A Polyhedral Study of Two-Period Relaxations for Big-Bucket Lot-Sizing Problems: Zero Setup Case

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In this paper, we investigate the two-period subproblems proposed by Akartunalı et al. (2014) for big-bucket lot-sizing problems, which have shown a great potential for obtaining strong bounds for these problems. In particular, we study the polyhedral structure of the mixed integer sets related to two relaxations of these subproblems for the special case of zero setup times, derive several families of valid inequalities and present their facet-defining conditions. Then we discuss the separation problems associated with these valid inequalities and propose exact separation algorithms. Finally, we investigate the computational strength of these cuts when they are integrated into a cutting plane framework. Our computational experiments indicate they can be indeed very effective improving lower bounds substantially.

Key words: Lot-Sizing; Integer Programming; Polyhedral Analysis; Valid Inequalities.

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1. Introduction

Lot-sizing has been a very active research area since the the seminal paper of [33]. In summary, the lot-sizing problem aims to determine a production plan detailing how much to produce and stock in each time period of the planning horizon, given manufacturing system limitations such as capacities, and customer orders and/or forecasted demand. It is a crucial area for manufacturing companies because it has a strong impact on their performance in terms of customer service quality and operating costs, and hence it has drawn attention both from researchers and practitioners. In this paper, we focus on the multi-item lot-sizing problems with *big bucket* capacities, i.e., each resource is shared by multiple items and more than one type of item can be produced in any time period. In particular, we will present a two-period relaxation of this problem, and study the polyhedral properties of two relaxations of this subproblem. Our main contributions are (i) several families of new valid inequalities and their facet-defining properties for the relaxations of the two-period relaxation, (ii) their extensions to the space of the two-period relaxation, and (iii) separation algorithms designed for all different cases. Our computational experiments show that the proposed inequalities have great potential to strengthen the lower bounds significantly.

1.1 A Formulation for Big-Bucket Lot-Sizing Problem

Adapting the notation of [2], we present next the mathematical formulation of the multi-item lot-sizing problem with big bucket capacities. We let NT, NI and NK indicate the number of *periods*, *items*, and *machine types*, respectively. We represent the production, setup, and inventory variables for item i in period t by x_t^i, y_t^i, and s_t^i, respectively.

$$\min \sum_{t=1}^{NT}\sum_{i=1}^{NI} f_t^i y_t^i + \sum_{t=1}^{NT}\sum_{i=1}^{NI} h_t^i s_t^i \tag{1}$$

$$\text{s.t. } x_t^i + s_{t-1}^i - s_t^i = d_t^i \qquad t \in \{1,\dots,NT\}, i \in \{1,\dots,NI\} \tag{2}$$

$$\sum_{i=1}^{NI}(a_k^i x_t^i + ST_k^i y_t^i) \le C_t^k \qquad t \in \{1,\dots,NT\}, k \in \{1,\dots,NK\} \tag{3}$$

$$x_t^i \le M_t^i y_t^i \qquad t \in \{1,\dots,NT\}, i \in \{1,\dots,NI\} \tag{4}$$

$$y \in \{0,1\}^{NT\text{x}NI}; x, s \ge 0 \tag{5}$$

The objective function (1) minimizes total cost, where f_t^i and h_t^i indicate the setup and inventory cost coefficients, respectively. The flow balance constraints (2) ensure that the demand for each item i in period t, denoted by d_t^i, is satisfied. We note that the model can be generalized to involve multiple levels as in [2], however, we omit this for the sake of simplicity. The big bucket capacity constraints (3) ensure that the capacity C_t^k of machine k is not exceeded in time period t, where a_k^i and ST_k^i represent the per unit production time and setup time for item i, respectively. The constraints (4) guarantee that the setup variable is equal to 1 if production occurs, where M_t^i represents the maximum number of item i that can be produced in period t, based on the minimum of remaining cumulative demand and capacity available. Finally, the integrality and non-negativity constraints are given by (5).

1.2 Literature Review

Most lot-sizing problems are inherently difficult problems: from the theoretical complexity perspective, even a single-item problem with varying capacities is known to be $\mathcal{NP}$-hard [16], and the multi-item case with a single joint capacity (without setup times) is strongly $\mathcal{NP}$-hard [8].

Moreover, from a computational perspective, problems with multiple items and capacities, in particular of industrial scale, remain notoriously difficult to solve to optimality, often resulting in high duality gaps, as noted in the recent review of [7]. Therefore, there is a wide spectrum of research on lot-sizing problems, ranging from practically efficient heuristics (see, e.g., [29, 30, 15]) and metaheuristics (see, e.g., [18]) to mathematical programming techniques, which we discuss in more detail next due to their relevance to our study.

Because of their apparent complexity, most researchers in the mathematical programming community studied special cases of lot-sizing problems, which can still provide valuable insights on some inherent structures of more general and complicated problems and hence support the solution methodologies proposed. The exact approaches most often employed either defining valid inequalities (e.g., [3, 10, 25, 20]) or extended reformulations (e.g., [19, 14, 28, 34, 27]) for variants of single-item problem, some of which were also extended to multi-item problems, see, e.g., [24, 4]. On the other hand, there are also few studies using other techniques such as Lagrangian relaxation (e.g., [5]) and Dantzig-Wolfe decomposition (e.g., [6, 12]). The book of [26] provides a thorough review of different variants of lot-sizing problems, their complexities and various solution methods used. Most recently, there have been insightful polyhedral results on multi-level problems, such as the valid inequalities of [35], and the compact formulations of [32] for *small bucket* capacities, i.e., items do not share resources.

Despite this extensive literature, the research explicitly investigating complications arising from multiple items competing for the limited capacities inherent in big bucket problems is rather limited, and only few exceptions exist to the best of our knowledge. The polyhedral analysis of a single-period relaxation by [21, 22] provided us some insightful properties of this polyhedron including new valid inequalities. The study of [17] presented various decompositions of these problems and indicated that period decompositions provide stronger bounds, which is recently investigated further by the sophisticated branch-and-cut framework of [11] resulting in promising computational results with regards to gaps. The work of [31] obtained strong lower bounds, most often stronger than any previous results, by applying approximate extended reformulations only for a small number of periods. The extensive computational study of [2] noted the bottleneck in multi-item problems as the lack of a better understanding of the convex hull of single-machine, multi-period problems. This motivated the sophisticated framework of [1], where the smallest such problem, a two-period relaxation, is used to separate all violated inequalities by generating the extreme points of its convex hull, without pre-defining families of inequalities. The computational results of this study have shown great promise to significantly close duality gaps for big bucket problems in general, which motivated us to study such a two-period relaxation in thorough detail from a polyhedral perspective. In this paper, we present our work investigating the special case of zero setup times, and in a companion study [13], we study the general case with setup times and also propose a sophisticated and computationally efficient branch-and-cut framework.

In the next section, we will present the two-period relaxation X^{2PL}, originally proposed by [1], and study some of its polyhedral properties, including for our special case without setup costs. Then, in Section 3, we will present two relaxations of X^{2PL}, propose a number of valid inequalities for these relaxations and discuss their facet-defining properties. In Section 4, we will extend these inequalities to the original space of X^{2PL} and prove their validity. Next, we will present separation algorithms for all families of valid inequalities in Section 5. The strength of the inequalities proposed are tested computationally in Section 6, which show promising results for their effectiveness. We will conclude the paper with a discussion of possible extensions and generalizations.

2. A Two-Period Relaxation: X^{2PL}

In this section, we present the feasible region and some basic polyhedral properties of a two-period, single-machine relaxation of the multi-level, multi-item production planning problem with big bucket capacities, which we denote by X^{2PL}. As proposed by [1] for designing a computational framework generating local cutting planes, the motivation for this relaxation comes from the observation made by [2] that the single-machine, big-bucket capacities impose a bottleneck in realistic multi-item lot-sizing problems and that a two-period structure captures the most basic such model involving the interactions between periods as well. We let $I = \{1, \dots, NI\}$ indicate the set of items for the ease of notation.

$$x^i_{t'} \le \widetilde{M}^i_{t'} y^i_{t'} \qquad i \in I, t' = 1,2 \tag{6}$$

$$x^i_{t'} \le \widetilde{d}^i_{t'} y^i_{t'} + s^i \qquad i \in I, t' = 1,2 \tag{7}$$

$$x^i_1 + x^i_2 \le \widetilde{d}^i_1 y^i_1 + \widetilde{d}^i_2 y^i_2 + s^i \qquad i \in I \tag{8}$$

$$x^i_1 + x^i_2 \le \widetilde{d}^i_1 + s^i \qquad i \in I \tag{9}$$

$$\sum_{i \in I} (a^i x^i_{t'} + ST^i y^i_{t'}) \le \widetilde{C}_{t'} \qquad t' = 1,2 \tag{10}$$

$$x, s \ge 0, y \in \{0,1\}^{2 \times NI} \tag{11}$$

As we consider a single machine in this relaxation, we dropped the k index from this formulation for the sake of simplicity; otherwise all parameters are defined in the same fashion as in the original formulation of the problem. We note that, for a given time period t, the obvious choice for the "horizon" of this two-period subproblem would be $t+1$, i.e., $t' = 1, 2$ relate to the consecutive periods of $t, t+1$, respectively, of the original problem. In this case, parameters can be associated with the original parameters using the relations $\widetilde{M}_{t'}^{i} = M_{t+t'-1}^{i}$ and $\widetilde{C}_{t'} = C_{t+t'-1}$, for all i and $t' = 1, 2$. On the other hand, the cumulative demand parameter $\widetilde{d}_{t'}^{i}$ can be defined as $\widetilde{d}_{t'}^{i} = d_{t+t'-1,t+1}^{i}$, for all i and $t' = 1, 2$. In case of non-consecutive periods, these basic definitions remain in the same fashion, with the cumulative demand parameter $\widetilde{d}_{1}^{i}$ involving all demand from the start to the end of the horizon of the two-period subproblem.

Next, we remark some polyhedral results for X^{2PL}.

Proposition 2.1 ([1]) *W.l.o.g., we make the following assumptions:*

1. $0 < \widetilde{M}_{t'}^{i}, \qquad \forall i \in I, t' = 1, 2$
2. $ST^{i} < \widetilde{C}_{t'}, \qquad \forall i \in I, t' = 1, 2$

Then, $conv(X^{2PL})$ is full-dimensional.

Proposition 2.2 *The trivial facet-defining inequalities for $conv(X^{2PL})$ and their facet-defining conditions (if any) are:*

1. $x_{t'}^{i} \geq 0, i \in I, t' = 1, 2.$
2. $y_{t'}^{i} \leq 1, i \in I, t' = 1, 2.$
3. $s^{i} \geq 0, i \in I.$
4. $x_{t'}^{i} \leq \widetilde{M}_{t'}^{i} y_{t'}^{i}, i \in I, t' = 1, 2.$
5. $x_{t'}^{i} \leq \widetilde{d}_{t'}^{i} y_{t'}^{i} + s^{i}, i \in I, t' = 1, 2$ *(if $\widetilde{d}_{t'}^{i} < \widetilde{M}_{t'}^{i}$).*
6. $x_{1}^{i} + x_{2}^{i} \leq \widetilde{d}_{1}^{i} y_{1}^{i} + \widetilde{d}_{2}^{i} y_{2}^{i} + s^{i}, i \in I$ *(if $\widetilde{d}_{t'}^{i} < \widetilde{M}_{t'}^{i}, \forall t' \in \{1, 2\}$).*
7. $\sum_{i=1}^{NI} (a^{i} x_{t'}^{i} + ST^{i} y_{t'}^{i}) \leq \widetilde{C}_{t'}, t' = 1, 2$ *(if for $t' \in \{1, 2\}$, $\sum_{i=1}^{NI} (a^{i} \widetilde{M}_{t'}^{i} + ST^{i}) \geq \widetilde{C}_{t'} + (a^{k} \widetilde{M}_{t'}^{k} + ST^{k}), \forall k \in I$).*

We omit the proofs for the sake of simplicity of the presentation. In this paper, we will investigate the special case of $ST^{i} = 0, a^{i} = 1 \; \forall i \in I$, whereas we investigate in a companion paper [13] the polyhedral properties of the general case of non-zero setups as well as the design of a sophisticated and computationally efficient branch-and-cut framework. Next, we present some of the non-trivial facets of $conv(X^{2PL})$.

Proposition 2.3 *For $i \in I$,*

1. *The following inequality is valid for X^{2PL}:*

$$x_1^i + x_2^i \leq d_1^i y_1^i + d_2^i + s^i$$

Under the condition $a^i \widetilde{d}_{t'}^i + ST^i \leq \widetilde{C}_{t'}, \forall t' \in \{1, 2\}$, it defines a facet of $conv(X^{2PL})$.

2. *If $ST^i \leq \widetilde{C}_2$, then the following inequality is valid for X^{2PL}:*

$$x_1^i + x_2^i \leq \Big(\widetilde{d}_1^i - \Big(\frac{\widetilde{C}_2 - ST^i}{a^i}\Big)\Big) y_1^i + \Big(\frac{\widetilde{C}_2 - ST^i}{a^i}\Big) + s^i$$

If $a^i \widetilde{d}_1^i + ST^i \leq \widetilde{C}_1$ and $a^i \widetilde{d}_2^i + ST^i > \widetilde{C}_2$, then it is facet-defining for $conv(X^{2PL})$.

3. *If $a^i d_1^i + ST^i \leq \widetilde{C}_1$, then the following inequality is valid for X^{2PL}:*

$$x_1^i + x_2^i \leq d_1^i y_1^i + \Big(\widetilde{d}_1^i - \Big(\frac{\widetilde{C}_1 - ST^i}{a^i}\Big)\Big) y_2^i + \Big(\Big(\frac{\widetilde{C}_1 - ST^i}{a^i}\Big) - d_1^i\Big) + s^i$$

Under the condition $a^i \widetilde{d}_1^i + ST^i > \widetilde{C}_1$ and $a^i \widetilde{d}_2^i + ST^i \leq \widetilde{C}_2$, it defines a facet of $conv(X^{2PL})$.

4. *If $a^i \widetilde{d}_1^i + 2ST^i \leq \widetilde{C}_1 + \widetilde{C}_2$, then the following inequality is valid for X^{2PL}:*

$$\begin{aligned} x_1^i + x_2^i \leq & \Big(\widetilde{d}_1^i - \Big(\frac{\widetilde{C}_2 - ST^i}{a^i}\Big)\Big) y_1^i + \Big(\widetilde{d}_1^i - \Big(\frac{\widetilde{C}_1 - ST^i}{a^i}\Big)\Big) y_2^i \\ & + \Big(\Big(\frac{\widetilde{C}_1 - ST^i}{a^i}\Big) + \Big(\frac{\widetilde{C}_2 - ST^i}{a^i}\Big) - \widetilde{d}_1^i\Big) + s^i \end{aligned}$$

If $a^i \widetilde{d}_1^i + ST^i > \widetilde{C}_1$ and $a^i \widetilde{d}_2^i + ST^i > \widetilde{C}_2$, then this inequality is facet-defining for $conv(X^{2PL})$.

The proof is straightforward and omitted here for the sake of simplicity of the presentation.

In the next section, we establish two relaxations of X^{2PL} and study their polyhedral structures. We first present the known facet-defining inequalities and then we derive several new classes of valid inequalities such as cover and reverse cover inequalities for those mixed integer sets and establish their facet-defining conditions.

3. Polyhedral Analysis of the Relaxations of X^{2PL}

First, we make necessary definitions for the remainder of the paper.

Definition 3.1 *For a given t:*

- *A* cover *of I for period t is a set S_t such that $\lambda_t = \sum_{i \in S_t} \widetilde{d}_t^i - \widetilde{C}_t > 0$.*

- *A* reverse cover *of I for period t is a set $S_t \neq \emptyset$ such that $\mu_t = \widetilde{C}_t - \sum_{i \in S_t} \widetilde{d}_t^i > 0$.*

- *For a given reverse cover S_t and a non-empty set $T_t' \subseteq I \backslash S_t$ such that $\sum_{i \in T_t'} \widetilde{M}_t^i \geq \mu_t$, we define the* reverse slack *as $\xi_t = \sum_{i \in T_t'} \widetilde{M}_t^i - \mu_t$.*

- *We define the set S_t^+ of* strictly positive cover elements *as follows:*

$$S_t^+ = \begin{cases} \{i \in S_t | \widetilde{d}_t^i > \lambda_t\} & \text{if } S_t \text{ is a cover.} \\ \{i \in S_t | \widetilde{d}_t^i > \xi_t\} & \text{if } S_t \text{ is a reverse cover.} \end{cases}$$

We also define the positive maximum function as $(b)^+ = \max\{b, 0\}$.

First, for a given t, we define the following relaxation, denoted by PIR_0, for our two-period problem, X^{2PL}, since it is studied in the literature by various researchers.

$$x^i \leq M^i y^i, \quad \forall i \in I$$
$$\sum_{i \in I} x^i \leq C$$
$$x \geq 0, y \in \{0,1\}^{NI}$$

We dropped here all the t indices as well as $\sim$ for the sake of simplicity. We note that the Definition 3.1 remains valid in the same fashion that we use the same definitions for this relaxation with all the t indices as well as $\sim$ dropped.

Next, we present known facet-defining inequalities for PIR_0.

Proposition 3.1 (Flow cover inequalities [23]) *Let S be a cover, and $\sum_S M^i = C + \lambda$. Assume that $\overline{M} = \max_{i \in S} M^i > \lambda$. Then,*

$$\sum_{i \in S} x^i - \sum_{i \in S} (M^i - \lambda)^+ y^i \leq C - \sum_{i \in S} (M^i - \lambda)^+ \tag{12}$$

is valid and defines a facet of $conv(PIR_0)$. Moreover, for $L \subseteq I \backslash S$ and $\overline{M}^i = \max(M^i, \overline{M})$, the inequality

$$\sum_{i \in S \cup L} x^i - \sum_{i \in S} (M^i - \lambda)^+ y^i - \sum_{i \in L} (\overline{M}^i - \lambda) y^i \leq C - \sum_{i \in S} (M^i - \lambda)^+ \tag{13}$$

is valid and defines a facet of $conv(PIR_0)$ if $0 < \overline{M} - \lambda < M^i \leq \overline{M}$ holds $\forall i \in L$.

In addition to this known relaxation and its facet-defining inequalities, we present a second relaxation of X^{2PL} for a given t. We call this as PIR_1 and study important properties of it in the remainder of this section:

$$x^i \leq M^i y^i, \quad \forall i \in I$$
$$x^i \leq d^i y^i + s^i, \quad \forall i \in I$$
$$\sum_{i \in I} x^i \leq C$$
$$x, s \geq 0, y \in \{0,1\}^{NI}$$

Similar to the previous relaxation, we dropped here all the t indices as well as $\tilde{}$ for the sake of simplicity. We note that the Definition 3.1 remains valid in the same fashion that we use the same definitions for this relaxation with all the t indices as well as $\tilde{}$ dropped. First, we note some obvious properties of this polyhedron, including the full dimensionality of the convex hull of PIR_1 and trivial facets of $conv(PIR_1)$. These propositions can be easily proven and therefore, we omit detailed proofs here for the sake of readability.

Proposition 3.2 *dim(conv(PIR_1)) = 3 NI.*

Proposition 3.3 *The following inequalities are the trivial facets of PIR_1:*

1. $x^i \geq 0$, $\forall i \in I$.
2. $y^i \leq 1$, $\forall i \in I$.
3. $s^i \geq 0$, $\forall i \in I$.
4. $x^i \leq M^i y^i$, $\forall i \in I$.
5. $x^i \leq d^i y^i + s^i$ *if* $d^i < M^i, \forall i \in I$
6. $\sum_{i \in I} x^i \leq C$ *if* $\sum_{i \in I} M^i \geq C + M^k, \forall k \in I$.

In the next subsections, we define some new inequalities and establish their facet-defining conditions for PIR_1.

3.1 Cover Inequalities for the Zero Setup Case

Proposition 3.4 *Let S be a cover of I. Then the following inequality (called* ***cover inequality****) is valid for PIR_1:*

$$\sum_{i \in S} x^i - \sum_{i \in S} (d^i - \lambda)^+ y^i \leq \sum_{i \in S} s^i + C - \sum_{i \in S} (d^i - \lambda)^+ \tag{14}$$

Proof. First, we rewrite the cover inequality as follows:

$$\sum_{i\in S} x^i - \sum_{i\in S^+}(d^i-\lambda)y^i \le \sum_{i\in S} s^i + C - \sum_{i\in S^+}(d^i-\lambda)$$

Let $(x, y, s) \in PIR_1$, and define $T = \{i \in I | y^i = 1\}$. We consider the following two cases:
Case I: $|S^+\backslash T| = 0$. It implies that $y^i = 1$, $\forall i \in S^+$, and so the validity is simply followed by $\sum_{i\in I} x^i \le C$ and $s^i \ge 0$, $\forall i \in I$.
Case II: $|S^+\backslash T| \ge 1$. Since $|S^+ \cap T| + 1 \le |S^+|$ holds,

$$\begin{aligned}
&\sum_{i\in S} x^i - \sum_{i\in S^+}(d^i-\lambda)y^i \\
&\le \sum_{i\in S\cap T} x^i - \sum_{i\in S^+\cap T}(d^i-\lambda) \\
&\le \sum_{i\in S\cap T} d^i + \sum_{i\in S\cap T} s^i - \sum_{i\in S^+\cap T} d^i + |S^+\cap T|\lambda \\
&\le \sum_{i\in S\cap T} d^i + \sum_{i\in S} s^i + \sum_{i\in S^+\backslash T} d^i - \sum_{i\in S^+} d^i + |S^+\cap T|\lambda \\
&\le \sum_{i\in S} d^i + \sum_{i\in S} s^i - \sum_{i\in S^+} d^i - \lambda + \Big(|S^+\cap T| + 1\Big)\lambda \\
&= \sum_{i\in S} s^i + C - \sum_{i\in S^+} d^i + \Big(|S^+\cap T| + 1\Big)\lambda \\
&\le \sum_{i\in S} s^i + C - \sum_{i\in S^+} d^i + |S^+|\lambda \\
&= \sum_{i\in S} s^i + C - \sum_{i\in S^+}(d^i-\lambda)
\end{aligned}$$

where the second inequality uses the defining inequality $x^i \le d^i y^i + s^i$ and the property $y^i = 1, i \in T$, the third and fourth inequalities use the simple set property of $S^+ \cap T = S^+ \setminus (S^+\backslash T)$, and the fifth inequality uses the basic definition of λ. □

Next, we establish the conditions necessary for the simple cover inequality to be facet-defining.

Proposition 3.5 *Assume that $d^i < M^i$ holds $\forall i \in S$, and $|S^+| \ge 2$. Then, (14) defines a facet of conv(PIR_1).*

Proof. Let j^1, j^2 be any two members of S^+, and $\epsilon > 0$ be a sufficiently small number. We present the following $3NI$ points that satisfy $\sum_{i\in S} x^i - \sum_{i\in S}(d^i-\lambda)^+ y^i = \sum_{i\in S} s^i + C - \sum_{i\in S}(d^i-\lambda)^+$:

1. For every $i' \in S^+$, set $x^{i'} = 0 = y^{i'}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i'\}$, and set all other variables to zero. **($|S^+|$ points)**

2. For every $i' \in S^+$, set $x^{i'} = d^{i'} - \lambda$ and $y^{i'} = 1$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i'\}$, and set all other variables to zero. **($|S^+|$ points)**

3. For every $i' \in S \backslash S^+$, set $x^{i'} = 0 = y^{i'}$, set $x^{j^1} = d^{j^1} - \lambda + d^{i'}$ and $y^{j^1} = 1$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i', j^1\}$, and set all other variables to zero. **($|S \backslash S^+|$ points)**

4. For every $i' \in S \backslash S^+$, set $x^{i'} = 0$ and $y^{i'} = 1$, set $x^{j^2} = d^{j^2} - \lambda + d^{i'}$, and $y^{j^2} = 1$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i', j^2\}$, and set all other variables to zero. **($|S \backslash S^+|$ points)**

5. For every $i' \in S \backslash \{j^1\}$, set $x^{i'} = d^{i'} + \epsilon$, $y^{i'} = 1$ and $s^{i'} = \epsilon$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i', j^1\}$, and set all other variables to zero. **($|S| - 1$ points)**

6. Set $x^{j^1} = d^{j^1} + \epsilon$, $y^{j^1} = 1$ and $s^{j^1} = \epsilon$, set $x^{j^2} = 0 = y^{j^2}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{j^1, j^2\}$, and set all other variables to zero. **(1 point)**

7. For every $i' \in I \backslash S$, set $s^{i'} = \epsilon$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{j^1\}$, and set all other variables to zero. **($NI - |S|$ points)**

8. For every $i' \in I \backslash S$, set $x^{i'} = 0$ and $y^{i'} = 1$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{j^1\}$, and set all other variables to zero. **($NI - |S|$ points)**

9. For every $i' \in I \backslash S$, set $x^{i'} = \epsilon$ and $y^{i'} = 1$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{j^1\}$, and set all other variables to zero. **($NI - |S|$ points)**

It is easy to observe that these $3NI$ points are affinely independent. This suffices to prove the claim. □

This leads us to the next family of valid inequalities, as follows.

Proposition 3.6 *Let S be a cover of I and $K \subseteq I \backslash S$ such that $M^i \leq \bar{d}^i$ holds $\forall i \in K$, where $\bar{d} = max_{i \in S} d^i \geq \lambda$ and $\bar{d}^i = \max(d^i, \bar{d}), i \in K$. Then the following inequality (called **item-extended cover inequality**) is valid for PIR_1:*

$$\sum_{i \in S \cup K} x^i - \sum_{i \in S} (d^i - \lambda)^+ y^i - \sum_{i \in K} (\bar{d}^i - \lambda) y^i \leq \sum_{i \in S} s^i + C - \sum_{i \in S} (d^i - \lambda)^+ \tag{15}$$

Proof. First, using the definition of S^+, we rewrite the item-extended cover inequality as follows:

$$\sum_{i\in S\cup K} x^i - \sum_{i\in S^+}(d^i-\lambda)y^i - \sum_{i\in K}(\bar{d}^i-\lambda)y^i + \sum_{i\in S^+}(d^i-\lambda) \le \sum_{i\in S} s^i + C$$

Let $T=\{i\in I|y^i=1\}$, where we consider an $(x,y,s)\in PIR_1$. There are two cases to evaluate as follows:

Case I: $|S^+\backslash T|\le |K\cap T|$. We can establish the following relationships:

$$\begin{aligned}
&\sum_{i\in S\cup K} x^i - \sum_{i\in S^+}(d^i-\lambda)y^i - \sum_{i\in K}(\bar{d}^i-\lambda)y^i + \sum_{i\in S^+}(d^i-\lambda)\\
&= \sum_{i\in S\cap T} x^i + \sum_{i\in K\cap T} x^i - \sum_{i\in S^+\cap T}(d^i-\lambda) + \sum_{i\in S^+}(d^i-\lambda) - \sum_{i\in K\cap T}(\bar{d}^i-\lambda)\\
&\le C + \sum_{i\in S^+\backslash T}(d^i-\lambda) - \sum_{i\in K\cap T}(\bar{d}^i-\lambda) \le C + \sum_{i\in S^+\backslash T}(\bar{d}-\lambda) - \sum_{i\in K\cap T}(\bar{d}-\lambda)\\
&\le C + \sum_{i\in S} s^i + \Big(|S^+\backslash T| - |K\cap T|\Big)(\bar{d}-\lambda) \le C + \sum_{i\in S} s^i
\end{aligned}$$

where the first inequality follows the fact that $\sum_{(S\cap T)\cup(K\cap T)} x^i \le C$, the second inequality follows the fact that $d^i\le \bar{d}\le \bar{d}^i$, and the last inequality follows $|S^+\backslash T|\le |K\cap T|$ and $\bar{d}\ge\lambda$.

Case II: $|S^+\backslash T|\ge |K\cap T|+1$. We can establish the following relationships:

$$\begin{aligned}
&\sum_{i\in S\cup K} x^i - \sum_{i\in S^+}(d^i-\lambda)y^i - \sum_{i\in K}(\bar{d}^i-\lambda)y^i + \sum_{i\in S^+}(d^i-\lambda)\\
&= \sum_{i\in S\cap T} x^i + \sum_{i\in K\cap T} x^i - \sum_{i\in S^+\cap T}(d^i-\lambda) + \sum_{i\in S^+}(d^i-\lambda) - \sum_{i\in K\cap T}(\bar{d}^i-\lambda)\\
&\le \sum_{i\in S\cap T} d^i + \sum_{i\in S\cap T} s^i + \sum_{i\in K\cap T} x^i + \sum_{i\in S^+\backslash T}(d^i-\lambda) - \sum_{i\in K\cap T}(\bar{d}^i-\lambda)\\
&\le \sum_{i\in S\cap T} d^i + \sum_{i\in S} s^i + \sum_{i\in K\cap T} M^i + \sum_{i\in S^+\backslash T} d^i - \sum_{i\in K\cap T}\bar{d}^i - |S^+\backslash T|\lambda + |K\cap T|\lambda\\
&\le \sum_{i\in S} d^i - \lambda + \sum_{i\in S} s^i + \sum_{i\in K\cap T}(M^i-\bar{d}^i) + \Big(|K\cap T| - |S^+\backslash T| + 1\Big)\lambda \le C + \sum_{i\in S} s^i
\end{aligned}$$

where the first inequality follows $x^i\le d^iy^i+s^i$ and the property $y^i=1, i\in T$, the second inequality is due $x^i\le M^iy^i$, and the last inequality follows the definition of λ as well as non-positivity of $M^i-\bar{d}^i$ and $|K\cap T|-|S^+\backslash T|+1$, and $\lambda>0$. □

Proposition 3.7 *Let $0<\bar{d}-\lambda<d^i\le\bar{d}$ holds $\forall i\in K$, and the conditions of Proposition 3.5 hold. Then, (15) defines a facet of conv(PIR_1).*

Proof. First, note that the condition requires that $\bar{d}^i = \bar{d}$, $\forall i \in K$. Let $\bar{d} = d^{j^1}$, i.e., j^1 has the highest demand in set S. Next, we note that a majority of the affinely independent points provided in the proof of Proposition 3.5 can be used here: the first 7 sets of the points are valid, and the last two sets are valid for $i' \notin S \cup K$. Therefore we need $2|K|$ new points, which we present as follows, where $\epsilon > 0$ is a sufficiently small number:

1. For every $i' \in K$, set $x^{i'} = \bar{d} - \lambda$ and $y^{i'} = 1$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{j^1\}$, and set all other variables to zero. **($|K|$ points)**

2. For every $i' \in K$, set $x^{i'} = \bar{d} - \lambda + \epsilon$ for a sufficiently small $\epsilon > 0$ and $y^{i'} = 1$, set $x^{j^1} = 0 = y^{j^1}$, set $x^{j^2} = d^{j^2} - \epsilon$ and $y^{j^2} = 1$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{j^1, j^2\}$, and set all other variables to zero. **($|K|$ points)**

These $3NI$ affinely independent points suffice to prove the claim. $\square$

Example. Let $I = \{1, 2, 3\}$, and PIR_1 defined by:

$$x^1 \leq 14y^1, \quad x^2 \leq 10y^2, \quad x^3 \leq 11y^3$$
$$x^1 \leq 10y^1 + s^1, \quad x^2 \leq 6y^2 + s^2, \quad x^3 \leq 8y^3 + s^3$$
$$x^1 + x^2 + x^3 \leq 14$$

$S = \{1, 2\}$ is a cover and hence $\lambda = 16 - 14 = 2$. Then, we can generate a facet-defining cover inequality as follows:

$$x^1 + x^2 - (10 - 2)y^1 - (6 - 2)y^2 \leq s^1 + s^2 + 14 - 8 - 4$$
$$\implies x^1 + x^2 - 8y^1 - 4y^2 \leq s^1 + s^2 + 2$$

Another facet-defining cover inequality can be defined for cover $S = \{1, 3\}$ and $\lambda = 4$:

$$x^1 + x^3 - 6y^1 - 4y^3 \leq s^1 + s^3 + 4$$

For cover $S = \{1, 3\}$, note $\bar{d} = \max_{i \in S} d^i = 10 > \lambda = 4$. Letting $k = \{2\}$ (and hence $\bar{d}^2 = \max(10, 6) = 10$), we note the facet-defining item-extended cover inequality:

$$x^1 + \mathbf{x^2} + x^3 - 6y^1 - (\mathbf{10} - \mathbf{4})\mathbf{y^2} - 4y^3 \leq s^1 + s^3 + 4$$

where bold elements indicate all terms that are additional compared to the previous cover inequality. $\square$

3.2 Reverse Cover Inequalities for the Zero Setup Case

Proposition 3.8 *Let S be a reverse cover of I, $\overline{T} = I\backslash S$, and (T', T'') be a partition of $\overline{T}$ such that $T' \neq \emptyset$. Then the following inequality (called **reverse cover inequality**) is valid for PIR_1:*

$$\sum_{i \in S \cup T'} x^i - \sum_{i \in S}(d^i - \xi)^+ y^i - \sum_{i \in T'}(M^i - \xi)^+ y^i \leq \sum_{i \in S} s^i + C - \sum_{i \in S}(d^i - \xi)^+ - \sum_{i \in T'}(M^i - \xi)^+ \quad (16)$$

Proof. First, using the definition $T'^+ = \{i \in T' | M^i > \xi\}$, we rewrite the reverse cover inequality as follows:

$$\sum_{i \in S \cup T'} x^i - \sum_{i \in S^+}(d^i - \xi)y^i - \sum_{i \in T'^+}(M^i - \xi)y^i \leq \sum_{i \in S} s^i + C - \sum_{i \in S^+}(d^i - \xi) - \sum_{i \in T'^+}(M^i - \xi)$$

Let $(x, y, s) \in PIR_1$ and $T = \{i \in I | y^i = 1\}$. So we consider two cases as follows:
Case I: $|S^+\backslash T| + |T'^+\backslash T| = 0$. Then it implies $|S^+\backslash T| = |T'^+\backslash T| = 0$. Then we have

$$\begin{aligned}
&\sum_{i \in S \cup T'} x^i - \sum_{i \in S^+}(d^i - \xi)y^i - \sum_{i \in T'^+}(M^i - \xi)y^i + \sum_{i \in S^+}(d^i - \xi) + \sum_{i \in T'^+}(M^i - \xi) \\
&= \sum_{i \in S \cup T'} x^i - \sum_{i \in S^+ \cap T}(d^i - \xi) - \sum_{i \in T'^+ \cap T}(M^i - \xi) + \sum_{i \in S^+}(d^i - \xi) + \sum_{i \in T'^+}(M^i - \xi) \\
&\leq C + \sum_{i \in S^+\backslash T}(d^i - \xi) + \sum_{i \in T'^+\backslash T}(M^i - \xi) + \sum_{i \in S} s^i = C + \sum_{i \in S} s^i
\end{aligned}$$

where the inequality follows the basic set properties and capacity constraint.
Case II: $|S^+\backslash T| + |T'^+\backslash T| \geq 1$. Then we have

$$\begin{aligned}
&\sum_{i \in S \cup T'} x^i - \sum_{i \in S^+}(d^i - \xi)y^i - \sum_{i \in T'^+}(M^i - \xi)y^i + \sum_{i \in S^+}(d^i - \xi) + \sum_{i \in T'^+}(M^i - \xi) \\
&= \sum_{i \in S \cap T} x^i + \sum_{i \in T' \cap T} x^i - \sum_{i \in S^+ \cap T}(d^i - \xi) - \sum_{i \in T'^+ \cap T}(M^i - \xi) + \sum_{i \in S^+}(d^i - \xi) \\
&+ \sum_{i \in T'^+}(M^i - \xi) = \sum_{i \in S \cap T} x^i + \sum_{i \in T' \cap T} x^i + \sum_{i \in S^+\backslash T}(d^i - \xi) + \sum_{i \in T'^+\backslash T}(M^i - \xi) \\
&\leq \sum_{i \in S \cap T} d^i + \sum_{i \in S \cap T} s^i + \sum_{i \in T' \cap T} M^i + \sum_{i \in S^+\backslash T} d^i - \sum_{i \in S^+\backslash T} \xi + \sum_{i \in T'^+\backslash T} M^i \\
&- \sum_{i \in T'^+\backslash T} \xi \leq \sum_{i \in S} d^i + \sum_{i \in S} s^i + \sum_{i \in T'} M^i - |S^+\backslash T|\xi - |T'^+\backslash T|\xi + \xi - \xi
\end{aligned}$$

$$= \sum_{i \in S} s^i + C - \Big(|S^+ \backslash T| + |T'^+ \backslash T| - 1\Big)\xi \le C + \sum_{i \in S} s^i$$

where the first inequality follows $x^i \le d^i y^i + s^i$, $x^i \le M^i y^i$ and the property $y^i = 1, i \in T$, the second inequality follows basic set properties, the last equation follows the definition of ξ and finally the last inequality follows $|S^+ \backslash T| + |T'^+ \backslash T| \ge 1$ and $\xi \ge 0$. □

Proposition 3.9 *Let $\xi > 0$, and assume that for $T'^+ = \{i \in T' | M^i > \xi\}$, $|T'^+| \ge 1$ holds. Moreover, assume that $d^i < M^i$ holds $\forall i \in S$. Then, (16) defines a facet of conv(PIR_1).*

Proof. Let j^1 be any member of T'^+, and $\epsilon > 0$ be a sufficiently small number. We present the following $3NI$ points that satisfy (16) as an equation:

1. For every $i' \in S^+$, set $x^{i'} = 0 = y^{i'}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i'\}$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T'$, and set all other variables to zero. **($|S^+|$ points)**

2. For every $i' \in S^+$, set $x^{i'} = d^{i'} - \xi$ and $y^{i'} = 1$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i'\}$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T'$, and set all other variables to zero. **($|S^+|$ points)**

3. For every $i' \in T'^+$, set $x^{i'} = 0 = y^{i'}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{i'\}$, and set all other variables to zero. **($|T'^+|$ points)**

4. For every $i' \in T'^+$, set $x^{i'} = M^{i'} - \xi$, $y^{i'} = 1$ and $s^{i'} = (M^{i'} - \xi - d^{i'})^+$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{i'\}$, and set all other variables to zero. **($|T'^+|$ points)**

5. For every $i' \in S \backslash S^+$, set $x^{i'} = 0 = y^{i'}$, set $x^{j^1} = M^{j^1} - \xi + d^{i'}$, $y^{j^1} = 1$ and $s^{j^1} = (M^{j^1} - \xi + d^{i'} - d^{j^1})^+$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i'\}$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{j^1\}$, and set all other variables to zero. **($|S \backslash S^+|$ points)**

6. For every $i' \in S \backslash S^+$, set $x^{i'} = 0$ and $y^{i'} = 1$, set $x^{j^1} = M^{j^1} - \xi + d^{i'}$, $y^{j^1} = 1$ and $s^{j^1} = (M^{j^1} - \xi + d^{i'} - d^{j^1})^+$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i'\}$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{j^1\}$, and set all other variables to zero. **($|S \backslash S^+|$ points)**

7. For every $i' \in T' \backslash T'^+$, set $x^{i'} = 0 = y^{i'}$, set $x^{j^1} = M^{j^1} - \xi + M^{i'}$, $y^{j^1} = 1$ and $s^{j^1} = (M^{j^1} - \xi + M^{i'} - d^{j^1})^+$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{i', j^1\}$, and set all other variables to zero. **($|T' \backslash T'^+|$ points)**

8. For every $i' \in T' \backslash T'^{+}$, set $x^{i'} = 0$ and $y^{i'} = 1$, set $x^{j^1} = M^{j^1} - \xi + M^{i'}$, $y^{j^1} = 1$ and $s^{j^1} = (M^{j^1} - \xi + M^{i'} - d^{j^1})^+$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{i', j^1\}$, and set all other variables to zero. **($|T' \backslash T'^{+}|$ points)**

9. For every $i' \in S$, set $x^{i'} = d^{i'} + \epsilon$, $y^{i'} = 1$ and $s^{i'} = \epsilon$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i'\}$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{j^1\}$, and set all other variables to zero. **($|S|$ points)**

10. Set $x^{j^1} = 0 = y^{j^1}$ and $s^{j^1} = \epsilon$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{j^1\}$, and set all other variables to zero. **(1 point)**

11. For every $i' \in T''$, set $s^{i'} = \epsilon$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{j^1\}$, and set all other variables to zero. **($|T''|$ points)**

12. For every $i' \in T' \backslash \{j^1\}$, set $x^{i'} = M^{i'}$, $y^{i'} = 1$ and $s^{i'} = (M^{i'} - d^{i'})^+ + \epsilon$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{i', j^1\}$, and set all other variables to zero. **($|T'| - 1$ points)**

13. For every $i' \in T''$, set $x^{i'} = 0$ and $y^{i'} = 1$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{j^1\}$, and set all other variables to zero. **($|T''|$ points)**

14. For every $i' \in T''$, set $x^{i'} = \epsilon$ and $y^{i'} = 1$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{j^1\}$, and set all other variables to zero. **($|T''|$ points)**

These $3NI$ points are built in a similar fashion as those presented in the proof of Proposition 3.5 and their affine independence is straightforward. □

Proposition 3.10 *Let S be a reverse cover of I, $\overline{T} = I \backslash S$, (T', T'') be a partition of $\overline{T}$, and $K \subseteq T''$. We define*

$$p_i = \begin{cases} d^i & : i \in S \\ M^i & : i \in T' \end{cases}$$

and $\bar{p} = \max_{i \in S \cup T'} p^i \geq \xi$. We also define $\bar{p}^i = \max(M^i, \bar{p}), i \in K$. Then the following inequality *(called **item-extended reverse cover inequality**) is valid for PIR_1:*

$$\sum_{i\in S\cup T'\cup K} x^i - \sum_{i\in S}(d^i-\xi)^+y^i - \sum_{i\in T'}(M^i-\xi)^+y^i - \sum_{i\in K}(\bar{p}^i-\xi)y^i \leq \sum_{i\in S} s^i + C - \sum_{i\in S}(d^i-\xi)^+ - \sum_{i\in T'}(M^i-\xi)^+ \quad (17)$$

Proof. Let $(x,y,s)\in PIR_1$. Using the definitions $T=\{i\in I|y^i=1\}$ and $T'^+=\{i\in T'|M^i>\xi\}$ as before, we consider two cases as follows:

Case I: $|S^+\backslash T|+|T'^+\backslash T|\leq |K\cap T|$. First, we rewrite (17) using S^+ and T'^+, and then can derive the following:

$$\begin{aligned}
&\sum_{i\in S\cup T'\cup K} x^i - \sum_{i\in S^+}(d^i-\xi)y^i - \sum_{i\in T'^+}(M^i-\xi)y^i - \sum_{i\in K}(\bar{p}^i-\xi)y^i + \sum_{i\in S^+}(d^i-\xi)\\
&+\sum_{i\in T'^+}(M^i-\xi) = \sum_{i\in S\cup T'\cup K} x^i - \sum_{i\in S^+\cap T}(d^i-\xi) - \sum_{i\in T'^+\cap T}(M^i-\xi) - \sum_{i\in K\cap T}(\bar{p}^i-\xi)\\
&+\sum_{i\in S^+}(d^i-\xi) + \sum_{i\in T'^+}(M^i-\xi) \leq C + \sum_{i\in S^+\backslash T}(d^i-\xi) + \sum_{i\in T'^+\backslash T}(M^i-\xi)\\
&-\sum_{i\in K\cap T}(\bar{p}^i-\xi) \leq C + \sum_{i\in S} s^i + \sum_{i\in S^+\backslash T}(\bar{p}-\xi) + \sum_{i\in T'^+\backslash T}(\bar{p}-\xi) - \sum_{i\in K\cap T}(\bar{p}-\xi)\\
&= C + \sum_{i\in S} s^i + \Big(|S^+\backslash T|+|T'^+\backslash T|-|K\cap T|\Big)(\bar{p}-\xi) \leq C + \sum_{i\in S} s^i
\end{aligned}$$

where the first equation simply exploits T, the first inequality uses the aggregate capacity constraint, the second inequality uses the definitions of $\bar{p}$, $\bar{p}^i$ and p^i, and the last inequality follows $|S^+\backslash T|+|T'^+\backslash T|\leq |K\cap T|$ and $\bar{p}\geq\xi$.

Case II: $|S^+\backslash T|+|T'^+\backslash T|\geq |K\cap T|+1$. First, we rewrite (17) using S^+ and T'^+, and then can derive the following:

$$\begin{aligned}
&\sum_{i\in S\cup T'\cup K} x^i - \sum_{i\in S^+}(d^i-\xi)y^i - \sum_{i\in T'^+}(M^i-\xi)y^i - \sum_{i\in K}(\bar{p}^i-\xi)y^i + \sum_{i\in S^+}(d^i-\xi)\\
&+\sum_{i\in T'^+}(M^i-\xi) = \sum_{i\in S\cap T} x^i + \sum_{i\in T'\cap T} x^i + \sum_{i\in K\cap T} x^i - \sum_{i\in S^+\cap T}(d^i-\xi)\\
&\qquad -\sum_{i\in T'^+\cap T}(M^i-\xi) - \sum_{i\in K\cap T}(\bar{p}^i-\xi) + \sum_{i\in S^+}(d^i-\xi) + \sum_{i\in T'^+}(M^i-\xi) \leq \sum_{i\in S\cap T} d^i\\
&+\sum_{i\in S\cap T} s^i + \sum_{i\in T'\cap T} M^i + \sum_{i\in K\cap T} M^i + \sum_{i\in S^+\backslash T}(d^i-\xi) + \sum_{i\in T'^+\backslash T}(M^i-\xi)
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i\in K\cap T}(\bar{p}^i-\xi)=\sum_{i\in S\cap T}d^i+\sum_{i\in S\cap T}s^i+\sum_{i\in T'\cap T}M^i+\sum_{i\in K\cap T}M^i+\sum_{i\in S^+\backslash T}d^i\\
&\qquad -|S^+\backslash T|\xi+\sum_{i\in T'^+\backslash T}M^i-|T'^+\backslash T|\xi-\sum_{i\in K\cap T}\bar{p}^i+|K\cap T|\xi\leq\sum_{i\in S}d^i+\sum_{i\in S}s^i\\
&+\sum_{i\in T'}M^i+\sum_{i\in K\cap T}(M^i-\bar{p}^i)+\Big(|K\cap T|-|S^+\backslash T|-|T'^+\backslash T|+1\Big)\xi-\xi\leq C+\sum_{i\in S}s^i
\end{aligned}
$$

where the first inequality follows $x^i \leq d^i y^i + s^i$, $x^i \leq M^i y^i$ and the property $y^i = 1, i \in T$, and the last inequality follows $M^i \leq \bar{p}^i, \forall i \in K$ and $|S^+\backslash T|+|T'^+\backslash T| \geq |K\cap T|+1$ as well as the definition and nonnegativity of ξ. □

Proposition 3.11 *Assume that the conditions presented in Proposition 3.9 hold. Moreover, let $0 < \bar{p} - \xi < M^i \leq \bar{p}$ hold $\forall i \in K$. Then, (17) defines a facet of conv(PIR_1).*

Proof. First, note that the condition requires that $\bar{p}^i = \bar{p}$, $\forall i \in K$. Next, we note that a majority of the affinely independent points provided in the proof of Proposition 3.9 are also valid for this proof: we can use the first 12 sets of the points without any change, and the last two sets are valid for $i' \in T''\backslash K$. Therefore we need to present $2|K|$ new points, which we list as follows. For these points, we let $\bar{p} = p^{j^1}$, i.e., j^1 has the highest p^i value in set $S \cup T'$, and also define $\epsilon > 0$, which is a sufficiently small number. We also identify another element $j^2 \in S \cup T'$, where $j^2 \in S$ if $j^1 \in T'^+$ or $j^2 \in T'$ if $j^1 \in S^+$.

1. For every $i' \in K$, set $x^{i'} = \bar{p} - \xi$, $y^{i'} = 1$ and $s^{i'} = (\bar{p} - \xi - d^{i'})^+$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S\backslash\{j^1\}$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T'\backslash\{j^1\}$, and set all other variables to zero. **($|K|$ points)**

2. For every $i' \in K$, set $x^{i'} = \bar{p} - \xi + \epsilon$, $y^{i'} = 1$ and $s^{i'} = (\bar{p} - \xi - d^{i'})^+ + \epsilon$, set $x^{j^1} = 0 = y^{j^1}$, set $x^{j^2} = M^{j^2} - \epsilon$, $y^{j^2} = 1$ and $s^{j^2} = (M^{j^2} - \epsilon - d^{j^2})^+$ if $j^2 \in T'$ or $x^{j^2} = d^{j^2} - \epsilon$, $y^{j^2} = 1$ and $s^{j^2} = (d^{j^2} - \epsilon - d^{j^2})^+$ if $j^2 \in S$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S\backslash\{j^1, j^2\}$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T'\backslash\{j^1, j^2\}$, and set all other variables to zero. **($|K|$ points)**

These $3NI$ affinely independent points suffice to prove the claim. □

Example (Continued). Recall the PIR_1 defined in previos xxample. Consider reverse cover $S = \{1\}$ and $T' = \{3\}$. Hence $\xi = 10 + 11 - 14 = 7$. Then, we can generate a facet-defining reverse cover inequality as follows:

$$x^1 + x^3 - (10-7)y^1 - (11-7)y^3 \leq s^1 + 14 - 3 - 4$$
$$\implies x^1 + x^3 - 3y^1 - 4y^3 \leq s^1 + 7$$

With reverse cover $S = \{1\}$ and $T' = \{3\}$, we note that $p^1 = 10, p^3 = 11$. Hence, $\bar{p} = \max_{i\in S\cup T'} p^i = 11 > \xi = 7$. Let $K = \{2\}$ (and hence $\bar{p}^2 = \max(11, 10) = 11$). Then, we can derive the facet-defining item-extended reverse cover inequality:

$$x^1 + \mathbf{x^2} + x^3 - 3y^1 - (\mathbf{11} - \mathbf{7})\mathbf{y^2} - 4y^3 \leq s^1 + 7$$

where bold elements indicate all terms that are additional compared to the previous reverse cover inequality. Using PORTA [9], we can identify 6 facet-defining reverse cover inequalities and 3 facet-defining item-extended reverse inequalities for this set. □

Proposition 3.12 *Let S be a reverse cover of I, $\overline{T} = I\backslash S$, and (T', T'') be a partition of $\overline{T}$ such that $T' \neq \emptyset$ and $d^i \leq M^i$ holds $\forall i \in T'$. Then the following inequality (called **reverse cover inequality (type 2)**) is valid for PIR_1:*

$$\sum_{i\in S\cup T'} x^i - \sum_{i\in S\cup T'} (d^i - \xi)^+ y^i \leq \sum_{i\in S} s^i + C - \sum_{i\in S\cup T'} (d^i - \xi)^+ \tag{18}$$

Proof. First, using the definition $T'^+ = \{i \in T' | d^i > \xi\}$, we rewrite the reverse cover inequality (type 2) as follows:

$$\sum_{i\in S\cup T'} x^i - \sum_{i\in S^+\cup T'^+} (d^i - \xi)y^i \leq \sum_{i\in S} s^i + C - \sum_{i\in S^+\cup T'^+} (d^i - \xi)$$

Let $(x, y, s) \in PIR_1$ and $T = \{i \in I | y^i = 1\}$. So we consider two cases as follows:
Case I: $|S^+\backslash T| + |T'^+\backslash T| = 0$. Then it implies $|S^+\backslash T| = |T'^+\backslash T| = 0$. Then we have

$$\begin{aligned}
&\sum_{i\in S\cup T'} x^i - \sum_{i\in S^+} (d^i - \xi)y^i - \sum_{i\in T'^+} (d^i - \xi)y^i + \sum_{i\in S^+} (d^i - \xi) + \sum_{i\in T'^+} (d^i - \xi) \\
&= \sum_{i\in S\cup T'} x^i - \sum_{i\in S^+\cap T} (d^i - \xi) - \sum_{i\in T'^+\cap T} (d^i - \xi) + \sum_{i\in S^+} (d^i - \xi) + \sum_{i\in T'^+} (d^i - \xi) \\
&\leq C + \sum_{i\in S^+\backslash T} (d^i - \xi) + \sum_{i\in T'^+\backslash T} (d^i - \xi) + \sum_{i\in S} s^i = C + \sum_{i\in S} s^i
\end{aligned}$$

where the only inequality follows the basic set properties and capacity constraint.

Case II: $|S^+ \backslash T| + |T'^+ \backslash T| \geq 1$. Then we have

$$\begin{aligned}
&\sum_{i \in S \cup T'} x^i - \sum_{i \in S^+} (d^i - \xi) y^i - \sum_{i \in T'^+} (d^i - \xi) y^i + \sum_{i \in S^+} (d^i - \xi) + \sum_{i \in T'^+} (d^i - \xi) \\
&= \sum_{i \in S \cap T} x^i + \sum_{i \in T' \cap T} x^i - \sum_{i \in S^+ \cap T} (d^i - \xi) - \sum_{i \in T'^+ \cap T} (d^i - \xi) + \sum_{i \in S^+} (d^i - \xi) \\
&+ \sum_{i \in T'^+} (d^i - \xi) = \sum_{i \in S \cap T} x^i + \sum_{i \in T' \cap T} x^i + \sum_{i \in S^+ \backslash T} (d^i - \xi) + \sum_{i \in T'^+ \backslash T} (d^i - \xi) \\
&\leq \sum_{i \in S \cap T} d^i + \sum_{i \in S \cap T} s^i + \sum_{i \in T' \cap T} M^i + \sum_{i \in S^+ \backslash T} d^i - \sum_{i \in S^+ \backslash T} \xi + \sum_{i \in T'^+ \backslash T} d^i - \sum_{i \in T'^+ \backslash T} \xi \\
&\leq \sum_{i \in S} d^i + \sum_{i \in S} s^i + \sum_{i \in T'} M^i - |S^+ \backslash T| \xi - |T'^+ \backslash T| \xi + \xi - \xi \\
&= C + \sum_{i \in S} s^i - \Big(|S^+ \backslash T| + |T'^+ \backslash T| - 1 \Big) \xi \leq C + \sum_{i \in S} s^i
\end{aligned}$$

where the first inequality follows $x^i \leq d^i y^i + s^i$, $x^i \leq M^i y^i$ and the property $y^i = 1, i \in T$, the second inequality follows $d^i \leq M^i, i \in T'$ and basic set properties, the last equation follows the definition of ξ and finally the last inequality follows $|S^+ \backslash T| + |T'^+ \backslash T| \geq 1$ and $\xi \geq 0$. □

Proposition 3.13 *Let S be a reverse cover of I, $\overline{T} = I \backslash S$ and (T', T'') be a partition of $\overline{T}$ such that $T' \neq \emptyset$. Assume that $K \subseteq I \backslash (S \cup T')$, and that $d^i \leq M^i$ holds $\forall i \in T'$. We define $\bar{d} = \max_{i \in S \cup T'} d^i \geq \xi$, and $\bar{d^i} = \max(d^i, \bar{d})$, $i \in K$, and let $M^i \leq \bar{d^i}$, $\forall i \in K$. Then the following inequality (called **item-extended reverse cover inequality (type 2)**) is valid for PIR_1:*

$$\sum_{i \in S \cup T' \cup K} x^i - \sum_{i \in S \cup T'} (d^i - \xi)^+ y^i - \sum_{i \in K} (\bar{d^i} - \xi) y^i \leq \sum_{i \in S} s^i + C - \sum_{i \in S \cup T'} (d^i - \xi)^+ \quad (19)$$

Proof. Let $(x, y, s) \in PIR_1$. Using the definitions $T = \{i \in I | y^i = 1\}$ and $T'^+ = \{i \in T' | d^i > \xi\}$ as before, we consider two cases as follows:

Case I: $|S^+ \backslash T| + |T'^+ \backslash T| \leq |K \cap T|$. First, we rewrite (19) using S^+ and T'^+, and then can derive the following:

$$\begin{aligned}
&\sum_{i \in S \cup T' \cup K} x^i - \sum_{i \in S^+} (d^i - \xi) y^i - \sum_{i \in T'^+} (d^i - \xi) y^i - \sum_{i \in K} (\bar{d^i} - \xi) y^i + \sum_{i \in S^+} (d^i - \xi) \\
&+ \sum_{i \in T'^+} (d^i - \xi) = \sum_{i \in S \cup T' \cup K} x^i - \sum_{i \in S^+ \cap T} (d^i - \xi) - \sum_{i \in T'^+ \cap T} (d^i - \xi) - \sum_{i \in K \cap T} (\bar{d^i} - \xi)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i\in S^+}(d^i-\xi)+\sum_{i\in T'^+}(d^i-\xi)=\sum_{i\in S\cup T'\cup K}x^i+\sum_{i\in S^+\setminus T}(d^i-\xi)+\sum_{i\in T'^+\setminus T}(d^i-\xi)\\
&-\sum_{i\in K\cap T}(\bar{d^i}-\xi)\le C+\sum_{i\in S^+\setminus T}(\bar{d}-\xi)+\sum_{i\in T'^+\setminus T}(\bar{d}-\xi)-\sum_{i\in K\cap T}(\bar{d}-\xi)+\sum_{i\in S}s^i\\
&=C+\sum_{i\in S}s^i+\Big(|S^+\setminus T|+|T'^+\setminus T|-|K\cap T|\Big)(\bar{d}-\xi)\le C+\sum_{i\in S}s^i
\end{aligned}
$$

where the first inequality uses the aggregate capacity constraint, and the last inequality follows $|S^+\setminus T|+|T'^+\setminus T|\le|K\cap T|$ and $\bar{d}\ge\xi$.

Case II: $|S^+\setminus T|+|T'^+\setminus T|\ge|K\cap T|+1$. First, we rewrite (19) using S^+ and T'^+, and then can derive the following:

$$
\begin{aligned}
&\sum_{i\in S\cup T'\cup K}x^i-\sum_{i\in S^+}(d^i-\xi)y^i-\sum_{i\in T'^+}(d^i-\xi)y^i-\sum_{i\in K}(\bar{d^i}-\xi)y^i+\sum_{i\in S^+}(d^i-\xi)\\
&+\sum_{i\in T'^+}(d^i-\xi)=\sum_{i\in S\cap T}x^i+\sum_{i\in T'\cap T}x^i+\sum_{i\in K\cap T}x^i-\sum_{i\in S^+\cap T}(d^i-\xi)-\sum_{i\in T'^+\cap T}(d^i-\xi)\\
&-\sum_{i\in K\cap T}(\bar{d^i}-\xi)+\sum_{i\in S^+}(d^i-\xi)+\sum_{i\in T'^+}(d^i-\xi)\le\sum_{i\in S\cap T}d^i+\sum_{i\in S\cap T}s^i\\
&+\sum_{i\in T'\cap T}M^i+\sum_{i\in K\cap T}M^i+\sum_{i\in S^+\setminus T}(d^i-\xi)+\sum_{i\in T'^+\setminus T}(d^i-\xi)-\sum_{i\in K\cap T}(\bar{d^i}-\xi)\\
&=\sum_{i\in S\cap T}d^i+\sum_{i\in S\cap T}s^i+\sum_{i\in T'\cap T}M^i+\sum_{i\in K\cap T}M^i+\sum_{i\in S^+\setminus T}d^i-|S^+\setminus T|\xi+\sum_{i\in T'^+\setminus T}d^i\\
&-|T'^+\setminus T|\xi-\sum_{i\in K\cap T}\bar{d^i}+|K\cap T|\xi\le\sum_{i\in S}d^i+\sum_{i\in S}s^i+\sum_{i\in T'}M^i+\sum_{i\in K\cap T}(M^i-\bar{d^i})\\
&+\Big(|K\cap T|-|S^+\setminus T|-|T'^+\setminus T|+1\Big)\xi-\xi\le C+\sum_{i\in S}s^i
\end{aligned}
$$

where the first inequality follows $x^i\le d^iy^i+s^i$, $x^i\le M^iy^i$ and the property $y^i=1,i\in T$, and the last inequality follows $M^i\le\bar{d^i},\forall i\in K$ and $|S^+\setminus T|+|T'^+\setminus T|\ge|K\cap T|+1$ as well as the definition and nonnegativity of ξ. $\square$

In Section 5, we will describe separation algorithms for these inequalities described here. Next, we will discuss how we can extend the results of this section to the space of the two-period relaxation of X^{2PL}.

4. Valid Inequalities in the Original Space of X^{2PL}

Here, we recall the "original" space defined earlier and denoted as X^{2PL}. This section aims to extend the inequalities developed for PIR_1 in the previous section to the original space of X^{2PL}. Note that we introduce here again all the t indices as well as $\widetilde{\ }$, which were dropped in the previous section for the sake of simplicity, since they will be part of the discussion here.

Corollary 4.1 *Let $t \in \{1,2\}$, and S_t be a cover of I for period t. Then the following inequality (mapped from the cover inequality in relaxed space PIR_1) is valid for X^{2PL}:*

$$\sum_{i\in S_t} x_t^i - \sum_{i\in S_t}(\widetilde{d}_t^i - \lambda_t)^+ y_t^i \leq \sum_{i\in S_t} s^i + \widetilde{C}_t - \sum_{i\in S_t}(\widetilde{d}_t^i - \lambda_t)^+ \tag{20}$$

The proof is straightforward as it follows the same logic as the proof of Proposition 3.4. We can also extend this inequality as follows.

Proposition 4.1 *Let $t, t' \in \{1,2\}$, $t \neq t'$, and S_t be a cover of I for period t. In addition, assume $L_{t'} \subseteq S_t$. Then the following inequality (called **period-extended cover inequality**) is valid for X^{2PL}:*

$$\sum_{i\in S_t} x_t^i + \sum_{i\in L_{t'}} x_{t'}^i - \sum_{i\in S_t}(\widetilde{d}_t^i - \lambda_t)^+ y_t^i - \sum_{i\in L_{t'}} \widetilde{d}_{t'}^i y_{t'}^i \leq \sum_{i\in S_t} s^i + \widetilde{C}_t - \sum_{i\in S_t}(\widetilde{d}_t^i - \lambda_t)^+ \tag{21}$$

Proof. Let $(x,y,s) \in X^{2PL}$, and $T_k = \{i \in I | y_k^i = 1\}$, for $k \in \{1,2\}$. Then we consider two cases as follows:

Case I: $|S_t^+ \backslash T_t| = 0$. Then, we have

$$\begin{aligned}
&\sum_{i\in S_t} x_t^i + \sum_{i\in L_{t'}} x_{t'}^i - \sum_{i\in S_t}(\widetilde{d}_t^i - \lambda_t)^+ y_t^i - \sum_{i\in L_{t'}} \widetilde{d}_{t'}^i y_{t'}^i + \sum_{i\in S_t}(\widetilde{d}_t^i - \lambda_t)^+ \\
&= \sum_{i\in S_t\cap T_t} x_t^i + \sum_{i\in L_{t'}\cap T_{t'}} x_{t'}^i - \sum_{i\in S_t^+\cap T_t}(\widetilde{d}_t^i - \lambda_t) - \sum_{i\in L_{t'}\cap T_{t'}} \widetilde{d}_{t'}^i + \sum_{i\in S_t^+}(\widetilde{d}_t^i - \lambda_t) \\
&\leq \widetilde{C}_t + \sum_{i\in L_{t'}\cap T_{t'}} \widetilde{d}_{t'}^i + \sum_{i\in L_{t'}\cap T_{t'}} s^i + \sum_{i\in S_t^+\backslash T_t}(\widetilde{d}_t^i - \lambda_t) - \sum_{i\in L_{t'}\cap T_{t'}} \widetilde{d}_{t'}^i \leq \widetilde{C}_t + \sum_{i\in S_t} s^i
\end{aligned}$$

where the first inequality follows the capacity constraint in period t and $x_{t'}^i \leq \widetilde{d}_{t'}^i y_{t'}^i + s^i$ and the property $y_{t'}^i = 1, i \in T_{t'}$, and the second inequality the fact that $L_{t'} \subseteq S_t$.

Case II: $|S_t^+ \backslash T_t| \geq 1$. Then, we have

$$\sum_{i\in S_t} x_t^i + \sum_{i\in L_{t'}} x_{t'}^i - \sum_{i\in S_t}(\widetilde{d}_t^i - \lambda_t)^+ y_t^i - \sum_{i\in L_{t'}} \widetilde{d}_{t'}^i y_{t'}^i + \sum_{i\in S_t}(\widetilde{d}_t^i - \lambda_t)^+$$

$$
\begin{aligned}
&= \sum_{i \in S_t} x_t^i + \sum_{i \in L_{t'}} x_{t'}^i - \sum_{i \in S_t^+ \cap T_t} (\widetilde{d}_t^i - \lambda_t) - \sum_{i \in L_{t'} \cap T_{t'}} \widetilde{d}_{t'}^i + \sum_{i \in S_t^+} (\widetilde{d}_t^i - \lambda_t) \\
&= \sum_{i \in S_t \setminus L_{t'}} x_t^i + \sum_{i \in L_{t'}} (x_t^i + x_{t'}^i) + \sum_{i \in S_t^+ \setminus T_t} (\widetilde{d}_t^i - \lambda_t) - \sum_{i \in L_{t'} \cap T_{t'}} \widetilde{d}_{t'}^i \leq \sum_{i \in (S_t \setminus L_{t'}) \cap T_t} \widetilde{d}_t^i \\
&+ \sum_{i \in (S_t \setminus L_{t'})} s^i + \sum_{L_{t'}} (\widetilde{d}_t^i y_t^i + \widetilde{d}_{t'}^i y_{t'}^i + s^i) + \sum_{i \in S_t^+ \setminus T_t} (\widetilde{d}_t^i - \lambda_t) - \sum_{i \in L_{t'} \cap T_{t'}} \widetilde{d}_{t'}^i \\
&= \sum_{i \in (S_t \setminus L_{t'}) \cap T_t} \widetilde{d}_t^i + \sum_{i \in (S_t \setminus L_{t'})} s^i + \sum_{i \in L_{t'} \cap T_t} \widetilde{d}_t^i + \sum_{i \in L_{t'} \cap T_{t'}} \widetilde{d}_{t'}^i + \sum_{i \in L_{t'}} s^i + \sum_{i \in S_t^+ \setminus T_t} \widetilde{d}_t^i \\
&- |S_t^+ \setminus T_t| \lambda_t - \sum_{i \in L_{t'} \cap T_{t'}} \widetilde{d}_{t'}^i \leq \sum_{i \in S_t} \widetilde{d}_t^i + \sum_{i \in S_t} s^i - |S_t^+ \setminus T_t| \lambda_t + \lambda_t - \lambda_t \\
&= \widetilde{C}_t + \sum_{i \in S_t} s^i + \Big(1 - |S_t^+ \setminus T_t|\Big) \lambda_t \leq \widetilde{C}_t + \sum_{i \in S_t} s^i
\end{aligned}
$$

where the first inequality uses $x_t^i \leq \widetilde{d}_t^i y_t^i + s^i$, $x_t^i + x_{t'}^i \leq \widetilde{d}_t^i y_t^i + \widetilde{d}_{t'}^i y_{t'}^i + s^i$, and the property $y_k^i = 1, i \in T_k, k \in \{1, 2\}$, the second inequality exploits the disjoint sets in the previous expression, the last equation uses the definition of λ_t, and finally the last inequality uses $\lambda_t > 0$ and $|S_t^+ \setminus T_t| \geq 1$. □

Next, we discuss item-extended cover inequalities and how they can be represented in the original space.

Corollary 4.2 *Let $t \in \{1, 2\}$ and S_t be a cover of I for period t. Let $K_t \subseteq I \setminus S_t$ such that $\widetilde{M}_t^i \leq \bar{d}_t^i$ holds $\forall i \in K_t$, where $\bar{d}_t = \max_{i \in S_t} \widetilde{d}_t^i \geq \lambda_t$ and $\bar{d}_t^i = \max(\bar{d}_t, \widetilde{d}_t^i)$. Then the following inequality (mapped from the item-extended cover inequality in relaxed space PIR_1) is valid for X^{2PL}:*

$$
\sum_{i \in S_t \cup K_t} x_t^i - \sum_{i \in S_t} (\widetilde{d}_t^i - \lambda_t)^+ y_t^i - \sum_{i \in K_t} (\bar{d}_t^i - \lambda_t) y_t^i \leq \sum_{i \in S_t} s^i + \widetilde{C}_t - \sum_{i \in S_t} (\widetilde{d}_t^i - \lambda_t)^+ \tag{22}
$$

The proof is straightforward as it follows the same logic as the proof of Proposition 3.6. We can also extend this inequality as follows.

Proposition 4.2 *Consider the same assumptions and definitions as in Corollary 4.2. In addition, let $t, t' \in \{1, 2\}$, $t \neq t'$, and $L_{t'} \subseteq S_t$. Then the following inequality (called* ***item-and-period-extended cover inequality****) is valid for X^{2PL}:*

$$
\begin{aligned}
\sum_{i \in S_t \cup K_t} x_t^i + \sum_{i \in L_{t'}} x_{t'}^i - \sum_{i \in S_t} (\widetilde{d}_t^i - \lambda_t)^+ y_t^i - \sum_{i \in K_t} (\bar{d}_t^i - \lambda_t) y_t^i - \sum_{i \in L_{t'}} \widetilde{d}_{t'}^i y_{t'}^i \leq \\
\sum_{i \in S_t} s^i + \widetilde{C}_t - \sum_{i \in S_t} (\widetilde{d}_t^i - \lambda_t)^+
\end{aligned} \tag{23}
$$

Proof. Let $(x,y,s) \in X^{2PL}$, and $T_k = \{i \in I | y_k^i = 1\}$, for $k \in \{1,2\}$. Then we consider two cases as follows:

Case I: $|S_t^+ \backslash T_t| \le |K_t \cap T_t|$. Then, we have

$$\sum_{i\in S_t\cup K_t} x_t^i + \sum_{i\in L_{t'}} x_{t'}^i - \sum_{i\in S_t}(\widetilde{d}_t^i - \lambda_t)^+ y_t^i - \sum_{i\in K_t}(\bar{d}_t^i - \lambda_t)y_t^i - \sum_{i\in L_{t'}} \widetilde{d}_{t'}^i y_{t'}^i$$

$$+\sum_{i\in S_t}(\widetilde{d}_t^i - \lambda_t)^+ = \sum_{i\in S_t\cup K_t} x_t^i + \sum_{i\in L_{t'}} x_{t'}^i - \sum_{i\in S_t^+}(\widetilde{d}_t^i - \lambda_t)y_t^i - \sum_{i\in K_t}(\bar{d}_t^i - \lambda_t)y_t^i$$

$$-\sum_{i\in L_{t'}} \widetilde{d}_{t'}^i y_{t'}^i + \sum_{i\in S_t^+}(\widetilde{d}_t^i - \lambda_t) = \sum_{i\in S_t\cup K_t} x_t^i + \sum_{i\in L_{t'}\cap T_{t'}} x_{t'}^i - \sum_{i\in S_t^+\cap T_t}(\widetilde{d}_t^i - \lambda_t)$$

$$-\sum_{i\in K_t\cap T_t}(\bar{d}_t^i - \lambda_t) - \sum_{i\in L_{t'}\cap T_{t'}} \widetilde{d}_{t'}^i + \sum_{i\in S_t^+}(\widetilde{d}_t^i - \lambda_t) \le \widetilde{C}_t + \sum_{i\in L_{t'}\cap T_{t'}} \widetilde{d}_{t'}^i + \sum_{i\in L_{t'}\cap T_{t'}} s^i$$

$$+\sum_{i\in S_t^+\backslash T_t}(\widetilde{d}_t^i - \lambda_t) - \sum_{i\in K_t\cap T_t}(\bar{d}_t^i - \lambda_t) - \sum_{i\in L_{t'}\cap T_{t'}} \widetilde{d}_{t'}^i \le \widetilde{C}_t + \sum_{i\in S_t} s^i$$

$$+\sum_{i\in S_t^+\backslash T_t}(\widetilde{d}_t^i - \lambda_t) - \sum_{i\in K_t\cap T_t}(\bar{d}_t^i - \lambda_t) \le \widetilde{C}_t + \sum_{i\in S_t} s^i + \sum_{i\in S_t^+\backslash T_t}(\bar{d}_t - \lambda_t)$$

$$-\sum_{i\in K_t\cap T_t}(\bar{d}_t - \lambda_t) = \widetilde{C}_t + \sum_{i\in S_t} s^i + \Big(|S_t^+\backslash T_t| - |K_t\cap T_t|\Big)(\bar{d}_t - \lambda_t) \le \widetilde{C}_t + \sum_{i\in S_t} s^i$$

where the first inequality follows the capacity constraint in period t, $x_{t'}^i \le \widetilde{d}_{t'}^i y_{t'}^i + s^i$ and the property $y_{t'}^i = 1, i \in T_{t'}$, the second inequality follows the fact that $L_{t'} \subseteq S_t$, the third inequality follows $\widetilde{d}_t^i \le \bar{d}_t \le \bar{d}_t^i$, and the last inequality uses $|S_t^+\backslash T_t| \le |K_t \cap T_t|$ and $\bar{d}_t \ge \lambda_t$.

Case II: $|S_t^+\backslash T_t| \ge |K_t \cap T_t| + 1$. Then, we have

$$\sum_{i\in S_t\cup K_t} x_t^i + \sum_{i\in L_{t'}} x_{t'}^i - \sum_{i\in S_t^+}(\widetilde{d}_t^i - \lambda_t)y_t^i - \sum_{i\in K_t}(\bar{d}_t^i - \lambda_t)y_t^i - \sum_{i\in L_{t'}} \widetilde{d}_{t'}^i y_{t'}^i + \sum_{i\in S_t^+}(\widetilde{d}_t^i - \lambda_t)$$

$$= \sum_{i\in S_t} x_t^i + \sum_{i\in L_{t'}} x_{t'}^i + \sum_{i\in K_t\cap T_t} x_t^i - \sum_{i\in S_t^+\cap T_t}(\widetilde{d}_t^i - \lambda_t) - \sum_{i\in K_t\cap T_t}(\bar{d}_t^i - \lambda_t) - \sum_{i\in L_{t'}\cap T_{t'}} \widetilde{d}_{t'}^i$$

$$+\sum_{i\in S_t^+}(\widetilde{d}_t^i - \lambda_t) = \sum_{i\in S_t\backslash L_{t'}} x_t^i + \sum_{i\in L_{t'}}(x_t^i + x_{t'}^i) + \sum_{i\in K_t\cap T_t} x_t^i + \sum_{i\in S_t^+\backslash T_t}(\widetilde{d}_t^i - \lambda_t)$$

$$-\sum_{i\in K_t\cap T_t}(\bar{d}_t^i - \lambda_t) - \sum_{i\in L_{t'}\cap T_{t'}} \widetilde{d}_{t'}^i \le \sum_{i\in (S_t\backslash L_{t'})\cap T_t} \widetilde{d}_t^i + \sum_{i\in S_t\backslash L_{t'}} s^i + \sum_{i\in L_{t'}\cap T_t} \widetilde{d}_t^i$$

$$+\sum_{i\in L_{t'}\cap T_{t'}} \widetilde{d}_{t'}^i + \sum_{i\in L_{t'}} s^i + \sum_{i\in K_t\cap T_t} \widetilde{M}_t^i + \sum_{i\in S_t^+\backslash T_t} \widetilde{d}_t^i - |S_t^+\backslash T_t|\lambda_t - \sum_{i\in K_t\cap T_t} \bar{d}_t^i$$

$$+|K_t\cap T_t|\lambda_t - \sum_{i\in L_{t'}\cap T_{t'}} \widetilde{d}_{t'}^i \le \sum_{i\in S_t} \widetilde{d}_t^i + \sum_{i\in S_t} s^i + \Big(|K_t\cap T_t| - |S_t^+\backslash T_t|\Big)\lambda_t + \lambda_t - \lambda_t$$

$$+\sum_{i\in K_t\cap T_t}(\widetilde{M}_t^i-\bar{d}_t^i)\le \widetilde{C}_t+\sum_{i\in S_t}s^i+\Big(|K_t\cap T_t|-|S_t^+\backslash T_t|+1\Big)\lambda_t\le \widetilde{C}_t+\sum_{i\in S_t}s^i$$

where the first inequality uses $x_t^i \le \widetilde{d}_t^i y_t^i + s^i$, $x_t^i + x_{t'}^i \le \widetilde{d}_t^i y_t^i + \widetilde{d}_{t'}^i y_{t'}^i + s^i$, $x_t^i \le \widetilde{M}_t^i y_t^i$, and the property $y_k^i = 1, i \in T_k, k \in \{1,2\}$, the second inequality exploits the disjoint sets in the previous expression, the third inequality uses the definition of λ_t and $\widetilde{M}_t^i \le \bar{d}_t^i$, and finally the last inequality uses $\lambda_t > 0$ and $|S_t^+\backslash T_t| \ge |K_t \cap T_t| + 1$. □

Next, we discuss reverse cover inequalities and how they can be represented in the original space of variables.

Corollary 4.3 *Let $t \in \{1,2\}$ and S_t be a reverse cover of I in period t. Let $\overline{T}_t = I\backslash S_t$, and (T_t', T_t'') be a partition of $\overline{T}_t$. Then the following inequality (mapped from the reverse cover inequality in relaxed space PIR_1) is valid for X^{2PL}:*

$$\sum_{i\in S_t\cup T_t'}x_t^i-\sum_{i\in S_t}(\widetilde{d}_t^i-\xi_t)^+y_t^i-\sum_{i\in T_t'}(\widetilde{M}_t^i-\xi_t)^+y_t^i\le \sum_{i\in S_t}s^i+\widetilde{C}_t-\sum_{i\in S_t}(\widetilde{d}_t^i-\xi_t)^+-\sum_{i\in T_t'}(\widetilde{M}_t^i-\xi_t)^+ \quad (24)$$

The proof is straightforward as it follows a similar logic to the proof of Proposition 3.8. We can also extend this inequality as follows.

Proposition 4.3 *Consider the same assumptions and definitions as in Corollary 4.3. In addition, let $L_{t'} \subseteq S_t$, where $t, t' \in \{1,2\}$ and $t \ne t'$. Then the following inequality (called **period-extended reverse cover inequality**) is valid for X^{2PL}:*

$$\sum_{i\in S_t\cup T_t'}x_t^i+\sum_{i\in L_{t'}}x_{t'}^i-\sum_{i\in S_t}(\widetilde{d}_t^i-\xi_t)^+y_t^i-\sum_{i\in T_t'}(\widetilde{M}_t^i-\xi_t)^+y_t^i-\sum_{i\in L_{t'}}\widetilde{d}_{t'}^i y_{t'}^i\le \sum_{i\in S_t}s^i+\widetilde{C}_t-\sum_{i\in S_t}(\widetilde{d}_t^i-\xi_t)^+-\sum_{i\in T_t'}(\widetilde{M}_t^i-\xi_t)^+ \quad (25)$$

We omit the proof since it follows a similar logic to the proof of Proposition 3.8 and Proposition 4.1.

Next, we discuss item-extended reverse cover inequalities and how they can be represented in the original space.

Corollary 4.4 *Let $t \in \{1,2\}$ and S_t be a reverse cover of I for time period t. Let $\overline{T}_t = I \backslash S_t$, (T'_t, T''_t) be a partition of $\overline{T}_t$, and $K_t \subseteq T''_t$. We define*

$$p_t^i = \begin{cases} \widetilde{d}_t^i & : i \in S_t \\ \widetilde{M}_t^i & : i \in T'_t \end{cases}$$

and $\bar{p}_t = \max_{i \in S_t \cup T'_t} p_t^i \geq \xi_t$. We also define $\bar{p}_t^i = \max(\widetilde{M}_t^i, \bar{p}_t), i \in K_t$. Then the following inequality (mapped from the item-extended reverse cover inequality in relaxed space PIR_1) is valid for X^{2PL}:

$$\sum_{i \in S_t \cup T'_t \cup K_t} x_t^i - \sum_{i \in S_t} (\widetilde{d}_t^i - \xi_t)^+ y_t^i - \sum_{i \in T'_t} (\widetilde{M}_t^i - \xi_t)^+ y_t^i - \sum_{i \in K_t} (\bar{p}_t^i - \xi_t) y_t^i \leq \sum_{i \in S_t} s^i + \widetilde{C}_t - \sum_{i \in S_t} (\widetilde{d}_t^i - \xi_t)^+ - \sum_{i \in T'_t} (\widetilde{M}_t^i - \xi_t)^+ \quad (26)$$

The proof is straightforward as it follows a similar logic to the proof of Proposition 3.10. We can also extend this inequality further as follows.

Proposition 4.4 *Consider the same assumptions and definitions as in Corollary 4.4. In addition, let $L_{t'} \subseteq S_t$, where $t, t' \in \{1,2\}$ and $t \neq t'$. Then, the following inequality (called **item-and-period-extended cover inequality**) is valid for X^{2PL}:*

$$\sum_{i \in S_t \cup T'_t} x_t^i + \sum_{i \in K_t \cup L_{t'}} x_t^i - \sum_{i \in S_t} (\widetilde{d}_t^i - \xi_t)^+ y_t^i - \sum_{i \in T'_t} (\widetilde{M}_t^i - \xi_t)^+ y_t^i - \sum_{i \in K_t} (\bar{p}_t^i - \xi_t) y_t^i - \sum_{i \in L_{t'}} \widetilde{d}_{t'}^i y_{t'}^i \leq \sum_{i \in S_t} s^i + \widetilde{C}_t - \sum_{i \in S_t} (\widetilde{d}_t^i - \xi_t)^+ - \sum_{i \in T'_t} (\widetilde{M}_t^i - \xi_t)^+ \quad (27)$$

We omit the proof here since it is very similar to the proof of Proposition 4.3. Next, we investigate the extendability of reverse cover inequalities of type 2 to the original space of variables.

Corollary 4.5 *Let $t \in \{1,2\}$ and S_t be a reverse cover of I for time period t. Let $\overline{T}_t = I \backslash S_t$, and (T'_t, T''_t) be a partition of $\overline{T}_t$, and $\widetilde{d}_t^i \leq \widetilde{M}_t^i$ holds $\forall i \in T'_t$. Then the following inequality (mapped from the reverse cover inequality (type 2) in relaxed space PIR_1) is valid for X^{2PL}:*

$$\sum_{i \in S_t \cup T'_t} x_t^i - \sum_{i \in S_t \cup T'_t} (\widetilde{d}_t^i - \xi_t)^+ y_t^i \leq \sum_{i \in S_t} s^i + \widetilde{C}_t - \sum_{i \in S_t \cup T'_t} (\widetilde{d}_t^i - \xi_t)^+ \quad (28)$$

The proof is straightforward as it follows a similar logic to the proof of Proposition 3.12. We can also extend this inequality further as follows.

Proposition 4.5 *Consider the same assumptions and definitions as in Corollary 4.5. In addition, let $L_{t'} \subseteq S_t$, where $t, t' \in \{1,2\}$ and $t \neq t'$. Then, the following inequality (called* ***period-extended reverse cover inequality (type 2)****) is valid for X^{2PL}:*

$$\sum_{i \in S_t \cup T'_t} x^i_t + \sum_{i \in L_{t'}} x^i_{t'} - \sum_{i \in S_t \cup T'_t} (\widetilde{d}^i_t - \xi_t)^+ y^i_t - \sum_{i \in L_{t'}} \widetilde{d}^i_{t'} y^i_{t'} \leq \sum_{i \in S_t} s^i + \widetilde{C}_t - \sum_{i \in S_t \cup T'_t} (\widetilde{d}^i_t - \xi_t)^+ \quad (29)$$

We omit the proof since it follows a similar logic to the proof of Proposition 3.12 and Proposition 4.1. Finally, we investigate the extendability of item-extended reverse cover inequalities of type 2 to the original space of variables.

Corollary 4.6 *Let $t \in \{1,2\}$ and S_t be a reverse cover of I for time period t. Let $\overline{T}_t = I \backslash S_t$, (T'_t, T''_t) be a partition of $\overline{T}_t$, and $K_t \subseteq I \backslash (S_t \cup T'_t)$. We define $\bar{d}_t = \max_{i \in S_t \cup T'_t} \widetilde{d}^i_t \geq \xi_t$ and $\bar{d}^i_t = \max(\widetilde{d}^i_t, \bar{d}_t)$, $i \in K_t$. Assume that $\widetilde{d}^i_t \leq \widetilde{M}^i_t$ holds $\forall i \in T'_t$, and $\widetilde{M}^i_t \leq \bar{d}^i_t$ holds $\forall i \in K_t$. Then the following inequality (mapped from the item-extended reverse cover inequality (type 2) in relaxed space PIR_1) is valid for X^{2PL}:*

$$\sum_{i \in S_t \cup T'_t \cup K_t} x^i_t - \sum_{i \in S_t \cup T'_t} (\widetilde{d}^i_t - \xi_t)^+ y^i_t - \sum_{i \in K_t} (\bar{d}^i_t - \xi_t) y^i_t \leq \sum_{i \in S_t} s^i + \widetilde{C}_t - \sum_{i \in S_t \cup T'_t} (\widetilde{d}^i_t - \xi_t)^+ \quad (30)$$

The proof is straightforward as it follows a similar logic to the proof of Proposition 3.13. We can also extend this inequality further as follows.

Proposition 4.6 *Consider the same assumptions and definitions as in Corollary 4.6. In addition, let $L_{t'} \subseteq S_t$, where $t, t' \in \{1,2\}$ and $t \neq t'$. Then, the following inequality (called* ***item-and-period-extended reverse cover inequality (type 2)****) is valid for X^{2PL}:*

$$\begin{aligned}\sum_{i \in S_t \cup T'_t \cup K_t} x^i_t + \sum_{i \in L_{t'}} x^i_{t'} - \sum_{i \in S_t \cup T'_t} (\widetilde{d}^i_t - \xi_t)^+ y^i_t - \sum_{i \in K_t} (\bar{d}^i_t - \xi_t) y^i_t - \sum_{i \in L_{t'}} \widetilde{d}^i_{t'} y^i_{t'} \leq \\ \sum_{i \in S_t} s^i + \widetilde{C}_t - \sum_{i \in S_t \cup T'_t} (\widetilde{d}^i_t - \xi_t)^+ \quad (31)\end{aligned}$$

We omit the proof here since it is very similar to the proof of Proposition 4.5.

In the next section, we describe in detail the separation procedures for all the inequalities discussed in this section.

5. Separation Algorithms for Relaxations and Original Space

The purpose of this section is to describe in detail the separation problems associated with all the families of inequalities defined in the previous two sections of the paper. Since the main purpose of this paper is to investigate the strength of the cuts generated, we focus on defining exact separation algorithms, which are computationally tested in the next section, rather than their computational efficiency, which is addressed in a companion study [13] in thorough detail. Here, we will follow the same structure and order of the previous two sections: we will firstly define separation problems associated with families of inequalities defined for the relaxations of the problems, and then for those associated with the original space. In the remainder of this section, we let $(\bar{x}, \bar{y}, \bar{s})$ to represent a fractional solution vector in the associated space that is to be cut off. We also note that w.l.o.g., we assume all problem parameters to be integer valued.

5.1 Separation in the Relaxation Space

We start first with the family of cover inequalities as defined by (14). First, we note that we can rewrite these inequalities as follows:

$$\sum_{i \in S} \left(x^i + (d^i - \lambda)^+ (1 - y^i) - s^i \right) \leq C$$

Since S must be a cover, for a given value of $\lambda > 0$, we can find the most violated inequality (if any) by solving the following knapsack problem:

$$f_\lambda = \max \left\{ \sum_{j \in I} \tau_j(\lambda) w_j | \sum_{i \in I} d^i w_i = C + \lambda, w \in \{0,1\}^{NI} \right\} \tag{32}$$

where $\tau_j(\lambda) = \bar{x}^j + (d^j - \lambda)^+ (1 - \bar{y}^j) - \bar{s}^j$. If $f_\lambda > C$, then a violated cover inequality is identified for the specified λ. We note that since $\lambda \in \mathbb{Z}_+$, one can solve this separation problem for any value of $\lambda \in [1, \sum_{i \in I} d^i - C]$.

Next, we discuss the separation procedure for the family of item-extended cover inequalities as defined by (15). Note that we can rewrite these inequalities as follows:

$$\sum_{i \in S} \left(x^i + (d^i - \lambda)^+ (1 - y^i) - s^i \right) + \sum_{i \in K} \left(x^i - (\bar{d}^i - \lambda) y^i \right) \leq C$$

We note that for a given cover S, if $\bar{d} \geq \lambda$, then we can define the set K as follows:

$$K = \{ i \in I \backslash S | \bar{x}^i - (\bar{d}^i - \lambda) \bar{y}^i > 0, M^i \leq \bar{d}^i \}$$

Therefore, one can generate covers using the procedure defined for the simple cover inequalities, and then heuristically generate the set K. We also note that this is a similar approach to the one proposed by [23] (p.854) for flow cover inequalities. Finally, we note that such a procedure is applied for the separation of the known inequalities (12) and (13).

Next, we discuss the separation procedure for the family of reverse cover inequalities as defined by (16). First note that we can rewrite these inequalities as follows:

$$\sum_{i\in S}\Big(x^i+(d^i-\xi)^+(1-y^i)-s^i\Big)+\sum_{i\in T'}\Big(x^i+(M^i-\xi)^+(1-y^i)\Big)\leq C$$

For a given value of ξ, we define the following IP for the separation:

$$\begin{aligned}
f_\xi = \max & \sum_{i\in I}\Big(\bar{x}^i+(d^i-\xi)^+(1-\bar{y}^i)-\bar{s}^i\Big)u_i+\sum_{i\in I}\Big(\bar{x}^i+(M^i-\xi)^+(1-\bar{y}^i)\Big)v_i\\
\text{s.t.} \;& \sum_{i\in I}d^iu_i+\sum_{i\in I}M^iv_i=C+\xi\\
& C>\sum_{i\in I}d^iu_i\\
& u_i+v_i\leq 1,\ \forall i\in I\\
& u_i,v_i\in\{0,1\},\ \forall i\in I
\end{aligned}$$

Here, u_i and v_i variables indicate whether an item i belongs to set S or T' respectively. The first two constraints ensure that $\sum_{i\in I}v_i\geq 1$, i.e., $T'\neq\emptyset$. A violated reverse cover inequality is found if $f_\xi>C$. Similar to the process for simple cover inequalities, since $\xi\in\mathbb{Z}_+$, one can solve this separation problem for any value of $\xi\in[1,\sum_{i\in I}\max\{d^i,M^i\}-C]$.

Next, we discuss the separation procedure for the family of item-extended reverse cover inequalities as defined by (17). First, we note that these inequalities can be rewritten as follows:

$$\begin{aligned}
&\sum_{i\in S}\Big(x^i+(d^i-\xi)^+(1-y^i)-s^i\Big)+\sum_{i\in T'}\Big(x^i+(M^i-\xi)^+(1-y^i)\Big)\\
&\qquad\qquad+\sum_{i\in K}\Big(x^i-(\bar{p}^i-\xi)y^i\Big)\leq C
\end{aligned}$$

We note that for given sets of S and T', when the condition $\bar{p}\geq\xi$ holds, we can define the set K as follows:

$$K=\{i\in I\backslash(S\cup T')|\bar{x}^i-(\bar{p}^i-\xi)\bar{y}^i>0\}$$

Similar to the procedure described for item-extended cover inequalities, one can solve the separation problem for reverse cover inequalities to identify sets S and T' first and use this procedure to identify the set K.

Next, we note that the separation procedures for the family of reverse cover inequalities (type 2) as defined by (18) and for the family of item-extended reverse cover inequalities (type 2) as defined by (19) are very similar to the separation procedures of the reverse cover inequalities (16) and of the item-extended reverse cover inequalities (17), respectively. Therefore, we omit a detailed description here for the sake of avoiding repetition.

5.2 Separation in the Original Space

We start this section with the separation of the period-extended cover inequalities in the original space as defined by (21). First, we rewrite these inequalities as follows:

$$\sum_{i \in S_t} \left(x_t^i + (\widetilde{d}_t^i - \lambda_t)^+ (1 - y_t^i) - s^i \right) + \sum_{i \in L_{t'}} \left(x_{t'}^i - \widetilde{d}_{t'}^i y_{t'}^i \right) \leq \widetilde{C}_t$$

where $S_t \subseteq I$ and $L_{t'} \subseteq S_t$, $t \neq t'$.

We note that for a given t and fixed $\lambda_t > 0$, we can solve the following separation problem:

$$\begin{aligned} &\max \sum_{i \in I} \left(\bar{x}_t^i + (\widetilde{d}_t^i - \lambda_t)^+ (1 - \bar{y}_t^i) - \bar{s}^i \right) u_i + \sum_{i \in I} \left(\bar{x}_{t'}^i - \widetilde{d}_{t'}^i \bar{y}_{t'}^i \right) v_i \\ &\text{s.t. } \sum_{i \in I} \widetilde{d}_t^i u_i = \widetilde{C}_t + \lambda_t; \quad v_i \leq u_i \ \ \forall i \in I; \quad u_i, v_i \in \{0,1\} \ \ \forall i \in I \end{aligned}$$

If the optimal value of the problem is strictly greater than $\widetilde{C}_t$, then a violated inequality is identified.

Next, we discuss the separation procedure for the family of item-and-period-extended cover inequalities in the original space as defined by (23). Note that we can rewrite these inequalities as follows:

$$\sum_{i \in S_t} \left(x_t^i + (\widetilde{d}_t^i - \lambda_t)^+ (1 - y_t^i) - s^i \right) + \sum_{i \in L_{t'}} \left(x_{t'}^i - \widetilde{d}_{t'}^i y_{t'}^i \right) + \sum_{i \in K_t} \left(x_t^i - (\bar{d}_t^i - \lambda_t) y_t^i \right) \leq \widetilde{C}_t$$

We note that for given sets of S_t and $L_{t'}$, if $\bar{d}_t \geq \lambda_t$ holds, then we can define the set K_t as follows:

$$K_t = \{ i \in I \backslash S_t | \bar{x}_t^i - (\bar{d}_t^i - \lambda_t) \bar{y}_t^i > 0, \widetilde{M}_t^i \leq \bar{d}_t^i \}$$

Therefore, one can generate the sets of S_t and $L_{t'}$ using the procedure defined for the period-extended cover inequalities, and then heuristically generate the set K_t.

Next, we note that the separation procedures for the period-extended reverse cover inequalities in the original space as defined by (25) and for the item-and-period-extended reverse cover inequalities in the original space as defined by (27) follow a very similar logic to the separation procedures of the period-extended cover inequalities in the original space as defined by (21) and of the item-and-period-extended cover inequalities in the original space as defined by (23), respectively. Therefore, we omit a detailed description here for the sake of simplicity. Similarly, we also omit the details for the separation algorithms of period-extended reverse cover (type 2) and item-and-period-extended reverse cover (type 2) inequalities in the original space, defined by (29) and (31), respectively, as they follow the same logic as well.

6. Computational Results

In this section, we present numerical results indicating the strength of the various cuts proposed earlier. We note that our primary aim here is not necessarily to build a practically efficient computational framework, which we are currently addressing in a companion paper [13], but instead to exhaustively generate all violated inequalities by exact separation (as discussed in Section 5) to measure their practical strength and effectiveness. All the separation algorithms and mathematical models are implemented and executed using the Mosel language of FICO® Xpress Optimization Suite (Mosel version 3.6.0, Xpress-MP v7.7 released in 2014) on a PC with Intel® Core i5 3.10GHz processor and 4GB RAM, where all possible two-period relaxations, both consecutive and non-consecutive, were considered.

In order to test the effectiveness of the cuts proposed, we have generated 240 random test instances in total, which we will describe in detail next. First of all, we note that exact separation is computationally expensive, causing issues with available memory or prohibitively long times when the problem size became bigger than $NT = 12$ and $NI = 10$, so that we set the maximum size to these values. We also note that even with this maximum size, computational times can be extensive. On the other hand, we have set the minimum size to $NT = 2$ and $NI = 3$, in order to capture the simplest problem with the two-period, multi-item structure. We have varied NT and NI values with intervals growing exponentially, in order to capture the variety created by the fact that the problem complexity grows exponentially (rather than using equal length intervals), resulting in 16 different NT, NI combinations. On the other hand, we have considered low, medium and high demand variability for a good mix of problems, randomly generating d_t^i parameters in the intervals of $[10, 20]$, $[10, 40]$ and $[10, 60]$, respectively. This results in 48 combinations, where for each combination, we have generated 5 test instances. The capacities in each period are generated as a random variable from the interval $[0.8 \times NI \times mid_t, 1.2 \times NI \times mid_t]$, where mid_t indicates the median demand in that interval. Finally, we note that the holding costs h_t^i are randomly generated from the interval $[0.1, 1]$ and the setup costs f_t^i takes a value of $\{1, 10, 50\}$, each with probability of $\frac{1}{3}$, in order to generate a good mix of low and high setup cost items (and in between).

Next, we present the computational results for low, medium and high demand variability, in Tables 1, 2 and 3, respectively. In each table, the first column indicates the combination NT, NI, followed by the columns indicating average **I**nitial **G**ap and the percentage **G**ap **C**losed for 5 instances with all the violated cuts generated. Note that the initial gap is based on the strengthened LP relaxation with all violated (ℓ, S) inequalities added a priori, which are known to be very effective in practice for multi-item problems, see, e.g., [2]. In the remainder of the tables, the columns indicate the total number of cuts generated of each type for 5 instances, in the following order: **C**over (14), **I**tem-extended **C**over (15), **P**eriod-extended **C**over (21), **I**tem-and-**P**eriod-extended **C**over (23), **R**everse cover (16), **I**tem-extended **R**everse cover (17), **P**eriod-extended **R**everse cover (25), **I**tem-and-**P**eriod-extended **R**everse cover (27), **R**everse cover type **2** (18), **I**tem-extended **R**everse cover type **2** (19), **P**eriod-extended **R**everse cover type **2** (29), **I**tem-and-**P**eriod-extended **R**everse cover type **2** (31), and **F**low **C**over (13).

Table 1: Average closed gaps and number of cuts generated of each type for test problems with $d_t^i \in [10, 20]$.

NT,			**# Cuts generated**												
NI	**IG**	**GC**	**C**	**IC**	**PC**	**IPC**	**R**	**IR**	**PR**	**IPR**	**R2**	**IR2**	**PR2**	**IPR2**	**FC**
2,3	18.01	52.88	5	0	0	0	0	0	0	0	0	0	0	0	0
2,4	18.03	50.52	5	0	0	0	0	0	0	0	0	0	0	0	1
2,6	9.85	48.30	7	0	0	0	0	0	0	0	0	0	0	0	0
2,10	6.01	34.52	2	0	0	0	0	0	0	0	0	0	0	0	0
3,3	17.40	38.79	5	0	0	0	10	0	0	0	3	0	0	0	0
3,4	14.24	28.99	1	0	0	0	9	0	0	0	4	0	0	0	0
3,6	12.65	27.68	5	2	0	0	12	1	0	0	2	0	0	0	2
3,10	10.06	15.60	0	0	0	0	21	5	0	0	8	0	0	0	2
6,3	19.39	24.97	8	3	4	0	31	3	0	0	9	0	0	0	3
6,4	15.35	19.94	6	0	1	0	26	4	1	1	4	0	0	0	3
6,6	8.93	15.41	6	4	0	0	16	13	0	0	7	0	0	0	9
6,10	8.40	7.49	4	2	0	0	21	9	0	0	4	0	0	0	7
12,3	14.45	19.60	12	5	13	1	45	5	6	0	9	0	2	0	0
12,4	13.81	16.19	8	4	6	2	64	18	9	2	9	0	1	0	3
12,6	15.22	17.62	10	0	1	0	46	13	0	0	5	0	0	0	0
12,10	7.46	5.19	5	1	0	0	144	20	0	0	1	0	0	0	1
Ave=	**13.08**	**26.48**	**6**	**1**	**2**	**0**	**28**	**6**	**1**	**0**	**4**	**0**	**0**	**0**	**2**

As the results in Tables 1-3 indicate, the cuts can close on average more than 25% of the initial gap. As one could naturally expect, the average gap closed by the cuts deteriorates when either the number of items or the number of periods is increased, where this deterioration seems more sensitive to the increase in the numbers of items than to the increase in the numbers of periods. When the number of items increases, the problem resembles more the structure of an uncapacitated problem, the convex hull of which can be effectively described by the (ℓ, S) inequalities and hence there is little room for improvement by other cuts. This can be indeed consistently observed from the average initial gaps for all the problems with 10 items. On the other hand, as the number of periods increases, the problem becomes further away from the "ideal" two-period problem, for which these cuts are originally derived. However, we note that when all instances with 10 items are taken out, even the average gap closed for the instances with 12 periods is 22.5%, which is a substantial improvement and also only slightly lower than 23.8%, the average gap closed for the instances with 6 periods and 3/4/6 items. In order to test this claim, we have also experimented with a limited number of instances with 24 periods (due to their extensive computational times). These preliminary tests indicated an almost identical gap improvement compared to instances with 12 periods.

Table 2: Average closed gaps and number of cuts generated of each type for test problems with $d_t^i \in [10, 40]$.

NT,			**# Cuts generated**												
NI	**IG**	**GC**	**C**	**IC**	**PC**	**IPC**	**R**	**IR**	**PR**	**IPR**	**R2**	**IR2**	**PR2**	**IPR2**	**FC**
2,3	11.15	59.49	5	0	0	0	0	0	0	0	0	0	0	0	0
2,4	16.63	42.27	7	2	0	0	1	0	0	0	1	0	0	0	1
2,6	10.19	46.62	7	0	0	0	0	0	0	0	0	0	0	0	0
2,10	5.88	10.97	3	0	0	0	0	0	0	0	0	0	0	0	1
3,3	20.20	40.74	7	0	0	0	10	0	0	0	5	0	0	0	0
3,4	10.15	30.43	4	0	0	0	12	0	0	0	4	0	0	0	0
3,6	10.67	25.30	7	1	0	0	14	4	0	0	2	0	0	0	5
3,10	4.79	29.51	5	0	0	0	41	11	0	0	11	1	0	0	0
6,3	11.23	28.40	12	2	3	0	22	3	1	0	7	0	0	0	1
6,4	12.28	20.90	7	2	4	0	31	7	0	0	4	0	0	0	1
6,6	9.09	17.79	3	0	2	0	36	18	1	1	2	0	0	0	0
6,10	7.06	17.02	7	1	0	0	50	12	3	0	7	0	3	0	5
12,3	10.52	36.73	25	1	20	3	71	5	18	1	18	0	3	0	4
12,4	13.82	25.20	16	1	7	0	46	8	7	0	6	0	0	0	1
12,6	6.30	17.65	10	0	7	0	77	12	10	3	9	0	0	0	2
12,10	5.21	3.32	4	0	0	1	366	22	2	1	9	0	0	0	4
Ave=	**10.32**	**28.27**	**8**	**1**	**3**	**0**	**49**	**6**	**3**	**0**	**5**	**0**	**0**	**0**	**2**

The results also indicate which types of inequalities are more inherent for different sizes of problems. A small number of cover inequalities seem to close substantial gaps for the 2-period problems almost only on their own, and the number of these inequalities do not vary much as the problem size gets bigger. On the other hand, the number of reverse cover inequalities generated increases significantly as the number of items and periods increase, making them most often generated inequalities in our framework, hence also pointing to where an computationally efficient framework can focus on. Finally, reverse covers of type 2 seem to be least violated, which is not surprising considering the fact that they are not facet-defining for the single-period relaxation PIR_1. Another interesting aspect the results point at is the fact that the number of item- and period-extended versions of the cuts remain small compared to the “simple” versions of these cuts; note that the **IPR2** cuts were even never violated for any of these 240 instances. Finally, we note that we generated the flow cover inequalities after our proposed cuts and the improvement made by these cuts is negligible, often simply zero.

Table 3: Average closed gaps and number of cuts generated of each type for test problems with $d_t^i \in [10, 60]$.

NT,			**# Cuts generated**												
NI	**IG**	**GC**	**C**	**IC**	**PC**	**IPC**	**R**	**IR**	**PR**	**IPR**	**R2**	**IR2**	**PR2**	**IPR2**	**FC**
2,3	14.61	44.77	5	0	0	0	0	0	0	0	0	0	0	0	0
2,4	16.73	37.21	7	2	0	0	2	0	0	0	2	0	0	0	1
2,6	9.42	33.50	8	0	0	0	0	0	0	0	0	0	0	0	0
2,10	7.01	33.81	3	0	0	0	0	0	0	0	0	0	0	0	0
3,3	20.26	41.64	8	0	0	0	13	0	0	0	6	0	0	0	3
3,4	8.80	38.19	6	0	0	0	10	3	0	0	4	0	0	0	2
3,6	10.92	31.77	6	0	0	0	19	5	0	0	4	0	0	0	0
3,10	6.82	27.63	10	0	0	0	69	6	0	0	19	5	0	0	2
6,3	11.98	33.10	15	1	7	0	27	3	6	0	5	0	2	0	2
6,4	7.79	30.42	15	1	2	0	38	3	2	0	2	0	1	0	1
6,6	6.16	23.55	4	0	2	0	32	19	6	4	8	0	2	0	0
6,10	2.03	15.56	1	0	1	0	55	28	0	0	10	2	1	0	2
12,3	11.20	35.19	19	1	22	4	78	6	9	0	17	0	2	0	1
12,4	12.43	19.87	15	3	11	1	71	7	10	0	8	0	1	0	0
12,6	8.69	14.13	9	0	4	1	60	21	9	4	2	0	0	0	0
12,10	5.99	7.18	9	0	1	0	614	16	2	0	6	0	0	0	0
Ave=	**10.03**	**29.22**	**9**	**1**	**3**	**0**	**68**	**7**	**3**	**1**	**6**	**0**	**1**	**0**	**1**

Finally, we make a remark on the effect of the proposed cuts when the demand variability changes. As the tables clearly indicate, the cuts (in particular reverse cover inequalities and its variants) are more often violated when the demand variability increases: these are also the instances when our cuts make more of an impact for the amount of the gap closed. This makes intuitive sense that reverse covers are more flexible than covers and hence a higher demand variability will be able to generate more violated inequalities of this type.

7. Conclusions

In this paper, we investigated a two-period subproblem of the big bucket lot-sizing problem from a theoretical perspective. In particular, we have identified various families of valid inequalities for a relaxation of this subproblem in the special case of zero setup times, described their facet-defining properties, and we have also extended these inequalities to the original space of the two-period subproblem. The computational results indicated significant potential for improving lower bounds, and we are currently investigating this thoroughly in a companion study in two immediate directions: i) understanding polyhedral characteristics of the general case with non-zero setup times and identifying further valid inequalities, and ii) designing a branch-and-cut framework with routines generating cutting planes of both zero and non-zero setup time settings in realistic times for multi-item lot-sizing problems.

The theoretical results we presented in this paper can be extended to other MIP problems thanks to the commonality of the mixed integer sets inherent in different problems. We have already noted various studies on the polyhedron of the PIR_0, the single node fixed charge set, which is a common mixed integer set in various MIP problems. On the other hand, the structure of PIR_1 poses different challenges and opportunities, and it is worth investigating further its link to other mixed integer sets. Finally, there is also immediate interest in investigating if and how our understanding of the two-period subproblems can be further extended to more sophisticated subproblems, e.g., a k-period subproblem. As [31] observed in their framework, even limiting it to the values of $k = 3$ and $k = 4$, there is substantial potential to develop a thorough understanding of the complex lot-sizing problems, which we plan to study in the future.

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