



Axioma Research Paper  
No.

**February 19, 2015**

---

## Multi-period portfolio optimization with alpha decay

The traditional Markowitz MVO approach is based on a single-period model. Single period models do not utilize any data or decisions beyond the rebalancing time horizon with the result that their policies are *myopic* in nature. For long-term investors, multi-period optimization offers the opportunity to make *wait-and-see* policy decisions by including approximate forecasts and long-term policy decisions beyond the rebalancing time horizon. We consider portfolio optimization with a composite alpha signal that is composed of a short-term and a long-term alpha signal. The short-term alpha better predicts returns at the end of the rebalancing period but it decays quickly, i.e., it has less memory of its previous values. On the other hand, the long-term alpha has less predictive power than the short-term alpha but it decays slowly. We develop a simple two stage multi-period model that incorporates this alpha model to construct the optimal portfolio at the end of the rebalancing period. We compare this model with the traditional single-period MVO model on a simulated example from Israelov & Katz [12] and also a large strategy with realistic constraints and show that the multi-period model tends to generate portfolios that are likely to have a better realized performance.

# Multi-period portfolio optimization with alpha decay

Kartik Sivaramakrishnan, Vishv Jeet, and Dieter Vandembussche

## 1 Introduction

The traditional Markowitz MVO approach is based on a single-period setting that limits its applicability to long-term investors with liabilities and goals at different times in the future. Single-period optimization does not use any data and decisions beyond the rebalancing time horizon with the result that its policies are *myopic* in nature. More importantly, single-period optimization does not account for the fact that a decision made in a rebalancing today affects decisions made in future rebalancings. This is a drawback even if the rebalancing is done periodically in light of new information. A multi-period model has several stages with the first stage representing the current rebalancing and stages two and beyond representing future rebalancings. The model allows the simultaneous rebalancing of a single portfolio across different periods (extending beyond the rebalancing time horizon) in a single optimization problem. Each stage in the model has its own data forecasts, objectives, and policy decisions and the stages in successive periods are coupled together. In most cases, we are only interested in the first-stage solution since the solution to the remaining stages can be updated as new information becomes available. The first-stage solution to the multi-period model incorporates a *wait-and-see* feature, since it is aware of future decisions and data forecasts through the other stages in the model.

One issue with multi-period models is that they require forecast information beyond the rebalancing time horizon that is subject to greater uncertainty. A second issue is that the resulting model is complex and requires greater solution time. Despite these issues, multi-period models have generated a lot of interest in the financial community since they attempt to model the future *dynamics* of the portfolio taking into account data and decisions beyond the rebalancing time horizon. This leads to better solutions implemented today than would be possible with a *myopic* single-period model.

Multi-period models have found several applications in finance. The following list is by no means exhaustive.

1. **Trade scheduling:** This is a common problem that is handled by large sell-side firms and investment banks. A trade scheduling problem aims to get from an initial portfolio to a target portfolio in stages while trading off the risk and market-impact across all these stages. The risk term in the objective forces the trading to happen quickly (move

slowly and the market will move you!). The market-impact term in the objective slows down the trading to avoid the market prices moving in an unfavorable fashion (move quickly and you will move the market!). One must consider *implementation shortfall*, which is defined as the difference between the prevailing portfolio price and the effective execution price for the proposed trade schedule. In a seminal paper, Almgren & Chriss [1] minimize the first two moments of implementation shortfall in a Markowitz mean-variance framework to determine the optimal trade schedule. They show that the first moment of the implementation shortfall models the market-impact and the second moment of implementation shortfall models risk. A similar model is also considered in Grinold & Kahn [10]. Multi-period models that only trade-off market-impact and time varying alpha are considered in Bertsimas & Lo [4]. The market-impact is usually a non-linear power function (quadratic, three-halves, five-thirds) of the transactions (see Almgren et al. [2] for more details). Almgren & Chriss consider a quadratic market-impact model and do not include realistic constraints (such as long-only) for the different stages in their multi-period model. Consequently, their model resembles the classical LQR problem from optimal control and they are able to derive a closed-form solution for their model by solving an algebraic Riccati equation.

2. **Transition management and dynamic benchmark tracking:** This is a passive management framework where the aim is to track a benchmark or target portfolio closely over time while minimizing the tracking error and market-impact across several stages. Some of the stages can also include potential cash flows and portfolio injections. This is a common problem faced by large ETF providers who receive cash inflows and portfolios from investors over time and need to track a dynamic index closely. Blake et al. [5] provide a good overview of multi-period optimization models in this area.
3. **Tax-optimization:** Taxes are special transaction cost models that penalize sales. Moreover, short-term gains are taxed at a substantially larger rate than long-term gains. Multi-period models can be used to delay the sale of assets in the portfolio in order to take advantage of the long-term rates. The multi-period tax optimization is especially challenging since the tax liability is a function of both the current share price as well as the prevailing price when the asset was purchased. Multi-period portfolio taxes models are considered in DeMiguel & Uppal [6] and more recently in Haugh et al. [11].
4. **Alpha-decay:** This is the application that we consider in this paper. In a typical portfolio application, the composite alpha signal is a composition of signals with different strengths and decay rates. Usually, the signals that carry more information decay quickly. In an influential paper, Garleanu & Pedersen [7] consider an unconstrained infinite-dimensional multi-period model that trades off risk, return, and quadratic market-impact. This model provides an opportunity to trade-off the strength of the fast decaying signals and the persistence (memory) of the slow decaying signals. Garleanu & Pedersen construct an optimal policy for their model using dynamic programming. We discuss their model in some detail in Section 3.

We consider a simple variant of the alpha-decay use case in this paper. The investor has an alpha that is composed of two signals: (a) a short-term signal that better predicts the returns at the end of the rebalancing time horizon but decays quickly, i.e., the signal in the next rebalancing period is very different to the one in the current period, and (b) a long-term signal that has more memory, i.e., the signal in the next rebalancing period is similar to the one in the current period but is poorer at predicting the realized returns at the end of the rebalancing period than the short-term signal. A short-term investor who only chases the short-term signal would incur high turnover as he rebalances his portfolio over time. This will eat into his returns. The long-term investor who only chases the long-term signal is giving up the valuable alpha in the short-term signal. So the dilemma faced by the investor is this: How do I trade-off the strength of the short-term signal and the slow-decay of the long-term signal while satisfying all my other mandates? We consider a simple two-stage multi-period model that is embedded in a *rolling-horizon* backtester to address this question. The backtester solves a two-stage multi-period model in each period and returns the first-stage portfolio as the final holdings. These holdings are then rolled-forward using the actual realized returns and a new two-stage multi-period model is solved in the next period and so on. To avoid any confusion, we will use *periods* to represent the different time periods (iterations) in the backtest and *stages* to represent the different rebalancings in a point-in-time multi-period model. Our model is inspired by the work in Garleanu & Pedersen [7] but we also consider realistic portfolio constraints in each stage of the model. We must emphasize that one needs an optimizer to solve a multi-period model with realistic constraints. The model no longer admits a closed form solution and also specialized approaches such as dynamic programming no longer apply.

There has been a lot of other work in the alpha-decay case. Israelov and Katz [12] develop an interesting *informed trading* heuristic that uses the short-term alpha signal to time the portfolio's trades; specifically, they continue with a trade only if both short-term and the long-term components of the composite alpha signal agree on the direction of the trade and otherwise cancel it. They test their heuristic on a realistic example and a simulated example with promising results. Grinold [8, 9], Qian et al. [13], and Sneddon [14] each solve a static auxiliary problem that trades-off signal strength and time decay to determine the signal weights. These weights are then used in a single-period model to generate the portfolio holdings.

This paper is organized as follows: Section 2 introduces multi-period portfolio models. Section 3 gives a brief overview of the Garleanu-Pedersen model that is the inspiration for our work. Section 4 presents our rolling-horizon two-stage multi-period algorithm for the two alpha problem. Section 5 compares the performance of our multi-period algorithm with a single-period backtester on a simulated example that is taken from Israelov & Katz [12]. Section 6 presents our computational experiences with the multi-period algorithm on a large strategy with realistic constraints. Section 7 presents our conclusions. The technical appendix gives the mathematical details of some of our main results.

## 2 Multi-period models

Consider the following two-stage multi-period model

$$\begin{aligned}
\max \quad & (\alpha^1)^T w^1 - \delta \text{TC}(\Delta w^1) - \gamma (w^1)^T \Sigma w^1 + E[\alpha^2]^T w^2 - \delta \text{TC}(\Delta w^2) - \gamma (w^2)^T \Sigma w^2 \\
\text{s.t.} \quad & w^2 = (1+r)w^1 + \Delta w^2, \\
& w^1 = w^0 + \Delta w^1, \\
& w^1 \in \mathcal{C}_1, \\
& w^2 \in \mathcal{C}_2
\end{aligned} \tag{1}$$

that trades off risk, return, and transaction costs across two stages. The variables  $w^1$  and  $w^2$  represent the final holdings of the portfolio and  $\Delta w^1$  and  $\Delta w^2$  represent the trades made in the first and second stages of the model, respectively. The initial holdings  $w^0$  are known. The first set of equations describe the final holdings of the portfolio in the second stage as the rolled-forward holdings from the first stage plus the trades made in the second stage. Note that these equations actually *couple* stages one and two. The third and the fourth set of equations represent the constraints that the portfolio must satisfy in the first and second stages of the model. We will hereafter refer to these set of constraints as *independent* constraints since they only involve the holding and the trade variables of a stage.

The inputs to the multi-period model include:

1.  $w^0$  - the initial portfolio holdings vector.
2.  $\alpha^1$  - the alpha vector for the first stage.
3.  $E[\alpha^2]$  - the expected alpha vector for the second stage. The expected alpha depends on the choice of our alpha uncertainty model and we present some details when we consider our two alpha model in Section 4.
4.  $\text{TC}(\Delta w^1)$  and  $\text{TC}(\Delta w^2)$  that represent the transaction cost (TC) models for the first and second stages, respectively. Each TC model is associated with a coefficient  $\delta$  in the objective function.
5. The risk model  $\Sigma$  and the risk aversion coefficient  $\gamma$ . For simplicity, we will assume that both stages use the same risk model and risk aversion coefficient.
6. The estimate  $r$  for the *roll-forward* returns. The portfolios generated by the multi-period optimization are sensitive to this choice!
7. The independent constraints in the first and second stages.

The first stage of the multi-period model represents the current rebalancing that the investor is interested in. The length of this stage corresponds to the rebalancing time horizon that the investor would consider in a single-period setting. The investor knows that she will be rebalancing their portfolio in the future and the second stage of the multi-period model represents this future rebalancing with its own data estimates and decisions. Note that the

decisions made in the second stage are a function of the decisions made in the first stage and the data for the second stage. The objective terms in the second stage of the multi-period model are usually *discounted* in practice as they represent rebalancings that are farther out in time. Note that  $w^2$  really represents the expected second stage portfolio holdings since the alpha for this stage is not known precisely. We are especially interested in the first-stage holdings  $w^1$  since the holdings from the second stage can be updated as more information becomes available. We want to know how the data and the decisions in the second stage of the model help us make *informed* decisions in the first stage. Finally, we must emphasize that it is hard to come with good roll-forward estimates for the multi-period model and we will assume that  $r = 0$  throughout this paper.

### 3 The Garleanu-Pedersen model

We briefly describe the Garleanu-Pedersen multi-period portfolio model [7] that incorporates alpha-decay in this section. Consider a composite alpha signal over  $n$  assets that is composed of  $k$  factors with varying decay rates. The composite alpha in stage  $j$  of the multi-period model is given by

$$\alpha^j = Bf^j \quad (2)$$

where the factors evolve as

$$f^j = (I - \Phi)f^{j-1} + \epsilon^j \quad (3)$$

where  $\Phi$  is a diagonal matrix of factor decay-rates and  $\epsilon^j$  is a random variable with zero mean and finite variance. Garleanu and Pedersen consider the following unconstrained infinite dimensional multi-period portfolio problem to determine the trades  $(w^1, w^2, \dots)$

$$\max_{w^1, w^2, \dots} E_1 \left( \sum_t -\frac{\delta}{2} (\Delta w^t)^T \Sigma (\Delta w^t) + \left( (\alpha^t)^T w^t - \frac{\gamma}{2} (w^t)^T \Sigma (w^t) \right) \right) \quad (4)$$

that trades off the present value of all the future excess expected return (alpha), risk, return, and transaction costs.<sup>1</sup> The expectation is taken over the random  $\alpha^t$ . The risk is represented by the variance of the portfolio returns, where  $\Sigma$  is the asset covariance matrix. The transaction costs are modeled by a quadratic function with the costs proportional to risk. This assumption enables us to derive a simple closed form solution for the model. The positive scalars  $\gamma, \delta$  represent the risk and TC aversion parameters, respectively. Using the law of total expectation, we can write the objective function in (4) as

$$E_{\alpha^1} \left( \text{OBJ}^1 + \left( E_{\alpha^2|\alpha^1} \left( \text{OBJ}^2 + \left( E_{\alpha^3|\alpha^1, \alpha^2} \left( \text{OBJ}^3 + \dots \right) \right) \right) \right) \right) \quad (5)$$

where

$$\text{OBJ}^j = -\frac{\delta}{2} (\Delta w^j)^T \Sigma (\Delta w^j) + (\alpha^j)^T w^j - \frac{\gamma}{2} (w^j)^T \Sigma (w^j)$$

<sup>1</sup>We are neglecting the discount term in the Garleanu-Pedersen model to simplify the exposition.

is the objective function in stage  $j$ . The outer expectation  $E_{\alpha^1}$  in (5) is the expectation over the random  $\alpha^1$ , the inner expectation  $E_{\alpha^2|\alpha^1}$  is the conditional expectation of  $\alpha^2$  with  $\alpha^1$  known, and  $E_{\alpha^3|\alpha^1,\alpha^2}$  is the conditional expectation of  $\alpha^3$  with  $\alpha^1$  and  $\alpha^2$  both known and so on.

The Markowitz portfolio at stage  $j$  that only trades off risk and return during this period is given by

$$MV^j = (\gamma\Sigma)^{-1}Bf^j \quad (6)$$

Consider an investor who performs the following single-period optimization

$$\max_{w^j} (\alpha^j)^T w^j - \frac{\gamma}{2} (w^j)^T \Sigma (w^j) - \frac{\delta}{2} (\Delta w^j)^T \Sigma (\Delta w^j) \quad (7)$$

in stage  $j$  with initial holdings  $w^{j-1}$  where  $\Delta w^j = (w^j - w^{j-1})$ . The optimal single-period portfolio for this investor is given by

$$w_{spo}^j = \frac{\delta}{\gamma + \delta} w^{j-1} + \frac{\gamma}{\gamma + \delta} MV^j. \quad (8)$$

Garleanu & Pedersen [7] use dynamic programming to derive an expression for the portfolio holdings in the different stages. They show that the optimal portfolio at stage  $j$  is

$$w^j = \left(1 - \frac{a}{\delta}\right) w^{j-1} + \frac{a}{\delta} TAR^j \quad (9)$$

where

$$a = \frac{\sqrt{\gamma^2 + 4\gamma\delta} - \gamma}{2} \quad (10)$$

is a positive scalar and

$$TAR^j = (\gamma\Sigma)^{-1} B \begin{bmatrix} \frac{f^{j,1}}{\left(1 + \frac{\phi^1 a}{\gamma}\right)} \\ \vdots \\ \frac{f^{j,k}}{\left(1 + \frac{\phi^k a}{\gamma}\right)} \end{bmatrix} \quad (11)$$

is the target portfolio that trades off risk and return. Comparing (8) and (9), we see that the optimal multi-period portfolio uses the target portfolio instead of the Markowitz portfolio at stage  $j$ . Moreover, the target portfolio is basically the Markowitz portfolio with each factor scaled down by its decay rate.

## 4 Two alpha multi-period model

We apply the two-stage multi-period model that we introduced in Section 2 to the two alpha case in this section. The composite alpha signal in the model is constructed from (a) a strong and rapidly decaying short-term signal, and (b) a weak and persistent long-term signal.

We embed this multi-period model in a *rolling-horizon* backtester. Our model resembles the *model predictive* approach (see Chapter 6 in Bertsekas [3]) to solving stochastic multi-stage problems where one (a) solves a sequence of deterministic two-stage multi-period models conditional on the available information, (b) implements only the first-stage solution, (c) refines the second-stage solution with new information as it becomes available. One can show that the multi-stage *rolling-horizon* backtester generates the same trades as the dynamic programming approach of Garleanu-Pedersen for the unconstrained case with quadratic transaction costs and the same number of stages. On the other hand, the rolling-backtester can incorporate realistic constraints and different transaction cost models (linear, piecewise-linear, three-halves) in the multi-period model, where closed-form solutions are not available and dynamic programming is no longer a viable approach. We must also emphasize that our rolling-horizon model is a heuristic approach to solving the stochastic multi-stage model with realistic constraints. It is however known to generate good solutions in practice (see Bertsekas [3]).

The alpha signal in stage  $j$  of the multi-period model at time  $t$  is given by

$$\alpha^{t,j} = \lambda_s^j \alpha_s^t + \lambda_l^j \alpha_l^t \quad (12)$$

where  $\alpha_s^t, \alpha_l^t$  are the short-term and long-term alpha signals and  $\lambda_s^j, \lambda_l^j$  are their associated weights. Furthermore the signal follow a first-order autoregressive process given by

$$\begin{aligned} \alpha_s^j &= \sigma_s \alpha_s^{j-1} + \epsilon_s^j, \\ \alpha_l^j &= \sigma_l \alpha_l^{j-1} + \epsilon_l^j, \end{aligned} \quad (13)$$

where  $\sigma_s, \sigma_l \in (0, 1)$  denote the short-term and long-term autocorrelations and  $\epsilon_s^t$  and  $\epsilon_l^t$  denote random short-term and long-term innovations with zero mean and finite variance. Note that (13) resembles the Garleanu-Pedersen alpha-decay model, where we assume that the uncertainty is baked into the signals themselves. The autocorrelations measure how close today's signal is to yesterday's signal while the innovations represent the new information flowing into the signals in each period. We have  $\lambda_s > \lambda_l$  since the short-term signal better predicts the immediate returns, i.e., the returns at the end of the previous period. On the other hand, we have  $\sigma_s < \sigma_l$  since the long-term signal has more memory. From (13), we can compute the conditional expected alpha in the next period given the current alpha as

$$\begin{aligned} E[\alpha_s^j | \alpha_s^{j-1}] &= \sigma_s \alpha_s^{j-1}, \\ E[\alpha_l^j | \alpha_l^{j-1}] &= \sigma_l \alpha_l^{j-1}. \end{aligned} \quad (14)$$

We are now ready to present our multi-period backtester.

### Multi-period backtester

1. Start the algorithm with a set of initial holdings  $w^0$ . Estimate the short-term and long-term alpha weights  $\lambda_s, \lambda_l$ , the risk and TC aversion parameters  $\gamma, \delta$ , and the risk model  $\Sigma$ . We will assume a quadratic TC model that is proportional to the risk and that the alpha weights, risk and TC aversion parameters, and risk model are fixed over time.
2. For period  $t$ , estimate the short-term alpha  $\alpha_s^t$  and the long-term alpha  $\alpha_l^t$ .
3. Solve the following two-stage multi-period model

$$\max_{w^1, w^2} \sum_{j=1}^2 E [\alpha^{t,j}]^T w^j - \frac{\gamma}{2} (w^j)^T \Sigma (w^j) - \frac{\delta}{2} (w^j - w^{j-1})^T \Sigma (w^j - w^{j-1}) \quad (15)$$

where  $w^0$  is known and

$$\begin{aligned} E [\alpha^{t,1}] &= \lambda_s \alpha_s^t + \lambda_l \alpha_l^t, \\ E [\alpha^{t,2}] &= \sigma_s \lambda_s \alpha_s^t + \sigma_l \lambda_l \alpha_l^t \end{aligned} \quad (16)$$

at the beginning of period  $t$ . We use (14) to derive the expression of the expected second stage alpha in the multi-period model. We hold the first-stage holding  $w^1$  throughout period  $t$ .

4. Calculate the net return in period  $t$  as the realized portfolio return minus transaction costs.
5. Set  $t = t + 1$ , the initial holding for the next period to be  $w^1$  rolled-forward, and return to Step 2.

A few comments are now in order:

1. We only implement the first-stage decision of the two period model. We are not interested in the second-stage decision as it can be refined as more information becomes available in the future. As we show below, the second-stage in the multi-period model induces a *wait-and-see* feature in the first-stage decision.
2. We show in the Technical Appendix that the first stage solution for (15) is

$$\begin{aligned} w^1 &= \underbrace{\psi \delta w^0}_{\text{Initial portfolio}} \\ &+ \underbrace{\psi \Sigma^{-1} \left( \left( 1 + \frac{\delta \sigma_s}{\delta + \gamma} \right) \lambda_s \alpha_s^t \right)}_{\text{ST Signal strength component}} \\ &+ \underbrace{\psi \Sigma^{-1} \left( \left( 1 + \frac{\delta \sigma_l}{\delta + \gamma} \right) \lambda_l \alpha_l^t \right)}_{\text{LT Time-decay component}} \end{aligned} \quad (17)$$

where

$$\psi = \frac{\gamma + \delta}{(\gamma + \delta)^2 + \delta\gamma}.$$

This shows that the first stage solution is a non-negative combination of three portfolios:

- (a) Initial portfolio - to reduce transaction costs.
  - (b) Signal strength component - this component is the sum of the Markowitz portfolios for the short-term signal (trading-off risk and short-term return) over the two stages. It emphasizes the better predictive power of the short-term signal.
  - (c) Time-decay component - this component is the sum of the Markowitz portfolios for the long-term signal (trading-off risk and long-term return) over the two stages. It emphasizes the slow decay of the long-term signal.
3. We compare our multi-period backtester with a single-period backtester. The single-period backtester follows all the steps of the rolling multi-period algorithm except that it uses different weights on the short-term and long-term signals, a different TC aversion parameter, and solves the following single-period model

$$\max_w (\bar{\lambda}_s \alpha_s^t + \bar{\lambda}_l \alpha_l^t)^T w - \frac{\gamma}{2} w^T \Sigma w - \frac{\bar{\delta}}{2} (w - w^0)^T \Sigma (w - w^0) \quad (18)$$

in Step 3 of the algorithm. One can easily show that the solution to this model in period  $t$  is

$$w^{spo} = \frac{\bar{\delta}}{\bar{\delta} + \gamma} w^0 + \Sigma^{-1} (\bar{\lambda}_s \alpha_s^t + \bar{\lambda}_l \alpha_l^t). \quad (19)$$

4. Comparing (17) and (18), we show in the Technical Appendix that one can recover the first-stage solution of the multi-period algorithm in a single-period backtesting framework if we choose the single-period TC aversion and short-term and long-term weights as

$$\begin{aligned} \bar{\delta} &= \left( \frac{\gamma + \delta}{\gamma + 2\delta} \right) \delta, \\ \bar{\lambda}_s &= \psi (\gamma + \bar{\delta}) \left( 1 + \frac{\sigma_s \delta}{\delta + \gamma} \right) \lambda_s, \\ \bar{\lambda}_l &= \psi (\gamma + \bar{\delta}) \left( 1 + \frac{\sigma_l \delta}{\delta + \gamma} \right) \lambda_l. \end{aligned} \quad (20)$$

For these choice of parameters the single-period backtester is equivalent to the multi-period algorithm. As we show in the Technical Appendix, the single-period backtester has to increase the weight on the long-term signal in order to recover the multi-period solution.

The interesting question is what happens to the first stage solution when one adds infinitely many stages to the model in (15). When the number of stages is very large, we can use the Garleanu-Pedersen result to show that the first stage solution at time  $t$  converges to

$$w^1 = \left(1 - \frac{a}{\bar{\delta}}\right) w^0 + \frac{a}{\bar{\delta}} \Sigma^{-1} \left( \frac{\lambda_s}{\gamma + a(1 - \sigma_s)} \alpha_s^t + \frac{\lambda_l}{\gamma + a(1 - \sigma_l)} \alpha_l^t \right) \quad (21)$$

where

$$a = \frac{\sqrt{\gamma^2 + 4\gamma\rho} - \gamma}{2}.$$

Comparing this expression with (18), one can show that the single-period backtester is equivalent to the multi-period algorithm with infinitely many stages by choosing

$$\begin{aligned} \bar{\delta} &= \frac{\gamma}{a} \left(1 - \frac{a}{\bar{\delta}}\right) \delta, \\ \bar{\lambda}_s &= \frac{a\lambda_s}{\delta(\gamma + a(1 - \sigma_s))}, \\ \bar{\lambda}_l &= \frac{a\lambda_l}{\delta(\gamma + a(1 - \sigma_l))}. \end{aligned} \quad (22)$$

Given that one can replicate the multi-period setting in a single-period framework, why should one be interested in a multi-period portfolio model? The answer is that the equivalence between the multi-period and single-period setups holds only under limited settings. Some settings may require changing more single-period data parameters and there are several cases where the equivalence breaks down.

1. Consider a two period model where one maximizes the two-term alpha subject to constraints that impose upper bounds on risk and TC in the two stages. In this case, one can use the duality theory of convex optimization (see the Technical Appendix for details) to rewrite this problem as (15) where  $\gamma$  and  $\delta$  are the optimal duals to the risk and TC constraints. These duals will vary over time and so an equivalent single-period framework would have to dynamically change the TC aversion and more importantly the short-term and long-term signal weights over time.
2. If the strategy contains realistic long-only constraints or combinatorial constraints such as cardinality and threshold constraints, then it is impossible to find a set of single-period parameters for which the single-period backtester is equivalent to the rolling multi-period algorithm.

We would like to reiterate that multi-period optimization allows the investor to naturally trade-off signal strength and decay in one consolidated optimization problem. The signal weights in the first stage of the model can be chosen entirely on signal strength. These weights are appropriately down-weighted in the later stages of the model based on the signal decay; the short-term signal weight in each of these stages gets downweighted more than the corresponding long-term signal weight.

## 5 Single period vs Multiperiod: Simulated results

This section describes the simulation that we used to compare the multi-period and single-period algorithms. We use the methodology from the simulated example in the Israelov-Katz paper [12] to generate the short-term and the long-term alphas. Both backtests are run over  $T = 10,000$  periods with a portfolio universe of 10 assets. We briefly describe the alpha generation process and refer the reader to [12] for more details.

1. The realized returns are simulated from a normal distribution with zero mean, 20 percent annualized volatility, and 0.5 correlation between the assets.
2. The short-term and long-term alphas are generated from two independent exponentially weighted moving averages (EWMA) (with appropriate half-lives) of the realized returns plus noise. The EWMA construction proceeds backwards in time so that the alphas can explain future returns. The short-term and long-term alphas describe a first-order autoregressive AR(1) process

$$\begin{aligned}\alpha_s^t &= \frac{2}{3}\alpha_s^{t-1} + \epsilon_s^t, \\ \alpha_l^t &= \frac{260}{261}\alpha_l^{t-1} + \epsilon_l^t,\end{aligned}\tag{23}$$

with autocorrelations  $\sigma_s = \frac{2}{3}$  and  $\sigma_l = \frac{260}{261}$ , respectively.

3. The short-term and long-term alphas are then converted into z-scores (cross-sectionally across the 10 assets).

The short-term alpha is designed to forecast returns over a short-time horizon; over this short horizon it forecasts returns better than the long-term alpha. However, the short-alpha signal decays quickly; the daily decay rate is  $\frac{1}{3}$  for the 10 assets. The long-term alpha signal is designed to forecast returns over a longer time horizon with a decay rate of  $\frac{1}{261}$ . Figure 1 plots the autocorrelations (250 time lags) of the fast and the slow moving alphas for asset 1. This plot shows the fast (slow) decay of the short-term (long-term) alpha signals. The autocorrelation plots for the remaining nine assets are similar.

Consider the *Information Coefficient* (IC) (see Qian et al. [13]) of the short-term and the long-term alpha signals which at  $t$  is defined as the correlation between the alpha signal at the start of period  $t$  and the realized returns at the end of period  $t$ . Similarly, we define the *lagged IC* as the correlation between the alpha signal at the beginning of period  $t$  and the realized returns at a future period say  $t+l$  where  $l > 0$ . The IC measures the strength of the alpha-signal in predicting returns, while the lagged IC measures how this predictive power decays over time. Define the IR of an alpha signal as the annualized ratio of the mean IC to the standard deviation of IC. Table 1 presents the IRs for the short-term and long-term alphas with time lags of  $0, \dots, 6$ . The IR of the short-term alpha for a lag of 0 is 1.46, that is twice that of the long-term alpha. This shows that the short-term alpha better predicts

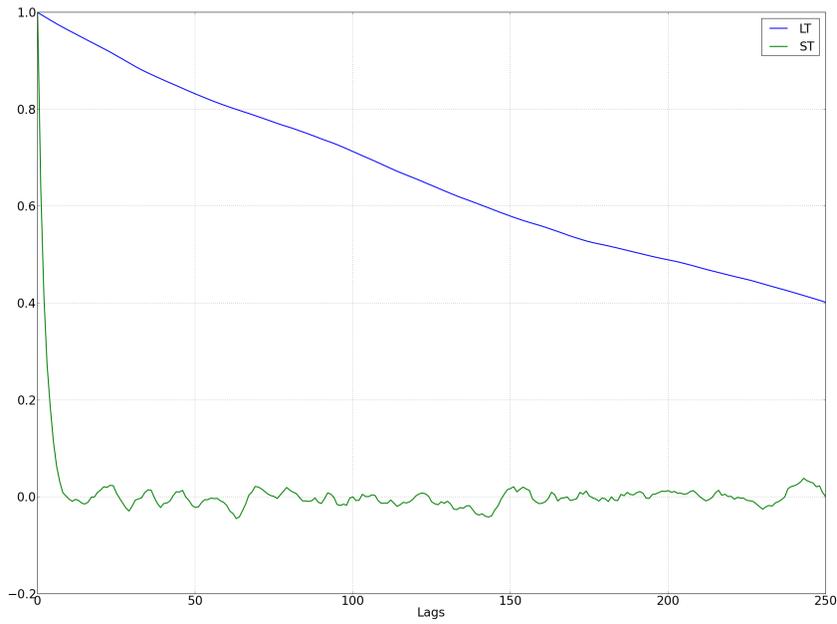


Figure 1: Short-term and long-term alpha autocorrelations

Signal	Information ratio						
	0	1	2	3	4	5	6
ST-Alpha	1.46	0.826	0.441	0.291	0.322	0.221	-0.145
LT-Alpha	0.73	0.73	0.73	0.73	0.73	0.73	0.73

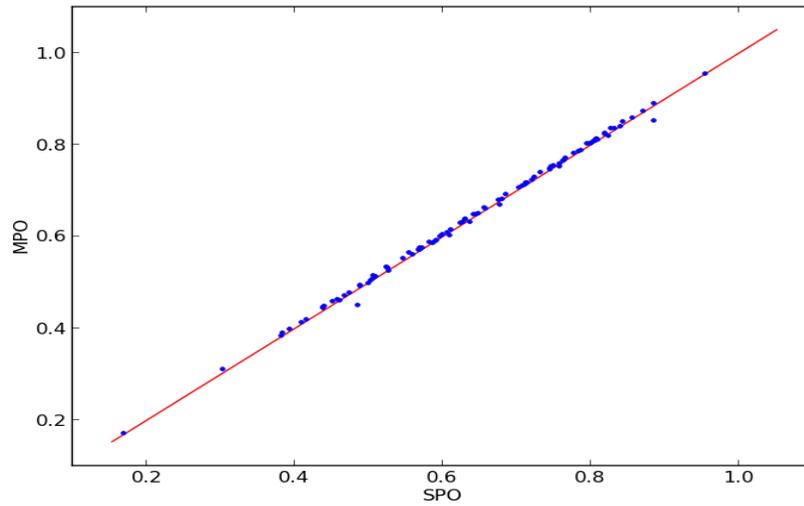
Table 1: IRs of the alpha signals with different time lags

immediate returns. On the other hand, the predictive power of the short-term alpha falls quickly as we increase the number of lags. With a lag of more than 7 periods, the predictive power of the short-term alpha drops to zero. The long-term alpha, however, retains most of its predictive ability as one increases the number of lags.

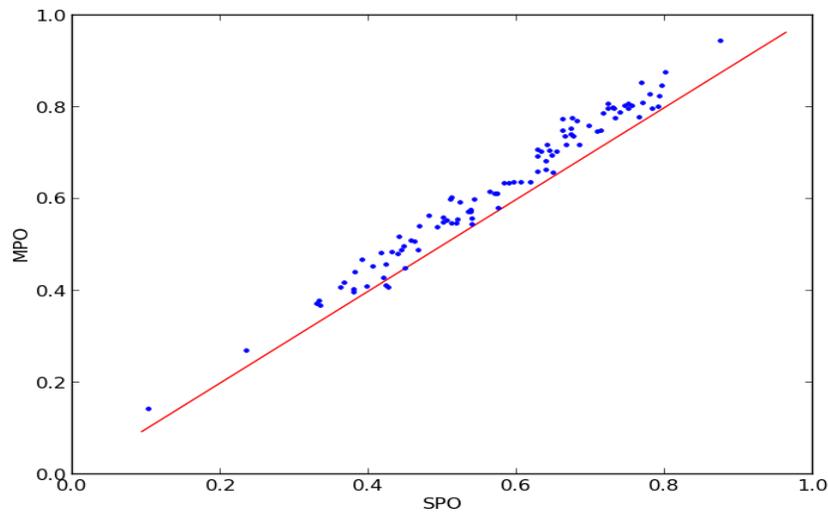
We now describe the multi-period and the single-period backtest setups on this two-alpha model. The multi-period backtester solves a two stage model in each period.

1. Generate 100 different time series of returns and alphas using the EWMA construction in equation (23) with different seeds.
2. For each seed, we run the multi-period and single-period backtests with 10 assets covering 10000 periods where one solves the following model in each period.
  - Maximize two-term alpha with the long-term weights varying from 10 – 100%. For the multi-period model, the short-term and long-term alphas in the second stage are down-weighted by their autocorrelations as described in equation (15).
  - Dollar-neutral and risk bound of 10%. These constraints are applied to both stages of the multi-period model.
  - We run two cases:
    - Case 1 has a  $\frac{3}{2}$  market-impact term in the objective, this market-impact term is applied to both stages of the multi-period model
    - Case 2 has a round-trip turnover constraint that is applied to both stages of the multi-period model. Moreover, in Case 2, we also fix the leverage of the dollar-neutral portfolio to the reference size, i.e., the long and the short sides of the portfolio sum up to the reference size of the portfolio.
  - Fix the weight on the market-impact term in Case 1 and the round-trip turnover bound in Case 2.

Note that we run a frontier using the single-period and multi-period approaches by varying the relative weights on the short-term and the long-term signals. The upper exhibit of Figure 2 presents a scatter plot of the maximum Sharpe ratios obtained using the two approaches with the MI objective (Case 1) for the 100 seeds. Note that for each seed, the maximum Sharpe ratio is the best Sharpe ratio obtained over the signal weight frontier, where we vary the weight on the long-term signal from 10 – 100%. We have also included the 45 degree line as a reference in this exhibit. All dots above the 45 degree line represent scenarios where the multi-period setup returned better maximum Sharpe ratios than the single-period setup. By comparing the best Sharpe ratio of both approaches across the frontier, we want to see whether we can recover the multi-period solution in a single-period setting. We see that most of the dots are clustered across the 45 degree line indicating that both frontier backtests produce similar maximum Sharpe ratios. This is not surprising given that the model in Case 1 is very close to the unconstrained models that we considered in Section 4. There we made the case that it is possible to recover the multi-period solution in a



(a) Max Sharpe ratios



(b) Average Sharpe ratios

Figure 2: SPO vs MPO: Case 1 - Market Impact Objective

single-period framework by appropriately changing some of the parameters in the single-period model - notably the weight on the short-term and the long-term signals. Loosely speaking we can interpret the result in the upper exhibit of Figure 2 as follows: In a simple strategy, if a portfolio manager is good at picking the alpha weights, then they can recover the performance of multi-period backtest in a single-period setting. The lower exhibit of Figure 2 presents a scatter plot of the average Sharpe ratios obtained using the single-period and multi-period frontier backtests with the MI objective (Case 1) for the 100 seeds. This plot tells a different story; most of the dots are above and some are well above the 45 degree reference line indicating that the multi-period backtester is able to deliver better Sharpe ratios on average. Loosely speaking we can interpret these results as follows: If a portfolio manager is average at picking the alpha weights, then the multi-period setting with its in-built alpha-decay feature is more likely to deliver a better realized performance. The upper exhibit of Figure 3 presents a scatter plot of the maximum Sharpe ratios obtained using the single-period and multi-period frontier backtests with the turnover constraint and enforcing leverage (Case 2) for the 100 seeds. About 70% of the dots are above the 45 degree line indicating the multi-period backtester has a better realized performance when more realistic and complicated combinatorial constraints are present in the model. In this case, it is more difficult to achieve the multi-period performance in a single-period setting by just varying the weights on the alpha signals. The lower exhibit of Figure 3 presents a scatter plot of the average Sharpe ratios obtained using the two approaches for Case 2 over the 100 seeds. Almost all the dots are above the 45 degree line. Moreover, a greater proportion of these dots are above the 45 degree line than in the lower exhibit of Figure 2 indicating that on average the multi-period backtester is more likely to deliver a better Sharpe ratio with a more complicated strategy.

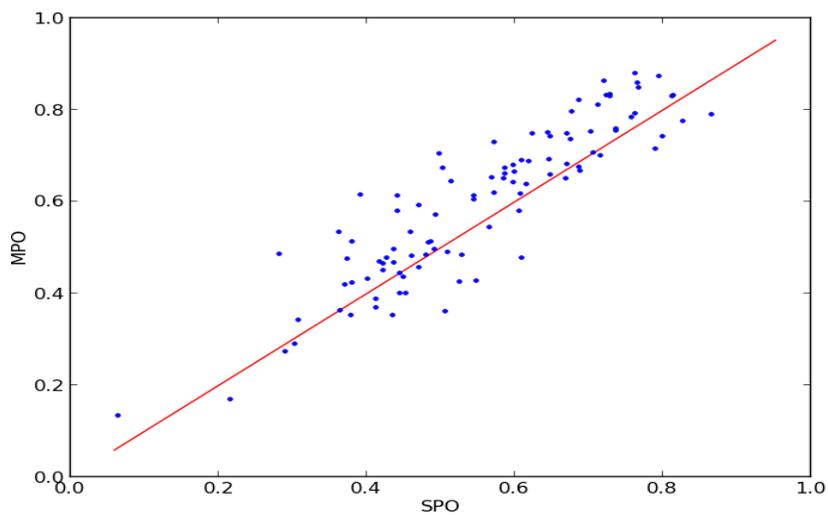
## 6 Single period vs Multiperiod: Realistic backtests

This section describes the realistic backtest that we used to compare the multi-period and single-period algorithms. The short-term and the long-term alphas are obtained from two independent exponentially weighted moving averages (with a half-life of 0.3 months for the short-term signal and 6 months for the long-term signal) of the 24 month forward-looking S&P 500 realized returns plus noise. They are then standardized to z-scores. We generate 30 different series of the short-term and long-term signals by varying the noise. Table 2 presents

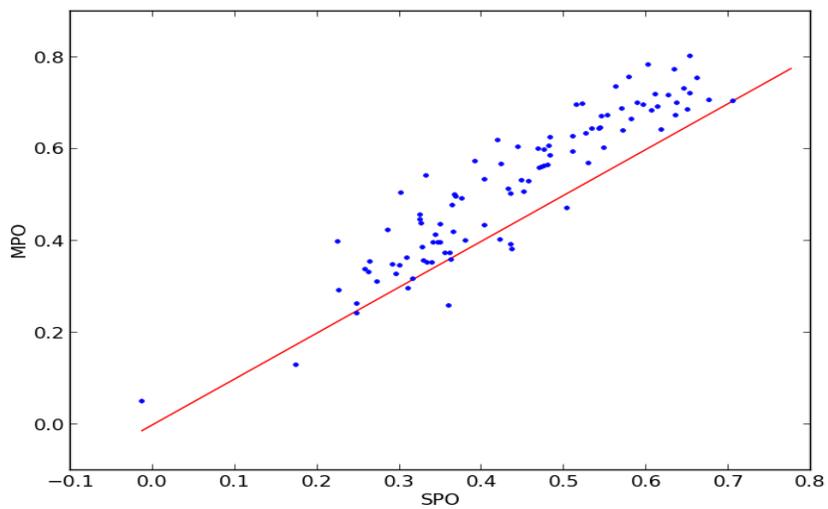
Signal	Information ratio			
	0	1	2	3
ST-Alpha	1.81	0.30	-0.06	0.11
LT-Alpha	1.24	1.15	0.94	0.82

Table 2: IRs of the alpha signals with different time lags

the IRs of the alpha signals with different time lags (each lag corresponds to a month).



(a) Max Sharpe ratios



(b) Average Sharpe ratios

Figure 3: SPO vs MPO: Case 2 - Turnover constraint and forcing leverage

The short-term signal better predicts the immediate returns than the long-term signal but its predictive power decays rapidly as we increase the number of lags. In particular, the short-term alpha loses its predictive power within two months. The autocorrelations for the short-term and long-term signals are 0.1 and 0.9, respectively.

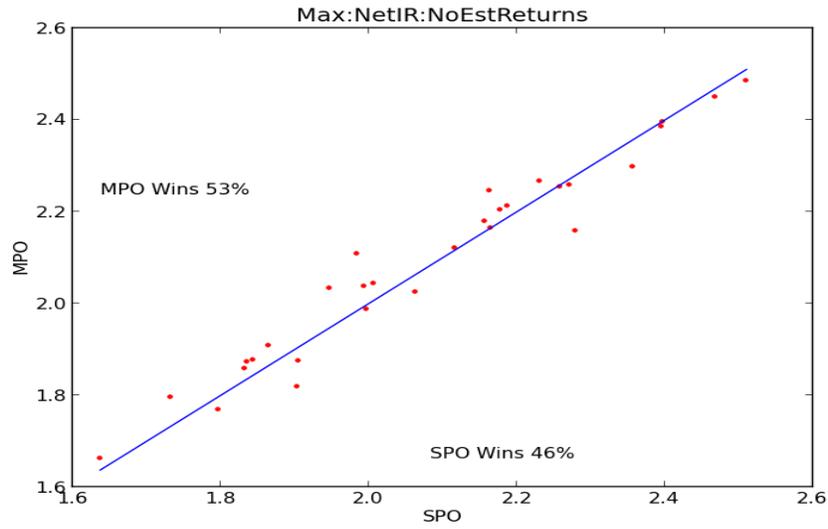
We now run the following backtest with the single-period and multi-period algorithms:

1. Monthly backtest run from January 1997 to January 2012.
2. Maximize two-term alpha with the long-term weights varying from 10 – 100%.
3. Our investment universe includes all the assets in the S&P 500.
4. Portfolio is long-only and fully-invested.
5. Tracking error of 3% with respect to the S&P 500.
6. Turnover limit of 10%.
7. Limit absolute asset trades to 5% of the asset's 20-day ADV.
8. Use 10 bps for buys/sells when computing the net realized returns in each period.

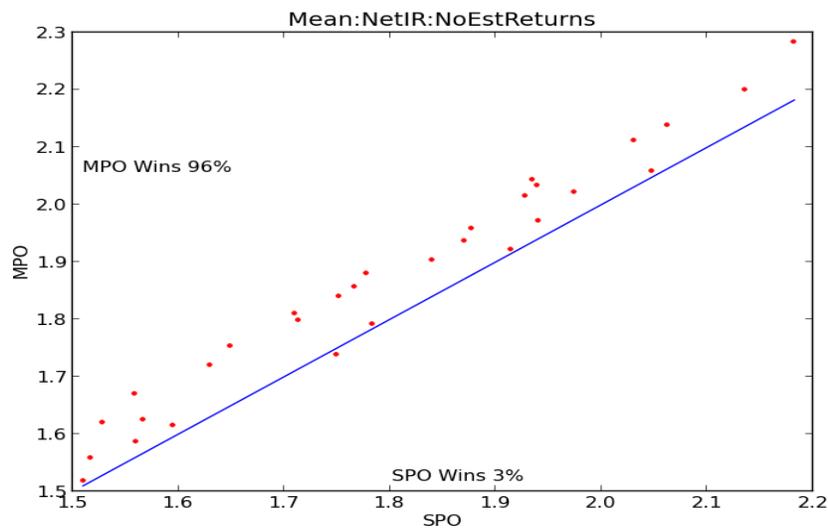
Our multi-period backtester has the following features:

1. It solves a two stage multi-period model in each rebalancing.
2. The short-term and long-term alphas are down-weighted by their autocorrelations in the second period. More specifically, the short-term signal weight in the second period is 0.1 times its weight in the first period. The long-term signal weight in the second period is 0.9 times its weight in the first period.
3. The multi-period model has a copy of the realistic constraints in both stages.
4. We continue to assume that the roll-forward return estimates are zero.

The upper exhibit of Figure 4 presents a scatter plot of the maximum single-period and multi-period Sharpe ratios for the 30 seeds. About 53% of the dots lie above the 45 degree reference line showing that the multi-period backtester performs slightly better than the single-period backtester over these 30 seeds. The lower exhibit of Figure 4 presents a scatter plot of the mean Sharpe ratios for the 30 seeds. About 96% of the dots are now above the 45 degree line. This shows that the multi-period backtester performs much better than the single-period backtester when the portfolio manager is "average" at picking the signal weights.



(a) Max Sharpe ratios



(b) Mean Sharpe ratios

Figure 4: SPO vs MPO: Realistic setup

## 7 Conclusions

We consider a portfolio construction model with a composite alpha signal that is composed of a short-term and a long-term alpha signal. This is a classical problem in investing and alpha signals that can be classified as short-term or long-term are already part of the arsenal of most quantitative researchers. We develop a simple two-stage multi-period model that incorporates such an alpha model and generates a more informed first-stage decision with the available information. The first-stage decision also incorporates a *wait-and-see* feature since this decision is aware of data and decisions beyond the rebalancing time horizon through the second stage in the multi-period model. We embed the two-stage multi-period model in a *rolling-horizon* backtester and compare this algorithm with the traditional single-period backtester on a simulated example from Israelov & Katz [12] and also a sizable, more realistic strategy. We show that the multi-period algorithm generates portfolios with a better realized performance in both sets of experiments.

It is clear that the multi-period algorithm implicitly favors the long-term signal in the optimization and one can attempt to reproduce this in the single-period backtester by assigning more weight to the long-term signal. This is why we generated frontiers for the single-period and multi-period approaches by varying the relative weights on the short-term and the long-term signals. By comparing the best (over the frontier) Sharpe ratio of each approach, one can see how close one can get to the multi-period portfolio performance by using the single-period approach. We observe that with the more complex, more constrained strategies, even the best Sharpe ratio with the single-period model is generally not that close to the best Sharpe ratio obtained via the multi-period model. This indicates that one cannot recover the multi-period portfolios by choosing a fixed adjustment (through time) to the signal weights in the single-period approach. One can still hope for a time varying adjustment of the single-period signal weights that would allow a single-period model achieve the benefits of the multi-period approach. However, the complexity of finding such weight adjustments in realistic strategies makes this impractical.

We also used a scaled version of the short-term alpha as an estimate for the *roll-forward* returns. We found that the performance of the multi-period approach was (a) similar to the zero roll-forward estimate case that we reported in the paper, and (b) was not very sensitive to the choice of these *roll-forward* returns. We also performed an additional experiment, where we use the true forward looking returns in the roll-forward estimate, but we did NOT use those true returns in the alpha estimates. In this case, we found that the performance of the multi-period approach drastically improved. Obviously, this is not of practical use but we do think that coming up with good *roll-forward* estimates that improve the performance of the multi-period algorithm is a worthy topic for future investigation.

To summarize, multi-period optimization allows the investor to naturally trade-off signal strength and decay in one consolidated optimization problem. The signal weights in the first stage of the model can be chosen entirely on signal strength. These weights are appropriately down-weighted in the later stages of the model based on the signal decay; the short-term signal weight in each of these stages gets down-weighted more than the corresponding long-term signal weight. An optimizer is necessary to solve the multi-period model with realistic

constraints and our computational experience indicates that the solution time taken by the *rolling-horizon* two stage multi-period algorithm is comparable to a single-period backtester in practice.

## References

- [1] R. ALMGREN AND N. CHRISS, *Optimal execution of portfolio transactions*, Journal of Risk (Winter 2000/2001), pp. 5-39.
- [2] R. ALMGREN, C. THUM, E. HAUPTMANN, AND H. LI, *Direct estimation of equity market impact*, Risk, 18(2005), pp. 57-62.
- [3] D. BERTSEKAS, *Dynamic Programming and Optimal Control*, Volume I, Athena Scientific, Nashua, NH 03061-0805, 2012.
- [4] D. BERTSIMAS AND A. LO, *Optimal control of execution costs*, Journal of Financial Markets, 1(1998), pp. 1-50.
- [5] C. BLAKE, D. PEITRICH, AND A. ULITSKY, *The right tool for the job: Using Multi-period optimization in transitions*, Trading, Summer 2003, pp. 33-37.
- [6] V. DEMIGUEL AND R. UPPAL, *Portfolio investment with the exact tax basis via non-linear programming*, Management Science, 51(2005), pp. 277-290.
- [7] N. GARLEANU AND L.H. PEDERSEN, *Dynamic trading with predictable returns and transaction costs*, Journal of Finance, 68(6), December 2013, pp 2309-2339.
- [8] R. GRINOLD, *Dynamic portfolio analysis*, The Journal of Portfolio Management, Fall 2007, pp. 12-26.
- [9] R. GRINOLD, *Signal weighting*, The Journal of Portfolio Management, Summer 2010, pp. 1-11.
- [10] R.C. GRINOLD AND R.N. KAHN, *Active Portfolio Management*, 2nd edition, McGraw-Hill, New York, NY, 2000.
- [11] M. HAUGH, G. IYENGAR, AND C. WANG, *Tax-aware dynamic asset allocation*, Technical Report, Department of IE and OR, Columbia University, March 2014. Available at [http://papers.ssrn.com/sol3/papers.cfm?abstract\\_id=2374966](http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2374966)
- [12] R. ISRAELOV AND M. KATZ, *To Trade or Not to Trade? Informed trading with Short-Term signals for Long-Term investors*, Financial Analysts Journal, 67(5), 2011, pp. 23-36.
- [13] E. QIAN, E.H. SORENSEN, AND R. HUA, *Information horizon, portfolio turnover, and optimal alpha models*, The Journal of Portfolio Management, 34(Fall 2007), 2007, pp. 27-40.

- [14] L. SNEDDON, *The tortoise and the hare: Portfolio dynamics for active managers*, The Journal of Investing, Winter 2008, pp. 106-111.

## Technical Appendix: Recovering the first-stage MPO solution in an SPO framework

We compare simple unconstrained two-stage multi-period (MPO) and single-period (SPO) models that trade off risk, return, and transaction costs. The alpha signal for the two models are constructed from short-term and long-term alpha signals.

We assume that the alpha signal for the two models has two components:

1. A strong but fast decaying alpha signal  $\alpha_s$ .
2. A weak but slowly decaying alpha signal  $\alpha_l$ .

To obtain analytic solutions, we consider a quadratic transaction cost model

$$\text{TC}(\Delta w) = \frac{1}{2} (\Delta w)^T \Lambda \Delta w, \quad (24)$$

with  $\Lambda = \delta \Sigma$ , where  $\Sigma$  is the covariance matrix that defines risk. These assumptions greatly simplify the expressions for the optimal SPO and MPO portfolios in the following discussions.

Consider the following two-stage MPO model

$$\max_{w^1, w^2} (\alpha^1)^T w^1 + E[\alpha^2]^T w^2 - \sum_{k=1}^2 \left( \frac{\gamma}{2} (w^k)^T \Sigma w^k - \frac{\delta}{2} (\Delta w^k)^T \Sigma \Delta w^k \right) \quad (25)$$

with  $\Delta w^2 = w^2 - w^1$  and  $\Delta w^1 = w^1 - w^0$  where  $w^0$  is known. The alphas for the two stages are given by

$$\begin{aligned} \alpha^1 &= \lambda_s \alpha_s + \lambda_l \alpha_l, \\ E[\alpha^2] &= \sigma_s \lambda_s \alpha_s + \sigma_l \lambda_l \alpha_l \end{aligned} \quad (26)$$

where  $\lambda_s, \lambda_l$  represent the weights of the short-term and long-term signals and  $\sigma_s, \sigma_l$  represent the short-term and long-term autocorrelations. We will assume that

1.  $\lambda_s > \lambda_l$  since the short-term signal better explains the realized returns at the end of the rebalancing period.
2.  $\sigma_s < \sigma_l$  since the long-term signal decays more slowly.

The optimality conditions for (25) are

$$\begin{aligned} E[\alpha^2] - \gamma \Sigma w^2 - \delta \Sigma \Delta w^2 &= 0, \\ \alpha^1 - \gamma \Sigma w^1 - \delta \Sigma \Delta w^1 + \delta \Sigma \Delta w^2 &= 0. \end{aligned} \quad (27)$$

The first equation in (27) gives

$$w^2 = \frac{\delta}{\gamma + \delta} w^1 + \frac{1}{\gamma + \delta} \Sigma^{-1} E[\alpha^2]. \quad (28)$$

Substituting this expression for  $w^2$  in the second equation in (27) in turn and simplifying, we have

$$w^1 = a\Sigma^{-1}\alpha^1 + a\delta w^0 + a\frac{\delta}{\delta + \gamma}\Sigma^{-1}E[\alpha^2] \quad (29)$$

where

$$a = \frac{\gamma + \delta}{(\gamma + \delta)^2 + \delta\gamma}, \quad (30)$$

and

$$\begin{aligned} MV^1 &= \frac{1}{\gamma}\Sigma^{-1}\alpha^1, \\ &= \frac{1}{\gamma}\Sigma^{-1}(\lambda_l\alpha_l + \lambda_s\alpha_s), \\ E[MV^2] &= \frac{1}{\gamma}\Sigma^{-1}E[\alpha^2], \\ &= \frac{1}{\gamma}\Sigma^{-1}(\sigma_l\lambda_l\alpha_l + \sigma_s\lambda_s\alpha_s) \end{aligned} \quad (31)$$

are the first and second stage Markowitz portfolios.

This shows that the optimal first-stage MPO portfolio is a non-negative combination of three portfolios:

1. Initial portfolio (to reduce transaction costs).
2. Markowitz portfolio from the first stage that heavily weights the short-term alpha since this alpha has better predictive power (signal strength component).
3. The expected Markowitz portfolio from the second stage that overweights the long-term alpha since this component decays more slowly (alpha-decay component).

We are interested in constructing the first-stage multi-period solution in a SPO model by only changing:

1. The weight  $\delta > 0$  on the transaction cost term.
2. The weights  $\lambda_s > 0$  on the short-term signal and  $\lambda_l > 0$  on the long-term signal in the composite alpha.

Consider the SPO model

$$\max_w \left( (\alpha)^T w - \frac{\gamma}{2} (w)^T \Sigma w \right) - \frac{\delta}{2} (\Delta w)^T \Sigma \Delta w \quad (32)$$

where  $\Delta w = w - w^0$  with  $w^0$  known and the alpha signal is given by

$$\alpha = \bar{\lambda}_s \alpha_s + \bar{\lambda}_l \alpha_l. \quad (33)$$

Note that the SPO model uses the same risk model and risk aversion parameter as the MPO model. It can be easily shown that the optimal solution is given by

$$w_{SPO} = \frac{\gamma}{\bar{\delta} + \gamma} MV^1 + \frac{\bar{\delta}}{\bar{\delta} + \gamma} w^0 \quad (34)$$

where

$$\begin{aligned} MV^1 &= \frac{1}{\gamma} \Sigma^{-1} \alpha^1, \\ &= \frac{1}{\gamma} \Sigma^{-1} (\bar{\lambda}_s \alpha_s^1 + \bar{\lambda}_l \alpha_l^1) \end{aligned} \quad (35)$$

is the unconstrained Markowitz portfolio that only trades off risk and return.

This shows that the SPO optimal portfolio is a convex combination of two portfolios:

1. Initial portfolio (to reduce transaction costs).
2. Markowitz portfolio that heavily weights the short-term alpha since this alpha has better predictive power.

We want to choose  $\bar{\delta}$ ,  $\bar{\lambda}_s$ , and  $\bar{\lambda}_l$  so that we recover the optimal first-stage MPO portfolio (29) in the SPO solution (34).

Equating (29) and (34) and simplifying, we have

$$\begin{aligned} \bar{\delta} &= \frac{(\gamma + \delta)\delta}{(\gamma + 2\delta)}, \\ \bar{\lambda}_s &= a(\gamma + \bar{\delta}) \left(1 + \frac{\sigma_s \delta}{\delta + \gamma}\right) \lambda_s, \\ \bar{\lambda}_l &= a(\gamma + \bar{\delta}) \left(1 + \frac{\sigma_l \delta}{\delta + \gamma}\right) \lambda_l, \end{aligned} \quad (36)$$

where the scalar  $a$  is given in (10). These are the values of the SPO parameters that help one recover the first-stage optimal MPO solution in the SPO setting.

Note that

$$\begin{aligned} \frac{\bar{\lambda}_s}{\bar{\lambda}_l} &= \left( \frac{1 + \frac{\sigma_s \delta}{\delta + \gamma}}{1 + \frac{\sigma_l \delta}{\delta + \gamma}} \right) \frac{\lambda_s}{\lambda_l} \\ &< \frac{\lambda_s}{\lambda_l} \end{aligned}$$

since  $\sigma_l > \sigma_s$ . This indicates that the SPO model has to increase the weight on the long-term signal in order to capture its slow time decay.

Now consider the following MPO model

$$\begin{aligned}
 \max_{w^1, w^2} \quad & (\alpha^1)^T w^1 + E[\alpha^2]^T w^2 - \sum_{k=1}^2 \frac{\gamma}{2} (w^k)^T Q w^k \\
 \text{s.t.} \quad & \frac{1}{2} (\Delta w^1)^T \Sigma (\Delta w^1) \leq \kappa, \\
 & \frac{1}{2} (\Delta w^2)^T \Sigma (\Delta w^2) \leq \kappa
 \end{aligned} \tag{37}$$

where the transaction costs are modeled as constraints.

Using the duality theory of convex optimization, this problem can be equivalently written as

$$\max_{w^1, w^2} \quad (\alpha^1)^T w^1 + E[\alpha^2]^T w^2 - \sum_{k=1}^2 \frac{\gamma}{2} (w^k)^T Q w^k - \frac{\delta_1^*}{2} (\Delta w^1)^T \Sigma (\Delta w^1) - \frac{\delta_2^*}{2} (\Delta w^2)^T \Sigma (\Delta w^2)$$

where  $\delta_1^*$  and  $\delta_2^*$  are the optimal dual variables for the two TC constraints. We can also use the procedure described earlier to determine the SPO parameters needed to recover the optimal first-stage MPO portfolio. Note that the procedure needs to be reworked a bit since  $\delta_1^*$  and  $\delta_2^*$  are different.

Moreover, if one were to embed the SPO and the MPO models in a backtest, then one would need to modify the TC aversion and the weights on the short-term and long-term signals in the SPO model during each period of the backtest in order to consistently reproduce the optimal first-stage MPO portfolio.



**Flexible is better.®**

**US and Canada: 212-991-4500**

**Europe: +44 20 7856 2451**

**Asia: +852-8203-2790**

**New York Office**

Axioma, Inc.  
17 State Street  
Suite 2550  
New York, NY 10004  
**Phone:** 212-991-4500  
**Fax:** 212-991-4539

**Hong Kong Office**

Axioma, (HK) Ltd.  
Unit B, 17/F, Entertainment Building  
30 Queen's Road Central  
Hong Kong  
**Phone:** +852-8203-2790  
**Fax:** +852-8203-2774

**Singapore Office**

Axioma, (Asia) Pte Ltd.  
30 Raffles Place  
#23-00 Chevron House  
Singapore 048622  
**Phone:** +65 6233 6835  
**Fax:** +65 6233 6891

**Atlanta Office**

Axioma, Inc.  
8800 Roswell Road  
Building B, Suite 295  
Atlanta, GA 30350  
**Phone:** 678-672-5400  
**Fax:** 678-672-5401

**San Francisco Office**

Axioma, Inc.  
201 Mission Street  
Suite #2230  
San Francisco, CA 94105  
**Phone:** 415-614-4170  
**Fax:** 415-614-4169

**Geneva Office**

Axioma CH  
Rue du Rhone 69, 2nd Floor  
1207 Geneva, Switzerland  
**Phone:** +33 611 96 81 53

**London Office**

Axioma, (UK) Ltd.  
30 Crown Place London, EC2A 4EB  
**Phone:** +44 (0) 20 7856 2451  
**Fax:** +44 (0) 20 3006 8747

**Sales: [sales@axioma.com](mailto:sales@axioma.com)**

**Client Support: [support@axioma.com](mailto:support@axioma.com)**

**Careers: [careers@axioma.com](mailto:careers@axioma.com)**