Perfect dimensional ratios and optimality of some empirical numerical standards

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Abstract

Experience and observations often underlie some widely used numerical characteristics. The problem is in the extent to which such characteristics are optimal. The paper presents results of theoretical analysis of the most frequently used numerical characteristics regarding the number of classes in classification systems, of the base of the number system, and of the level of confidence in the various estimates. Conceptions of informational optimality, harmony and balance were used as criteria in the analysis. The study was also aimed at determining the allowable deviation from the optimal values (mathematical constants) in the identification of the manifestation of harmony, balance and optimality. The predominance of manifestation of mathematical harmony in nature and human activity has been explained using the estimation of optimization reliability.

1. Introduction

Some commonly used numerical characteristics have had no theoretical explanation. In particular, their listing includes: integer 7 as the classificational characteristic in various scientific aspects and in practical purposes, and even in spiritual sphere; integer 10 - the base of decimal number system; 95% - the level of confidence to various estimates, including measurement results. Among often used characteristics of classificational character one can also place integers 3 and 4, which characterize Test Uncertainty Ratios 3:1 and 4:1 respectively, which are most popular in measurement practice.

Versatility and practicality of all these standards stimulates their consideration in terms of simple quantitative conceptual categories of optimality, harmony and balance used in the paper.

There is another problem, which initially shall be and is solved in the paper. It is permissible deviations from numeric constants of optimality, harmony and balance in the assessment of their quantitative manifestations in nature and in human practice.

Clearly, the resolution of above problems lies in the sphere of optimization. The paper discusses the optimization criterion for the simplest system of two components of quality of the same dimensionality, no matter how the quality is named. If considering the components in terms of grouping by parts of complete quality, this system represents the classification according to dichotomy. The optimization method used that is based on the principles of qualimetry and information theory. A combination of these principles enabled to pass to the weights of components as of the relative characteristics of their individual contribution into quality. The usage of normalized weights as analogs of probabilities allowed considering them as arguments for informational criterion of optimization.

In connection with the above problems the paper elucidates the reason of predominant manifestation of information harmony in nature and human practice as compared with information optimality and balance. To this end the analysis was performed basing on geometrical simulation of information optimality.

It should be noted the used dichotomic approach, focused to numerical standards, represents the extreme case of a common informational optimization in selecting characteristics of quality, and of optimizing the accuracy of measurement [1] or estimations in analyzing multi-component system.

2. Optimal classification index

In terms of qualimetry [2] the contributions of components to system's quality are characterized by means of their weights (K). A two-component system is described by K_{max} and K_{ψ} – the weight of the maximal and of the lesser contribution to quality respectively. Relative values of weights: $K_{max}(\rho) = 1/(1 + \rho)$ and $K_{\psi}(\rho) = \rho/(1 + \rho)$. The ratio $\rho = K_{\psi}/K_{max}$ may be named as *classification index*, which represents the initial object of optimization.

When considering relative weights as of analogs of probabilities, the optimization is performed using principles borrowed from information theory. In so doing, the optimal index of classification (also called informatively optimal accuracy coefficient) is determined as $\rho_o = K_{\psi o}/K_{max} = 1/2\pi \approx 0.159$, where $\psi_o = \psi_o(\rho)$ is the necessary and sufficient fraction of integer 2.

Index ρ_o may be qualified as the fundamental informational constant, tightly bound with the mathematical constant π, and therefore ρ_o expresses the so-called information cycle [1]. The substantiation of ρ_o is performed using information entropy regarding the system of weights that in a simplest case, with which we are dealing, consists in solving the following equations system:

$$\begin{cases} \rho_o = arg\ min[\psi_o(\rho) - 1.5 = 0]; & (1) \\ \psi(\rho) = exp[- K_{max}(\rho) \ln K_{max}(\rho) - K_{\psi o}(\rho) \ln K_{\psi o}(\rho)] & (2) \end{cases}$$

where $\psi(\rho) = 1.5 = \psi_o$ is true for the most uncertain classification situation (50% confidence) about allowing or ignoring the lesser component.

The advantage in application of expression (2) is clarified in Appendix 1 being based also on the results of analysis performed in section 6.

In a quality estimation $(1 - \rho_o)$ indicates on information necessity and sufficiency regarding the component that is characterized by a maximal weight, while ρ_o reflects the boundary in absence of informational redundancy relative to the second component. Mainly the optimization is being associated with some informational redundancy, its quantitative characteristic should be determined and proven, and this is also the aim of further consideration.

3. Perfect dimensional ratios and permissible deviations from optimality

Ratio ρ_o, symbolizing informational optimality, is one of three so-called perfect dimensional ratios (PDR). Two others: the golden ratio $\Phi = (\sqrt{5} - 1)/2 \approx 0.618$ – fundamental constant symbolizing mathematical harmony, and the dimensional balance which we denote $\lambda = 0.5$.

All manifestations of perfect dimensional ratios in nature and human practice are always characterized by deviations from their constants (C_{PDR}). The problem consists in the absence of substantiated permissible range of the deviations. When considering a constant C_{PDR} as optimum value, the problem in question relates to the determination of optimization limits. Apparently, this problem is not confined to PDR, but is significant in the aspect of optimization in general; hence the approach proposed below may be useful in this respect also.

The problem can be resolved by analogy with the method of ρ_o estimation. In so doing, if $\pm 0.5\delta$ is the variation of estimation error in determining C_{PDR}, one may prove that the permissible range of deflection from C_{PDR} equals the optimal error (δ_o) that is defined as follows:

$$\delta_o = arg\ min\{\psi_o(\delta) = exp[-K_{max}(\delta) ln\ K_{max}(\delta) - K_{\psi o}(\delta) ln\ K_{\psi o}(\delta)] = 1.5\} = \rho_o C_{PDR}, \quad (3)$$

where $K_{max}(\delta) = (C_{PDR} - 0.5\delta)/(C_{PDR} + 0.5\delta)$; (4)

$K_{\psi o}(\delta) = \delta/(C_{PDR} + 0.5\delta)$ (5)

Thus, PDR values with deviations $\pm 0.5\rho_o C_{PDR}$ (approximately ±8% of any constant) are listed in Table 1. These values can be taken as tolerances by the criterion of acceptability in determining the conformity to the requirements of the optimality, harmony and balance.

Table 1: The list and expressions of perfect numerical ratios

Dimensional concept	Expression of optimal limitation for PDR	Numerical values with tolerances	
		C_{PDR}	$1 - C_{PDR}$
Optimality	$\rho_o (1 \pm 0.5\ \rho_o)$	0.159 ± 0.013	0.841 ± 0.013
Harmony	$\Phi (1 \pm 0.5\ \rho_o)$	0.618 ± 0.049	0.382 ± 0.049
Balance	$\lambda (1 \pm 0.5\ \rho_o)$	0.5 ± 0.04	0.5 ± 0.04

While harmony is the widely known concept, more rare manifestations of balance and optimality have not yet attracted noticeable attention. At the same time harmony and balance are fundamentally associated via relations of adjacent Fibonacci numbers, which start with pure balance. Interestingly, in the succession of these relations after 0.5 the next number is 0.667, which is the boundary permissible deviation from Φ. The comparison of this phenomenon with result of calculation of limiting value $[1 - \Phi(1 + 0.5\rho_o)]/\Phi(1 + 0.5\rho_o) = \lambda$ is one of substantiations of its informational nature in terms of optimization. Besides, in terms of optimal PDR limitations the interrelation between harmony and balance is easy illustrated by means of equations: $\lambda/(1 + \lambda) = 0.333 = 1 - \Phi(1 + 0.5\ \rho_o)$, and $1/(1 + \lambda) = 0.667 = 1 - \Phi(1 - 0.5\ \rho_o)$.

Most PDR manifestations can be expected amongst evolutionary-steady systems in non-animate nature. Critical temperatures of water are an example of such systems which are dealt with in the Appendix 2.

Comparatively frequently PDR manifestations are detectable in systems being still on the stage of evolution. For example, the PDR manifestations for normal biomedical parameters of blood, listed in [3], consist in 63% cases for Φ, 21% for λ, and 16% for ρ_o, as has shown in [4].

4. Geometrical simulation of information optimality

Geometrically numerical ratios $\rho = 1/x$ (where $x > 0$) can be treated in terms of n-dimensional models ($n = 1, 2, 3, \ldots$). We shall deal with simplest unitary models, so that $(1/x)^{1/n} = 1$. If $1/x = \rho_o = \text{const}$, one may pass to the function $y = \rho_o^{1/n}$ or else (through classification duality) the function $y' = 1 - \rho_o^{1/n}$. In our habitual world, for instance, geometrically the ratios can be treated in terms of one-, two- or three-dimensional model, i.e. a line ($n = 1$), a section ($n = 2$), or a volume ($n = 3$) respectively that is exemplified in Appendix 3.

The calculating analysis for determining dimensions meeting the requirements of information optimality when using the unitary models is carried out by the criterion of permissible relative deviation (≈ 8%) of y or $(1 - y^{'})$ from the respective C_{PDR}. Results of calculation are presented in Table 2. Six types of dimensional models were found out that satisfy the above condition.

Table 2: Results of calculation estimates: y, $y^{'}$ and deviations from C_{PDR}

Calculated Estimates	Results estimation for n-dimensional models					
	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 10$	$n = 11$
$y = \rho_o^{1/n}$	0.159	0.399	0.542	0.631	0.832	0.846
$y^{'} = 1 - \rho_o^{1/n}$	0.841	0.601	0.458	0.369	0.168	0.154
Deviation from C_{PDR} (%)	0	3	8	2	6	3

Special integers X_o as functions of n and PDR constants, followed by rounding off ($\mathcal{R}$), were determined with such an approach by using the following equations:

for n = 1, 3, and 4 as $X_{o(PDR)} = \mathcal{R}\{1/C_{PDR}^{n}\}$; (6)

for n = 2, 10, and 11 as $X_{o(1\text{-}PDR)} = \mathcal{R}\{1/(1 - C_{PDR})^{n}\}$ (7)

Results of X_o calculation are presented in Table 3. They, in particular, demonstrate the existence of three specific integers: 6, 7, and 8 that correspond to the perfect dimensional ratios and, as demonstrated in the next section, have a bearing to optimizing the numbers of classification levels.

Table 3: Results of X_o calculation

C_{PDR}	$1 - C_{PDR}$	X_o					
		n =1	n =2	n =3	n =4	n =10	n =11
ρ_o	$1 - \rho_o$	6				6	7
Φ	$1 - \Phi$		7		7		
λ	$1 - \lambda$			8			

The following outcomes are also obtained:

(a) other than two-dimensional model the harmony is peculiar to four-dimensional model. This result can be of use for the studies of four-dimensional space, and of four-dimensionalism;

(b) in addition to one-dimensional model, the informational optimality is specific to models with ten and eleven dimensions that need separate consideration.

5. Optimal classification integers and optimal base for number system

The objective tendency to optimization in nature and human practice allows expecting that the "magical" by Miller's definition [5] number 7, as well as integer 10 meet the requirements of informational optimality as the base characteristic of classification and as the base of number system respectively. As the first step aimed to examine of this tendency one can analyze the correspondence of classificational integers to PDR. In this case the optimization regarding respective integers (X_o) is carried out by the following equations:

$X_{o(\rho o)} = \mathcal{R}\{arg\ min|\rho_o - (1/x)/(1 - 1/x)| = 0\};$ (8)

$X_{o(\Phi)} = \mathcal{R}\{arg\ min|(1 - \Phi) - (1/x)/(1 - 1/x)| = 0\};$ (9)

$X_{o(\lambda)} = \mathcal{R}\{arg\ min|\lambda - (1/x)/(1 - 1/x)| = 0\}$ (10)

Graphically the results of calculation are presented in Fig. 1, where $\Delta(C_{PDR})$ is function of x that in the equations (8), (9) and (10) is denoted as the absolute value.

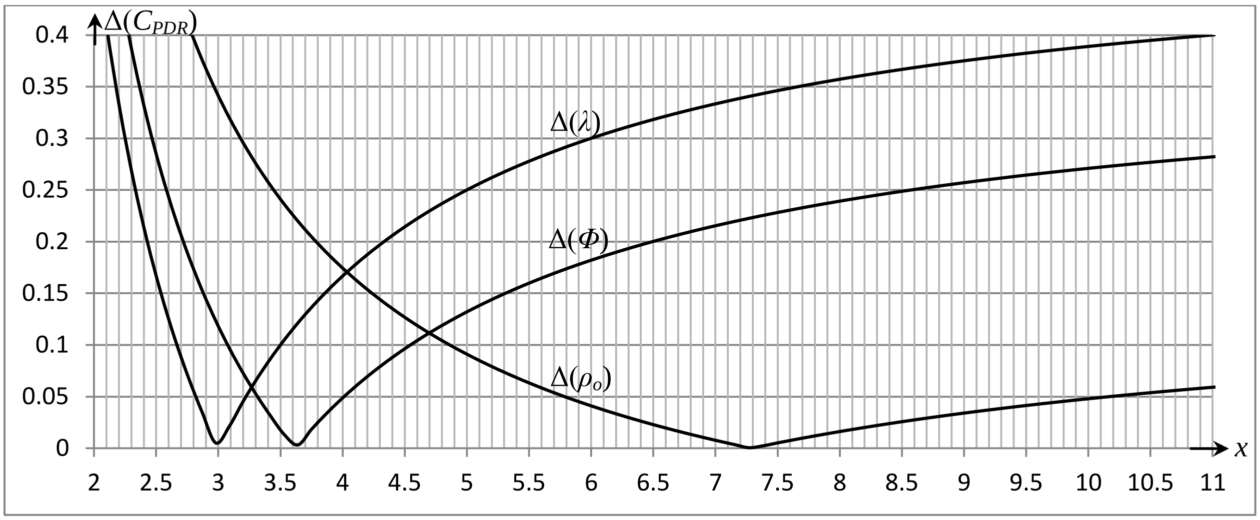

Fig.1: Graphical illustration of classificational optimality of $X_{o(\rho o)} = 7$ (for $x_{o(\rho o)} = 7.28$), of harmony $X_{o(\Phi)} = 4$ (for $x_{o(\Phi)} = 3.64$) and of balance $X_{o(\lambda)} = 3$ (for $x_{o(\lambda)} = 3$)

First of all one can analyze these outcomes in terms of Test Uncertainty Ratios (TUR). Obtained $X_{o(\Phi)} = 4$ and $X_{o(\lambda)} = 3$ are the proof the practical reasonability of these integers in terms of dimensional balance and harmony, explaining the popularity of their usage as TUR. It should also be noted the current trend requires to apply to this end only integer 4 [6] that definitely is preferable due to the evolutionary value of harmony, compared with balance.

Both the integers are applicable for the unified TUR estimation, irrespective of considered quality system. In terms of TUR, integer $X_o(\rho_o) = 7$ is of principle different: it is optimal for the quality components, possessing maximal weights. It is easy to be convinced that, turning to the argument x, the unified estimation in regard to $X_{o(\Phi)} = 4$ refers to averaged value $x_{o(\Phi)} = 0.5x_{o(\rho o)}$ in a linear model of weights. In particular, all this means that the real optimization demands individual estimation of TUR for each component depending of its contribution into system's quality. Some ways of achieving this have been discussed in [4].

Turning back to general classification problem, the highly important result is $X_o(\rho_o) = 7$ which points onto the unique property of “magical seven” as optimal integer for classification. This needs more detailed consideration.

Various approaches may lie in forming a classification system, including the establishing of rational or optimal number of classes that sometimes are also called levels of classification. Meanwhile, number 7 is often used to this end, and as the character of integrity, completeness and, in a hierarchical sense classificational periodicity in many spheres of science and practice [7]. It allows expecting that this integer meets the requirement of informational optimality.

When considering of any distinguishing features, number of which forms complete cycle of classification, the whole number characterizing the cycle may, supposedly, be associated with the integer 6 as the rounding off value of 2π. Since this integer refers to the number of intervals of length (the number of radii along the length of loop), the classification cycle is characterized as the seven points, bounding six intervals. Depending on the kind of classification, the number of the points or the intervals might be attributed to the levels of classification. Clearly, the usage of informational cycle allows us to claim that such an approach meets the requirements of informational optimality, and this is more detailed below.

Now there are two interrelated types in presenting the base of classification as integer number of classes ($N_{cl} > 1$): firstly when the N_{cl} equals to one of natural numbers, and second when the classes are identified as intervals, which number on one less than in the first case.

When using of permissible limits of informational optimality, the optimal classification integers for the first (N_{1cl}) and the second (N_{2cl}) cases respectively can be obtained as the results of solution the following equations:

$$N_{1cl} = \mathcal{R}\{1/[\rho_o(1 - 0.5\rho_o)]\} = 7; \quad (11)$$

$$N_{2cl} = \mathcal{R}\{1/[\rho_o(1 + 0.5\rho_o)]\} = 6 \quad (12)$$

The comparison of these outcomes with those obtained in section 4 when applying to $n = 1$, 2 and 3 allows asserting:

a) the coincidence of $X_{o(\rho_o)}$ and $X_{o(1-\rho_o)}$ with optimal classification integer N_{2cl}, as well as $X_{o(\Phi)}$ and $X_{o(1-\Phi)}$ with optimal classification integer N_{1cl};

b) the deflection of $X_{o(\lambda)} = X_{o(1-\lambda)} = 8$ from optimal classification integers that indicates the evident drawback of dimensional balance for applying it in classification systems.

In line with expectation and numerous examples in science and practice the most evident result is $N_{1cl} = 7$. This shows the unique property of the number seven to be considered as optimal for the base of classification.

Incidentally, integer $N_{2cl} = 6$ known in mathematics as the first perfect number. Remarkably, its unique features in connection to PDR were investigated and shown in [8], which results are briefly presented in Appendix 4. Both classification integers have been used for creating The Universal Scheme of Accuracy's Classification [9].

As for a base of numeration system, logically, the optimality of this characteristic somehow is related to the optimal base of classification. In terms of optimization the relation between a classification and a grouping in number system consists of conditions of necessity and sufficiency regarding above mentioned integers. Thus, basing on the described in section 2 method of informational optimization, one can determine the improper fraction close to the optimal base of numeration (b_o) when considering ψ_o as the conventional unit of classification that meets proportionality $b_o/[K_{\psi o}(\rho)]^{-1} = 2/\psi_o$, where $\mathcal{R}\{[K_{\psi o}(\rho)]^{-1}\} = N_{1cl}$. Therefore integer b_o is determined as following:

$$b_o = \mathcal{R}\{2(1 + \rho_o)/\rho_o\psi_o\} = 10 \quad (13)$$

This result coincides to $X_{o(1-\rho_o)}$ for the 10-dimensional geometrical model of informational optimality. The presence of coincidence between $(b_o + 1) = 11$ and $X_{o(1-\rho o)}$ for 11-dimensional model is easily explained in terms of the mentioned above two types of classification, so that there is the analogy between N_{1cl}, and $(b_o + 1)$, and between b_o and N_{2cl}. Significantly, this analogy is accompanied by rather convincing dimensional harmony between the characteristics since $N_{1cl}/(b_o + 1) = 0.636$, and $N_{2cl}/b_o = 0.6$, that is about ± 3% of Φ.

It should be noted that there are some other ways of substantiating the optimality of decimal numeration by the method of informational optimality, in particular by using Benford's probabilities. Such an approach enabled discovering the tight relation of b_o with the optimal level of confidence in classificational aspect, that is demonstrated in the next section.

6. Optimal levels of confidence

In order to consider percentage levels of confidence (LC) in terms of PDR one should pass from probabilistic to dimensional characteristics. As known, an interval representing LC is convenient to characterize by the value ($\mathcal{E}_{LC}$), which one can name dimensional factor of confidence:

$$\mathcal{E}_{LC} = 1/(1 - \mathrm{LC}/100), \tag{14}$$

In terms of metrology, for instance, there is one chance from the number equal to the factor that the measured value lies outside the interval.

It follows from the informational approach that $\mathcal{E}_{50} = 1/(1 - 50/100) = 2$ corresponds to the most uncertain classificational situation on allowing or ignoring of the lesser component in a two-component system. Then according to expression (14) and considering the ratio $\mathcal{E}_{50}/\mathcal{E}_{LC\rho o} = \rho_o$, the factor $\mathcal{E}_{LC\rho o} = 12.6$ refers to the minimum permissible level of confidence $\mathrm{LC}_{\rho o} = (1 - 0.5\rho_o)*100\% = 92\%$.

Correspondingly, the level of confidence LC_x as function of any ρ_x is determined as follows:

$$\mathrm{LC_x} = (1 - 0.5\rho_x)*100\% \tag{15}$$

The increasing of information redundancy ($\rho_x < \rho_o$) leads to increase of confidence ($\mathrm{LC_x} > \mathrm{LC}_{\rho o}$). In terms of dimensional perfection the harmonization of information redundancy is achieved when $\mathcal{E}_{LC\rho o}/\mathcal{E}_{LC\Phi} = \Phi$, that is $\mathcal{E}_{LC\Phi} = 20.4$, and $\mathrm{LC}_{\Phi} = 95\%$.

The further increasing of information redundancy leads to achieving the dimensional balance, resulting in $\mathcal{E}_{LC\rho o}/\mathcal{E}_{LC\lambda} = \lambda$, that is $\mathcal{E}_{LC\lambda} = 25.2$, and $\mathrm{LC}_{\lambda} = 96\%$. As proven in Appendix 5, this value represents the maximally permissible (by informational optimality) level of confidence.

Turning to expression (2) and to Table 1, one can be convinced that boundary levels of confidence $\mathrm{LC}_{\rho o}$ and LC_{λ} relate to minimum and maximum possible redundancy of information, i.e. when $K_{\psi o}(\rho)/K_{max}(\rho) = \rho_o = 1/2\pi$ and $K_{\psi o}(\rho)/K_{max}(\rho) = 0.5\rho_o = 1/4\pi$ respectively.

Clearly, the optimal level of confidence $\mathrm{LC_o}$, if it exists, shall correspond to some optimal information redundancy, i.e. $92\% < \mathrm{LC_o} < 96\%$. If consider $\mathrm{LC}_{\rho o}$, LC_{Φ}, and LC_{λ} as classification indices, then classificational integers for $\mathrm{LC}_{\rho o}$ and LC_{λ} are determined as $\mathcal{R}\{1/\rho_o\} = 6$ and $\mathcal{R}\{1/0.5\rho_o\} = 13$ respectively.

The critical integer $z_o = \mathcal{R}\{1/\rho_{or}\}$is inherent to the optimal redundancy ($6 < z_o < 13$), and can be determined by using Benford's Law [10]. According to this method any integer z, greater than one, can be used as classification integer. The system will employ z different digits. Benford's probability $P(d)$ regarding any number d from 1 to (z –1) is calculated as follows:

$$P\,(d) = log_z\,(1 + 1/d) \tag{16}$$

These probabilities form a complete group of independent events, i.e. their sum = 1; and a logarithmic sequence has obvious classification character. One can consider the system of two components $P_{min} = P[d = (z - 1)]$ and $P_{max} = P(d = 1)$ Benford's probabilities. For their ratio the optimization criterion and the index of optimal classification z_o (that is the logarithmic base) are determined as follows:

$$(P_{min}/P_{max})_o = P[d = (z - 1)]/P(d = 1) = \rho_o(1 \pm 0.5\rho_o), \quad (17)$$
$$z_o = \mathcal{R}\{arg\ min\ |log_z[1 + 1/(z - 1)]/log_z(1 + 1) - (1/2\pi)|\} = 10 \quad (18)$$

Now, coming back to expression (15) and assuming $\rho_x = 1/z_o$, the optimally redundant level of confidence one may determine as follows:

$$LC_o = (1 - 0.5/z_o)*100\% = 95\% \quad (19)$$

Clearly the equality $z_o = b_o$ points to the additional substantiation of optimality of decimal system and in that it conforms with the optimal level of confidence.

It should be noted, according to expression (15) and to the upper boundary ρ_o, the practically admissible minimum level of confidence is determined as $LC_{min} = \mathcal{R}\{1 - 0.5\rho_o(1 + 0.5\rho_o)\}*100\% = 91\%$. Then, in practice the maximum limitation for the classification criteria lies in the range from $\rho_{min} = 2(1 - LC_\lambda/100) = 0.08$ to $\rho_{max} = 2(1 - LC_{min}/100) = 0.18$.

7. Predominance of dimensional harmony

Inasmuch as Φ most often occurs in nature and human practice, this points to prevailing role of harmonious ratio. Now, using the information approach, we will discuss the possible reason for the dimensional harmony's prevalence in comparison with dimensional optimality and balance.

Turning to results of geometrical simulation, one may numerically analyze interposition of $y_1 = \rho_o^{1/n}$, $y_2 = [\rho_o (1 - 0.5\ \rho_o)]^{1/n}$, $y_3 = [\rho_o (1 + 0.5\ \rho_o)]^{1n}$ and respective optimal limitation for PDRs. Graphically the outcomes of analysis are illustrated in Fig. 2.

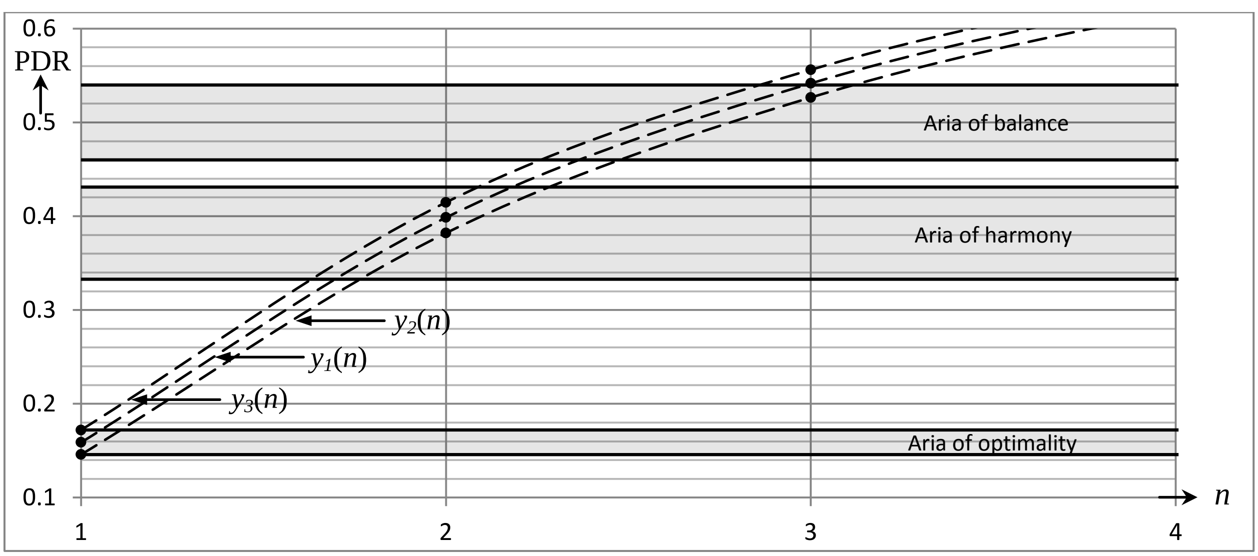

Fig.2: Interposition of $y_i(n)$ and respective optimal limitations for PDR

The predominance of one of PDR may be defined via the quantitative index of maximum *optimization reliability* (R_{av}), using the criterion of minimal estimation error (δ_{av}) which, in turn, is determined by means of absolute average relative difference between $y_i(n)$ and C_{PDR}. In so doing, the maximum reliability $max\ R_{av}(\delta_{av})$ in percentage is determined as follows:

$$max\ R_{av}(\delta_{av}) = [1 - min\ \delta_{av}(n;C_{PDR})]*100\%, \quad (20)$$

where $\delta_{av}(n;C_{PDR}) = [1/(3C_{PDR})] \sum_{i=1}^{n} | y_i(n) - C_{PDR} | \quad (21)$

Reliability Rav(PDR) regarding a certain PDR should not be less than 92% that corresponds to permissible δav = 8 %. Using expressions (7), (21), calculation results graphically are illustrated in Fig.3.

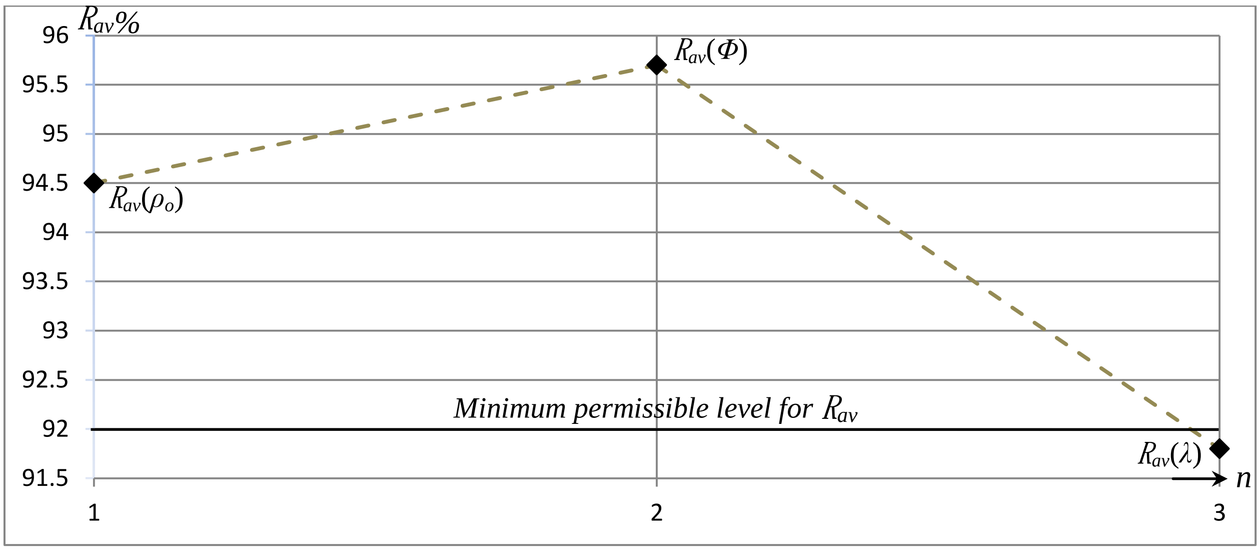

Fig.3: Graphical illustration of harmony's advantage in optimization reliability

Fig.3 shows the maximum estimate for $R_{av}(\Phi)$ regarding dimensional harmony together with the minimum estimate $R_{av}(\lambda)$ for dimensional balance. Besides, $R_{av}(\lambda) < 92\%$ that, in turn, indicates onto a certain unreliability of dimensional balance.

8. Conclusions

1. Tolerances for PDR constants in the evaluation of information optimality, harmony and balance have been determined for their proper practical application. For each PDR the permissible deviation about 8% of the constant is found.
2. The optimality of integers 7 and 6 as base characteristics in forming classification systems, as well as of integer 10 as the base of number system was demonstrated.
3. The advantage of the integer 4 as the characteristic of unified TUR (4:1), and its compliance to dimensional harmony, has been proven.

4. The optimality of 95% level of confidence, and its correspondence to dimensional harmony was proven. It was also demonstrated that 92% and 96% represent informatively permissible levels of confidence, which correspond to dimensional optimality and balance respectively.

5. The notable peculiarity of 95% level of confidence is that in field of estimating levels of confidence a manifestation of mathematical harmony is adequate to informational optimality.

6. Regarding optimization reliability, it is shown that harmony represents the optimal relation amongst perfect dimensional ratios, i.e. is the more desirable and, hence, most often occurs in Nature and in human activity. The consideration in terms of optimization reliability allows assuming that dimensional balance is the natural stage in the evolutional achievement of dimensional harmony as the most stable PDR.

7. The spectacular outcome of the study is in equality and in equivalence of critical integer z_o, inherent to the optimal redundancy of information on the one hand, and of the base of decimal number system m_o on the other hand, since logically they have the same classificational content.

8. Appendixes are supplementing or illustrate certain theses, or they independently represent results of studies aimed at substantiating essential problems concerning the determination of informative components, and of the informational limitation of level of confidence.

Appendix 1: Novelty in determining of informatively necessary components

The dividing of quality components onto informative and non-informative leads one to regard informative components in two aspects. On the one hand, such a grouping represents the classification according to dichotomy that allows considering the equivalence of informative components. On the other hand, informative components are different by weights, i.e. by the quantity of information they are contributing into the system's quality that, in particular, requires individual approach in optimizing the accuracy of their measurement or estimation.

From these considerations, formula (2), concerning two-component system originates from the general expression $\psi_{eo} = exp[-\sum_{j=1}^{m} K_j \ln K_j]$ for any number (m) of components in multi-component quality system. The expression is based on the principle of equivalence, i.e. equality between the total entropy of system and the entropy of ψ_{eo} informatively necessary components in the assumption that by the fact of grouping all ψ_{eo} components are equivalent by weights. This new approach is completely different from and is the alternative to the commonly used principle of proportionality $\psi_{po} = (m/\ln m)*exp[-\sum_{j=1}^{m} K_j \ln K_j]$, according to which the ratio ψ_{po}/m equates to the ratio between the entropy of real system and the entropy of m components in the assumption that all m components are equivalent by weights.

Results of comparison these approaches for the case under consideration (m = 2) are presented in Table 4, where $\delta_\psi = [1 - \psi(\rho)/1.5]*100\%$ according to the boundary condition of confidence 50% is defined as the estimation error in determining of informatively necessary components.

Table 4: Comparison approaches in determining $\psi(\rho_o)$ for a two-component system

Used approach	Expression for $\psi(\rho)$	$\rho_o = arg\ min[\psi(\rho) - 1.5 = 0]$	$\psi(\rho_o)$	δ_ψ
Equivalence	$\psi_e(\rho) = exp\{-[1/(1+\rho)]\ln[1/(1+\rho)]- -[\rho/(1+\rho)]\ln[\rho/(1+\rho)]\}$	0.159	1.49	0.7%
Proportionality	$\psi_p(\rho) = (2/\ln 2)\{-[1/(1+\rho)]\ln[1/(1+\rho)]- -[\rho/(1+\rho)]\ln[\rho/(1+\rho)]\}$	0.274	1.15	23%

As follows from Table 4, for the two-component system the principle of equivalence is characterized by significant advantage in estimation accuracy compared with the principle of proportionality. It should be emphasized that unlike the proportionality, for which estimation error is because of fundamental methodological drawback, the error for the principle equivalence is only due to the uncertainty of calculation (roundings off).

In passing from two-component to the multi-component system, it is expedient to consider a linear model of weights when $K_{j(l)} = 2(m + 1 - j)/m(m + 1)$ that for the analysis has proven the optimal [1]. In so doing, and taking into account the cited in section 6 of practically permissible criteria (ρ_{min}, ρ_{max}) for the selection of quality components, the graphical illustration of comparison two principles under consideration is shown in Fig. 4.

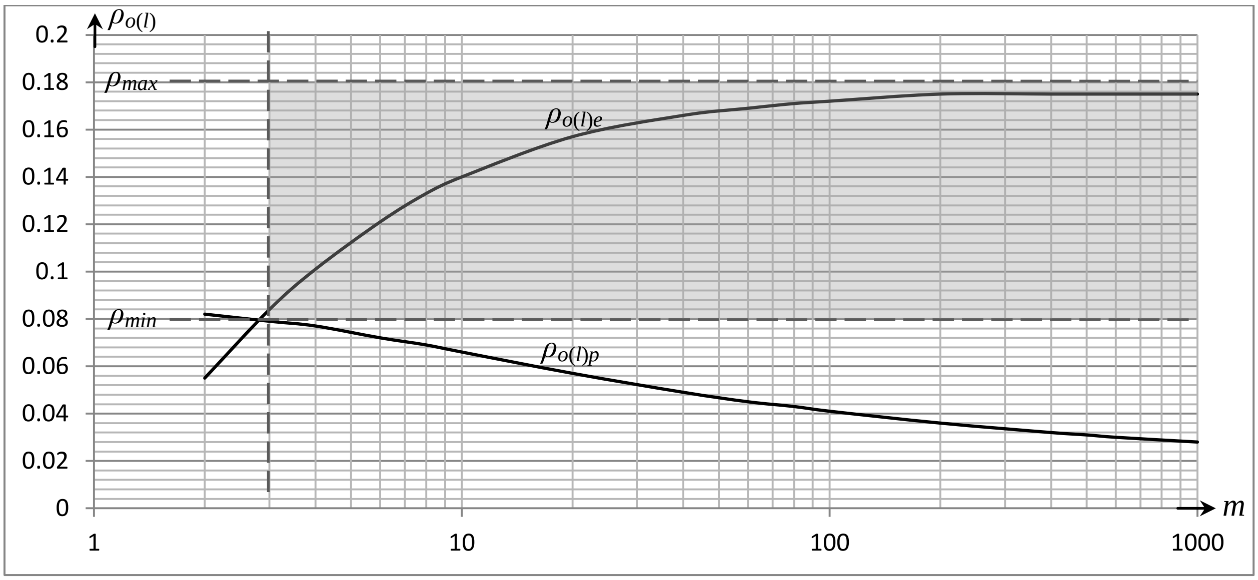

Fig.4: Graphical illustration of comparing of two principles of selecting informative components of multi-component system (m = 3, 4, …) with the linear distribution of weights by using the following functions: $\rho_{o(l)e} = (1 - \psi_{o(l)e}/m)$ and $\rho_{o(l)p} = (1 - \psi_{o(l)p}/m)$

The comparison data in Fig. 4 convincingly demonstrates the complete fulfillment with requirements of permissible criteria regarding the principle of equivalence ($\rho_{min} < \rho_{o(l)e} < \rho_{max}$), and on the other hand the complete disparity as regards the principle of proportionality ($\rho_{o(l)e} < \rho_{min}$) for a multi-component system of quality. Thus, the analysis points onto the essential advantage of the principle of equivalence over the principle of proportionality.

Appendix 2: PDR manifestations in the system of temperatures of water

Some of specific temperature points represent indicators of boundaries for physical states of substances. Hypothetically these temperatures are to be located on the temperature scale in conformity with permissible limits of dimensional perfectness. Such temperature points on the temperature scale of water are shown in Fig 5, and data of analysis and calculating, aimed at defining of their dimensional perfectness – in Table 5.

```
-273                                                  -48 -41     0                100
|------------------------------------------------------|----|--------|-------------------|--------►°C
t_o                                                    t_f  t_1    t_2                 t_3
```

Fig. 5: Temperature scale of water

The following specific temperatures are designated on the scale:

t_0 = the absolute zero temperature for all substances;

t_1 = the supercooled water which is its "homogenous nucleation temperature" – the lowest temperature at which the ice crystallization rate can be measured as water is freezing. Below t_1, ice is crystallizing too fast for any property of the remaining liquid [11];

t_2 = the melting temperature of water (triple point at 0.6117 kPa);

t_3 = the boiling temperature of water;

t_f = the lowest temperature when supercooled liquid water becomes ice completely [11].

Table 5: Relations of temperature intervals regarding specific temperatures of water, and the identification to PDR

Temperature intervals ratio	r_i	$1 - r_i$	Identification to PDR	Deviation from PDR |%|
$r_1 = (t_3 – t_2)/(t_2 – t_0)$	0.366	0.634	Φ	2.6
$r_2 = (t_3 – t_1)/(t_1 – t_0)$	0.608	0.392	Φ	1.6
$r_3 = (t_3 – t_f)/(t_f – t_0)$	0.658	0.342	Φ	6.5
$r_4 = (t_2 – t_1)/(t_3 – t_2)$	0.410	0.590	Φ	4.5
$r_5 = (t_2 – t_f)/(t_3 – t_2)$	0.48	0.52	λ	4.0
$r_6 = (t_1 – t_f)/(t_2 – t_1)$	0.171	0.829	ρ_o	7.5
$r_7 = (t_1 – t_f)/(t_2 – t_f)$	0.146	0.854	ρ_o	8.0

Table 5 illustrates: informatively significant ratios regarding the specific temperatures of water correspond to dimensional harmony or balance or optimality within the permissible PDR values.

Appendix 3: Illustration of PDR simulation for three dimensions

The graphical illustration of unitary dimensional models is shown in Fig. 6. Results of calculations regarding respective functions $y = \rho_o^{1/n}$ and $y' = 1 – \rho_o^{1/n}$ are presented in Table 6.

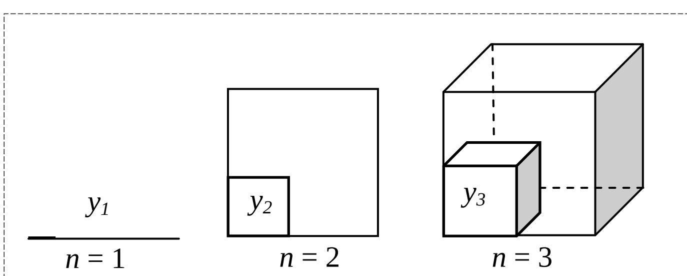

Fig. 6: Example of unitary dimensional models

Table 6: y and y' values

n	$y = \rho_o^{1/n}$	$y' = 1 - \rho_o^{1/n}$
1	0.159	0.841
2	0.399	0.601
3	0.54	0.46

Data of Table 6, combined with data of Table 1, once more convinces in the existence of obvious connection between informational optimality on the one hand and the harmony and balance on the other hand. Besides, one can note the significant difference in deviation from C_{PDR}: 2.8% for two-dimensional and 8% for three-dimensional model respectively that indicates onto predominance of harmony, compared with balance.

Appendix 4: Perfection of proper fractions for the first perfect number

In number theory a perfect number is known as is a positive integer that is equal to the sum of its proper positive divisors, that is, the sum of its positive devisers, excluding the number itself. Integer 6 is the first in the list of perfect numbers, because 1, 2, and 3 are its proper positive divisors, and 1 + 2 + 3 = 6. Accordingly 1/2, 1/3, 1/6, 2/3, 2/6, and 3/6 are proper fractions for the first perfect number.

The location of proper fractions of the first perfect number within permissible ranges of PDR is illustrated in Table 7.

Table 7: Proper fractions of 6 within permissible PDR deviations

PDR	1/2	1/3	1/6	2/6	3/6	2/3
$\rho_o(1 \pm 0.5\rho_o)$			*			
$\Phi(1\pm0.5\rho_o)$ or $[1 - \Phi(1\pm0.5\rho_o)]$		*		*		*
$\lambda(1\pm0.5\,\rho_o)$ or $[1 - \lambda(1\pm0.5\,\rho_o)]$	*				*	

Clearly, all fractions which can be created using the number 6 correspond to PDR. At the same time, one can be convinced of the fact that, excepting 6, there are no perfect numbers: 6, 28, 496, 8128, … , possessing such quality. That is why the six is to a far greater extent the perfect number than all others.

Appendix 5: Informational limitation for the level of confidence

To find the upper limitation of LC, we will recourse to the level of confidence regarding the necessity of information redundancy (LCS) as a subsidiary characteristic. Clearly, the condition 0 < LCS < 50% is true in the permissible range of informational redundancy.

Inasmuch as the $LC_{\rho o}$ = 92% corresponds to the criterion of not redundant necessity and sufficiency, in the dimensional interpretation LCS_x for any LC_x can be determined as function of the ratio $\mathcal{E}_{LCx}/\mathcal{E}_{92}$ as follows:

$$LCS_x = (1 - 0.5\mathcal{E}_{LCx}/\mathcal{E}_{92})*100\%. \quad (22)$$

Proceeding from the condition mentioned above, when $LCS_x = 0$, the maximum permissible $\mathcal{E}_{LCx} = 25.5 = \mathcal{E}_{LC\lambda}$ that matches the maximum permissible level of confidence LC_λ = 96%, QED.

In the consideration of this result concerning the quality of multi-component system, the interesting is to focus on components with weights $K_j < K_{\psi o}$. When using the equation $LC_{jo} = (1 - \rho_{jo}/\mathcal{E}_{50})*100\% = (1 - 0.5\,\rho_{jo})*100\% = (1 - 0.5\,K_{\psi o}/K_j)*100\%$, one can argue that the decreasing of the quality reaches 0% of LC_{jo} for K_j when $K_j = 0.5K_{\psi o}$. A further redundancy increase leads to the failure of the informational criterion of classification and formally results in a negative confidence level for the components with weights $K_j < 0.5K_{\psi o}$.

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