A polyhedral study of binary polynomial programs

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Abstract

We study the polyhedral convex hull of a mixed-integer set $\mathcal{S}$ defined by a collection of multilinear equations of the form $y_I = \prod_{i \in I} x_i$ over the $0-1$-cube. Such sets appear frequently in the factorable reformulation of mixed-integer nonlinear optimization problems. In particular, the set $\mathcal{S}$ represents the feasible region of a linearized unconstrained binary polynomial optimization problem. We define an equivalent hypergraph representation of the mixed-integer set $\mathcal{S}$, which enables us to derive several families of facet-defining inequalities, structural properties, and lifting operations for its convex hull in the space of the original variables. Our theoretical developments extend several well-known results from the Boolean quadric polytope and the cut polytope literature, paving a way for devising novel optimization algorithms for nonconvex problems containing multilinear sub-expressions.

Key words: binary polynomial optimization; polyhedral relaxations; multilinear functions; cutting planes; lifting

1 Introduction

We consider a box-constrained mixed-integer multilinear optimization problem of the form

$$\max \left\{ \sum_{I \in \mathcal{I}} c_I \prod_{i \in I} x_i : \ x_i \in [0,1] \ \forall i \in J_1, \ x_i \in \{0,1\} \ \forall i \in J_2 \right\}, \tag{ML}$$

where $\mathcal{I}$ is a family of subsets of $\{1,\dots,n\}$, and c_I, $I \in \mathcal{I}$ are nonzero real-valued coefficients. Moreover, the index sets J_1 and J_2 form a partition of $\{1,\dots,n\}$. We refer to $r = \max\{|I| : I \in \mathcal{I}\}$ as the degree of Problem (ML).

Problem (ML) subsumes several well-known $\mathcal{N}P$-hard optimization problems. For instance, by letting $J_1 = \emptyset$, (ML) reduces to Pseudo-Boolean optimization (c.f. [13] for an extensive literature review). In addition, since $x_i^p = x_i$, for $p \in \mathbb{Z}_+$ and $x_i \in \{0,1\}$, in this case,

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Problem (ML) is equivalent to unconstrained binary polynomial optimization. In particular, if $r = 2$, then we obtain the well-studied binary quadratic optimization or the max-cut problem (c.f. [5, 21, 6, 7, 11]). More generally, it is well-known that any real-valued function in binary variables can be rewritten as a multilinear function in the same variables. Thus, Problem (ML) subsumes any unconstrained binary nonlinear optimization problem [22]. At the other end of the spectrum, by letting $J_2 = \emptyset$ and noting that multilinear functions are closed under scaling and shifting of variables, Problem (ML) is equivalent to maximizing a multilinear function over a box. The latter problem has been studied extensively by the global optimization community (c.f. [1, 32, 34, 33, 28, 37, 27, 4, 15]).

A standard technique to tackle Problem (ML) is to first introduce a new variable y_I for every product term $\prod_{i \in I} x_i$ with $|I| \geq 2$ and obtain an equivalent optimization problem in the lifted space (x, y):

$$\begin{aligned} \max \quad & \sum_{I \in \mathcal{I}, |I|=1} c_I x_I + \sum_{I \in \mathcal{I}, |I|>1} c_I y_I \\ \text{s.t.} \quad & y_I = \prod_{i \in I} x_i \qquad \forall I \in \mathcal{I},\ |I| \geq 2 \\ & x_i \in [0, 1] \qquad \forall i \in J_1 \\ & x_i \in \{0, 1\} \qquad \forall i \in J_2. \end{aligned} \tag{EML}$$

Subsequently, the feasible region of Problem (EML) is replaced by a convex relaxation and the resulting problem is solved to obtain an upper bound on the optimal value of Problem (ML). A widely-used method to convexify the above problem is to relax the nonconvex region defined by each term $y_I = \prod_{i \in I} x_i$ over the unit hypercube by its convex hull [20]. Crama [16] derives conditions under which the upper bound given by this so-called standard linearization is equal to the optimal value of the original problem (EML). However, in general, the standard linearization can lead to very weak bounds [27, 4].

A key observation toward constructing a sharper relaxation for Problem (EML) is that for any vector $c \in \mathbb{R}^{\mathcal{I}}$, there exists an optimal solution that is attained at a vertex of the unit hypercube (see, e.g., [38]). It then follows that the convex hull of the feasible region of (EML) is a polytope and the projection of its vertices onto the space of x variables is given by $\{0, 1\}^n$ (c.f. [36]). Consequently, the objective function of Problem (EML) can be equivalently optimized over the following binary set

$$\mathcal{S} = \Big\{(x, y) : y_I = \prod_{i \in I} x_i, I \in \mathcal{I},\ |I| \geq 2,\ x \in \{0, 1\}^n\Big\}. \tag{1}$$

Throughout this paper, we refer to the set $\mathcal{S}$ as the *Multilinear set.*

The general convexification techniques developed over the past few decades by Sherali and Adams [35], Lovàsz and Schrijver [26], Balas, Ceria and Cornuéjols [3], Parrilo [31], and Lasserre [24] provide automated mechanisms for the generation of sharp relaxations for mixed-integer polynomial optimization problems in an extended space. The general idea is to construct hierarchies of successive polyhedral or semidefinite relaxations, whose projection onto the space of original variables converges to the convex hull of the feasible set. For a nonconvex set with a polyhedral convex hull such as the Multilinear set, these techniques result in an exact description of the convex hull after a finite number of steps.

For quadratic sets, i.e., $r = 2$, the convex hull of $\mathcal{S}$ is the *Boolean quadric polytope*, defined by Padberg. In [29], Padberg studies various structural properties of the Boolean quadric polytope and derives several families of facet-defining inequalities as well as lifting operations for this polytope. Moreover, a significant amount of research has been devoted to studying the facial structure of the cut polytope [8, 19, 12]. It is well-known that every Boolean quadric polytope is the image of a cut polytope under a bijective linear transformation [17]. These theoretical developments have had a significant impact on the performance of branch-and-cut based algorithms for mixed-integer quadratic optimization problems [18, 40, 23, 25].

However, for a Multilinear set $\mathcal{S}$ with $r > 2$, similar polyhedral studies are rather scarce. In the special case where $r = n$ and the set $\mathcal{I}$ contains all subsets of $\{1, \ldots, n\}$, a complete linear description of the convex hull of the Multilinear set has been derived by several authors independently (cf. [39, 35, 30]). In [39], Ursic considers the Multilinear set with $\mathcal{I}$ containing all subsets of $\{1, \ldots, n\}$ of cardinality between 2 and r for some $r \geq 2$. The author refers to the corresponding polyhedral convex hull as the binomial polytope, studies some of its fundamental properties and identifies several families of facets of this polytope. In [14], the authors propose a reduction scheme to reformulate a binary polynomial optimization problem to a quadratic one in a higher dimensional space in order to make use of the existing separation algorithms for the Boolean quadric polytope and the cut polytope. The proposed approach is most effective when the original instance is reducible; that is, every set in $\mathcal{I}$ containing more than one element is a union of two other sets in $\mathcal{I}$. Otherwise, all such variables are added to the model to make the Multilinear set reducible. In [10], the authors review some quadratization techniques for higher degree multilinear optimization problems and demonstrate their usefulness in some computer vision applications.

Our work is mainly inspired by Padberg's results on the Boolean quadric polytope. We consider the Multilinear set $\mathcal{S}$ defined by (1) with the degree r greater than two, and refer to its polyhedral convex hull as the *Multilinear polytope* (MP). We study the facial structure of the Multilinear polytope in the space of the original variables (x, y). In contrast to all earlier studies detailed above [39, 30], we fully recognize the sparsity in the problem structure; that is, we do not make any assumptions on the structure of the set $\mathcal{I}$. To this end, we define an equivalent hypergraph representation for Multilinear set. Recall that a *hypergraph* G is a pair (V, E) where $V = V(G)$ is the set of nodes of G, and $E = E(G)$ is a set of subsets of V of cardinality at least two, called the edges of G (see Berge [9] for an introduction to hypergraphs). Throughout this paper, we consider hypergraphs without loops (edges containing a single node) and parallel edges (multiple edges containing the same set of nodes). With any hypergraph G, we associate a the Multilinear set S_G defined as follows:

$$\mathcal{S}_G = \Big\{ z \in \{0, 1\}^d : z_e = \prod_{v \in e} z_{\{v\}}, \ e \in E(G) \Big\}, \tag{2}$$

where $d = |V(G)| + |E(G)|$. We denote by MP_G the polyhedral convex hull of $\mathcal{S}_G$. Note that the variables $z_{\{v\}}$, $v \in V(G)$ in (2) correspond to the variables x_i, $i = 1, \ldots, n$ in (1) and the variables z_e, $e \in E(G)$ in (2) correspond to the variables y_I, $I \in \mathcal{I}$ in (1).

Lifting is a widely-used methodology to construct valid inequalities for high-dimensional sets starting from inequalities valid for simpler subsets of the original set. More formally, in our context, consider two Multilinear sets $\mathcal{S}_G$ and $\mathcal{S}_{G'}$ as defined by (2), and suppose that $\mathcal{S}_{G'}$ is obtained from $\mathcal{S}_G$ by letting $z_{\{v\}} = 0$ or $z_{\{v\}} = 1$ for some $v \in V(G)$ and/or by projecting out some variables z_e, $e \in E(G)$. Denote by $az \leq \alpha$ a valid inequality for $\text{MP}_{G'}$. Lifting $az \leq \alpha$ means finding a pair $(\bar{a}, \bar{\alpha})$ such that $\bar{a}z \leq \bar{\alpha}$ is a valid inequality for the Multilinear polytope MP_G, where $\bar{a}$ is obtained by adding new coordinates to a, after possibly changing some of its coefficients. Given a facet-defining inequality for $\text{MP}_{G'}$, it is often desirable to generate a facet of MP_G via lifting.

In this paper, we develop the theory of various lifting operations for the Multilinear polytope. First, we consider the so called *zero-lifting* operation, whereby we let $\bar{a} = (a, 0)$ and $\bar{\alpha} = \alpha$. As we will show later, in this case, without loss of generality, we can assume that the set $\mathcal{S}_{G'}$ is obtained by letting $z_{\{v\}} = 0$ for all $v \in V(G) \setminus V(G')$. Subsequently, we characterize cases for which the zero-lifting of a facet-defining inequality for $\text{MP}_{G'}$ defines a facet of MP_G. Our results generalize Padberg's zero-lifting theorems for the Boolean quadric polytope to sets containing high-degree multilinears. In principle, one could start from multiple inequalities that are facet-defining for distinct low-dimensional sets and lift them simultaneously to obtain a facet of the Multilinear polytope. For instance, given a hypergraph G, consider a partition of its nodes defined as $V(G) = V_1 \cup V_2$. For $i = 1, 2$, let G_i denote a hypergraph containing edges of the form $e \cap V_i$ for all $e \in E(G)$ such that $e \setminus V_i \neq \emptyset$. Starting from two inequalities that induce facets of MP_{G_i}, for $i = 1, 2$, under certain assumptions, we obtain a valid inequality for MP_G by multiplying these facet-defining inequalities and linearizing the resulting relation. Subsequently, we derive conditions under which the new inequality defines a facet of MP_G. In [39], the author defines a similar lifting operation for hypergraphs containing all edges up to a certain cardinality. We then study a different lifting operation, for which the Multilinear set $\mathcal{S}_{G'}$ is generated by fixing certain variables in $\mathcal{S}_G$ to one. As we detail later, the hypergraph G' is obtained by "removing" some nodes from the hypergraph G. Together with the known families of facet-defining inequalities for the Boolean quadric polytope, the proposed lifting operations enable us to construct many types of facet-defining inequalities for sets containing higher degree multilinears. These cutting planes can be embedded in general-purpose global solvers to enhance the quality of existing relaxations for nonconvex problems containing multilinear sub-expressions.

The structure of the paper is as follows. In Section 2, we establish a number of fundamental properties of the Multilinear polytope, which will be used in the rest of the paper. We develop the theory of zero-lifting for the Multilinear polytope in Section 3 and investigate a number of special cases for which the general assumptions are either satisfied or can be significantly simplified. In Section 4, we introduce a facet generation framework, in which certain facets of the Multilinear polytope can be obtained by multiplying and linearizing facet-defining inequalities of simpler Multilinear polytopes. Lifting via node addition is the subject of Section 5. Finally, conclusions are offered in Section 6.

2 Basic properties of the Multilinear polytope

In this section, we establish a number of fundamental properties of the Multilinear polytope that are essential for the subsequent developments. We start by introducing some graph-theoretic terminology which will be used throughout the paper. Let $G = (V, E)$ be a hypergraph. The *rank* of G is the maximum cardinality of an edge in E. An important special case is when E consists of all subsets of V of cardinality between 2 and r, for some $r \geq 2$. We refer to such a hypergraph as a *rank-r full hypergraph*, and we denote it by $K^{n,r}$, where $n = |V(K^{n,r})|$. Moreover, in this case, we denote the associated Multilinear set (2) and its convex hull by $\mathcal{S}^{n,r}$ and $\mathrm{MP}^{n,r}$, respectively. In particular, the set $\mathrm{MP}^{n,2}$ represents the well-studied Boolean quadric polytope on complete graphs [29]. A rank-n full hypergraph with n nodes is said to be *complete*.

A hypergraph $G' = (V', E')$ is a *partial hypergraph* of G, if $V' \subseteq V$ and $E' \subseteq E$. Given a subset V' of V, the *section hypergraph* of G induced by V' is the partial hypergraph $G' = (V', E')$, where $E' = \{e \in E : e \subseteq V'\}$. A subset $V' \subseteq V$ is called *inducing*, if for every $e \in E$ with $|e \cap V'| \geq 2$, we have $e \cap V' \in E$. The *support hypergraph* of a valid inequality $az \leq \alpha$ for MP_G, is the hypergraph $G(a)$, where $V(G(a)) = \{v \in V : a_{\{v\}} \neq 0\} \cup \{v \in V : \exists e \in E \text{ s.t. } v \in e, a_e \neq 0\}$, and $E(G(a)) = \{e \in E : a_e \neq 0\}$. For notational simplicity, we define $L(G) = \{\{v\} : v \in V(G)\}$; furthermore, for any vector z having a component $z_{\{v\}}$ corresponding to a node v, we write z_v instead of $z_{\{v\}}$. Finally, given $U \subseteq V$, throughout the paper, w^U denotes the $(|V| + |E|)$-vector having entries one corresponding to nodes in U and edges $e \in E$ such that $e \subseteq U$, and the remaining entries equal to zero.

We begin by determining the dimension of the Multilinear polytope.

Proposition 1. *The Multilinear polytope MP_G is full-dimensional for every hypergraph G, i.e.,* $\dim(MP_G) = |V(G)| + |E(G)|$.

Proof. The set of $|V(G)| + |E(G)| + 1$ vectors w^U, for every $U \in \{\emptyset\} \cup L(G) \cup E(G)$ are affinely independent vectors in MP_G. □

In particular, Proposition 1 implies that $\dim(\mathrm{MP}^{n,r}) = \sum_{i=1}^{r} \binom{n}{i}$. Clearly, any inequality of the form $z_p \geq 0$, with $p \in L(G) \cup E(G)$, is valid for MP_G. The next proposition provides the condition under which $z_p \geq 0$ defines a facet of MP_G.

Proposition 2. *Let G be a hypergraph and let $p \in L(G) \cup E(G)$. Then the inequality $z_p \geq 0$ is facet-defining for MP_G if and only if there exists no $e \in E(G)$ such that $e \supset p$.*

Proof. Suppose that there exists no edge $e \in E(G)$ such that $e \supset p$. Then the vectors w^U, for $U \in \{\emptyset\} \cup L(G) \cup E(G) \setminus \{p\}$, are $|V(G)| + |E(G)|$ affinely independent vectors in $\mathcal{S}_G$ that satisfy $z_p = 0$. Thus, $z_p \geq 0$ defines a facet of MP_G. Conversely, suppose that there exists an edge $e \in E(G)$ such that $e \supset p$. Then the two inequalities $z_e \leq z_p$ and $z_e \geq 0$ are valid for MP_G and together imply $z_p \geq 0$, contradicting the assumption that $z_p \geq 0$ is facet-defining. □

The above result implies that $z_v \geq 0$, for some $v \in V(G)$, defines a facet of MP_G if and only if v is an isolated node. If the hypergraph G of Proposition 2 is a rank-r full hypergraph, we have the following characterization:

Proposition 3. *The inequality $z_e \geq 0$, with $e \in E(K^{n,r})$, defines a facet of $MP^{n,r}$ if and only if $|e| = r$.*

Let G' be a partial hypergraph of a hypergraph G. It is simple to verify that the projection of every vertex of MP_G onto the space of $\mathrm{MP}_{G'}$ is a vertex of $\mathrm{MP}_{G'}$, and each vertex of $\mathrm{MP}_{G'}$ can be obtained in this way. The following propositions explain the relationship between the Multilinear polytopes MP_G and $\mathrm{MP}_{G'}$ and are consequences of this basic fact.

Proposition 4. *Let G' be a partial hypergraph of a hypergraph G. Then $MP_{G'}$ is obtained from MP_G by projecting out the variables z_v with $v \notin V(G')$, and z_e with $e \notin E(G')$.*

Given a valid inequality $az \leq \alpha$ for MP_G, its *restriction* $\tilde{a}z \leq \alpha$ to $\mathrm{MP}_{G'}$ is obtained by discarding from a all components a_v, with $v \in V(G) \setminus V(G')$ and a_e, with $e \in E(G) \setminus E(G')$. If G' is a section hypergraph of G, the projection in Proposition 4 can be done in a trivial manner.

Proposition 5. *Let G' be a section hypergraph of a hypergraph G, and let $MP_G = \{z : a^j z \leq \alpha_j,\ j \in J\}$. Then $MP_{G'} = \{z : \tilde{a}^j z \leq \alpha_j,\ j \in J\}$.*

Proof. The polytope $\mathrm{MP}_{G'}$ can be obtained from the face of MP_G defined by $z_v = 0$, for $v \in V(G) \setminus V(G')$, and $z_e = 0$, for $e \in E(G) \setminus E(G')$, by projecting out variables z_v, for $v \in V(G) \setminus V(G')$ and variables z_e, for $e \in E(G) \setminus E(G')$. □

By Proposition 5, if G' is a section hypergraph of G, then the restriction to $\mathrm{MP}_{G'}$ of a valid inequality for MP_G is valid for $\mathrm{MP}_{G'}$. Note that if G' contains the support hypergraph of $az \leq \alpha$ as a partial hypergraph, then the restriction $\tilde{a}z \leq \alpha$ of $az \leq \alpha$ is obtained by discarding only zero components from a, therefore the two inequalities are identical.

Proposition 6. *Let $az \leq \alpha$ be a valid inequality for MP_G, and let G' be a partial hypergraph of G containing $G(a)$ as a partial hypergraph. Then the restriction $\tilde{a}z \leq \alpha$ of $az \leq \alpha$ to $MP_{G'}$ is valid for $MP_{G'}$. Moreover, if $az \leq \alpha$ is facet-defining for MP_G, then $\tilde{a}z \leq \alpha$ is facet-defining for $MP_{G'}$.*

Proof. Let $\tilde{z}$ be a vertex of $\mathrm{MP}_{G'}$, and let $\bar{z}$ be a vertex of MP_G whose projection onto the space of $\mathrm{MP}_{G'}$ is $\tilde{z}$. Then the validity of $\tilde{a}\tilde{z} \leq \alpha$ follows from the validity of $a\bar{z} \leq \alpha$.

Assume now that $az \leq \alpha$ is facet-defining for MP_G. Then there exists a set Z of $|V(G)| + |E(G)|$ affinely independent vertices of MP_G that satisfy $az = \alpha$. Let $\tilde{Z}$ be the projection of Z onto the space of $\mathrm{MP}_{G'}$. The points in $\tilde{Z}$ are vertices of $\mathrm{MP}_{G'}$, and they all satisfy $\tilde{a}z = \alpha$. Moreover they contain $|V(G')| + |E(G')|$ affinely independent vectors. □

Next, we present a *switching operation* for the Multilinear polytope $\mathrm{MP}^{n,r}$ that enables us to convert valid linear inequalities into other valid linear inequalities that induce faces of the same dimension. A similar operator has been introduced by several authors independently for Boolean quadric and cut polytopes (c.f. [29, 8]) and for $\mathrm{MP}^{n,r}$ [39].

Consider the hypergraph $K^{n,r}$, and let $d = \sum_{i=1}^{r} \binom{n}{i}$. For any $U \subseteq V(K^{n,r})$, consider the mapping $\psi_U : \mathbb{R}^d \to \mathbb{R}^d$ given by

$$\psi_U(z_v) = \begin{cases} 1 - z_v & \text{if } v \in U \\ z_v & \text{if } v \in V(K^{n,r}) \setminus U \end{cases} \tag{3}$$

$$\psi_U(z_e) = \sum_{\substack{W \subseteq e \cap U \\ |W| \text{ even}}} z_{(e \setminus U) \cup W} - \sum_{\substack{W \subseteq e \cap U \\ |W| \text{ odd}}} z_{(e \setminus U) \cup W} \qquad e \in E(K^{n,r}), \tag{4}$$

where we define $z_\emptyset = 1$.

By definition, $z_e = \prod_{v \in e} z_v$ for every $z \in \{0,1\}^d$ and $e \in E(K^{n,r})$. It follows that

$$\psi_U(z_e) = \prod_{v \in e \cap U} (1 - z_v) \prod_{v \in e \setminus U} z_v \qquad \forall z \in \{0,1\}^d,\ e \in E(K^{n,r}). \tag{5}$$

It is simple to verify that ψ_U is a non-singular affine transformation as it can be written as $\psi_U(z) = Az + b$, where $A \in \mathbb{R}^{d \times d}$ is a lower triangular matrix whose diagonal entries are either 1 or -1. Moreover, for any $W \subseteq V(K^{n,r})$, we have $\psi_U(w^W) = w^{W \Delta U}$, where $W \Delta U = (W \setminus U) \cup (U \setminus W)$ denotes the symmetric difference of W and U. In particular, ψ_U maps $\text{MP}^{n,r}$ onto itself. It follows that the image of a facet-defining inequality for $\text{MP}^{n,r}$ under ψ_U is also facet-defining for $\text{MP}^{n,r}$.

Consequently, by Proposition 3, the following inequalities define facets of $\text{MP}^{n,r}$:

$$\psi_U(z_e) \geq 0 \qquad \forall e \in E(K^{n,r}) \text{ with } |e| = r,\ \forall U \subseteq V(K^{n,r}). \tag{6}$$

The mapping ψ_U can also be defined for more general hypergraphs. For any graph G, and $U \subseteq V(G)$, ψ_U is always a non-singular affine transformation. However, for general hypergraphs this is not always the case because $(e \setminus U) \cup W$ might not be an edge of G for some $U \subseteq V(G)$ and $W \subseteq U$. Thus, we cannot directly utilize the switching operator to obtain new facets of MP_G from the existing ones. In [14], the authors characterize the hypergraphs G for which ψ_U is a non-singular affine transformation for every $U \subseteq V(G)$ (see Theorem 4.3 in [14]).

The next theorem follows by a result proven by Sherali and Adams [35] (see also [39, 30]).

Theorem 1 (Theorem 2 in [35]). *The Multilinear polytope $MP^{n,r}$ with $n = r$ is given by facet-defining inequalities* (6).

In practice, however, we often have $n \gg r$ for which $\text{MP}^{n,r}$ has a far more complex structure. The following result follows directly from Theorem 1 and Proposition 6.

Corollary 1. *Let $az \leq \alpha$ define a facet of $MP^{n,r}$ and assume that its support hypergraph has r nodes. Then, $az \leq \alpha$ is of the form* (6).

Note that by Theorem 1, the support hypergraph of any facet-defining inequality for $\text{MP}^{n,r}$ has at least r nodes.

We conclude this section by presenting a technical lemma that will be used to prove our main lifting theorems in Sections 3 and 4. Let G be a hypergraph and let G' be a partial hypergraph of G. Denote by $az \leq \alpha$ a facet-defining inequality for $\text{MP}_{G'}$ and let $bz \leq \beta$ denote a valid inequality for MP_G. Suppose that for any point in $\mathcal{S}_G$ whose restriction to $\mathcal{S}_{G'}$ satisfies $az = \alpha$, we have $bz = \beta$. The following lemma establishes the relationship between the coefficients of the two inequalities.

Lemma 1 (Proportionality Lemma). *Let $G = (V, E)$ be a hypergraph and let $G' = (V', E')$ be a partial hypergraph of G with the following properties*

- *V' is an inducing subset of V,*

- *any edge of the form $e \cap V'$ for some $e \in E$ with $|e \cap V'| \geq 2$ and $e \setminus V' \neq \emptyset$ is present in E'.*

Let $az \leq \alpha$ denote a facet-defining inequality for $MP_{G'}$ and let $Q_a = \{q \in \{\emptyset\} \cup L' \cup E' : a_q \neq 0\}$, where we define $L' = L(G')$ and $a_\emptyset = -\alpha$. Let $bz \leq \beta$ be a valid inequality for MP_G that is satisfied tightly by any point whose restriction to $\mathcal{S}_{G'}$ satisfies $az = \alpha$. Then:

1. *Let U be a nonempty subset of $V \setminus V'$, and $P_U = \{p \in L(G) \cup E(G) : p \setminus V' = U\}$. Then, the following cases arise:*

   (i) if $\{p \setminus U : p \in P_U\} \supseteq Q_a$, then there exists $\lambda_U \in \mathbb{R}$ such that $b_p = \lambda_U a_{p \setminus U}$ for all $p \in P_U$ with $p \setminus U \in Q_a$ and $b_p = 0$ for all $p \in P_U$ with $p \setminus U \notin Q_a$.

   (ii) otherwise, $b_p = 0$ for all $p \in P_U$.

2. *If in addition, we have $b_e = 0$ for all $e \in E \setminus E'$ with $e \subseteq V'$, then $b_p = \lambda a_p$ for all $p \in \{\emptyset\} \cup L' \cup E'$ for some $\lambda \geq 0$, where we define $b_\emptyset = -\beta$.*

Proof. We start by proving part 1. Define $V'' = V \setminus V'$. Let $\tilde{z}^i$, $i = 1, \ldots, k$, denote all points in $\mathcal{S}_{G'}$ satisfying $az = \alpha$. We lift these points to a set of points in $\mathcal{S}_G$ by letting $z_v = 0$ for all $v \in V''$ and computing z_e, $e \in E \setminus E'$ accordingly. Substituting the lifted points in $bz = \beta$, yields

$$\sum_{p \in L' \cup E'} b_p \tilde{z}^i_p + \sum_{e \in \bar{E}'} b_e \prod_{v \in e} \tilde{z}^i_v = \beta \qquad \forall i = 1, \ldots, k, \tag{7}$$

where $\bar{E}' = \{e \in E \setminus E' : e \subseteq V'\}$. Next, for every nonempty $U \subseteq V''$, we lift $\tilde{z}^i$, $i = 1, \ldots, k$, to a set of points in $\mathcal{S}_G$ by setting $z_v = 1$ for all $v \in U$, $z_v = 0$ for all $v \in V'' \setminus U$ and computing the variables z_e, $e \in E \setminus E'$ accordingly. Define $\tilde{P}_U$ as the (disjoint) union of sets P_W, for $W \subseteq U$, $W \neq \emptyset$; i.e., $\tilde{P}_U = \{p \in L(G) \cup E(G) : p \cap V'' \subseteq U,\ p \cap V'' \neq \emptyset\}$, where the set P_W is defined in the statement of the theorem. Substituting these points in $bz = \beta$, yields:

$$\sum_{p \in L' \cup E'} b_p \tilde{z}^i_p + \sum_{e \in \bar{E}'} b_e \prod_{v \in e} \tilde{z}^i_v + \sum_{p \in \tilde{P}_U} b_p \tilde{z}^i_{p \setminus U} = \beta \qquad \forall i = 1, \ldots, k, \tag{8}$$

where we define $\tilde{z}^i_\emptyset = 1$. Note that by the two properties of the hypergraph G' given in the statement, for each $p \in \tilde{P}_U$, we have $p \setminus U \in \{\emptyset\} \cup L' \cup E'$. From (7) and (8), it follows that

$$\sum_{p \in \tilde{P}_U} b_p \tilde{z}^i_{p \setminus U} = 0 \qquad \forall i = 1, \ldots, k. \tag{9}$$

We now prove that the following is valid for all nonempty $U \subseteq V''$.

$$\sum_{p \in P_U} b_p \tilde{z}^i_{p \setminus U} = 0 \qquad \forall i = 1, \ldots, k. \tag{10}$$

We show it by induction on $|U|$, the base case being $|U| = 1$; i.e., $U = \{u\}$ for some $u \in V''$. In this case, (9) simplifies to (10) since we have $\tilde{P}_U = P_U$. Next, we proceed to the inductive step. Namely, we show that if (10) holds for all $U \subseteq V''$ with cardinality between 1 and δ, then the same condition is valid for all U with cardinality $\delta+1$. Consider (9) for a subset U of cardinality $\delta+1$. We have

$$\sum_{p \in \tilde{P}_U} b_p \tilde{z}^i_{p \setminus U} = \sum_{p \in P_U} b_p \tilde{z}^i_{p \setminus U} + \sum_{U' \subset U} \sum_{p \in P_{U'}} b_p \tilde{z}^i_{p \setminus U'} \qquad \forall i = 1, \dots, k.$$

By induction we have $\sum_{p \in P_{U'}} b_p \tilde{z}^i_{p \setminus U'} = 0$ for all $U' \subset U$. Thus the above system simplifies to (10). Therefore, relation (10) is valid for all nonempty $U \subseteq V''$.

Recall that $\tilde{z}^i$, $i = 1, \dots, k$, denote all the points in $\mathcal{S}_{G'}$ satisfying the facet-defining inequality $az \leq \alpha$ tightly and in addition these points satisfy $\sum_{p \in P_U} b_p z_{p \setminus U} = 0$. It follows that for a given $U \subseteq V''$, the equation $\sum_{p \in P_U} b_p z_{p \setminus U} = 0$ is a scaling of $az - \alpha = 0$. From the definition of P_U, it follows that for every $q \in Q_a$, there exists at most one $p \in P_U$ such that $q = p \setminus U$. Note that such a property does not hold for $\tilde{P}_U$, in general. Therefore, the following cases arise:

(i) if $\{p \setminus U : p \in P_U\} \supseteq Q_a$, then there exists $\lambda_U \in \mathbb{R}$ such that $b_p = \lambda_U a_{p \setminus U}$ for all $p \in P_U$ with $p \setminus U \in Q_a$ and $b_p = 0$ for all $p \in P_U$ with $p \setminus U \notin Q_a$.

(ii) otherwise, $b_p = 0$ for all $p \in P_U$.

We now proceed to part 2 of the lemma. Suppose that $b_e = 0$ for all $e \in \bar{E}'$. In this case (7) simplifies to

$$\sum_{p \in L' \cup E'} b_p \tilde{z}^i_p = \beta, \quad \forall i = 1, \dots, k.$$

Since $az \leq \alpha$ defines a facet of $\text{MP}_{G'}$ and $\tilde{z}^i$, $i = 1, \dots, k$ denote all points in $\mathcal{S}_{G'}$ satisfying this facet tightly, we conclude that $b_p = \lambda a_p$, for all $p \in L' \cup E'$ and $\beta = \lambda \alpha$ for some $\lambda \in \mathbb{R}$. Let $\tilde{G}$ denote the section hypergraph of G induced by V'. By Proposition 5, the restriction of $bz \leq \beta$ to $\text{MP}_{\tilde{G}}$ is a valid inequality for $\text{MP}_{\tilde{G}}$. Moreover $bz \leq \beta$ has zero coefficients corresponding to edges not in E'. Thus, by Proposition 6, the restriction of $bz \leq \beta$ to $\text{MP}_{G'}$ is a valid inequality for $\text{MP}_{G'}$. Hence $\lambda \geq 0$. □

We are often interested in cases for which the valid inequality $bz \leq \beta$ defines a facet of MP_G. To this end, we need to make additional assumptions on the structure of the hypergraphs G and G'. In the following two sections, we study two important instances for which the inequality $bz \leq \beta$ defines a facet of MP_G.

3 Zero-Lifting

In this section, we develop the zero-lifting theorem for the Multilinear polytope. As we will detail later, our result serves as the generalization of Padberg's zero-lifting theorem for the Boolean quadric polytope. Let G' be a partial hypergraph of a hypergraph G. If $az \leq \alpha$ is a valid inequality for $\text{MP}_{G'}$, by Proposition 4, we can obtain a valid inequality $\bar{a}z \leq \alpha$ for MP_G, by introducing zero coefficients for the additional variables as follows: $\bar{a}_p = a_p$ for all $p \in L(G') \cup E(G')$ and $\bar{a}_p = 0$, otherwise. We refer to $\bar{a}z \leq \alpha$ as the *zero-lifting* of $az \leq \alpha$ to MP_G. In the sequel, we say that an inequality $az \leq \alpha$ is *nontrivial*, if the vector a has at least one nonzero component.

Now suppose that $az \leq \alpha$ defines a facet of $\text{MP}_{G'}$. We are interested in characterizing cases for which the zero-lifting of $az \leq \alpha$ defines a facet of MP_G. Let $\bar{G}$ be the section hypergraph of G induced by $V(G')$. If the zero-lifting of $az \leq \alpha$ to $\text{MP}_{\bar{G}}$ does not define a facet of $\text{MP}_{\bar{G}}$, then, by Proposition 6, its zero-lifting to MP_G is not facet-defining for MP_G. Thus, in the following, without loss of generality, we assume that the partial hypergraph G' is a section hypergraph of G.

Given a hypergraph G and a monomial $a_{p_1,\ldots,p_t} \prod_{i=1}^t z_{p_i}^{n_i}$, for $p_1, \ldots, p_t \subseteq V(G)$, we define its *linearization* as $a_{p_1,\ldots,p_t} z_{p_1 \cup \cdots \cup p_t}$. More generally, given a polynomial inequality, we define its *linearization* as the linear inequality obtained by replacing each monomial term with its linearization as defined above. The above linearization can be performed only if for every nonzero term $a_{p_1,\ldots,p_t} z_{p_1 \cup \cdots \cup p_t}$, we have $p_1 \cup \cdots \cup p_t \in L(G) \cup E(G)$. Note that each binary vector satisfies a polynomial inequality if and only if it satisfies its linearization. We will make use of the following proposition to prove our main lifting result:

Proposition 7. *Let G' be a partial hypergraph of G and let $az \leq \alpha$ be a valid inequality for $MP_{G'}$. Assume there exists a nonempty subset U of $V(G)$ that satisfies the following conditions:*

(i) for every nonzero coefficient a_p, $p \in L(G') \cup E(G')$, we have $p \cup U \in E(G)$,

(ii) if α is nonzero, then $U \in L(G) \cup E(G)$,

(iii) the linearization of the inequality obtained via multiplying $az \leq \alpha$ by $\prod_{v \in U} z_v$ is nontrivial and is different from $az \leq \alpha$.

Then, the zero-lifting of $az \leq \alpha$ is not facet-defining for MP_G.

Proof. By multiplying $az \leq \alpha$ by $\prod_{v \in U} z_v$ and by $1 - \prod_{v \in U} z_v$, and using (i),(ii) to linearize the resulting relations, we obtain two distinct valid linear inequalities for MP_G, whose sum is $az \leq \alpha$. This implies that $az \leq \alpha$ is not facet-defining for MP_G. □

In the sequel, we say that a valid inequality $az \leq \alpha$ for $\text{MP}_{G'}$ is *maximal* for MP_G, if there exists no $U \subseteq V(G)$ for which conditions (i)-(iii) of Proposition 7 are satisfied. Now suppose that G' is a section hypergraph of G and $V(G')$ is an inducing subset of $V(G)$. In the following lemma, we use these additional assumptions to derive a simpler criterion to check the maximality of a facet of $\text{MP}_{G'}$ for MP_G.

Lemma 2. *Let G be a hypergraph and let G' be a section hypergraph of G such that $V(G')$ is an inducing subset of $V(G)$. Let the inequality $az \leq \alpha$ define a facet of $MP_{G'}$. Then $az \leq \alpha$ is not maximal for MP_G if and only if conditions (i),(ii) of Proposition 7 are satisfied for some nonempty $U \subseteq V(G) \setminus V(G')$.*

Proof. First assume that conditions (i),(ii) of Proposition 7 are satisfied for some nonempty $U \subseteq V(G) \setminus V(G')$. Then the linearized inequalities obtained via multiplying $az \leq \alpha$ by $\prod_{v \in U} z_v$ and $1 - \prod_{v \in U} z_v$ are nontrivial and are different from the original inequality; i.e., condition (iii) of Proposition 7 is automatically satisfied. Hence $az \leq \alpha$ is not maximal for MP_G.

Now assume that $az \leq \alpha$ is not maximal for MP_G, and let U be a nonempty subset of $V(G)$ satisfying conditions (i)-(iii). We will show that the set $W = U \setminus V(G')$ satisfies (i),(ii). First we show that the inequality obtained via multiplying $az \leq \alpha$ by $\prod_{v \in U \cap V(G')} z_v$ is identical to $az \leq \alpha$. By assumption G' is a section hypergraph of G and the subset $V(G')$ is inducing; it follows that for every nonzero a_p, $p \in L(G') \cup E(G')$, we have $(p \cup U) \cap V(G') \in L(G') \cup E(G')$. Moreover, if $\alpha \neq 0$, then $U \cap V(G') \in L(G') \cup E(G')$. Therefore, the inequality $\ell_{G'}$ obtained by multiplying $az \leq \alpha$ by $\prod_{v \in U \cap V(G')} z_v$, can be linearized and is valid for $\text{MP}_{G'}$. Since $az \leq \alpha$ defines a facet of $\text{MP}_{G'}$ and $\ell_{G'}$ is satisfied tightly by every point $z \in \mathcal{S}_{G'}$ that satisfies $az = \alpha$, it follows that $\ell_{G'}$ can be obtained by multiplying $az \leq \alpha$ by a scalar $\lambda \geq 0$. By the nontriviality assumption in condition (iii), we get $\lambda > 0$.

Therefore, the inequality obtained by multiplying $az \leq \alpha$ by $\prod_{v \in W} z_v$, coincides with the one obtained by multiplying $az \leq \alpha$ by $\prod_{v \in U} z_v$. In particular, $W \neq \emptyset$. Now, since $W \cap V(G') = \emptyset$, for every nonzero coefficient a_p, $p \in L(G') \cup E(G')$, the linearization of the inequality obtained by multiplying $az \leq \alpha$ by $\prod_{v \in W} z_v$ contains the term $a_p z_{p \cup W}$, thus we have $p \cup W \in E(G)$. Similarly, if α is nonzero, we get $W \in L(G) \cup E(G)$. □

We are now in a position to present conditions under which the zero-lifting of a facet-defining inequality for $\text{MP}_{G'}$ is facet-defining for MP_G.

Theorem 2 (Zero-lifting Theorem)**.** *Let G be a hypergraph, let G' be a section hypergraph of G such that $V(G')$ is inducing, and suppose that $az \leq \alpha$ is a facet-defining inequality for $MP_{G'}$. Then the zero-lifting of $az \leq \alpha$ defines a facet of MP_G if and only if it is maximal for MP_G.*

Proof. The necessity of the maximality assumption follows from Proposition 7. Thus, we now show sufficiency. Assume that the zero-lifting $\bar{a}z \leq \alpha$ of $az \leq \alpha$ is maximal for MP_G. Denote by $bz \leq \beta$ a nontrivial valid inequality for $\mathcal{S}_G$ that is satisfied tightly by all points in $\mathcal{S}_G$ satisfying $\bar{a}z = \alpha$. We show that the two inequalities $\bar{a}z \leq \alpha$ and $bz \leq \beta$ coincide up to a positive scaling, implying that $\bar{a}z \leq \alpha$ defines a facet of MP_G.

It is simple to verify that all assumptions of Lemma 1 are satisfied, including the one in Part 2, since G' is a section hypergraph of G, which implies $\{e \in E(G) \setminus E(G') : e \subseteq V(G')\} = \emptyset$. Hence, we have $b_p = \lambda a_p$ for all $p \in L(G') \cup E(G')$ and $\beta = \lambda \alpha$ for some $\lambda \geq 0$.

Let U denote a nonempty subset of $V(G) \setminus V(G')$. Define $P_U = \{p \in L(G) \cup E(G) : p \setminus V(G') = U\}$ and $Q_a = \{p \in \{\emptyset\} \cup L(G') \cup E(G') : a_p \neq 0\}$, where $a_\emptyset = -\alpha$. Since the inequality $az \leq \alpha$ is maximal for MP_G, Lemma 2 implies that $\{p \setminus U : p \in P_U\} \not\supseteq Q_a$. Consequently, by Part 1 of Lemma 1, we have $b_p = 0$ for all $p \in P_U$. As the above argument applies to every nonempty subset U of $V(G) \setminus V(G')$, and $\cup_{U \subseteq V(G) \setminus V(G'), U \neq \emptyset} P_U = (L(G) \setminus L(G')) \cup (E(G) \setminus E(G'))$, we obtain $b_v = 0$ for all $v \in V(G) \setminus V(G')$, and $b_e = 0$ for all $e \in E(G) \setminus E(G')$. Hence $(b, \beta) = \lambda(\bar{a}, \alpha)$ for some $\lambda \geq 0$. By assumption, $bz \leq \beta$ is nontrivial. Thus $\lambda > 0$, and the theorem follows. □

3.1 Consequences of the Zero-lifting Theorem for Multilinear sets with a special structure

In the reminder of this section, we consider the sets $\mathcal{S}_G$ with certain structures for which the assumptions of the zero-lifting Theorem are either trivially satisfied or can be simplified significantly.

Suppose that the hypergraph G in the statement of Theorem 2 is a rank-r hypergraph. It follows that, if the facet-defining inequality $az \leq \alpha$ for $\text{MP}_{G'}$ has a nonzero coefficient corresponding to an edge e of G' of cardinality r, then it is maximal for MP_G. To see this, first note that by Lemma 2 to check maximality, it suffices to consider nonempty subsets $U \subseteq V(G) \setminus V(G')$. It then follows that $|e \cup U| > r$. Thus $e \cup U$ is not an edge of G, and by definition $az \leq \alpha$ is maximal for MP_G.

The following lemma implies that, if the section hypergraph G' is rank-r full, then each facet of $\text{MP}_{G'}$ contains at least one nonzero coefficient corresponding to an edge of cardinality r.

Lemma 3. *If $az \leq \alpha$ defines a facet of $MP^{n,r}$, then $a_e \neq 0$ for at least one $e \in E(K^{n,r})$ with $|e| = r$.*

Proof. By contradiction assume that $a_e = 0$ for all $e \in E(K^{n,r})$ with $|e| = r$, and let $G' = G(a)$. By Proposition 6, the restriction $\tilde{a}z \leq \alpha$ of $az \leq \alpha$ to $\text{MP}_{G'}$ is valid for $\text{MP}_{G'}$. Let f be an edge of maximum cardinality in $E(G')$, and let $v \in V(K^{n,r}) \setminus f$. Conditions (i),(ii) of Proposition 7 are satisfied for $U = \{v\}$, thus, by Lemma 2, $az \leq \alpha$ is not facet-defining for $\text{MP}^{n,r}$, which is a contradiction. □

By Lemma 3 and Theorem 2, the following result is immediate.

Corollary 2. *Let G be a rank-r hypergraph that contains $K^{n,r}$ as a section hypergraph. Then the zero-lifting of every facet-defining inequality for $MP^{n,r}$ is facet-defining for MP_G.*

Proof. Since G is a rank-r hypergraph, $V(K^{n,r})$ is an inducing subset of $V(G)$. In addition by Lemma 3, every facet of $\text{MP}^{n,r}$ has a nonzero coefficient corresponding to an edge of $K^{n,r}$ of rank r, implying its maximality for MP_G. Thus, by Theorem 2 the result follows. □

In particular, for rank-r full hypergraphs, we have the following:

Corollary 3. *The zero-lifting of every facet of $MP^{n,r}$ defines a facet of $MP^{n',r}$ for all $n' > n$.*

Interestingly, for quadratic sets, i.e., MP_G where G is a graph, the results of Theorem 2 and Corollary 3 simplify to the lifting theorems of Padberg (see Corollary 2 and Theorem 3 in [29]). Clearly, the inducing and maximality assumptions of Theorem 2 are trivially satisfied for quadratic sets, but add further restrictions on the lifting operation when the Multilinear set contains higher degree multilinear terms.

More generally, for a hypergraph containing a complete partial hypergraph, we can state the following lifting result:

Corollary 4. *Let the hypergraph G contain a complete partial hypergraph G'. The zero-liftings of all facet-defining inequalities for $MP_{G'}$ are facet-defining for MP_G if and only if there exists no edge $e \in E(G)$ such that $e \supset V(G')$.*

Proof. Since the partial hypergraph G' is a complete hypergraph, $V(G')$ is an inducing subset of $V(G)$. By Lemma 3, for every facet $az \leq \alpha$ of $\text{MP}_{G'}$, the coefficient a_f, where $f = V(G')$, is nonzero. Hence, if G does not have an edge e of the form $e \supset V(G')$, we conclude that all facets of $\text{MP}_{G'}$ are maximal for MP_G and consequently are facet-defining for it by Theorem 2.

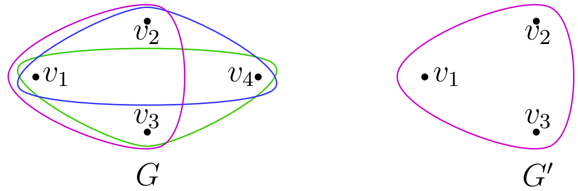

Figure 1: The hypergraphs G and G' defined in Example 1 to demonstrate the necessity of the inducing assumption for Theorem 2

Now suppose that G contains an edge e such that $e \supset V(G')$. Let $\tilde{V} = e \setminus V(G')$. By Proposition 2, the inequality $z_f \geq 0$ with $f = V(G')$ defines a facet of $\text{MP}_{G'}$. After multiplying both sides of this inequality by the nonnegative factor $\prod_{v \in \tilde{V}} z_v$ and $1 - \prod_{v \in \tilde{V}} z_v$ and linearizing the resulting relations we obtain $z_e \geq 0$ and $z_f - z_e \geq 0$ both of which are valid inequalities for MP_G and their sum is given by $z_f \geq 0$. Thus, $z_f \geq 0$ does not define a facet of MP_G. This completes the proof. □

Suppose that the section hypergraph G' defined in the statement of Theorem 2 consists of a non-isolated node $\bar{v}$. Clearly, in this case, $V(G')$ is an inducing subset of $V(G)$. The convex hull of $\mathcal{S}_{G'}$ is the line segment defined by the two facets $z_{\bar{v}} \geq 0$ and $z_{\bar{v}} \leq 1$. To characterize the cases for which the zero-lifting of these two inequalities are facet-defining for MP_G, by Theorem 2, it suffices to examine their maximality for MP_G. Since $\bar{v}$ is not an isolated node, by Proposition 2, $z_{\bar{v}} \geq 0$ does not define a facet of MP_G. In the following corollary we characterize the conditions under which $z_{\bar{v}} \leq 1$ defines a facet of MP_G.

Corollary 5. *Let G be a hypergraph and let $\bar{v} \in V(G)$. Then $z_{\bar{v}} \leq 1$ defines a facet of MP_G if and only if the following conditions are satisfied:*

(i) every edge containing $\bar{v}$ has cardinality at least three,

(ii) for every two edges $f, g \in E(G)$ with $f \supset g$, we have $f \setminus g \neq \{\bar{v}\}$.

Proof. By Theorem 2, $z_{\bar{v}} \leq 1$ does not define a facet of MP_G if and only if there exists a nonempty $U \subseteq V(G)$ satisfying conditions (i) and (ii) of Proposition 7. The existence of such a set is equivalent to the existence of an edge of cardinality two containing $\bar{v}$, if $|U| = 1$; and is equivalent to the existence of $f, g \in E(G)$ with $f \supset g$, and $f \setminus g = \{\bar{v}\}$, if $|U| \geq 2$. □

The result of Corollary 5 implies that $z_v \leq 1$ is not facet-defining for the Boolean quadric polytope, whereas, it defines a facet of a set MP_G, where G is a k-uniform hypergraph with $k \geq 3$. Recall that a *k-uniform hypergraph* is a hypergraph such that all its edges have cardinality k.

Before proceeding further, we demonstrate that the inducing assumption on the partial hypergraph G' of G is required for the validity of Theorem 2 via a simple example. More precisely, if $V(G')$ is not an inducing subset of $V(G)$, it is possible that a facet-defining inequality for $\text{MP}_{G'}$ is maximal for MP_G, and its zero-lifting does not define a facet of MP_G. For notational simplicity, in the following examples, given a node v_i, we write z_i instead of z_{v_i}. Similarly, given an edge $\{v_i, v_j, v_k\}$, we write z_{ijk} instead of $z_{\{v_i,v_j,v_k\}}$.

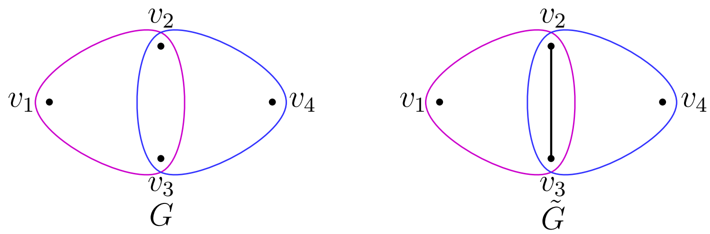

Figure 2: The hypergraphs G and $\tilde{G}$ of Example 2 demonstrating that in certain cases, the inducing assumption of Theorem 2 can be relaxed.

Example 1. *Consider the Multilinear set* $\mathcal{S}_G$ *with* $G = (V, E)$*,* $V = \{v_1, v_2, v_3, v_4\}$ *and,* $E = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}\}$ *(see Figure 1). Let* $V' = \{v_1, v_2, v_3\}$ *and denote by* G' *the section hypergraph of* G *induced by* V'*. The subset* V' *is not inducing since the edges* $\{v_1, v_2\}$ *and* $\{v_1, v_3\}$ *are not present in* E*. The inequality* $z_{123} \leq z_1$ *is a facet of* $MP_{G'}$ *and it contains a nonzero coefficient corresponding to an edge of maximum degree and is therefore maximal for* MP_G*. We now show that* $z_{123} \leq z_1$ *is not a facet of* MP_G *by providing two valid inequalities for* MP_G *that together imply* $z_{123} \leq z_1$*. Consider the expression* $\ell = z_i z_j - z_i z_k + z_j z_k$ *for distinct* $i, j, k \in \{1, \ldots, 4\}$*, where* z *is a binary vector. It is simple to check that* $\ell \leq 1$*. Now consider* $\ell_1 = z_2 z_3 - z_2 z_4 + z_3 z_4$ *and* $\ell_2 = z_2 z_3 - z_3 z_4 + z_2 z_4$*. Multiplying* $\ell_1 \leq 1$ *and* $\ell_2 \leq 1$ *by* z_1 *and linearizing the resulting inequalities we obtain* $z_{123} - z_{124} + z_{134} \leq z_1$ *and* $z_{123} - z_{134} + z_{124} \leq z_1$*. The sum of these two inequalities is* $z_{123} \leq z_1$*, showing that such inequality is not facet-defining for* MP_G*.* $\diamond$

Thus, in general, the inducing assumption on the partial hypergraph G' is required for the validity of Theorem 2. However, for various structured hypergraphs or specific classes of facets, the result of Theorem 2 remains valid even when the inducing assumption is not satisfied. The next example demonstrates that in certain cases, we can combine the result of Proposition 4 and Theorem 2 to utilize the lifting operation for the hypergraphs that do *not* satisfy the inducing assumption.

Example 2. *Consider the hypergraph* G *shown in Figure 2, with* $V(G) = \{v_1, v_2, v_3, v_4\}$ *and* $E(G) = \{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\}$*. The set* $V' = \{v_1, v_2, v_3\}$ *is not inducing since* $\{v_2, v_3\}$ *does not belong to* $E(G)$*. Now, let* $\tilde{G}$ *be the hypergraph obtained from* G *by adding the edge* $\{v_2, v_3\}$*. In this case,* V' *is an inducing subset of* $V(\tilde{G})$*. Let* G'' *denote the section hypergraph of* $\tilde{G}$ *induced by* V'*. The inequality* $z_{123} \leq z_1$ *defines a facet of* $MP_{G''}$ *and by Theorem 2 is facet-defining for* $MP_{\tilde{G}}$*. By Proposition 6, it follows that* $z_{123} \leq z_1$ *defines a facet of* MP_G *as well, since its coefficient corresponding to the edge* $\{v_2, v_3\}$ *is zero.* $\diamond$

More generally, we have the following result:

Corollary 6. *Let* G *be a hypergraph and let* G' *be a section hypergraph of* G*. Denote by* $az \leq \alpha$ *a facet of* $MP_{G'}$ *that is maximal for* MP_G*. Denote by* G'' *the hypergraph obtained from* G' *by adding all edges of the form* $e \cap V(G')$*, where* $e \in E(G)$*. If the zero-lifting of* $az \leq \alpha$ *to* $MP_{G''}$ *is facet-defining for* $MP_{G''}$*, then its zero-lifting to* MP_G *defines a facet of* MP_G*.*

Proof. Follows directly from Proposition 6 and Theorem 2. □

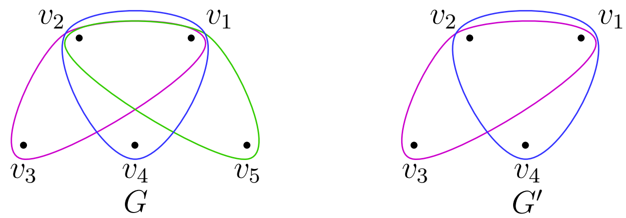

Figure 3: The hypergraphs G and G' defined in Example 3 demonstrating that the converse of Corollary 6 does not hold in general

Finally, the following example demonstrates that the converse of Corollary 6 does not hold in general; that is, if $az \leq \alpha$ does not define a facet of $\mathrm{MP}_{G''}$, its zero-lifting may still be facet-defining for MP_G.

Example 3. *Consider the hypergraph G, where $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_5\}\}$ (see Figure 3). Denote by G' the section hypergraph of G induced by $V' = \{v_1, v_2, v_3, v_4\}$. It can be verified that the inequality*

$$z_3 - z_{123} + z_{124} \leq 1 \tag{11}$$

and its zero-lifting define facets of $MP_{G'}$ and MP_G, respectively. Clearly, V' is not an inducing subset of $V(G)$ since $\{v_1, v_2\}$ does not belong to $E(G)$. Let $\tilde{G}$ be the hypergraph obtained from G by adding the edge $\{v_1, v_2\}$ and let G'' denote the section hypergraph of $\tilde{G}$ induced by V'. We now show that inequality (11) is not facet-defining for $MP_{G''}$ by providing two valid inequalities for $MP_{G''}$ that together imply (11). Denote by H_1 and H_2 the section hypergraphs of G'' induced by the subsets $V_1' = \{v_1, v_2, v_3\}$ and $V_2' = \{v_1, v_2, v_4\}$, respectively. It is simple to verify that the inequalities $z_3 + z_{12} - z_{123} \leq 1$ and $-z_{12} + z_{124} \leq 0$ define facets of MP_{H_1} and MP_{H_2}, respectively. Hence, their zero liftings are valid inequalities for $MP_{G''}$. In addition, adding the two inequalities yields (11). Therefore, the assumption of Corollary 6 is not always required for the validity of the zero-lifting operation. $\diamond$

4 Lifting via facet multiplication

Let G_1 and G_2 denote two partial hypergraphs of a hypergraph G with $V(G_1) \cap V(G_2) = \emptyset$ and let the inequalities $az + \alpha \geq 0$ and $bz + \beta \geq 0$ define facets of the Multilinear polytopes MP_{G_1} and MP_{G_2}, respectively. Suppose that the two inequalities are not maximal for MP_G implying that their zero-liftings are not facet-defining for MP_G. In particular, assume that for every nonzero a_p, $p \in L(G_1) \cup E(G_1)$ and b_q, $q \in L(G_2) \cup E(G_2)$, we have $p \cup q \in E(G)$. Clearly, the linearization of the relation $(az+\alpha)(bz+\beta) \geq 0$ is a valid inequality for MP_G. We are interested in characterizing the cases for which this inequality defines a facet of MP_G. In the special case where G, G_1, G_2 are all complete hypergraphs and $V(G) = V(G_1) \cup V(G_2)$, by Theorem 1 and relation (5), the linearization of any inequality obtained by multiplying two facet-defining inequalities of MP_{G_1} and MP_{G_2} defines a facet of MP_G. Moreover, the collection of all such inequalities characterizes MP_G. In this section, we consider this lifting operation for general sparse hypergraphs. In fact, as we will demonstrate in the following theorem, such lifting operation is valid in a more general setting, namely G_1 and G_2 are auxiliary hypergraphs which are not necessarily partial hypergraphs of G.

Theorem 3. *Let G be a hypergraph and consider a partition of the nodes of G defined as $V(G) = V_1 \cup V_2$. Let*

$$E_i = \{e \cap V_i : e \in E(G),\ |e \cap V_i| \geq 2,\ e \setminus V_i \neq \emptyset\} \qquad i = 1, 2. \tag{12}$$

Define the hypergraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Let the inequalities $az + \alpha \geq 0$ and $bz + \beta \geq 0$ define facets of MP_{G_1} and MP_{G_2}, respectively. Finally, suppose that for every nonzero a_p, $p \in \{\emptyset\} \cup L(G_1) \cup E_1$ and every nonzero b_q, $q \in \{\emptyset\} \cup L(G_2) \cup E_2$, we have $p \cup q \in \emptyset \cup L(G) \cup E(G)$, where $a_\emptyset = \alpha$ and $b_\emptyset = \beta$. Then the linearization of the relation

$$(az + \alpha)(bz + \beta) \geq 0,$$

given by

$$\sum_{p \in \emptyset \cup L(G_1) \cup E_1} \sum_{q \in \emptyset \cup L(G_2) \cup E_2} a_p b_q z_{p \cup q} \geq 0, \tag{13}$$

defines a facet of MP_G.

Proof. Let $L_1 = L(G_1)$ and $L_2 = L(G_2)$. We start by defining a hypergraph $\tilde{G}$ obtained by adding to the hypergraph G all edges $e \in E_i$, $i = 1, 2$ as defined by (12) that are not present in $E(G)$; i.e., $V(\tilde{G}) = V(G)$ and $E(\tilde{G}) = E(G) \cup E_1 \cup E_2$. The key to this construction is that V_1 and V_2 are inducing subsets of $V(\tilde{G})$, whereas they are not inducing subsets of $V(G)$, in general. Subsequently, we prove that the zero-lifting of inequality (13) defines a facet of $\text{MP}_{\tilde{G}}$. It then follows from Proposition 6 that inequality (13) is facet-defining for MP_G as well, since by assumption its support hypergraph is a partial hypergraph of G.

Clearly, inequality (13) is valid for $\text{MP}_{\tilde{G}}$ as G_1 and G_2 are partial hypergraphs of $\tilde{G}$. Denote by $cz + \gamma \geq 0$ a nontrivial valid inequality for $\text{MP}_{\tilde{G}}$ that is satisfied tightly by the set of all points in $\mathcal{S}_{\tilde{G}}$ that satisfy inequality (13) tightly. We show that the two inequalities coincide up to a positive scaling, which in turn implies inequality (13) defines a facet of $\text{MP}_{\tilde{G}}$. By construction, any point in $\mathcal{S}_{\tilde{G}}$ whose restriction to $\mathcal{S}_{G_1}$ (resp. $\mathcal{S}_{G_2}$) satisfies $az + \alpha = 0$ (resp. $bz + \beta = 0$), satisfies inequality (13) tightly. To characterize the relationship between the coefficients of $az + \alpha \geq 0$ and $cz + \gamma \geq 0$, we first employ the result of Lemma 1 with $G' = G_1$ and where U is a nonempty subset of V_2. As defined in the statement of Lemma 1 we have $Q_a = \{p \in \{\emptyset\} \cup L_1 \cup E_1 : a_p \neq 0\}$ and $P_U = \{w \in L(\tilde{G}) \cup E(\tilde{G}) : w \cap V_2 = U\}$. Since V_1 is an inducing subset of $V(\tilde{G})$, by Part 1 of Lemma 1, for each $U \subseteq V_2$, we have:

(1.1) if $p \cup U \in P_U$ for all $p \in Q_a$, then there exists $\lambda_U \in \mathbb{R}$ such that $c_{p \cup U} = a_p \lambda_U$ for all $p \in Q_a$, and $c_{p \cup U} = 0$ for all $p \cup U \in P_U$ with $p \notin Q_a$,

(1.2) otherwise, $c_{p \cup U} = 0$ for all $p \subseteq V_1$ such that $p \cup U \in P_U$.

Symmetrically, we use Lemma 1 with $G' = G_2$ and where U is a nonempty subset of V_1. With these new choices of G' and U in the statement of Lemma 1, we have $Q_b = \{q \in \{\emptyset\} \cup L_2 \cup E_2 : b_q \neq 0\}$ and $P_U = \{w \in L(\tilde{G}) \cup E(\tilde{G}) : w \cap V_1 = U\}$. By Part 1 of Lemma 1, for each $U \subseteq V_1$, we obtain:

(2.1) if $U \cup q \in P_U$ for all $q \in Q_b$, then there exists $\mu_U \in \mathbb{R}$ such that $c_{U \cup q} = \mu_U b_q$ for all $q \in Q_b$, and $c_{U \cup q} = 0$ for all $U \cup q \in P_U$ with $q \notin Q_b$, (2.2) otherwise, $c_{U\cup q} = 0$ for all $q \subseteq V_2$ such that $U \cup q \in P_U$.

To characterize the coefficients of $cz+\gamma \geq 0$, we partition $L(\tilde{G}) \cup E(\tilde{G})$ into the following subsets and analyze each separately: (i) $E_{1,2}$ containing any edge $e \in E(\tilde{G})$ whose intersection with both sets V_1 and V_2 is nonempty, (ii) $\bar{E}_1 = \{e \in E(\tilde{G}) : V_1 \supseteq e,\ e \notin E_1\}$ and $\bar{E}_2 = \{e \in E(\tilde{G}) : V_2 \supseteq e,\ e \notin E_2\}$, (iii) $L_1 \cup E_1$ and $L_2 \cup E_2$.

Consider an edge $e \in E_{1,2}$; define $p = e \cap V_1$ and $q = e \cap V_2$. By our assumption on the structure of $\tilde{G}$, it follows that $p \in L_1 \cup E_1$ and $q \in L_2 \cup E_2$. We show that for some $\mu_p \in \mathbb{R}$ and $\lambda_q \in \mathbb{R}$

$$c_e = \begin{cases} a_p\lambda_q = \mu_p b_q & \text{if } p \in Q_a \setminus \{\emptyset\},\ q \in Q_b \setminus \{\emptyset\} \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

First, let $p \in Q_a \setminus \{\emptyset\}$ and $q \in Q_b \setminus \{\emptyset\}$. By assumption, for every nonzero $a_{\tilde{p}}$, $\tilde{p} \in L_1 \cup E_1$ and every nonzero $b_{\tilde{q}}$, $\tilde{q} \in L_2 \cup E_2$, we have $\tilde{p} \cup \tilde{q} \in E(\tilde{G})$. Consequently, for any $q \in Q_b \setminus \{\emptyset\}$, we have $p \cup q \in E(\tilde{G})$ for all $p \in Q_a$ and by (1.1), we obtain $c_e = a_p\lambda_q$. Similarly, for any $p \in Q_a \setminus \{\emptyset\}$, we have $p \cup q \in E(\tilde{G})$ for all $q \in Q_b$ and by (2.1), we obtain $c_e = \mu_p b_q$. Finally, if $p \notin Q_a$ (resp. $q \notin Q_b$), then by (1.1-1.2) (resp. (2.1-2.2)), we have $c_e = 0$. Combining these arguments, we obtain (14).

Next we characterize the coefficients c_e of $cz + \gamma \geq 0$ for all $e \in \bar{E}_1 \cup \bar{E}_2$, where the subsets $\bar{E}_1$ and $\bar{E}_2$ are as defined above. Since $P_e = \{e\}$ for all $e \in \bar{E}_1 \cup \bar{E}_2$, by (1.2) and (2.2) above we have

$$c_e = 0 \qquad \forall e \in \bar{E}_1 \cup \bar{E}_2. \tag{15}$$

Finally, we characterize the remaining coefficients of $cz + \gamma \geq 0$; i.e., c_w for all $w \in L_1 \cup E_1 \cup L_2 \cup E_2$. To this end, we utilize the result of Part 2 of Lemma 1 by first letting $G' = G_1$ and using the fact that $c_e = 0$ for all $e \in \bar{E}_1$. It then follows that

$$c_p = \eta a_p \qquad \forall p \in \{\emptyset\} \cup L_1 \cup E_1, \tag{16}$$

for some $\eta \geq 0$, where for notational simplicity we define $c_\emptyset = \gamma$. Symmetrically,

$$c_q = \zeta b_q \qquad \forall q \in \{\emptyset\} \cup L_2 \cup E_2, \tag{17}$$

for some $\zeta \geq 0$.

To summarize, let us define $Q_{a,b} = \{p \cup q : p \in Q_a, q \in Q_b\}$; i.e., $Q_{a,b}$ consists of those elements of $\{\emptyset\} \cup L(\tilde{G}) \cup E(\tilde{G})$ whose corresponding coefficients in inequality (13) are nonzero. Then, for any $w \in \{\emptyset\} \cup L(\tilde{G}) \cup E(\tilde{G})$, by relations (14) – (17), we have

$$c_w = \begin{cases} a_p\lambda_q = \mu_p b_q & \text{if } w = p \cup q, \text{ such that } p \cup q \in Q_{a,b} \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

where we define $\mu_p = \zeta$, if $p = \emptyset$ and $\lambda_q = \eta$, if $q = \emptyset$.

Denote by Q_c the set containing those elements of $\{\emptyset\} \cup L(\tilde{G}) \cup E(\tilde{G})$ whose corresponding coefficients in $cz + \gamma \geq 0$ are nonzero. By (18), $Q_c \subseteq Q_{a,b}$. We now show that $Q_c = Q_{a,b}$. To do so, it suffices to prove that for any nonzero a_p and nonzero b_q as defined in (18), the coefficient $c_{p\cup q}$ is nonzero as well. Assume the contrary by letting $c_{\tilde{p}\cup\tilde{q}} = 0$ for some

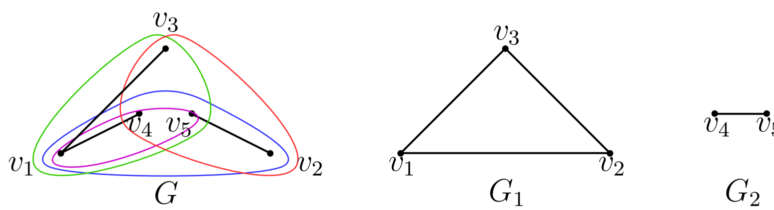

Figure 4: Hypergraphs G, G_1, G_2 of Example 4 to demonstrate the lifting scheme introduced in Theorem 3. Namely, a facet of MP_G can be obtained by multiplying and linearizing certain facet-defining inequalities of MP_{G_1} and MP_{G_2}

$\tilde{p} \cup \tilde{q} \in Q_{a,b}$. Since $a_{\tilde{p}} \neq 0$ and $b_{\tilde{q}} \neq 0$, by (18), we have $\mu_{\tilde{p}} = \lambda_{\tilde{q}} = 0$. It follows that $c_{\tilde{p}\cup q} = \mu_{\tilde{p}} b_q = 0$ for all $q \in Q_b$. By (18), $c_{\tilde{p}\cup q}$ can be equivalently written as $c_{\tilde{p}\cup q} = a_{\tilde{p}}\lambda_q$ and since by assumption $a_{\tilde{p}} \neq 0$, it follows that $\lambda_q = 0$ for all $q \in Q_b$. Consequently, $c_{p\cup q} = a_p\lambda_q = 0$ for all $p \cup q \in Q_{a,b}$; i.e., $cz + \gamma \geq 0$ simplifies to the trivial inequality $0 \geq 0$, which gives us a contradiction. Thus, we conclude that $c_{p\cup q}$ is nonzero, whenever both a_p and b_q are nonzero, implying $Q_c = Q_{a,b}$.

Therefore, without loss of generality, we can assume that μ_p and λ_q are nonzero for all nonzero $c_{p\cup q}$ as defined in (18). As a result, we can factorize μ_p as $\mu_p = \nu_{p,q} a_p$ for some nonzero $\nu_{p,q}$. By (18), it follows that

$$c_{p\cup q} = \nu_{p,q} a_p b_q \qquad \forall p \cup q \in Q_c. \tag{19}$$

Finally, consider two elements in Q_c of the form $p\cup\tilde{q}$ and $p\cup\hat{q}$. Using relations (18) and (19) for $c_{p\cup\tilde{q}}$ and $c_{p\cup\hat{q}}$ yields $\mu_p = \nu_{p,\tilde{q}} a_p = \nu_{p,\hat{q}} a_p$. Therefore, $\nu_{p,q} = \nu$ for all $p \cup q \in Q_c$; i.e., $cz+\gamma \geq 0$ coincides with inequality (13) up to a scaling. In addition, since both inequalities are valid for $\text{MP}_{\tilde{G}}$, it follows that ν is positive. This in turn implies (13) defines a facet of $\text{MP}_{\tilde{G}}$. Hence, by Proposition 6, inequality (13) is facet-defining for MP_G. □

In [39], the author considers the Multilinear polytope $\text{MP}^{n,r}$ and derives some conditions under which certain facets of this polytope can be obtained from multiplying and linearizing facet-defining inequalities of MP^{n_1,r_1} and MP^{n_2,r_2} with $n = n_1 + n_2$ and $r = r_1 + r_2$. By a recursive application of Theorem 3, we can construct certain facets of MP_G from the facets of k simpler polytopes MP_{G_i}, $i = \{1, \ldots, k\}$. Next, we demonstrate the applicability of the above lifting operation via a simple example.

Example 4. *Consider the hypergraph $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4, v_5\}$ and*

$$E = \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_5\}, \{v_1, v_4, v_5\}, \{v_1, v_2, v_4, v_5\}, \{v_1, v_3, v_4, v_5\}, \{v_2, v_3, v_4, v_5\}\}.$$

See Figure 4. Define a partition of the nodes of G as $V = V_1 \cup V_2$, where $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5\}$. Define the two hypergraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ as in Theorem 3; i.e., $E_1 = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\}$ and $E_2 = \{\{v_4, v_5\}\}$. Now consider the facets of MP_{G_1} and MP_{G_2} given by $z_{12} + z_{13} \leq z_1 + z_{23}$ and $z_{45} \geq 0$, respectively. Then, by Theorem 3, the inequality

$$z_{1245} + z_{1345} \leq z_{145} + z_{2345}$$

is facet-defining for MP_G. ◇

We should remark that the converse of Theorem 3 does not hold in the following sense; let $V(G) = V_1 \cup V_2$ be any partition of the nodes of the hypergraph G and let G_1 and G_2 be the corresponding hypergraphs as defined in Theorem 3. Suppose that the inequality $dz + \delta \geq 0$ defines a facet of MP_G and can be obtained by linearizing $(az + \alpha)(bz + \beta) \geq 0$, where $az + \alpha \geq 0$ and $bz + \beta \geq 0$ are valid inequalities for MP_{G_1} and MP_{G_2}, respectively. Then, these inequalities are not necessarily facet-defining for the corresponding polytopes. In addition, it might not be possible to obtain $dz + \delta \geq 0$ by multiplying and linearizing two (other) facet-defining inequalities of MP_{G_1} and MP_{G_2}. We demonstrate this fact via a simple example:

Example 5. *Consider the hypergraph $G = (V, E)$ with $V = \{v_1, v_2, v_3\}$ and $E = \{\{v_1, v_2, v_3\}\}$ and consider a facet of MP_G given by $z_1 - z_{123} \geq 0$. Define a partition of the nodes of G as $V = V_1 \cup V_2$ with $V_1 = \{v_1\}$ and $V_2 = \{v_2, v_3\}$. Construct the two hypergraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ as defined in Theorem 3; i.e., $E_1 = \emptyset$ and $E_2 = \{\{v_2, v_3\}\}$. The inequality $z_1 - z_{123} \geq 0$ can be obtained by linearizing the relation $z_1(1 - z_{23}) \geq 0$. While $z_1 \geq 0$ defines a facet of MP_{G_1}, the inequality $1 - z_{23} \geq 0$ is not facet-defining for MP_{G_2} as it is implied by $z_2 - z_{23} \geq 0$ and $1 - z_2 \geq 0$, both of which are valid inequalities for MP_{G_2}. It is simple to check that $z_1 - z_{123} \geq 0$ cannot be obtained by multiplying and linearizing any two facet-defining inequalities of MP_{G_1} and MP_{G_2}. In addition, it can be verified that there exists no partition of the nodes of G that can be utilized along with Theorem 3 to generate the facet-defining inequality $z_1 - z_{123} \geq 0$.* $\diamond$

Now suppose that the hypergraph $G = (V, E)$ defined in Theorem 3 is a rank-$(r + 1)$ full hypergraph $K^{n,r+1}$. Let $V_1 = \{\tilde{v}\}$ for some $\tilde{v} \in V$ and let $V_2 = V \setminus \{\tilde{v}\}$. Define G_1 to be the graph corresponding to the node $\tilde{v}$ and G_2 to be the rank-r full hypergraph with the node set V_2. Then clearly all assumptions of Theorem 3 are satisfied and hence linearizations of

$$z_{\tilde{v}}(bz + \beta) \geq 0$$

and

$$(1 - z_{\tilde{v}})(bz + \beta) \geq 0$$

define facets of $\mathrm{MP}^{n,r+1}$ for any facet-defining inequality $bz + \beta \geq 0$ of MP_{G_2}.

More generally, by defining G to be a rank-$(r + \delta)$ full hypergraph for some $\delta \geq 1$, G_1 to be a complete hypergraph with δ nodes, G_2 to be a rank-r full hypergraph and utilizing Theorem 1 and Corollary 3, we obtain:

Corollary 7. *Let $bz + \beta \geq 0$ denote a facet-defining inequality for $MP^{n,r}$. Let W denote a subset of nodes of $K^{n,r}$ of cardinality $\delta \geq 1$ such that $W \cap V(G(b)) = \emptyset$. For any $U \subseteq W$, denote by ψ_U the switching operator as defined by relations* (3) *and* (4). *Then the linearization of any relation of the form*

$$\psi_U(z_W)(bz + \beta) \geq 0,$$

defines a facet of $MP^{n,r'}$, where $r' = r + \delta$.

The above result provides a systematic procedure to construct facets for the convex hull of nonconvex sets containing higher degree multilinears from the facets of those containing lower degree multilinear terms. For instance, various classes of facet-defining inequalities for the Boolean quadric polytope have been identified in the literature (c.f. [29, 12]). The result of Corollary 7 enables us to convert these facets into facets of higher degree Multilinear polytopes, as demonstrated by the following example.

Example 6. *Consider the Boolean quadric polytope QP_G defined over a complete graph G with $n := |V(G)|$. It is well-known that the triangle inequalities defined as*

$$\begin{aligned} &z_{ij} + z_{ik} \leq z_i + z_{jk} \\ &z_i + z_j + z_k - z_{ij} - z_{ik} - z_{jk} \leq 1, \end{aligned}$$

for all distinct $i, j, k \in \{1, \ldots, n\}$, are facet-defining for QP_G (c.f. [29]). Then, by Corollary 7, the following inequalities obtained by multiplying the triangle inequalities by z_l and $1 - z_l$, $l \in \{1, \ldots, n\} \setminus \{i, j, k\}$ and linearizing the resulting relations, define facets of $MP^{n,3}$:

$$\begin{aligned} &z_{ijl} + z_{ikl} \leq z_{il} + z_{jkl} \\ &z_{ij} + z_{ik} + z_{il} + z_{jkl} \leq z_i + z_{jk} + z_{ijl} + z_{ikl} \\ &z_{il} + z_{jl} + z_{kl} - z_{ijl} - z_{ikl} - z_{jkl} \leq z_l \\ &z_i + z_j + z_k + z_l - z_{ij} - z_{ik} - z_{jk} - z_{il} - z_{jl} - z_{kl} + z_{ijl} + z_{ikl} + z_{jkl} \leq 1, \end{aligned}$$

for all distinct $i, j, k, l \in \{1, \ldots, n\}$. $\diamond$

4.1 Characterization of structured Multilinear polytopes via facet multiplication

Denote by G_1 and G_2 two hypergraphs with $V(G_1) \cap V(G_2) = \emptyset$, and suppose that $\mathrm{MP}_{G_1} = \{z : a^i z + \alpha_i \geq 0, \ \forall \ i \in I\}$ and $\mathrm{MP}_{G_2} = \{z : b^j z + \beta_j \geq 0, \ \forall \ j \in J\}$. We define the *multiplication hypergraph* $G_1 \times G_2$ of G_1 and G_2 as the hypergraph with node set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{p \cup q : p \in L(G_1) \cup E(G_1), \ q \in L(G_2) \cup E(G_2)\}$. Let the polytope $\mathcal{P}_{G_1 \times G_2}$ be defined by the linearization of every relation of the form $(a^i z + \alpha_i)(b^j z + \beta_j) \geq 0$, for all $i \in I$ and $j \in J$. It is simple to verify that the polytope $\mathcal{P}_{G_1 \times G_2}$ is well-defined. Namely, $\mathcal{P}_{G_1 \times G_2}$ remains unchanged if any number of redundant inequalities are added to the descriptions of MP_{G_1} and MP_{G_2}. Clearly, $\mathcal{P}_{G_1 \times G_2} \supseteq \mathrm{MP}_{G_1 \times G_2}$. We are interested in identifying the cases for which $\mathrm{MP}_{G_1 \times G_2} = \mathcal{P}_{G_1 \times G_2}$. In the following, we investigate the relationship between the two polytopes $\mathcal{P}_{G_1 \times G_2}$ and $\mathrm{MP}_{G_1 \times G_2}$.

The next theorem shows that if one of the two hypergraphs, say G_2, is a single node, then $\mathcal{P}_{G_1 \times G_2} = \mathrm{MP}_{G_1 \times G_2}$. The proof technique used in Theorem 4 is similar to the disjunctive programming approach of Balas [2] who gives an extended formulation for the convex hull of the union of finitely many polytopes. In our case we are able to explicitly project such formulation, and characterize the convex hull in the space of the original variables.

Theorem 4. *Let G_1 be a hypergraph with $MP_{G_1} = \{z : a^i z + \alpha_i \geq 0, \ \forall i \in I\}$, and let G_2 be the graph corresponding to a single node $\tilde{v} \notin V(G_1)$. Then the polytope $MP_{G_1 \times G_2}$ is defined by the linearization of the following relations:*

$$(a^i z + \alpha_i) z_{\tilde{v}} \geq 0 \qquad (a^i z + \alpha_i)(1 - z_{\tilde{v}}) \geq 0,$$

for all $i \in I$.

Proof. Denote by $\bar{G}$ the multiplication hypergraph $G_1 \times G_2$. Consider the faces of $\mathrm{MP}_{\bar{G}}$ given by $F^0 = \{z \in \mathrm{MP}_{\bar{G}} : \; z_{\tilde{v}} = 0\}$ and $F^1 = \{z \in \mathrm{MP}_{\bar{G}} : \; z_{\tilde{v}} = 1\}$. From the definition of $\mathrm{MP}_{\bar{G}}$ it follows that

$$F^0 = \{z : z_{\tilde{v}} = 0, \text{ and } z_{p \cup \{\tilde{v}\}} = 0, \; z_p \in \mathrm{MP}_{G_1}, \; \forall p \in L(G_1) \cup E(G_1)\},$$

and

$$F^1 = \{z : z_{\tilde{v}} = 1, \text{ and } z_{p \cup \{\tilde{v}\}} = z_p, \; z_p \in \mathrm{MP}_{G_1}, \; \forall p \in L(G_1) \cup E(G_1)\}.$$

Since $\mathrm{MP}_{\bar{G}}$ is an integral polytope, any point $z \in \mathrm{MP}_{\bar{G}}$ can be written as $z = (1-\lambda)\tilde{z}^0 + \lambda \tilde{z}^1$ for some $0 \leq \lambda \leq 1$, where $\tilde{z}^0 \in F^0$ and $\tilde{z}^1 \in F^1$. Thus, $\mathrm{MP}_{\bar{G}}$ can be equivalently written as:

$$\begin{aligned}\mathrm{MP}_{\bar{G}} = \Big\{ z : z_{\tilde{v}} = \lambda, \; z_p = (1-\lambda) z^0_p + \lambda z^1_p, \; z_{p \cup \{\tilde{v}\}} = \lambda z^1_p, \; \forall p \in L(G_1) \cup E(G_1), \\ z^0, z^1 \in \mathrm{MP}_{G_1}, \; 0 \leq \lambda \leq 1 \Big\}. \qquad (20)\end{aligned}$$

Our objective is to derive an explicit description for $\mathrm{MP}_{\bar{G}}$ in the space of z variables. We start by eliminating λ from the description of $\mathrm{MP}_{\bar{G}}$ using $z_{\tilde{v}} = \lambda$.

Note that if $z_{\tilde{v}} = 0$ (i.e., $\lambda = 0$), then $z^1 \in \mathrm{MP}_{G_1}$ is redundant, and if $z_{\tilde{v}} > 0$, then $z^1 \in \mathrm{MP}_{G_1}$ is equivalent to

$$(a^i z^1 + \alpha_i) z_{\tilde{v}} \geq 0 \qquad \forall i \in I. \qquad (21)$$

Therefore, the constraint $z^1 \in \mathrm{MP}_{G_1}$ in (20) can be replaced with (21). Similarly, the constraint $z^0 \in \mathrm{MP}_{G_1}$ in (20) can be written as

$$(a^i z^0 + \alpha_i)(1 - z_{\tilde{v}}) \geq 0 \qquad \forall i \in I. \qquad (22)$$

By $z_{p \cup \{\tilde{v}\}} = z_{\tilde{v}} z^1_p$, inequalities (21) can be equivalently written as $\sum_p a^i_p z_{p \cup \{\tilde{v}\}} + \alpha_i z_{\tilde{v}} \geq 0$ for all $i \in I$, where $p \in L(G_1) \cup E(G_1)$, which is identical to the linearization of $(a^i z + \alpha_i) z_{\tilde{v}} \geq 0$ for all $i \in I$. In addition, from $z_{p \cup \{\tilde{v}\}} = z_{\tilde{v}} z^1_p$ and $z_p = (1 - z_{\tilde{v}}) z^0_p + z_{\tilde{v}} z^1_p$ for all $p \in L(G_1) \cup E(G_1)$, it follows that $(1 - z_{\tilde{v}}) z^0_p = z_p - z_{p \cup \{\tilde{v}\}}$. Hence, inequalities (22) are equivalently given by $\sum_p a^i_p (z_p - z_{p \cup \{\tilde{v}\}}) + (1 - z_{\tilde{v}}) \alpha_i \geq 0$ for all $i \in I$, and the latter system of inequalities is identical to the linearization of the system $(a^i z + \alpha_i)(1 - z_{\tilde{v}}) \geq 0$ for all $i \in I$. □

More generally, let G_2 be a complete hypergraph. Then, by a repeated application of Theorem 4, we conclude that also in this case $\mathcal{P}_{G_1 \times G_2} = \mathrm{MP}_{G_1 \times G_2}$:

Corollary 8. *Let G_1 be a hypergraph with $MP_{G_1} = \{z : a^i z + \alpha_i \geq 0, \; \forall i \in I\}$, and let G_2 be a complete hypergraph with $V(G_1) \cap V(G_2) = \emptyset$. Then the polytope $MP_{G_1 \times G_2}$ is defined by the linearization of the following relations:*

$$(a^i z + \alpha_i) \psi_U(z_{V(G_2)}) \geq 0 \qquad \forall i \in I, \; \forall U \subseteq V(G_2).$$

However, as we demonstrate in the following examples, if the hypergraphs G_1 and G_2 are both not complete, then $\mathrm{MP}_{G_1 \times G_2}$ is strictly contained in $\mathcal{P}_{G_1 \times G_2}$, in general.

Example 7. *Consider the two graphs G_1 and G_2 with $V(G_1) = \{v_1, v_2\}$, $E(G_1) = \{\emptyset\}$, $V(G_2) = \{v_3, v_4\}$ and $E(G_2) = \{\emptyset\}$. It follows that $MP_{G_1} = \{z : 0 \leq z_i \leq 1,\ i = 1, 2\}$, $MP_{G_2} = \{z : 0 \leq z_i \leq 1,\ i = 3, 4\}$ and $\mathcal{P}_{G_1 \times G_2} = \{z : z_{ij} \geq 0, z_i - z_{ij} \geq 0, z_j - z_{ij} \geq 0, z_{ij} - z_i - z_j + 1 \geq 0,\ (i, j) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}\}$. However, in this case, the multiplication hypergraph $G_1 \times G_2$ consists of a chordless cycle of length four; i.e., $V(G_1 \times G_2) = \{v_1, v_2, v_3, v_4\}$ and $E(G_1 \times G_2) = \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}\}$. It is well known that an inequality of the form $z_{13} + z_{14} + z_{23} \leq z_{24} + z_1 + z_3$ defines a facet of $MP_{G_1 \times G_2}$ (c.f. [29]), which is clearly not included in the description of $\mathcal{P}_{G_1 \times G_2}$. Thus, in this example $\mathcal{P}_{G_1 \times G_2} \subset MP_{G_1 \times G_2}$.* $\diamond$

In Example 7, both G_1 and G_2 are disconnected graphs. One might wonder if $\text{MP}_{G_1 \times G_2} = \mathcal{P}_{G_1 \times G_2}$ holds for any two connected hypergraphs G_1 and G_2. The following example shows that such a claim is not valid.

Example 8. *Consider the two hypergraphs G_1 and G_2 with $V(G_1) = \{v_1, v_2, v_3\}$, $E(G_1) = \{\{v_1, v_2, v_3\}\}$, $V(G_2) = \{v_4, v_5, v_6\}$ and, $E(G_2) = \{\{v_4, v_5, v_6\}\}$. Consider the multiplication hypergraph $G_1 \times G_2$ as defined above. It can be shown that an inequality of the form $-z_1 - z_4 + z_{14} + z_{16} + z_{34} - z_{36} + z_{123} + z_{456} - z_{1234} - z_{1456} \leq 0$ defines a facet of $MP_{G_1 \times G_2}$. However, it is simple to check that this inequality cannot be obtained by multiplying and linearizing any two facet-defining inequalities of MP_{G_1} and MP_{G_2}.* $\diamond$

Next, we utilize Theorem 4 to investigate the converse of the result considered in Corollary 7. More precisely, consider the Multilinear polytope $\text{MP}^{n,r}$, $r \geq 2$. For each facet-defining inequality $az + \alpha \geq 0$, denote by $U \subset V(K^{n,r})$ the set of nodes that are not present in $G(a)$. Then, by Corollary 7, multiplying $az + \alpha \geq 0$ by z_v (or $1 - z_v$) for each $v \in U$, and linearizing the resulting inequality gives a facet of $\text{MP}^{n,r+1}$. Denote by $\mathcal{P}^{n,r+1}$ the polytope defined by all such inequalities. We would like to characterize the relationship between $\text{MP}^{n,r+1}$ and $\mathcal{P}^{n,r+1}$. Such a result is of particular interest, as for instance it enables us to identify the structure of those facets of $\text{MP}^{n,r+1}$, $r \geq 2$ that cannot be obtained by lifting facets of the Boolean quadratic polytope via the above procedure, and as a result require different derivation techniques.

Before addressing the above question, we should remark that to construct the polytope $\mathcal{P}^{n,r+1}$, we multiply each facet-defining inequality of $\text{MP}^{n,r}$ by those variables that are not present in the support hypergraph of the corresponding facet. Consider a facet of $\text{MP}^{3,2}$ defined by $z_{12} + z_{13} \leq z_1 + z_{23}$. Multiplying this facet-defining inequality by z_3 and linearizing the resulting inequality yields $z_{123} \leq z_{23}$, which indeed defines a facet of $\text{MP}^{3,3}$. However, the same facet can also be obtained by considering another facet of $\text{MP}^{3,2}$ defined by $z_{12} \leq z_2$, multiplying this facet-defining inequality by z_3 and linearizing the resulting relation. In fact, the result of Theorem 4 implies that in general, it suffices to consider the nodes that are not present in the support hypergraphs of the facets of $\text{MP}^{n,r}$. To see this, consider a facet of $\text{MP}^{n,r}$ given by $az + \alpha \geq 0$ and let $\bar{v}$ denote a node that belongs to the hypergraph $G(a)$, such that the linearization of $z_{\bar{v}}(az + \alpha) \geq 0$ (or $(1\text{-}z_{\bar{v}})(az + \alpha) \geq 0$) defines a facet of $\text{MP}^{n,r+1}$. Clearly, the support hypergraph of the new facet is a partial hypergraph of the hypergraph $G_1 \times G_2$, where G_1 is the rank-r full hypergraph on the nodes different from $\bar{v}$, and G_2 is the graph corresponding to the single node $\bar{v}$. Therefore, by Theorem 4, the same facet can be obtained by multiplying a facet-defining inequality of $\text{MP}^{n-1,r}$ by $z_{\bar{v}}$ (or $(1\text{-}z_{\bar{v}})$), and subsequently linearizing it.

Now let us return to the question of the relationship between the two polytopes $\mathrm{MP}^{n,r+1}$ and $\mathcal{P}^{n,r+1}$, for $r \geq 2$. Let $G_1 = K^{n-1,r}$, and let G_2 be a graph corresponding to a single node not in $V(G_1)$. In this case, $G_1 \times G_2$ is a partial hypergraph of $K^{n,r+1}$, and in fact, the missing edges are precisely those rank-$(r+1)$ edges of $K^{n,r+1}$ contained in $V(G_1)$. It then follows by Theorem 4 that the polytope $\mathcal{P}^{n,r+1}$ contains any facet of $\mathrm{MP}^{n,r+1}$ whose support hypergraph is a partial hypergraph of $G_1 \times G_2$. In particular, if $n = r+1$, then we have $\mathrm{MP}^{n,n} = \mathcal{P}^{n,n}$. For the general case with $n > r+1$, by Theorem 4, we can state the following result:

Corollary 9. *Let $az + \alpha \geq 0$ denote a facet of $MP^{n,r+1}$ and denote by $\tilde{E}$ the set of all rank-$(r+1)$ edges in $G(a)$. Then $az + \alpha \geq 0$ can be obtained by linearizing a relation of the form $z_{\tilde{v}}(bz + \beta) \geq 0$ or of the form $(1 - z_{\tilde{v}})(bz + \beta) \geq 0$, where $bz + \beta \geq 0$ defines a facet of $MP^{n-1,r}$ and $\tilde{v} \notin V(G(b))$, if and only if $\tilde{v} \in \cap_{e \in \tilde{E}} e$.*

Proof. We first prove sufficiency of the condition. Let $az + \alpha \geq 0$ be obtained by linearizing a relation of the form $z_{\tilde{v}}(bz + \beta) \geq 0$ or of the form $(1 - z_{\tilde{v}})(bz + \beta) \geq 0$, where $bz + \beta \geq 0$ defines a facet of $\mathrm{MP}^{n-1,r}$ and $\tilde{v} \notin V(G(b))$. Then clearly all the edges of $G(a)$ of rank $(r+1)$ contain the node $\tilde{v}$.

Let $\tilde{v} \in \cap_{e \in \tilde{E}} e$. Then necessity of the condition follows by applying Theorem 4 to the rank-r full hypergraph G constructed on the $n-1$ nodes different from $\tilde{v}$. □

5 Lifting via node addition

In this section, we introduce a different lifting operation in which the Multilinear set $\mathcal{S}_{G'}$ is obtained by fixing certain independent variables in $\mathcal{S}_G$ to one; that is, we set $z_v = 1$ for some $v \in V(G)$. Equivalently, the hypergraph G' can be obtained from the hypergraph G by removing certain nodes of G. More precisely, given a node $\bar{v} \in V(G)$, we say that G' is obtained from G by *removing* $\bar{v}$, if $V(G') = V(G) \setminus \{\bar{v}\}$ and $E(G') = \{e \setminus \{\bar{v}\} : e \in E(G), |e \setminus \{\bar{v}\}| \geq 2\}$. This type of lifting can be used to obtain facets of sets containing higher degree multilinears from those with lower order ones.

As we detail in the following, our results are based on the key assumption that $\dim(\mathrm{MP}_G) = \dim(\mathrm{MP}_{G'}) + 1$, and this relation holds if and only if the hypergraph G' does not contain any loops or parallel edges; i.e., $\bar{e} \setminus \{\bar{v}\} \notin L(G) \cup E(G)$ for all edges $\bar{e}$ of G containing $\bar{v}$. This assumption is needed as otherwise the Multilinear polytope $\mathrm{MP}_{G'}$ is not full-dimensional. It then follows that there exist linearly independent inequalities defining the same facet of $\mathrm{MP}_{G'}$, in which case the lifting operations of this section are not well-defined.

Theorem 5. *Let G be a hypergraph, let $\bar{v} \in V(G)$, and let $\{\bar{e}_j : j \in J\}$ be the set of all edges containing $\bar{v}$. Let $e_j = \bar{e}_j \setminus \{\bar{v}\}$ for each $j \in J$ and suppose that $e_j \notin L(G) \cup E(G)$ for all $j \in J$. Let G' be the hypergraph obtained from G by removing the node $\bar{v}$. Denote by $az \leq \alpha$ a valid inequality for $MP_{G'}$. Define $\bar{J} = \{j \in J : a_{e_j} \neq 0\}$. Let $\tilde{v} \in V(G')$ with $a_{\tilde{v}} \geq 0$. Let $\{e_j : j \in \tilde{J}\}$ be the set of edges in $G(a)$ that contain $\tilde{v}$, and suppose that $\bar{J} = \tilde{J}$. Then the inequality*

$$\sum_{p \in L(G') \cup E(G') \setminus \{e_j : j \in \bar{J}\}} a_p z_p + \sum_{j \in \bar{J}} a_{e_j} z_{\bar{e}_j} + a_{\tilde{v}} z_{\bar{v}} \leq \alpha + a_{\tilde{v}} \tag{23}$$

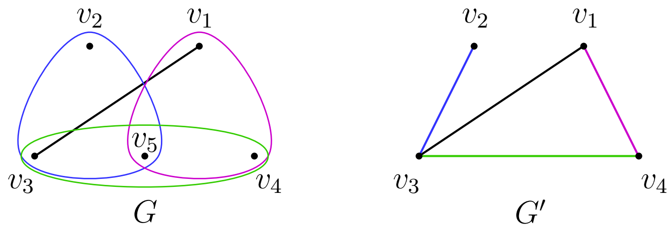

Figure 5: Hypergraphs G and G' of Example 9 demonstrating that certain facets of MP_G can be obtained from those of $\text{MP}_{G'}$ by employing the lifting operation defined in Theorem 5

is valid for MP_G*. Moreover, if* $az \leq \alpha$ *is facet-defining for* $MP_{G'}$ *and is different from* $z_{\tilde{v}} \leq 1$*, then* (23) *is facet-defining for* MP_G*.*

Proof. We start by establishing the validity of inequality (23) for every point $\bar{z} \in \mathcal{S}_G$. If $\bar{z}_{\bar{v}} = 1$, then the validity of (23) follows from $\bar{z}_{\bar{e}_j} = \bar{z}_{e_j}$ for all $j \in J$. Hence, let $\bar{z}_{\bar{v}} = 0$. In this case, if $\bar{z}_{\tilde{v}} = 1$, then the validity of (23) follows from the previous argument, the symmetry of the support hypergraph of inequality (23) with respect to $\bar{v}$ and $\tilde{v}$ (i.e., the two nodes are contained in the same set of edges of $G(a)$), and the fact that the coefficients of $z_{\bar{v}}$ and $z_{\tilde{v}}$ in (23) are identical. Thus, it suffices to show the validity of (23) if $\bar{z}_{\bar{v}} = \bar{z}_{\tilde{v}} = 0$. Let $\tilde{z} \in \mathcal{S}_{G'}$ be obtained from $\bar{z}$ by dropping $z_{\bar{v}}$ and computing $\tilde{y}_e$ accordingly for every $e \in E(G')$. It then follows that the value of the left hand side of inequality (23) at $\bar{z}$ is equal to the value of the left hand side of $az \leq \alpha$ at $\tilde{z}$. By assumption $a_{\tilde{v}} \geq 0$, hence inequality (23) is valid for MP_G.

We now show that if $az \leq \alpha$ is facet-defining for $\text{MP}_{G'}$, then inequality (23) defines a facet of MP_G. By assumption $e_j \notin L(G) \cup E(G)$ for all $j \in J$, implying $\dim(\text{MP}_G) = \dim(\text{MP}_{G'}) + 1$. Denote by z^i, $i = 1, \ldots, k$, all points in $\mathcal{S}_{G'}$ satisfying $az = \alpha$. We lift each of these points z^i to a point $\bar{z}^i$ in $\mathcal{S}_G$ by letting $\bar{z}^i_{\bar{v}} = 1$, and by computing $\bar{z}^i_e$ accordingly for every $e \in E(G)$. Clearly, $\bar{z}^i_{\bar{e}_j} = z^i_{e_j}$, for all $j \in J$, implying inequality (23) is satisfied tightly at these points. Since $az \leq \alpha$ defines a facet of $\text{MP}_{G'}$, there are at lest $|V(G')| + |E(G')|$ affinely independent points among $\bar{z}^i$, $i = 1, \ldots, k$.

To complete the proof, we need one additional point in $\mathcal{S}_G$ denoted by $\hat{z}$ that (i) cannot be written as an affine combination of $\bar{z}^i$, $i = 1, \ldots, k$, and (ii) satisfies (23) tightly. Clearly, any point with $\hat{z}_{\bar{v}} = 0$ satisfies condition (i), since $\bar{z}^i_{\bar{v}} = 1$ for all $i = 1, \ldots, k$. We choose a point $z^0 \in \mathcal{S}_{G'}$ with $az^0 = \alpha$, and $z^0_{\tilde{v}} = 0$. The existence of such a point follows from the assumption that the facet-defining inequality $az \leq \alpha$ is different from $z_{\tilde{v}} \leq 1$. Next, we lift z^0 to a point $\hat{z}$ in $\mathcal{S}_G$ by letting $\hat{z}_{\tilde{v}} = 1$, $\hat{z}_{\bar{v}} = 0$, and by computing $\hat{z}_e$ accordingly for every $e \in E(G)$. Note that $\hat{z}_{\bar{e}_j} = z^0_{e_j} = 0$ for every $j \in \bar{J} = \tilde{J}$. Therefore, $\hat{z}$ satisfies (23) tightly and as a result, inequality (23) is facet-defining for MP_G. □

The key assumption in Theorem 5 is the symmetric structure of the support hypergraph of inequality (23) with respect to the nodes $\tilde{v}$ and $\bar{v}$. Clearly, this assumption is satisfied in the special case where the hypergraph G is symmetric with respect to $\tilde{v}$ and $\bar{v}$, i.e., the two nodes are contained in the same set of edges of G. In the following example, we demonstrate the applicability of the above lifting operation.

Example 9. *Consider the hypergraph* G *defined as* $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ *and* $E(G) = \{\{v_1, v_3\}, \{v_1, v_4, v_5\}, \{v_2, v_3, v_5\}, \{v_3, v_4, v_5\}\}$ *(see Figure 5). We show that the following in-*

equalities are facet-defining for MP_G*:*

$$\begin{aligned} &-z_1 + z_{13} + z_{145} - z_{345} \leq 0 \\ &-z_3 + z_{13} - z_{145} + z_{345} \leq 0 \\ &z_1 + z_3 + z_4 + z_5 - z_{13} - z_{145} - z_{345} \leq 2. \end{aligned} \tag{24}$$

To see this, consider the graph G' *obtained by removing the node* $\bar{v} = v_5$ *from* G*; i.e.,* $V(G') = \{v_1, v_2, v_3, v_4\}$ *and* $E(G) = \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_3, v_4\}\}$*. First, note that the set of edges of* G' *corresponding to edges in* G *containing* v_5 *is* $E_J = \{\{v_1, v_4\}, \{v_2, v_3\}, \{v_3, v_4\}\}$*. Moreover, it is simple to verify that the following triangle inequalities are facet-defining for* $MP_{G'}$*:*

$$\begin{aligned} &-z_1 + z_{13} + z_{14} - z_{34} \leq 0 \\ &-z_3 + z_{13} - z_{14} + z_{34} \leq 0 \\ &z_1 + z_3 + z_4 - z_{13} - z_{14} - z_{34} \leq 1. \end{aligned} \tag{25}$$

The set of edges in E_J *with nonzero coefficients in each of the above inequalities is* $E_{\bar{J}} = \{\{v_1, v_4\}, \{v_3, v_4\}\}$*. Now let* $\tilde{v} = v_4$*. Clearly, in all three inequalities* (25)*, we have* $a_{\tilde{v}} \geq 0$*. Moreover, for all these inequalities the set of edges with nonzero coefficients containing* v_4 *is given by* $E_{\tilde{J}} = \{\{v_1, v_4\}, \{v_3, v_4\}\}$*. It follows that* $E_{\bar{J}} = E_{\tilde{J}}$*. Therefore, all assumptions of Theorem 5 are satisfied and inequalities* (24) *are facet-defining for* MP_G*.* $\diamond$

Next, we develop alternative lifting operations for cases that do not satisfy the assumptions of Theorem 5. We make use of the following proposition to present our next result.

Proposition 8. *Let* G *be a hypergraph, and let* $az \leq \alpha$ *be a facet-defining inequality for* MP_G*. Let* $\tilde{v} \in V(G(a))$*, and let* e_j*,* $j \in J$*, be the edges of* $G(a)$ *that contain* $\tilde{v}$*. Suppose that* $az \leq \alpha$ *is different from* $z_{\tilde{v}} \leq 1$*. If* $\sum_{j \in J} a_{e_j} z_{e_j} \geq 0$ *for every* $z \in \mathcal{S}_G$*, then* $a_{\tilde{v}} \leq 0$*.*

Proof. Since $az \leq \alpha$ is different from $z_{\tilde{v}} \leq 1$, it follows that there exists $\tilde{z} \in \mathcal{S}_G$ with $a\tilde{z} = \alpha$ and $\tilde{z}_{\tilde{v}} = 0$ (and $\tilde{z}_{e_j} = 0$ for every $j \in J$). Let $P = L(G) \cup E(G) \setminus \{\{\tilde{v}\}, e_j : j \in J\}$. We have

$$\sum_{p \in P} a_p \tilde{z}_p = \alpha. \tag{26}$$

Consider now the point $\bar{z}$ obtained from $\tilde{z}$ by setting $\bar{z}_{\tilde{v}} = 1$ and computing the corresponding $\bar{z}_e$ for every $e \in E(G)$. As $az \leq \alpha$ is valid for $\bar{z} \in \mathcal{S}_G$, we have $\sum_{p \in P} a_p \bar{z}_p + a_{\tilde{v}} + \sum_{j \in J} a_{e_j} \bar{z}_{e_j} \leq \alpha$. Since $\bar{z}_p = \tilde{z}_p$ for every $p \in P$, we obtain

$$\sum_{p \in P} a_p \tilde{z}_p + a_{\tilde{v}} + \sum_{j \in J} a_{e_j} \bar{z}_{e_j} \leq \alpha. \tag{27}$$

From (26) and (27) we get $a_{\tilde{v}} \leq -\sum_{j \in J} a_{e_j} \bar{z}_{e_j}$. As $\sum_{j \in J} a_{e_j} z_{e_j} \geq 0$ for every $z \in \mathcal{S}_G$, we conclude that $a_{\tilde{v}} \leq 0$. $\square$

Consider the hypergraphs G and G' defined in Theorem 5 and let $az \leq \alpha$ be facet-defining for $\text{MP}_{G'}$. In the next theorem, we introduce a lifting operation assuming that $\sum_{j \in J} a_{e_j} z_{e_j} \geq 0$ for every $z \in \mathcal{S}_{G'}$, where J corresponds to the index set of edges in $G(a)$ containing $\tilde{v}$. In this case, by Proposition 8 we have $a_{\tilde{v}} \leq 0$. Clearly, if $a_{\tilde{v}} < 0$, then the lifting technique of Theorem 5 cannot be utilized. The following lifting operation is applicable in many cases for which Theorem 5 cannot be applied.

Theorem 6. *Let G be a hypergraph, let $\bar{v} \in V(G)$, and let G' be obtained from G by removing $\bar{v}$. Denote by $\{\bar{e}_j : j \in J\}$ the set of all edges containing $\bar{v}$. Suppose that $\bar{e}_j \setminus \{\bar{v}\} \notin L(G) \cup E(G)$ for all $j \in J$. Let $az \leq \alpha$ denote a valid inequality for $MP_{G'}$ such that*

$$\sum_{j\in J} a_{e_j} z_{e_j} \geq 0 \qquad \forall z \in \mathcal{S}_{G'}, \tag{28}$$

where $e_j = \bar{e}_j \setminus \{\bar{v}\}$ for each $j \in J$. Then, the inequality

$$\sum_{e\in L(G')\cup E(G')\setminus\{e_j : j\in J\}} a_e z_e + \sum_{j\in J} a_{e_j} z_{\bar{e}_j} \leq \alpha, \tag{29}$$

is valid for MP_G. Denote by $\bar{J} = \{j \in J : a_{e_j} \neq 0\}$ and suppose that

$$\bigcap_{j\in\bar{J}} e_j \neq \emptyset. \tag{30}$$

If $az \leq \alpha$ is facet-defining for $MP_{G'}$, then inequality (29) is facet-defining for MP_G.

Proof. We start by establishing the validity of inequality (29) for MP_G. Let $\bar{z}$ be a feasible point in $\mathcal{S}_G$, and let $\tilde{z}$ be the corresponding point in $\mathcal{S}_{G'}$ obtained by dropping $\bar{z}_{\bar{v}}$ and computing the corresponding feasible components $\tilde{z}_e$, for all $e \in E(G')$. Two cases arise:

(i) $\bar{z}_{\bar{v}} = 1$; it then follows that $\bar{z}_{\bar{e}_j} = \tilde{z}_{e_j}$ for all $j \in J$ which in turn implies inequality (29) is valid at $\bar{z}$.

(ii) $\bar{z}_{\bar{v}} = 0$; in this case, substituting $\bar{z}$ in inequality (29) yields

$$\sum_{e\in L(G')\cup E(G')\setminus\{e_j : j\in J\}} a_e \bar{z}_e \leq \alpha. \tag{31}$$

By assumption, we have $\sum_{j\in J} a_{e_j} \tilde{z}_{e_j} \geq 0$. From $a\tilde{z} \leq \alpha$, it then follows that $\sum_{e\in L(G')\cup E(G')\setminus\{e_j : j\in J\}} a_e \tilde{z}_e$ α. Moreover, $\bar{z}_e = \tilde{z}_e$ for all $e \in L(G') \cup E(G') \setminus \{e_j : j \in J\}$. Hence, inequality (31) is valid.

We now show that if $az \leq \alpha$ is facet-defining for $\mathrm{MP}_{G'}$ and condition (30) is satisfied, then inequality (29) defines a facet of MP_G. Denote by z^i, $i = 1, \ldots, k$ the set of all points in $\mathcal{S}_{G'}$ satisfying $az = \alpha$. We lift each of these points z^i to a point $\bar{z}^i$ in $\mathcal{S}_G$ by letting $\bar{z}^i_{\bar{v}} = 1$, and by computing $\bar{z}^i_e$ accordingly, for each $e \in E(G)$. Clearly, $\bar{z}^i_{\bar{e}_j} = z^i_{e_j}$, for all $j \in J$. Hence, inequality (29) is satisfied tightly at these points. Since $az \leq \alpha$ is facet-defining for $\mathrm{MP}_{G'}$, the set $\{\bar{z}^i : i = 1, \ldots, k\}$ contains $|V(G')| + |E(G')|$ affinely independent points.

By assumption, $e_j \notin L(G) \cup E(G)$ for all $j \in J$. It follows that $\dim(\mathrm{MP}_G) = \dim(\mathrm{MP}_{G'}) + 1$. Consequently, to complete the proof, we need one additional point in $\mathcal{S}_G$ denoted by $\hat{z}$ that satisfies (29) tightly and cannot be written as an affine combination of the points $\bar{z}^i$, $i = 1, \ldots, k$. Clearly, any point $\hat{z}$ with $\hat{z}_{\bar{v}} = 0$ cannot be written as an affine combination of $\bar{z}^i$, $i = 1, \ldots, k$, since $\bar{z}^i_{\bar{v}} = 1$ for all $i = 1, \ldots, k$.

We first show that there exists a point $z^0 \in \mathcal{S}_{G'}$ with $az^0 = \alpha$, and $z^0_{e_j} = 0$ for every $j \in \bar{J}$. Let $\tilde{v} \in \cap_{j\in\bar{J}} e_j$. Note that by (30), a node $\tilde{v}$ always exists. Two cases arise: (i)

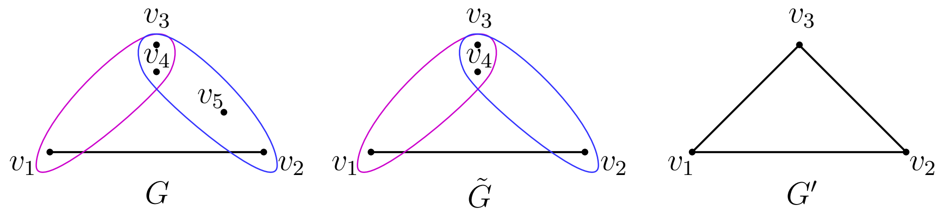

Figure 6: Hypergraphs G, $\tilde{G}$, and G' of Example 10 demonstrating that certain facets of MP_G can be obtained from those of $\mathrm{MP}_{G'}$ by employing the lifting operation defined in Theorem 6.

if $az \leq \alpha$ is different from $z_{\tilde{v}} \leq 1$, then there exists $z^0 \in \mathcal{S}_{G'}$ with $az^0 = \alpha$ and $z^0_{\tilde{v}} = 0$, implying $z^0_{e_j} = 0$ for every $j \in \bar{J}$, (ii) if $az \leq \alpha$ coincides with $z_{\tilde{v}} \leq 1$, then let z^0 be any point in $\mathcal{S}_{G'}$ with $z^0_{\tilde{v}} = 1$ and, for every $j \in \bar{J}$, $z^0_{v_j} = 0$ for a node $v_j \in e_j \setminus \{\tilde{v}\}$, which in turn implies $z^0_{e_j} = 0$ for every $j \in \bar{J}$. Next, we lift z^0 to a point $\hat{z}$ in $\mathcal{S}_G$ by letting $\hat{z}_{\bar{v}} = 0$, and by computing $\hat{z}_e$ accordingly, for each $e \in E$. Note that, since $z^0_{e_j} = 0$ for all $j \in \bar{J}$, it follows that $\hat{z}_{\bar{e}_j} = z^0_{e_j}$ for every $j \in \bar{J}$, and therefore $\hat{z}$ satisfies (29) tightly. Thus, inequality (29) is facet-defining for MP_G. □

Let us consider the case for which the assumptions of both Theorems 5 and 6 are satisfied. By Proposition 8, it then follows that $a_{\tilde{v}} = 0$, implying the two lifted inequalities defined by (23) and (29) are identical. We should remark that Theorem 6 relies on the assumption that $\sum_{j \in J} a_{e_j} z_{e_j} \geq 0$ for all $z \in \mathcal{S}_{G'}$. Clearly, this assumption is satisfied in the special case where $a_{e_j} \geq 0$ for all $j \in J$; i.e., the lifting operation of Theorem 6 can be utilized for the case in which the node to be removed is located at the intersection of edges of G whose corresponding coefficients in $az \leq \alpha$ are nonnegative. However, the assumption of Theorem 6 enables us to obtain facets in cases for which the latter nonnegativity assumption is not satisfied.

In the following example, we show the usefulness of the lifting operation defined in Theorem 6 to generate certain facets of a rank-3 hypergraph by lifting facets of a graph.

Example 10. *Consider the hypergraph G with $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{\{v_1, v_2\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4, v_5\}\}$ (see Figure 6). We claim that the following inequalities are facet-defining for MP_G:*

$$-z_3 - z_{12} + z_{134} + z_{2345} \leq 0, \qquad -z_4 - z_{12} + z_{134} + z_{2345} \leq 0. \tag{32}$$

To see this, consider the hypergraph $\tilde{G}$ obtained by removing node v_5 from G, and the hypergraph G' obtained by removing node v_4 from $\tilde{G}$. It is simple to verify that the following so called triangle inequality

$$-z_3 - z_{12} + z_{13} + z_{23} \leq 0 \tag{33}$$

defines a facet of $MP_{G'}$ (c.f. [29]). Since the coefficients of z_{13} and z_{23} in inequality (33) are nonnegative, by Theorem 6 and using a symmetry argument, it follows that the following inequalities are facet-defining for $MP_{\tilde{G}}$:

$$-z_3 - z_{12} + z_{134} + z_{234} \leq 0, \qquad -z_4 - z_{12} + z_{134} + z_{234} \leq 0. \tag{34}$$

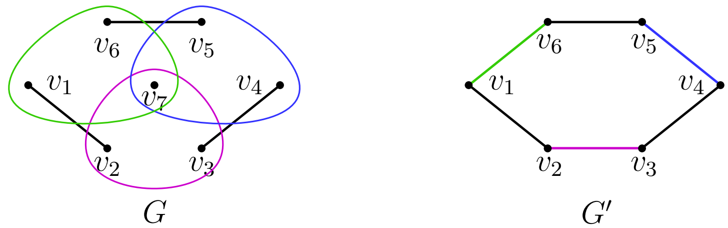

Figure 7: Hypergraphs G and G' of Example 11 demonstrating that the nonemptyness assumption defined by (30) in Theorem 6 is necessary, in general.

Again, since the coefficient of z_{234} *in both inequalities defined in* (34) *is nonnegative, we can utilize Theorem 6 to conclude that inequalities* (32) *are facet-defining for* MP_G*.* $\diamond$

It is important to note that the assumption defined by (30); i.e., requiring the existence of a node $\tilde{v}$ at the intersection of certain edges of G in Theorem 6 is weaker than the corresponding assumption in Theorem 5. Namely, while Theorem 6 requires the existence of $\tilde{v}$ at the intersection of those edges containing $\bar{v}$ whose corresponding coefficients in $az \leq \alpha$ are nonzero, Theorem 5 requires that, in addition, the node $\tilde{v}$ should not be contained in any other edge of $G(a)$. This in turn implies that inequality (29) is not necessarily symmetric with respect to $\bar{v}$ and $\tilde{v}$, whereas, inequality (23) has such a symmetric structure. In the following example, we demonstrate that Theorem 6 does not hold in general, if assumption (30) is not satisfied.

Example 11. *Consider the hypergraph* G *defined as* $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ *and* $E(G) = \{\{v_1, v_2\}, \{v_2, v_3, v_7\}, \{v_3, v_4\}, \{v_4, v_5, v_7\}, \{v_5, v_6\}, \{v_1, v_6, v_7\}\}$ *(see Figure 7). Let* $\bar{v} = v_7$*. For this example, the hypergraph* G' *obtained by removing node* v_7 *from* G *is a chordless cycle of length six. It is well-known that the following so-called odd cycle inequality defines a facet of* $MP_{G'}$ *(c.f. [29]):*

$$-z_{12} + z_{23} - z_{34} + z_{45} - z_{56} + z_{16} \leq 1.$$

Since the coefficients of z_{23}*,* z_{45} *and* z_{16} *in the above inequality are nonnegative, if we relax the assumption on the nonemptyness of the intersection of the corresponding edges in* G'*, by Theorem 6, one concludes that the following inequality defines a facet of* MP_G*:*

$$-z_{12} + z_{237} - z_{34} + z_{457} - z_{56} + z_{167} \leq 1. \tag{35}$$

We now show that the above inequality is not facet-defining for MP_G *by providing a valid inequality for* MP_G *that implies inequality* (35)*. Consider the expression on the left hand side of inequality* (35)*. We first compute the maximum value of this expression over* MP_G*; that is, we find the maximum of* $f = -z_1 z_2 - z_3 z_4 - z_5 z_6 + z_1 z_6 z_7 + z_2 z_3 z_7 + z_4 z_5 z_7$*, where* $z_v \in \{0, 1\}$ *for all* $v \in V(G)$*. Consider the following cases:*

(i) $z_7 = 0$*: in this case* f *simplifies to* $-z_1 z_2 - z_3 z_4 - z_5 z_6$ *whose maximum over* $\{0, 1\}^7$ *is equal to zero.*

(ii) $z_7 = 1$*: in this case, we have* $f = -z_1 z_2 + z_2 z_3 - z_3 z_4 + z_4 z_5 - z_5 z_6 + z_1 z_6$ *and it is simple to verify that* $f \leq 1$*.*

Thus, the following is a valid inequality for MP_G:

$$-z_{12} + z_{237} - z_{34} + z_{457} - z_{56} + z_{167} \leq z_7.$$

Clearly, the above inequality, together with $z_7 \leq 1$, implies (35) *and as result inequality* (35) *is not facet-defining for MP_G. Thus, we conclude that the lifting operation of Theorem 6 is not valid in general, if the nonemptyness assumption defined by* (30) *does not hold.* $\diamond$

In Theorem 6, if the node $\bar{v}$ is contained in a single edge $\bar{e}$, with $|\bar{e}| \geq 3$ and $\bar{e} \setminus \bar{v} \notin E(G)$, then the assumptions of the theorem simplify to $a_{\bar{e}\setminus\{\bar{v}\}} \geq 0$; i.e., the node $\bar{v}$ is restricted to be removed from an edge whose corresponding coefficient in the facet of $\mathrm{MP}_{G'}$ is nonnegative. In the following theorem, we consider the case where $\bar{v}$ is removed from an edge with a negative coefficient.

Theorem 7. *Let G be a hypergraph and let $\bar{v}$ be a node of G that is contained only in one edge $\bar{e} \in E(G)$. Suppose that $|\bar{e}| \geq 3$, and that $\bar{e} \setminus \bar{v} \notin E(G)$. Let G' be obtained from G by removing $\bar{v}$, and let $\tilde{e} = \bar{e} \setminus \{\bar{v}\}$. Let $az \leq \alpha$ denote a valid inequality for $MP_{G'}$ with $a_{\tilde{e}} < 0$. Then, the inequality*

$$\sum_{p \in L(G') \cup E(G') \setminus \{\tilde{e}\}} a_p z_p + a_{\tilde{e}} z_{\bar{e}} - a_{\tilde{e}} z_{\bar{v}} \leq \alpha - a_{\tilde{e}} \tag{36}$$

is valid for MP_G. Moreover, if $az \leq \alpha$ is facet-defining for $MP_{G'}$ and is different from $z_{\tilde{e}} \geq 0$, then (36) *is facet-defining for MP_G.*

Proof. We first establish the validity of inequality (36) for MP_G. Let $\bar{z}$ be a feasible point in $\mathcal{S}_G$. We show that inequality (36) is satisfied by $\bar{z}$. Let $\tilde{z}$ be the corresponding point in $\mathcal{S}_{G'}$ obtained by dropping $\bar{z}_{\bar{v}}$, and by computing the corresponding feasible $\tilde{z}_e$, for $e \in E(G')$. Note that $\bar{z}_p = \tilde{z}_p$ for every $p \in L(G') \cup E(G') \setminus \{\tilde{e}\}$.

First, let $\bar{z}_{\bar{v}} = 1$. In this case, the validity of inequality (36) follows from the fact that $\bar{z}_{\bar{e}} = \tilde{z}_{\tilde{e}}$:

$$\sum_{p \in L(G') \cup E(G') \setminus \{\tilde{e}\}} a_p \bar{z}_p + a_{\tilde{e}} \bar{z}_{\bar{e}} - a_{\tilde{e}} \bar{z}_{\bar{v}} = \sum_{p \in L(G') \cup E(G')} a_p \tilde{z}_p - a_{\tilde{e}} \leq \alpha - a_{\tilde{e}}.$$

Next, let $\bar{z}_{\bar{v}} = 0$. In this case, we have $\bar{z}_{\bar{e}} = 0$. Hence:

$$\sum_{p \in L(G') \cup E(G') \setminus \{\tilde{e}\}} a_p \bar{z}_p + a_{\tilde{e}} \bar{z}_{\bar{e}} - a_{\tilde{e}} \bar{z}_{\bar{v}} = \sum_{p \in L(G') \cup E(G') \setminus \{\tilde{e}\}} a_p \tilde{z}_p \leq \alpha - a_{\tilde{e}} \tilde{z}_{\tilde{e}} \leq \alpha - a_{\tilde{e}}.$$

The last inequality is valid since by assumption $a_{\tilde{e}} < 0$. This completes the proof of validity.

We now show that if $az \leq \alpha$ is facet-defining for $\mathrm{MP}_{G'}$, then inequality (36) defines a facet of MP_G. Denote by z^i, $i = 1, \ldots, k$, the set of all points in $\mathcal{S}_{G'}$ satisfying $az = \alpha$. We now convert each of these points to a point $\bar{z}^i \in \mathrm{MP}_G$, by letting $\bar{z}^i_{\bar{v}} = 1$ for all $i \in \{1, \ldots, k\}$ and computing $\bar{z}^i_e$ accordingly for every $e \in E(G)$. Clearly, these points satisfy inequality (36) tightly. Since $az \leq \alpha$ defines a facet of $\mathrm{MP}_{G'}$, the set $\{\bar{z}^i : i = 1, \ldots, k\}$ contains $|V(G')| + |E(G')|$ affinely independent points.

By assumption, $\tilde{e} \notin E(G)$, implying that $\dim(\text{MP}_G) = \dim(\text{MP}_{G'}) + 1$. Thus to complete the proof, we need one point in $\mathcal{S}_G$, denoted by $\hat{z}$, which satisfies (36) tightly and cannot be written as an affine combination of the points $\bar{z}^i$, $i = 1, \ldots, k$. We now choose a point, say $z^0 \in \mathcal{S}_{G'}$, satisfying $az = \alpha$ with $z^0_{\tilde{e}} = 1$. We can always assume that such a point exists, since otherwise the hyperplane $az = \alpha$ is contained in $z_{\tilde{e}} = 0$, which is in contradiction with the assumption that $az \leq \alpha$ defines a facet of $\text{MP}_{G'}$ different from $z_{\tilde{e}} \geq 0$. We now lift z^0 to a point $\hat{z} \in \mathcal{S}_G$ by letting $\hat{z}_{\bar{v}} = 0$ and $\hat{z}_{\bar{e}} = 0$. Clearly, this point satisfies (36) tightly and cannot be written as an affine combination of points in $\bar{z}^i$, $i = 1, \ldots, k$, since $z^i_{\bar{v}} = 1$ for all i. Thus, inequality (36) is facet-defining for MP_G. □

In the following example, we demonstrate the applicability of the lifting operation defined in Theorem 7.

Example 12. *Consider the hypergraph $G = (V, E)$ with $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3, v_4\}\}$. We argue that the following inequality*

$$- z_1 + z_4 + z_{12} + z_{13} - z_{234} \leq 1, \tag{37}$$

defines a facet of MP_G. To see this, consider the graph G' obtained by removing the node v_4 from G. It is simple to check that $-z_1 + z_{12} + z_{13} - z_{23} \leq 0$ defines a facet of $MP_{G'}$. Since the coefficient of z_{23} in this inequality is negative, by Theorem 7, the inequality (37) *is facet-defining for MP_G. For this example, Theorem 5 is not applicable since G' does not have a node of the form $\tilde{v}$, as defined in this theorem.* ◇

We conclude this section by presenting a family of facet-defining inequalities for hypergraphs with a certain structure. The proposed facets are obtained via a recursive application of the lifting operations introduced in this section.

Corollary 10. *Let $G = (V, E)$ be a hypergraph with edges $e_1, \ldots, e_t$, for some $t \geq 3$. Suppose that e_i, $i \in \{1, \ldots, t\}$ has nonempty intersections with e_{i-1} and e_{i+1} only, where we define $e_0 = e_t$ and $e_{t+1} = e_1$. In addition, each node is contained in at most two edges of G. Let M be a subset of E of odd cardinality. Denote by $S_1 \subseteq V(G)$ the set of nodes that are not contained in any edge in $E \setminus M$, and let $S_2 \subseteq V(G)$ denote a set of nodes that contains exactly one node in $e_i \cap e_{i+1}$ for every $i \in \{1, \ldots, t\}$ with $e_i, e_{i+1} \in E \setminus M$. Then the following inequality is facet-defining for MP_G:*

$$\sum_{v \in S_1} z_v - \sum_{e \in M} z_e - \sum_{v \in S_2} z_v + \sum_{e \in E \setminus M} z_e \leq k + \lfloor |M|/2 \rfloor, \tag{38}$$

where $k = |S_1| - |\{i \in \{1, \ldots, t\} : e_i, e_{i+1} \in M\}|$.

Proof. We start by defining the following auxiliary hypergraphs:

- the hypergraph G' is obtained by removing from G all nodes contained in exactly one edge e' of G with $e' \in M$; that is, all nodes in the set $\{v : v \in e' \text{ for some } e' \in M,\ v \notin e,\ \forall e \in E \setminus \{e'\}\}$ are removed from G.

- the hypergraph G'' is obtained by removing from G' the following nodes: (i) all nodes contained in exactly one edge in $E \setminus M$, (ii) for each $i \in \{1, \ldots, t\}$ with $e_i, e_{i+1} \in E \setminus M$, all the nodes in $e_i \cap e_{i+1} \setminus S_2$,

- the graph G''' is obtained by removing from G'' the following nodes: (i) for each $i \in \{1, \dots, t\}$ with $e_i, e_{i+1} \in M$, all the nodes but one in $e_i \cap e_{i+1}$, (ii) for each $i \in \{1, \dots, t\}$ with $e_i \in M$ and $e_{i+1} \in E \setminus M$ or $e_i \in E \setminus M$ and $e_{i+1} \in M$, all nodes but one in $e_i \cap e_{i+1}$.

Since by definition of G there is no node contained in more than two edges, it can be checked that there is a bijection among the edges of any pair of hypergraphs G, G', G'', G'''. For notational simplicity, in the following, we use the same notation for the edges in G, G', G'', G''' that are in a one-to-one correspondence.

By construction, the graph G''' is a chordless cycle of length t. Hence, the following so called odd-cycle inequality is facet-defining for $\text{MP}_{G'''}$ (see [8, 29]):

$$\sum_{v \in S_1 \cap V(G''')} z_v - \sum_{e \in M} z_e - \sum_{v \in S_2} z_v + \sum_{e \in E \setminus M} z_e \leq \lfloor |M|/2 \rfloor. \tag{39}$$

In inequality (39), all coefficients corresponding to the nodes at the intersection of two edges in M, and in the intersection of one edge in M and one in $E \setminus M$, are nonnegative. As a result, we can apply Theorem 5 recursively to obtain the following facet-defining inequality for $\text{MP}_{G''}$:

$$\sum_{v \in S_1 \cap V(G'')} z_v - \sum_{e \in M} z_e - \sum_{v \in S_2} z_v + \sum_{e \in E \setminus M} z_e \leq k' + \lfloor |M|/2 \rfloor, \tag{40}$$

where $k' = |S_1 \cap V(G'')| - |\{i \in \{1, \dots, t\} : e_i, e_{i+1} \in M\}|$.

Since in inequality (40), the coefficients corresponding to edges in $E \setminus M$ are nonnegative, we can recursively apply Theorem 6 and obtain the following facet-defining inequality for $\text{MP}_{G'}$:

$$\sum_{v \in S_1 \cap V(G')} z_v - \sum_{e \in M} z_e - \sum_{v \in S_2} z_v + \sum_{e \in E \setminus M} z_e \leq k' + \lfloor |M|/2 \rfloor. \tag{41}$$

Finally, observe that in inequality (41), all coefficients corresponding to edges in M are negative. Hence, by a recursive application of Theorem 7, we conclude that inequality (38) defines a facet of MP_G. □

6 Concluding remarks

We studied the convex hull of the Multilinear set defined by a collection of multilinear equations from a polyhedral point of view. We developed the theory of various types of lifting operations for this set, giving rise to many types of facet-defining inequalities in the space of the original variables. In particular, together with the known families of facet-defining inequalities for the Boolean quadric polytope, the proposed lifting techniques enable us to construct sharper polyhedral relaxations for mixed-integer nonlinear optimization problems containing multilinear sub-expressions. Devising efficient separation algorithms along with extensive computational experimentations with the proposed cutting planes is a subject of future research.

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