In each family, we solved 20 random instances for each size $n \in \{10, 20, 30, 40\}$ to compare the strength of different relaxations. For each instance, we set the cardinality constraints to have value $K = \lfloor 2n/5 \rfloor$. To precisely define the relaxations, let

$$P \stackrel{\text{def}}{=} \{(x, z) \in [0,1]^n \times [0,1]^n \mid e^T x = 1, e^T z \le K, \alpha^T x \ge \rho, x_j \le z_j \forall j \in N\}$$

be the continuous relaxation of the feasible region of (46). The natural continuous relaxation obtains root objective value

$$z_R \stackrel{\text{def}}{=} \min_{x,z} \{x^T Q x \mid (x, z) \in P\}.$$

To create a relaxation based on a diagonal matrix-splitting, we solve the SDP (44) to obtain an optimal solution d, and we let $Q = R + \operatorname{diag}(d)$. The strengthened relaxation obtains root objective value

$$z_D \stackrel{\text{def}}{=} \min_{x,z,v,t} \Big\{ v \mid \quad v \ge x^T R x + e^T t \\ (x, z) \in P \\ z_j t_j \ge d_j x_j^2 \ \forall j \in N \Big\}.$$

To create a relaxation based on a block-diagonal matrix splitting, we solve the SDP (45) to obtain an optimal solution B, and we let $Q = R + B$, with $B = LL^T$. The strengthened relaxation employing our new conic inequalities obtains root objective value

$$z_B \stackrel{\text{def}}{=} \min_{x,z,v,t,y} \Big\{ v \mid \quad v \ge x^T R x + e^T t \\ (x, z) \in P \\ L^T y = x \\ (y_j, y_{j+1}, t_j, t_{j+1}, z_j, z_{j+1}) \in \mathcal{T}(S(L_{jj}^{-T}, L_{j,j+1}^{-T}, L_{j+1,j+1}^{-T})) \ \forall j \in O \Big\}.$$

Note the dependence of the feasible region, specifically the sets S, on the inverse of the Cholesky factor of B.

We also create a relaxation based on performing the same block-diagonal matrix splitting used to compute z_B, but we replace $\mathcal{T}(S)$ (our approximation of $\operatorname{conv}(S)$), with $\operatorname{conv}(S)$ using the extended formulation described in Proposition 2.1. Specifically, we compute the value

$$z_{\text{conv}} \stackrel{\text{def}}{=} \min_{x,z,v,t,y} \Big\{ v \mid \quad v \ge x^T R x + e^T t \\ (x, z) \in P \\ L^T y = x \\ (y_j, y_{j+1}, t_j, t_{j+1}, z_j, z_{j+1}) \in \operatorname{conv}(S(L_{jj}^{-T}, L_{j,j+1}^{-T}, L_{j+1,j+1}^{-T})) \ \forall j \in O \Big\}.$$

Relaxations are compared with respect to their root optimality gap

$$\text{Gap}(z_?) \stackrel{\text{def}}{=} \frac{z_? - z^*}{z^*},$$

where $z_?$ is one of z_R, z_D, z_B, or z_{conv} depending on which relaxation is being evaluated.

Tables 5, 6, and 7 show the average root optimality gaps for the four different relaxation methods over the 20 randomly generated instances of each problem size.

n	Gap(z_R)	Gap(z_D)	Gap(z_B)	Gap(z_{conv})
10	39.10	38.89	38.19	38.08
20	28.73	28.64	28.71	28.69
30	23.85	23.76	23.73	23.71
40	18.43	18.43	18.43	18.43

Table 5: Average Gap(%) for Min Variance Portfolio Problem with ($\bar{\zeta} = 0, \bar{\gamma} = 0$)

n	Gap(z_R)	Gap(z_D)	Gap(z_B)	Gap(z_{conv})
10	39.68	12.07	12.25	10.69
20	31.96	16.09	19.25	18.13
30	29.14	15.66	20.17	18.39
40	25.46	15.73	18.82	17.57

Table 6: Average Gap(%) for Min Variance Portfolio Problem ($\bar{\zeta} = 100, \bar{\gamma} = 0$)

n	Gap(z_R)	Gap(z_D)	Gap(z_B)	Gap(z_{conv})
10	24.72	20.33	13.20	11.80
20	21.16	19.56	15.83	15.38
30	19.58	18.06	15.75	15.13
40	19.05	17.92	17.22	16.89

Table 7: Average Gap(%) for Min Variance Portfolio Problem ($\bar{\zeta} = 0, \bar{\gamma} = 100$)

Table 5 shows that strengthening the relaxation using the perspective reformulation or the new conic inequalities appears to have little or no impact for the instances with $\bar{\zeta} = 0, \bar{\gamma} = 0$. However, Tables 6 and 7 show that there are significant root gap improvements for instances in the families where Q contains elements specifically added to the diagonal or off-diagonal. The diagonal extraction method works best for instances that are the most diagonally dominant ($\bar{\zeta} = 100, \bar{\gamma} = 0$). For these instances, we even observe Gap(z_D) $<$ Gap(z_{conv}). Since the 2×2 block extraction technique is a generalization of the diagonal extraction technique, we would expect the reverse. However, the final root gap depends on the extracted matrix. We conjecture that Gap(z_D) $<$ Gap(z_{conv}) because we placed a significant weight ($s = 2$) on extracting off-diagonal elements in our extraction SDP (45). In Table 7, clearly the relaxations from block-diagonal extraction (z_B and z_{conv}) are the strongest. This meshes with our intuition, since we specifically studied and strengthened the sets arising from this reformulation. We are encouraged that the values for Gap(z_B) are close to Gap(z_{conv}). This indicates that the new conic inequalities (Ineq 1)-(Ineq 4) do a good job approximating $\text{conv}(S)$, at least in the direction of the objective function. Overall, we conclude that the block-extraction technique and proposed conic inequalities may be effective for instances where Q is close to block-diagonal.

5 Conclusions

In this work, we describe one possible way to strengthen the relaxation of non-separable quadratic MIPs with on-off constraints, like those arising in portfolio optimization, thereby extending the perspective reformulation. We describe a block-diagonal matrix extraction technique that we use to motivate an in-depth study of a low-dimensional mixed integer nonlinear set. We characterize the extreme points of the continuous relaxation of our set as well as the extreme points of its convex hull. We introduce four conic inequalities and provide evidence of the importance of these nonlinear inequalities in strengthening the continuous relaxation of our set. Computational results demonstrate that the proposed inequalities may be effective when the matrix Q is close to block-diagonal.

In future work, we plan to explore the computational impact of different block-extraction techniques and use the nonlinear cutting planes in a branch-and-bound algorithm to solve large practical instances. We also continue to work to characterize $\text{conv}(S)$ in the original space of variables.

References

[1] D. Bienstock. Computational study of a family of mixed-integer quadratic programming problems. *Mathematical Programming*, 74:121–140, 1996.

[2] J. Castro, A. Frangioni, and C. Gentile. Perspective reformulations of the CTA problem with ℓ_2 distances. *Operations Research*, 62(4):891909, 2014.

[3] S. Ceria and J. Soares. Convex programming for disjunctive optimization. *Mathematical Programming*, 86:595–614, 1999.

[4] H. Dong and J. Linderoth. On valid inequalities for quadratic programming with continuous variables and binary indicators. In *IPCO 2013: The Sixteenth Conference on Integer Programming and Combinatorial Optimization*, volume 7801, pages 169–180. Springer, 2013.

[5] D. P. Foster and E. I. George. The risk inflation criterion for multiple regression. *Annals of Statistics*, 22:1947–1975, 1994.

[6] A. Frangioni and C. Gentile. Perspective cuts for a class of convex 0-1 mixed integer programs. *Mathematical Programming*, 106:225–236, 2006.

[7] A. Frangioni and C. Gentile. SDP diagonalizations and perspective cuts for a class of nonseparable MIQP. *Operations Research Letters*, 35:181–185, 2007.

[8] A. Frangioni, C. Gentile, and F. Lacalandra. Tighter approximated MILP formulations for unit commitment problems. *IEEE Transactions on Power Systems*, 24(1):105113, 2009.

[9] J. Gao and D. Li. Cardinality constraint linear-quadratic optimal control. *IEEE Transactions on Automatic Control*, 56(8):1936–1941, 2011.

[10] R. E. Gomory, E. L. Johnson, and L. Evans. Corner polyhedra and their connection with cutting planes. *Mathematical Programming Series B*, 96:321–329, 2003.

[11] O. Günlük and J. Linderoth. Perspective relaxation of mixed integer nonlinear programs with indicator variables. *Mathematical Programming Series B*, 104:186–203, 2010.

[12] O. Günlük and J. T. Linderoth. Perspective reformulation and applications. In J. Lee and S. Leyffer, editors, *The IMA Volumes in Mathematics and its Applications*, volume 154, pages 61–92, 2012.

[13] Brady Hunsaker. *Measuring Facets of Polyhedra to Predict Usefulness in Branch-and-cut Algorithms*. PhD thesis, Georgia Institute of Technology, 2003.

[14] Fatma Kılınç-Karzan. On minimal inequalities for mixed integer conic programs. Technical report, Carnegie Mellon University, June 2013. GSIA Working Paper Number: 2013-E20, GSIA, Carnegie Mellon University, Pittsburgh, PA: `http://arxiv.org/pdf/1408.6922v1.pdf`.

[15] H. W. Kuhn. On the origin of the Hungarian method. In J. K. Lenstra, A. H. G. Rinnooy Kan, and A. Schrijver, editors, *History of Mathematical Programming: A Collection of Personal Reminiscences*, pages 77–81, Amsterdam, 1991. Elsevier. Postscript of the article.

[16] R. R. Meyer. Integer and mixed-integer programming models: General properties. *Journal of Optimization Theory and Applications*, 16:191–206, 1976.

[17] R. A. Stubbs. *Branch-and-Cut Methods for Mixed 0-1 Convex Programming*. PhD thesis, Northwestern University, December 1996.

[18] D. Wei, C. K. Sestok, and A. V. Oppenheim. Sparse filter design under a quadratic constraint: Low-complexity algorithms. *IEEE Transactions on Signal Processing*, 61:857–870, 2013.

[19] X. Zheng, X. Sun, and D. Li. Improving the performance of MIQP solvers for quadratic programs with cardinality and minimum threshold constraints: A semidefinite program approach. *INFORMS Journal on Computing*, 26(4):690–703, 2014.