A corrected semi-proximal ADMM for multi-block convex optimization and its application to DNN-SDPs

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Abstract

In this paper we propose a corrected semi-proximal ADMM (alternating direction method of multipliers) for the general p-block ($p \geq 3$) convex optimization problems with linear constraints, aiming to resolve the dilemma that almost all the existing modified versions of the directly extended ADMM, although with convergent guarantee, often perform substantially worse than the directly extended ADMM itself with no convergent guarantee. Specifically, in each iteration, we use the multi-block semi-proximal ADMM with step-size at least 1 as the prediction step to generate a good prediction point, and then make correction as small as possible for the middle $(p-2)$ blocks of the prediction point. Among others, the step-size of the multi-block semi-proximal ADMM is adaptively determined by the infeasibility ratio made up by the current semi-proximal ADMM step for the one yielded by the last correction step. For the proposed corrected semi-proximal ADMM, we establish the global convergence results under a mild assumption, and apply it to the important class of doubly nonnegative semidefinite programming (DNN-SDP) problems with many linear equality and/or inequality constraints. Our extensive numerical tests show that the corrected semi-proximal ADMM is superior to the directly extended ADMM with step-size $\tau = 1.618$ and the multi-block ADMM with Gaussian back substitution [12, 14]. It requires the least number of iterations for 70% test instances within the comparable computing time with that of the directly extended ADMM, and for about 40% tested problems, its number of iterations is only 67% that of the multi-block ADMM with Gaussian back substitution [12, 14].

Keywords: Multi-block convex optimization, corrected semi-proximal ADMM, global convergence, DNN-SDPs

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1 Introduction

Let $\mathbb{Z}_1,\ldots,\mathbb{Z}_p$ and $\mathbb{X}$ be the real finite dimensional vector spaces which are equipped with an inner product $\langle\cdot,\cdot\rangle$ and its induced norm $\|\cdot\|$. We consider the following multi-block convex minimization problem with linear constraints in which the objective function is a sum of p ($p\geq 3$) closed proper convex functions without overlapping variables:

$$\begin{aligned}&\min_{z_i\in\mathbb{Z}_i}\ \textstyle\sum_{i=1}^{p}\theta_i(z_i)\\ &\ \text{s.t.}\ \textstyle\sum_{i=1}^{p}\mathcal{A}_i^*z_i=c,\end{aligned}\tag{1.1}$$

where $\theta_i\colon\mathbb{Z}_i\to(-\infty,+\infty]$ for $i=1,\ldots,p$ are closed proper convex functions, $\mathcal{A}_i^*\colon\mathbb{Z}_i\to\mathbb{X}$ for $i=1,\ldots,p$ are the adjoint operator of $\mathcal{A}_i\colon\mathbb{X}\to\mathbb{Z}_i$, and $c\in\mathbb{X}$ is the given vector. Throughout this paper, we assume that problem (1.1) has an optimal solution.

There are many important cases that take the form of (1.1). One compelling example is the so-called robust PCA (principle component analysis) with noisy and incomplete data considered in [34, 30] and the low rank matrix required to be nonnegative, which can be modelled as (1.1) in which the objective function is the sum of the nuclear norm, the ℓ_1-norm, and the indicator function over the nonnegative orthant cone. Another prominent example comes from the matrix completion with or without fixed basis coefficients [7, 18, 19], for which the nuclear norm and the nuclear semi-norm penalized least squares convex relaxation problems exactly have a dual of the form (1.1). Another interesting example is the simultaneous minimization of the nuclear norm and ℓ_1-norm of some structured matrix, which arises frequently from the structured low-rank and sparse representation for image classification and subspace clustering [38, 36]. In Section 4 of this paper, we focus on the solution of model (1.1) from the doubly nonnegative semidefinite programming (DNN-SDP) problems with many linear equality and/or inequality constraints, which arise in convex relaxation for some difficult combinatorial optimization problems.

The alternating direction method of multipliers (ADMM for short) was first proposed by Glowinski and Marrocco [8] and Gabay and Mercier [9] for the convex problem

$$\min_{z_1\in\mathbb{Z}_1,z_2\in\mathbb{Z}_2}\Big\{\theta_1(z_1)+\theta_2(z_2)\mid\mathcal{A}_1^*z_1+\mathcal{A}_2^*z_2=c\Big\}.\tag{1.2}$$

Let $l_\sigma\colon\mathbb{Z}_1\times\mathbb{Z}_2\times\mathbb{X}\to(-\infty,+\infty]$ be the Lagrange function of model (1.2) defined by

$$l_\sigma(z_1,z_2;x):=\theta_1(z_1)+\theta_2(z_2)+\langle x,\mathcal{A}_1^*z_1+\mathcal{A}_2^*z_2-c\rangle+\frac{\sigma}{2}\big\|\mathcal{A}_1^*z_1+\mathcal{A}_2^*z_2-c\big\|^2,$$

where $\sigma>0$ is the penalty parameter. For a chosen initial point $(z_1^0,z_2^0,x^0)\in\operatorname{dom}\theta_1\times\operatorname{dom}\theta_2\times\mathbb{X}$, the ADMM consists of the following iteration steps

$$\begin{cases} z_1^{k+1}\in\mathop{\arg\min}\limits_{z_1\in\mathbb{Z}_1}\ l_\sigma(z_1,z_2^k;x^k), & \text{(1.3a)}\\ z_2^{k+1}\in\mathop{\arg\min}\limits_{z_2\in\mathbb{Z}_2}\ l_\sigma(z_1^{k+1},z_2;x^k), & \text{(1.3b)}\\ x^{k+1}=x^k+\tau\sigma\big(\mathcal{A}_1^*z_1^{k+1}+\mathcal{A}_2^*z_2^{k+1}-c\big), & \text{(1.3c)}\end{cases}$$

where $\tau \in (0, \frac{1+\sqrt{5}}{2})$ is a constant to control the step-size in (1.3c). The iterative scheme of ADMM actually embeds a Gaussian-Seidel decomposition into each iteration of the classical augmented Lagraigan method of Hestenes-Powell-Rockafellar [20, 21, 26], so that the challenging task, i.e. the exact solution or the approximate solution with a high precision of the Lagrangian minimization problem, is relaxed to several easy ones.

Motivated by the same philosophy, one naturally extends the above 2-block ADMM to the multi-block convex minimization problem (1.1) directly. Let $L_\sigma : \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_p \times \mathbb{X} \to (-\infty, +\infty]$ denote the augmented Lagrange function for model (1.1), defined by

$$L_\sigma(z_1, \ldots, z_p; x) := \textstyle\sum_{i=1}^p \theta_i(z_i) + \langle x, \sum_{i=1}^p \mathcal{A}_i^* z_i - c\rangle + \dfrac{\sigma}{2}\big\|\sum_{i=1}^p \mathcal{A}_i^* z_i - c\big\|^2$$

where $\sigma > 0$ is the penalty parameter. With a chosen initial point $(z_1^0, \ldots, z_p^0; x^0) \in \operatorname{dom}\theta_1 \times \cdots \times \operatorname{dom}\theta_p \times \mathbb{X}$, the multi-block ADMM consists of the iteration steps:

$$\begin{cases} z_1^{k+1} \in \underset{z_1 \in \mathbb{Z}_1}{\arg\min}\ L_\sigma(z_1, z_2^k, \ldots, z_p^k; x^k), \\ \qquad\qquad \vdots \\ z_i^{k+1} \in \underset{z_i \in \mathbb{Z}_i}{\arg\min}\ L_\sigma(z_1^{k+1}, \ldots, z_{i-1}^{k+1}, z_i, z_{i+1}^k, \ldots, z_p^k; x^k), & (1.4a) \\ \qquad\qquad \vdots \\ z_p^{k+1} \in \underset{z_p \in \mathbb{Z}_p}{\arg\min}\ L_\sigma(z_1^{k+1}, \ldots, z_{p-1}^{k+1}, z_p; x^k), \\ x^{k+1} = x^k + \tau\sigma\big(\sum_{i=1}^p \mathcal{A}_i^* z_i^{k+1} - c\big). & (1.4b) \end{cases}$$

Many numerical results have illustrated that the directly extended ADMM with $\tau > 1$ works very well in many cases (see, e.g., [33, 4, 12, 35, 27]). In particular, Wen et al. [33] have utilized the 3-block ADMM with $\tau = 1.618$ to develop an efficient software for solving some SDP problems of large sizes. However, it was shown very recently by Chen et al. [3] that in contrast to the 2-block ADMM, the directly extended 3-block ADMM may diverge even if τ is sufficiently small. This dashes any hope of using the directly extended multi-block ADMM without any modifications or any restrictions on θ_i or $\mathcal{A}_i$.

In fact, before the announcement of [3], some researchers have made serious attempts in correcting the possible divergent multi-block ADMM (see, e.g., [11, 15, 12, 13, 14, 5]). Among them, the multi-block ADMM with Gaussian back substitution [12] distinguishes itself for simplicity and generality. However, the recent numerical results reported in [27] indicate that the multi-block ADMM with Gaussian back substitution (ADMMG for short) requires more iterations and computing time than the directly extended ADMM with $\tau = 1.618$ for at least 75% test problems, and for 61.5% test problems it requires at least 1.5 times as many iterations as the latter does. Now the dilemma is that almost all modified versions of the directly extended ADMM, although with convergent guarantee, often perform substantially worse than the directly extended ADMM with no convergent guarantee. This paper will make an active attempt in getting out of the dilemma.

We observe that the ADMMG [12] in each iteration makes a correction on the iterate point yielded by the directly extended ADMM with the unit step-size to achieve the global convergence. As recognized by the authors in [12], the introduction of the correction step often destroys the good numerical performance of the directly extended ADMM. It is well known that the ADMM with $\tau = 1$ always requires more 20% to 50% iterations than the one with $\tau = 1.618$. Thus, there is a big possibility that the iterate points yielded by the directly extended ADMM with $\tau = 1$ are insufficient to overcome the negative influence of the correction step, which may interpret why the ADMMG even with little correction (i.e., the correction step-size α is as close to 1 as possible) usually has worse performance than the directly extended ADMM with the unit step-size. Motivated by the crucial observation, we propose a corrected ADMM for problem (1.1) by imposing suitable correction only on the middle $(p-2)$ blocks of the iterate point yielded by the multi-block semi-proximal ADMM with a large step-size, which is adaptively determined by the infeasibility ratio made up by the current semi-proximal ADMM step for the one yielded by the last correction step. Here, the multi-block semi-proximal ADMM, instead of the directly extended ADMM, is used to yield the prediction step just for the consideration that some subproblems involved in the directly extended ADMM are hard to solve but the proximal operators of the corresponding θ_i's are easy to obtain.

In contrast to the ADMMG [12] and the linearized ADMMG [14], our corrected semi-proximal ADMM do not make any correction for the pth block and the multiplier block of the prediction point. Although the correction step in [12, 14] would not make any correction for the two blocks if the correction step-size takes 1, the global convergence analysis there is not applicable to this extreme case. In addition, when the subproblems involved in the directly extended ADMM are easy to solve, one may set the semi-proximal operators to be zero, and then the corrected semi-proximal ADMM is using the directly extended ADMM to yield the prediction step. However, for this case, the linearized ADMMG [14] still uses a linearized version of the directly extended ADMM to yield the prediction step except that all $\mathcal{A}_i\mathcal{A}_i^*$ reduce to the identity and the proximal parameters are all set to be the smallest one $\sigma\|\mathcal{A}_i\mathcal{A}_i^*\|$. For the advantage of a semi-proximal term over a strongly convex proximal term, the interested readers may refer to [6, 27].

For the proposed corrected semi-proximal ADMM, we provide the global convergence analysis under a mild assumption for the operators $\mathcal{A}_i$'s, and apply it to the dual problems of five classes of doubly nonnegative SDP problems without linear inequality constraints and a class of doubly nonnegative SDP problems with many linear inequality constraints, which take the form of (1.1) with $p = 3$ and $p = 4$, respectively. Our extensive numerical experiments for total **671** test problems demonstrate that the corrected semi-proximal ADMM is superior to the directly extended ADMM with $\tau = 1.618$ and the ADMMG [12] and the linearized ADMMG [14], and it requires the least number of iterations for about 70% test instances within the comparable computing time with that of the directly extended ADMM. In particular, for about 40% test problems, the number of iterations of the corrected semi-proximal ADMM is at most 70% that of the ADMMG [12, 14].

In the rest of this paper, we say that a linear operator $\mathcal{T}\colon \mathbb{X} \to \mathbb{X}$ is positive semidefinite (respectively, positive definite) if $\mathcal{T}$ is self-adjoint and $\langle u, \mathcal{T}u\rangle \geq 0$ for any $u \in \mathbb{X}$ (respectively, $\langle u, \mathcal{T}u\rangle > 0$ for any $u \in \mathbb{X}\backslash\{0\}$), and write $\|u\|_{\mathcal{T}} = \sqrt{\langle u, \mathcal{T}u\rangle}$ for any $u \in \mathbb{X}$.

2 A corrected semi-proximal ADMM

Choose the positive semidefinite linear operators $\mathcal{T}_i \colon \mathbb{Z}_i \to \mathbb{Z}_i$ for $i = 1, 2, \ldots, p$ such that all $\mathcal{T}_i + \mathcal{A}_i\mathcal{A}_i^*$ are positive definite. Define the mapping $F \colon \mathbb{Z}_1 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_p \to \mathbb{X}$ by

$$F(z_1, z_2, \ldots, z_p) := \mathcal{A}_1^* z_1 + \mathcal{A}_2^* z_2 + \cdots + \mathcal{A}_p^* z_p - c. \tag{2.1}$$

Next we describe the detailed iteration steps of the corrected semi-proximal ADMM.

Algorithm 2.1 *(***Corrected semi-proximal ADMM***)*

(S.0) *Let $\sigma > 0, \alpha \in (0, 1)$ and $\overline{\tau} \in (0, 1)$ be given. Choose a suitable small $\varepsilon \in (0, 1/2)$, a starting point $\big(z_1^0, \ldots, z_p^0, x^0\big) \in \operatorname{dom}\theta_1 \times \cdots \times \operatorname{dom}\theta_p \times \mathbb{X}$, and $\tau_0 \in (1, 2)$. Set $\widetilde{z}_i^0 = z_i^0$ for $i = 1, 2, \ldots, p$. For $k = 0, 1, \ldots$, perform the kth iteration as follows.*

(S.1) *(**Semi-proximal ADMM**) Compute the following minimization problems*

$$\begin{cases} z_1^{k+1} = \underset{z_1 \in \mathbb{Z}_1}{\arg\min}\ L_\sigma\big(z_1, \widetilde{z}_2^k, \ldots, \widetilde{z}_p^k; x^k\big) + \frac{\sigma}{2}\|z_1 - \widetilde{z}_1^k\|_{\mathcal{T}_1}^2, \\ \quad\vdots \\ z_i^{k+1} = \underset{z_i \in \mathbb{Z}_i}{\arg\min}\ L_\sigma\big(z_1^{k+1}, \ldots, z_{i-1}^{k+1}, z_i, \widetilde{z}_{i+1}^k, \ldots, \widetilde{z}_p^k; x^k\big) + \frac{\sigma}{2}\|z_i - \widetilde{z}_i^k\|_{\mathcal{T}_i}^2, \\ \quad\vdots \\ z_p^{k+1} = \underset{z_p \in \mathbb{Z}_p}{\arg\min}\ L_\sigma\big(z_1^{k+1}, \ldots, z_{p-1}^{k+1}, z_p; x^k\big) + \frac{\sigma}{2}\|z_p - \widetilde{z}_p^k\|_{\mathcal{T}_p}^2, \end{cases} \tag{2.2}$$

and then update the Lagrange multiplier by the following formula

$$x^{k+1} = x^k + \tau_k \sigma\big(\mathcal{A}_1^* z_1^{k+1} + \mathcal{A}_2^* z_2^{k+1} + \cdots + \mathcal{A}_p^* z_p^{k+1} - c\big), \tag{2.3}$$

where

$$\tau_k := \begin{cases} \min(1+\delta_k, \tau_{k-1}) & \text{if } 1+\delta_k > \overline{\tau} \\ \overline{\tau} & \text{otherwise} \end{cases} \quad \text{for } k \geq 1 \tag{2.4}$$

with

$$\delta_k = \frac{\|F(z_1^{k+1}, \widetilde{z}_2^k, \ldots, \widetilde{z}_p^k)\|^2 - \varepsilon\big(\|F(z_1^{k+1}, \ldots, z_p^{k+1})\|^2 + \|\mathcal{A}_p^*(z_p^{k+1} - z_p^k)\|^2\big)}{\|F(z_1^{k+1}, \ldots, z_p^{k+1})\|^2}. \tag{2.5}$$

(S.2) *(**Correction step**) Set $\widetilde{z}_i^{k+1} = z_p^{k+1}, \widetilde{z}_1^{k+1} = z_1^{k+1}$ and $\widetilde{z}_i^{k+1}$ for $i = p-1, \ldots, 2$ as*

$$\widetilde{z}_i^{k+1} = \widetilde{z}_i^k + \alpha(z_i^{k+1} - \widetilde{z}_i^k) - \textstyle\sum_{j=i+1}^p \big(\mathcal{T}_i + \mathcal{A}_i\mathcal{A}_i^*\big)^{-1}\mathcal{A}_i\mathcal{A}_j^*(\widetilde{z}_j^{k+1} - \widetilde{z}_j^k). \tag{2.6}$$

(S.3) *Let $k \leftarrow k + 1$, and then go to Step (S.1).*

Since the positive semidefinite linear operators $\mathcal{T}_i$ for $i = 1, 2, \ldots, p$ are chosen such that all $\mathcal{T}_i + \mathcal{A}_i\mathcal{A}_i^*$ are positive definite, each subproblem in (S.1) is strongly convex, which implies that Algorithm 2.1 is well defined. An immediate choice for such $\mathcal{T}_i$ is $\varrho_i\mathcal{I} - \mathcal{A}_i\mathcal{A}_i^*$ with $\varrho_i \geq \|\mathcal{A}_i\mathcal{A}_i^*\|$. Notice that (S.1) of Algorithm 2.1 is using the multi-block semi-proximal ADMM to yield a prediction point, which can effectively deal with the case where the subproblems involved in (1.4a) of the directly extended ADMM do not have closed form solutions but the proximal operators of θ_i are easy to obtain. The semi-proximal ADMM is clearly proposed just in the recent paper [27], though the convergence of two-block semi-proximal ADMM was established in the earlier papers [37, 6]. When all $\mathcal{A}_i\mathcal{A}_i^*$ are positive definite, one may choose all $\mathcal{T}_i$ to be the zero operator and (S.1) of Algorithm 2.1 reduces to the directly extended ADMM with adaptive step-size.

Remark 2.1 *In contrast to the ADMMG in [12] and the linearized ADMMG in [14], Algorithm 2.1 introduces a step-size τ_k into the multiplier update (2.3), which is adaptively determined by formula (2.4)-(2.5). The δ_k defined in (2.5) actually characterizes the infeasibility ratio made up by the kth semi-proximal ADMM step for the one yielded by the $(k-1)$th correction step. When the constant ε is chosen to be sufficiently small, the ratio δ_k is always positive, and consequently the step-size τ_k is at least 1.*

Observe that the multiplier update in the ADMM is same as that of the augmented Lagrangian function method, while the latter is an approximate Newton direction when the penalty parameter is over a certain threshold (see [29]). This implies that the multiplier block is good. In addition, the block z_p^{k+1} from the semi-proximal ADMM is good since the negative influence of the last correction step on it is tiny after the first $(p$–$1)$ minimization of the semi-proximal ADMM. So, unlike the ADMMG [12] and the linearized ADMMG [14], Algorithm 2.1 does not impose any correction on the pth block and the multiplier block of the prediction point. Of course, the ADMMG would not make any correction for the pth block and the multiplier block of the prediction point if the correction step-size takes 1, but the convergence analysis there is not applicable to the extreme case.

Remark 2.2 *Now let us take a look at a special case with $p = 3$, where all $\mathcal{T}_i = 0$, θ_2 is a linear function, to say $\theta_2(z_2) = \langle b, z_2\rangle$ for some $b \in \mathbb{Z}_2$, and the operator $\mathcal{A}_2$ is surjective. Then, the correction step with the unit step-size reduces to*

$$\tilde{z}_3^{k+1} = z_3^{k+1},\ \tilde{z}_2^{k+1} = z_2^{k+1} - (\mathcal{A}_2\mathcal{A}_2^*)^{-1}\mathcal{A}_2\mathcal{A}_3^*(\tilde{z}_3^{k+1} - z_3^k)\ \text{ and }\ \tilde{z}_1^{k+1} = z_1^{k+1}.$$

In this case, the iterate $(z_1^{k+1}, z_2^{k+1}, z_3^{k+1})$ of Algorithm 2.1 is actually yielded by

$$\begin{cases} z_2^{k+\frac{1}{2}} = \underset{z_2\in\mathbb{Z}_2}{\arg\min}\ L_\sigma(z_1^k, z_2, z_3^k, x^{k-1}), & \text{(2.7a)}\\ z_1^{k+1} = \underset{z_1\in\mathbb{Z}_1}{\arg\min}\ L_\sigma(z_1, z_2^{k+\frac{1}{2}}, z_3^k, x^k), & \text{(2.7b)}\\ z_2^{k+1} = \underset{z_2\in\mathbb{Z}_2}{\arg\min}\ L_\sigma(z_1^{k+1}, z_2, z_3^k, x^k), & \text{(2.7c)}\\ z_3^{k+1} = \underset{z_3\in\mathbb{Z}_3}{\arg\min}\ L_\sigma(z_1^{k+1}, z_2^{k+1}, z_3, x^k), & \text{(2.7d)} \end{cases}$$

since it is easy to verify that $z_2^{k+\frac{1}{2}} = \widetilde{z}_2^k$. *If, in addition,* x^{k-1} *in* (2.7a) *is replaced by* x^k, *then the iterate in* (2.7a)-(2.7d) *is equivalent to that of the following two-block ADMM*

$$\begin{cases} z_1^{k+1} = \underset{z_1\in\mathbb{Z}_1}{\arg\min}\ L_\sigma\big(z_1, \phi(z_1, z_3^k, x^k), x^k\big), \\ z_3^{k+1} = \underset{z_3\in\mathbb{Z}_3}{\arg\min}\ L_\sigma\big(z_1^{k+1}, \phi(z_1^{k+1}, z_3^k, x^k), z_3, x^k\big) \end{cases} \tag{2.8}$$

with $\phi(z_1, z_3, x) := -(\mathcal{A}_2\mathcal{A}_2^*)^{-1}(\mathcal{A}_1^* z_1 + \mathcal{A}_3 z_3 - c) - \frac{1}{\sigma}(b_2 + \mathcal{A}_2 x)$, *and* (2.8) *is equivalent to*

$$\begin{cases} (z_1^{k+1}, z_2^{k+1}) = \underset{z_1\in\mathbb{Z}_1, z_2\in\mathbb{Z}_2}{\arg\min}\ L_\sigma\big(z_1, z_2, z_3^k, x^k\big) + \frac{\sigma}{2}\|z_1 - z_1^k\|_{\mathcal{T}}^2, \\ z_3^{k+1} = \underset{z_3\in\mathbb{Z}_3}{\arg\min}\ L_\sigma\big(z_1^{k+1}, z_2^{k+1}, z_3, x^k\big) \end{cases} \tag{2.9}$$

with $\mathcal{T} = \mathcal{A}_1\mathcal{A}_2^*(\mathcal{A}_2\mathcal{A}_2^*)^{-1}\mathcal{A}_2\mathcal{A}_1^*$. *The equivalence between the iterate schemes* (2.8) *and* (2.9) *is recently employed by Sun, Toh and Yang [27] and Li, Toh and Sun [17] to resolve a special case of (1.1) in which* $p = 3$ *and one of* θ_i*'s is linear or quadratic.*

Remark 2.3 *It is immediate to see that the iterate* $(\widetilde{z}_1^k, \ldots, \widetilde{z}_p^k, x^k)$ *yielded by Algorithm 2.1 satisfies* $\widetilde{z}_1^k \in \operatorname{dom}\theta_1$ *and* $\widetilde{z}_p^k \in \operatorname{dom}\theta_p$. *Thus, when the constraints* $z_1 \in \operatorname{dom}\theta_1$ *and* $z_p \in \operatorname{dom}\theta_p$ *are hard to be satisfied, the best solution order of the subproblems in (S.1) of Algorithm 2.1 should be as follows: to solve* z_1 *(or* z_p*) first and solve* z_p *(or* z_1*) last.*

3 Convergence analysis of Algorithm 2.1

We need the following constraint qualification where Ω denotes the feasible set of (1.1):

Assumption 3.1 *There exists a point* $(\widehat{z}_1, \ldots, \widehat{z}_p) \in \operatorname{ri}(\operatorname{dom}\theta_1 \times \cdots \times \operatorname{dom}\theta_p) \cap \Omega$.

Under Assumption 3.1, from [24, Corollary 28.2.2 & 28.3.1] and [24, Theorem 6.5 & 23.8], it follows that $(z_1^*, \ldots, z_p^*) \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_p$ is an optimal solution to problem (1.1) if and only if there exists a Lagrange multiplier $x^* \in \mathbb{X}$ such that

$$-\mathcal{A}_i x^* \in \partial\theta_i(z_i^*) \ \text{ for } i = 1, 2, \ldots, p \ \text{ and } \ \mathcal{A}_1^* z_1^* + \cdots + \mathcal{A}_p^* z_p^* - c = 0, \tag{3.1}$$

where $\partial\theta_i$ is the subdifferential mapping of θ_i. Moreover, any $x^* \in \mathbb{X}$ satisfying (3.1) is an optimal solution to the dual problem of (1.1). Notice that the subdifferential mapping of a closed proper convex function is maximal monotone by [25, Theorem 12.17]. Therefore, for each $i \in \{1, 2, \ldots, p\}$, there exists a positive semidefinite linear operator $\Sigma_i : \mathbb{Z}_i \to \mathbb{Z}_i$ such that for all $z_i, \overline{z}_i \in \operatorname{dom}\theta_i$, $u_i \in \partial\theta_i(z_i)$ and $\overline{u}_i \in \partial\theta_i(\overline{z}_i)$,

$$\theta_i(z_i) \geq \theta_i(\overline{z}_i) + \langle \overline{u}_i, z_i - \overline{z}_i\rangle + \frac{1}{2}\|z_i - \overline{z}_i\|_{\Sigma_i}^2 \text{ and } \langle u_i - \overline{u}_i, z_i - \overline{z}_i\rangle \geq \|z_i - \overline{z}_i\|_{\Sigma_i}^2. \tag{3.2}$$

First, we establish a technical lemma to deal with the cross terms of the iterates.

Lemma 3.1 *Let* $\{(z_1^k,\ldots,z_p^k,x^k)\}$ *and* $\{(\tilde{z}_1^k,\ldots,\tilde{z}_p^k)\}$ *be the sequences generated by Algorithm 2.1. Then, under Assumption 3.1, for any optimal solution* $(z_1^*,\ldots,z_p^*)\in\mathbb{Z}_1\times\cdots\times\mathbb{Z}_p$ *of (1.1) and the associated Lagrange multiplier* $x^*\in\mathbb{X}$*, we have*

$$\begin{aligned}
&2\sum_{i=2}^{p}\Big\langle \tilde{z}_i^k-z_i^*,\,\textstyle\sum_{j=2}^{i}\mathcal{A}_i\mathcal{A}_j^*(\tilde{z}_j^k-z_j^{k+1})+\mathcal{T}_i(\tilde{z}_i^k-z_i^{k+1})\Big\rangle\\
&\quad+\frac{2}{\tau_k\sigma^2}\langle x^k-x^{k+1},x^k-x^*\rangle+2\langle z_1^k-z_1^*,\,\mathcal{T}_1(z_1^k-z_1^{k+1})\rangle\\
&\geq \Big\|\sum_{i=2}^{p}\mathcal{A}_i^*(z_i^{k+1}-\tilde{z}_i^k)-\frac{1}{\tau_k\sigma}(x^{k+1}-x^k)\Big\|^2+\frac{1}{(\tau_k\sigma)^2}\|x^{k+1}-x^k\|^2\\
&\quad+\sum_{i=2}^{p}\|z_i^{k+1}-\tilde{z}_i^k\|^2_{\mathcal{A}_i\mathcal{A}_i^*+2\mathcal{T}_i}+2\|z_1^{k+1}-z_1^k\|^2_{\mathcal{T}_1}+\frac{2}{\sigma}\sum_{i=1}^{p}\|z_i^{k+1}-z_i^*\|^2_{\Sigma_i}\quad\forall k.
\end{aligned}\tag{3.3}$$

Proof: From the definition of z_i^{k+1} in equation (2.2), it follows that for $i=1,2,\ldots,p$,

$$-\mathcal{A}_i\Big[x^k+\sigma\big(\textstyle\sum_{j=1}^{i}\mathcal{A}_j^*z_j^{k+1}+\sum_{j=i+1}^{p}\mathcal{A}_j^*\tilde{z}_j^k-c\big)\Big]-\sigma\mathcal{T}_i(z_i^{k+1}-\tilde{z}_i^k)\in\partial\theta_i(z_i^{k+1}).\tag{3.4}$$

Since $-\mathcal{A}_ix^*\in\partial\theta_i(z_i^*)$ for $i=1,2,\ldots,p$, from equation (3.2) we have that

$$\begin{aligned}
&\Big\langle z_i^*-z_i^{k+1},\,\mathcal{A}_i\Big[x^k-x^*+\sigma\big(\textstyle\sum_{j=1}^{i}\mathcal{A}_j^*z_j^{k+1}+\sum_{j=i+1}^{p}\mathcal{A}_j^*\tilde{z}_j^k-c\big)\Big]\Big\rangle\\
&\quad+\sigma\langle z_i^*-z_i^{k+1},\,\mathcal{T}_i(z_i^{k+1}-\tilde{z}_i^k)\rangle\geq\|z_i^{k+1}-z_i^*\|^2_{\Sigma_i},\quad i=1,2,\ldots,p.
\end{aligned}$$

Substituting (2.3) into the last p inequalities successively yields that

$$\begin{aligned}
&\langle\mathcal{A}_i^*(z_i^*-z_i^{k+1}),\,x^{k+1}-x^*\rangle+\sigma\langle z_i^*-z_i^{k+1},\,\mathcal{T}_i(z_i^{k+1}-\tilde{z}_i^k)\rangle-\|z_i^{k+1}-z_i^*\|^2_{\Sigma_i}\\
&\geq\sigma\big\langle\mathcal{A}_i^*(z_i^*-z_i^{k+1}),(\tau_k-1)\big(\textstyle\sum_{j=1}^{i}\mathcal{A}_j^*z_j^{k+1}-c\big)+\sum_{j=i+1}^{p}\mathcal{A}_j^*\big(\tau_kz_j^{k+1}-\tilde{z}_j^k\big)\big\rangle
\end{aligned}\tag{3.5}$$

for $i=1,2,\ldots,p$. Now adding the term $\sigma\langle\mathcal{A}_i^*(z_i^*-z_i^{k+1}),\sum_{j=2}^{p}\mathcal{A}_j^*(\tilde{z}_j^k-\tau_kz_j^{k+1})\rangle$ to the both sides of the ith inequality in (3.5) for $i=1,2,\ldots,p$ yields that

$$\begin{aligned}
&\langle\mathcal{A}_i^*(z_i^*-z_i^{k+1}),\,x^{k+1}-x^*+\sigma\textstyle\sum_{j=2}^{p}\mathcal{A}_j^*(\tilde{z}_j^k-\tau_kz_j^{k+1})\rangle\\
&\geq\sigma\langle\mathcal{A}_i^*(z_i^*-z_i^{k+1}),\,(\tau_k-1)(\mathcal{A}_1^*z_1^{k+1}-c)+\textstyle\sum_{j=2}^{i}\mathcal{A}_j^*(\tilde{z}_j^k-z_j^{k+1})\rangle\\
&\quad+\sigma\langle z_i^*-z_i^{k+1},\,\mathcal{T}_i(\tilde{z}_i^k-z_i^{k+1})\rangle+\|z_i^{k+1}-z_i^*\|^2_{\Sigma_i},\quad i=1,2,\ldots,p.
\end{aligned}\tag{3.6}$$

Adding the p inequalities in (3.6) together, we have that the left hand side is equal to

$$\begin{aligned}
&\langle\textstyle\sum_{i=1}^{p}\mathcal{A}_i^*(z_i^*-z_i^{k+1}),\,x^{k+1}-x^*+\sigma\sum_{j=2}^{p}\mathcal{A}_j^*(\tilde{z}_j^k-\tau_kz_j^{k+1})\rangle\\
&=\langle c-\textstyle\sum_{i=1}^{p}\mathcal{A}_i^*z_i^{k+1},\,x^{k+1}-x^*+\sigma\sum_{j=2}^{p}\mathcal{A}_j^*(\tilde{z}_j^k-\tau_kz_j^{k+1})\rangle,
\end{aligned}$$

where the equality is due to $\sum_{i=1}^p \mathcal{A}_i^* z_i^* = c$; while the right hand side is equal to

$$\Big\langle c - \sum_{i=1}^p \mathcal{A}_i^* z_i^{k+1},\, \sigma(\tau_k - 1)(\mathcal{A}_1^* z_1^{k+1} - c)\Big\rangle + \sigma \sum_{i=2}^p \Big\langle z_i^* - z_i^{k+1},\, \textstyle\sum_{j=2}^i \mathcal{A}_i^* \mathcal{A}_j^* (\tilde{z}_j^k - z_j^{k+1})\Big\rangle$$
$$+ \sigma \sum_{i=1}^p \big\langle z_i^* - z_i^{k+1},\, \mathcal{T}_i(\tilde{z}_i^k - z_i^{k+1})\big\rangle + \sum_{i=1}^p \|z_i^{k+1} - z_i^*\|_{\Sigma_i}^2.$$

By combining the last two equations with inequality (3.6), it then follow that

$$\Big\langle c - \sum_{i=1}^p \mathcal{A}_i^* z_i^{k+1},\, x^{k+1} - x^* - \sigma(\tau_k - 1)(\mathcal{A}_1^* z_1^{k+1} - c) + \sigma \sum_{j=2}^p \mathcal{A}_j^* (\tilde{z}_j^k - \tau_k z_j^{k+1})\Big\rangle$$
$$\geq \sigma \sum_{i=2}^p \Big\langle z_i^* - z_i^{k+1},\, \textstyle\sum_{j=2}^i \mathcal{A}_i^* \mathcal{A}_j^* (\tilde{z}_j^k - z_j^{k+1})\Big\rangle$$
$$+ \sigma \sum_{i=1}^p \big\langle z_i^* - z_i^{k+1},\, \mathcal{T}_i(\tilde{z}_i^k - z_i^{k+1})\big\rangle + \sum_{i=1}^p \|z_i^{k+1} - z_i^*\|_{\Sigma_i}^2,$$

which, by noting that $c - \sum_{i=1}^p \mathcal{A}_i^* z_i^{k+1} = \frac{1}{\tau_k \sigma}(x^k - x^{k+1})$, can be equivalently written as

$$\frac{1}{\tau_k \sigma}\langle x^k - x^{k+1}, x^{k+1} - x^*\rangle + \sigma \sum_{i=2}^p \big\langle z_i^{k+1} - z_i^*,\, \textstyle\sum_{j=2}^i \mathcal{A}_i^* \mathcal{A}_j^* (\tilde{z}_j^k - z_j^{k+1})\big\rangle$$
$$\geq \frac{1}{\tau_k}\Big\langle x^{k+1} - x^k,\, \sum_{i=2}^p \mathcal{A}_i^* (\tilde{z}_i^k - \tau_k z_i^{k+1})\Big\rangle + \frac{\tau_k - 1}{\tau_k}\langle x^k - x^{k+1},\, \mathcal{A}_1^* z_1^{k+1} - c\rangle$$
$$+ \sigma \sum_{i=1}^p \big\langle z_i^* - z_i^{k+1},\, \mathcal{T}_i(\tilde{z}_i^k - z_i^{k+1})\big\rangle + \sum_{i=1}^p \|z_i^{k+1} - z_i^*\|_{\Sigma_i}^2. \tag{3.7}$$

Next we make a simplification for (3.7). The left hand side of (3.7) can be rewritten as

$$\sigma \sum_{i=2}^p \big\langle \tilde{z}_i^k - z_i^*,\, \textstyle\sum_{j=2}^i \mathcal{A}_i \mathcal{A}_j^* (\tilde{z}_j^k - z_j^{k+1})\big\rangle + \frac{1}{\tau_k \sigma}\langle x^k - x^{k+1}, x^k - x^*\rangle$$
$$+ \sigma \sum_{i=2}^p \big\langle z_i^{k+1} - \tilde{z}_i^k,\, \textstyle\sum_{j=2}^i \mathcal{A}_i \mathcal{A}_j^* (\tilde{z}_j^k - z_j^{k+1})\big\rangle - \frac{1}{\tau_k \sigma}\|x^{k+1} - x^k\|^2, \tag{3.8}$$

while the right hand side of (3.7) can be rearranged as follows

$$\frac{1}{\tau_k}\left[\sum_{i=2}^p \big\langle x^{k+1} - x^k,\, \mathcal{A}_i^* (\tilde{z}_i^k - z_i^{k+1})\big\rangle + (\tau_k - 1)\big\langle x^k - x^{k+1}, \sum_{i=1}^p \mathcal{A}_i^* z_i^{k+1} - c\big\rangle\right]$$
$$+ \sigma \sum_{i=1}^p \big\langle z_i^* - z_i^{k+1},\, \mathcal{T}_i(\tilde{z}_i^k - z_i^{k+1})\big\rangle + \sum_{i=1}^p \|z_i^{k+1} - z_i^*\|_{\Sigma_i}^2,$$

which, by using the equality $\sum_{i=1}^{p} \mathcal{A}_i^* z_i^{k+1} - c = \frac{1}{\tau_k \sigma}(x^{k+1} - x^k)$, is equivalent to

$$\frac{1}{\tau_k} \sum_{i=2}^{p} \langle x^{k+1} - x^k, \mathcal{A}_i^*(\widetilde{z}_i^k - z_i^{k+1}) \rangle - \frac{\tau_k - 1}{\tau_k^2 \sigma} \|x^{k+1} - x^k\|^2 + \sigma \sum_{i=1}^{p} \|z_i^{k+1} - \widetilde{z}_i^k\|_{\mathcal{T}_i}^2$$
$$+ \sigma \sum_{i=1}^{p} \langle z_i^* - \widetilde{z}_i^k, \mathcal{T}_i(\widetilde{z}_i^k - z_i^{k+1}) \rangle + \sum_{i=1}^{p} \|z_i^{k+1} - z_i^*\|_{\Sigma_i}^2. \tag{3.9}$$

Now, combining equations (3.8)-(3.9) with inequality (3.7), we obtain that

$$\begin{aligned}
&\sum_{i=2}^{p} \langle \widetilde{z}_i^k - z_i^*, \textstyle\sum_{j=2}^{i} \mathcal{A}_i \mathcal{A}_j^*(\widetilde{z}_j^k - z_j^{k+1}) \rangle + \frac{1}{\tau_k \sigma^2} \langle x^k - x^{k+1}, x^k - x^* \rangle \\
&\geq \sum_{i=2}^{p} \Big\langle \widetilde{z}_i^k - z_i^{k+1}, \textstyle\sum_{j=2}^{i} \mathcal{A}_i \mathcal{A}_j^*(\widetilde{z}_j^k - z_j^{k+1}) \Big\rangle + \frac{1}{(\tau_k \sigma)^2} \|x^{k+1} - x^k\|^2 \\
&\quad + \frac{1}{\tau_k \sigma} \sum_{i=2}^{p} \langle x^{k+1} - x^k, \mathcal{A}_i^*(\widetilde{z}_i^k - z_i^{k+1}) \rangle + \frac{1}{\sigma} \sum_{i=1}^{p} \|z_i^{k+1} - z_i^*\|_{\Sigma_i}^2 \\
&\quad + \sum_{i=1}^{p} \|z_i^{k+1} - \widetilde{z}_i^k\|_{\mathcal{T}_i}^2 + \sum_{i=1}^{p} \langle z_i^* - \widetilde{z}_i^k, \mathcal{T}_i(\widetilde{z}_i^k - z_i^{k+1}) \rangle \\
&= \frac{1}{2} \sum_{i=2}^{p} \|\mathcal{A}_i^*(z_i^{k+1} - \widetilde{z}_i^k)\|^2 + \frac{1}{2} \Big\| \sum_{i=2}^{p} \mathcal{A}_i^*(z_i^{k+1} - \widetilde{z}_i^k) - \frac{1}{\tau_k \sigma}(x^{k+1} - x^k) \Big\|^2 \\
&\quad + \frac{1}{2(\tau_k \sigma)^2} \|x^{k+1} - x^k\|^2 + \frac{1}{\sigma} \sum_{i=1}^{p} \|z_i^{k+1} - z_i^*\|_{\Sigma_i}^2 \\
&\quad + \sum_{i=1}^{p} \|z_i^{k+1} - \widetilde{z}_i^k\|_{\mathcal{T}_i}^2 + \sum_{i=1}^{p} \langle z_i^* - \widetilde{z}_i^k, \mathcal{T}_i(\widetilde{z}_i^k - z_i^{k+1}) \rangle.
\end{aligned}$$

This along with $\widetilde{z}_1^k = z_1^k$ implies the desired inequality. The proof is completed. $\square$

To establish the convergence results of Algorithm 2.1, we introduce the notations

$$w^* = (z_2^*, \ldots, z_p^*), \quad \mathcal{E}_i = \mathcal{A}_i \mathcal{A}_i^* + \mathcal{T}_i \ \text{ and } \ \mathcal{B}_i = \mathcal{A}_i \mathcal{E}_i^{-1} \mathcal{A}_i^* \ \text{ for } \ i = 2, 3, \ldots, p.$$

For each k, let $w^k = (z_2^k, z_3^k, \ldots, z_p^k)$ and $\widetilde{w}^k = (\widetilde{z}_2^k, \widetilde{z}_3^k, \ldots, \widetilde{z}_p^k)$. Define the linear operators $\mathcal{M}\colon \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_p \to \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_p$ and $\mathcal{H}\colon \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_p \to \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_p$, respectively, by

$$\mathcal{M} := \begin{bmatrix} \mathcal{E}_2 & 0 & \cdots & 0 & 0 \\ \mathcal{A}_3 \mathcal{A}_2^* & \mathcal{E}_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{A}_{p-1} \mathcal{A}_2^* & \mathcal{A}_{p-1} \mathcal{A}_3^* & \cdots & \mathcal{E}_{p-1} & 0 \\ \mathcal{A}_p \mathcal{A}_2^* & \mathcal{A}_p \mathcal{A}_3^* & \cdots & \mathcal{A}_p \mathcal{A}_{p-1}^* & \mathcal{E}_p \end{bmatrix}$$

and

$$\mathcal{H} := \begin{bmatrix} \mathcal{I} & \mathcal{E}_2^{-1}\mathcal{A}_2\mathcal{A}_3^* & \cdots & \mathcal{E}_2^{-1}\mathcal{A}_2\mathcal{A}_{p-1}^* & \mathcal{E}_2^{-1}\mathcal{A}_2\mathcal{A}_p^* \\ 0 & \mathcal{I} & \cdots & \mathcal{E}_3^{-1}\mathcal{A}_3\mathcal{A}_{p-1}^* & \mathcal{E}_3^{-1}\mathcal{A}_3\mathcal{A}_p^* \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathcal{I} & \mathcal{E}_{p-1}^{-1}\mathcal{A}_{p-1}^*\mathcal{A}_p \\ 0 & 0 & \cdots & 0 & \alpha\mathcal{I} \end{bmatrix}.$$

An elementary computation yields that the operator $\mathcal{G} := \mathcal{M}\mathcal{H}$ takes the form of

$$\begin{bmatrix} \mathcal{E}_2 & \mathcal{A}_2\mathcal{A}_3^* & \cdots & \mathcal{A}_2\mathcal{A}_{p-1}^* & \mathcal{A}_2\mathcal{A}_p^* \\ \mathcal{A}_3\mathcal{A}_2^* & \mathcal{A}_3(\mathcal{I}+\mathcal{B}_2)\mathcal{A}_3^*+\mathcal{T}_3 & \cdots & \mathcal{A}_3(\mathcal{I}+\mathcal{B}_2)\mathcal{A}_{p-1}^* & \mathcal{A}_3(\mathcal{I}+\mathcal{B}_2)\mathcal{A}_p^* \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{A}_p\mathcal{A}_2^* & \mathcal{A}_p(\mathcal{I}+\mathcal{B}_2)\mathcal{A}_3^* & \cdots & \mathcal{A}_p(\mathcal{I}+\sum_{j=2}^{p-2}\mathcal{B}_j)\mathcal{A}_{p-1} & \mathcal{A}_p(\alpha\mathcal{I}+\sum_{j=2}^{p-1}\mathcal{B}_j)\mathcal{A}_p^*+\alpha\mathcal{T}_p \end{bmatrix}.$$

It is not hard to verify that the self-adjoint linear operator $\mathcal{G}$ is positive definite.

Now we are in a position to establish the global convergence of Algorithm 2.1.

Theorem 3.1 *Suppose that Assumption 3.1 holds and the operators $\mathcal{T}_i$ for $i = 1, 2, \ldots, p$ are chosen such that $\mathcal{A}_i\mathcal{A}_i^* + \mathcal{T}_i$ are positive definite. Then, the following statements hold:*

(a) $\lim_{k\to+\infty} \|z_i^{k+1} - \widetilde{z}_i^k\| = 0$ *for* $i = 2, 3, \ldots, p$ *and* $\lim_{k\to+\infty} \|x^{k+1} - x^k\| = 0$.

(b) The sequences $\{(z_1^k, \ldots, z_p^k)\}$ and $\{(\widetilde{z}_1^k, \ldots, \widetilde{z}_p^k)\}$ converge to an optimal solution to (1.1), and $\{x^k\}$ converges to an optimal solution to the dual problem of (1.1).

Proof: Let $(z_1^*, \ldots, z_p^*) \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_p$ be an optimal solution to (1.1) and $x^* \in \mathbb{X}$ be the associated Lagrange multiplier. Then, the sequences $\{(z_1^k, \ldots, z_p^k, x^k)\}$ and $\{(\widetilde{z}_1^k, , \ldots, \widetilde{z}_p^k)\}$ generated by Algorithm 2.1 satisfies the inequality (3.3) of Lemma 3.1. By using the expression of the above linear operator $\mathcal{M}$, it is not hard to obtain that

$$\begin{aligned} & 2\sum_{i=2}^{p}\Big\langle \widetilde{z}_i^k - z_i^*,\, \textstyle\sum_{j=2}^{i}\mathcal{A}_i\mathcal{A}_j^*\big(\widetilde{z}_j^k - z_j^{k+1}\big) + \mathcal{T}_i(\widetilde{z}_i^k - z_i^{k+1})\Big\rangle \\ & = 2\big\langle \widetilde{w}^k - w^*, \mathcal{M}\big(\widetilde{w}^k - w^{k+1}\big)\big\rangle = 2\big\langle \widetilde{w}^k - w^*, \mathcal{M}\mathcal{H}\mathcal{H}^{-1}\big(\widetilde{w}^k - w^{k+1}\big)\big\rangle. \end{aligned} \tag{3.10}$$

From the expression of $\mathcal{H}$ and the corrected step of Algorithm 2.1, we can verify that

$$\mathcal{H}(\widetilde{w}^{k+1} - \widetilde{w}^k) = \alpha(w^{k+1} - \widetilde{w}^k), \tag{3.11}$$

which by the invertibility of $\mathcal{H}$ implies that $\widetilde{w}^k - \widetilde{w}^{k+1} = \alpha\mathcal{H}^{-1}(\widetilde{w}^k - w^{k+1})$. Hence,

$$\begin{aligned} & 2\big\langle \widetilde{w}^k - w^*, \mathcal{M}\mathcal{H}\mathcal{H}^{-1}(\widetilde{w}^k - w^{k+1})\big\rangle = 2\alpha^{-1}\big\langle \widetilde{w}^k - w^*, \mathcal{G}(\widetilde{w}^k - \widetilde{w}^{k+1})\big\rangle \\ & = \alpha^{-1}\|\widetilde{w}^k - w^*\|_{\mathcal{G}}^2 - \alpha^{-1}\|\widetilde{w}^{k+1} - w^*\|_{\mathcal{G}}^2 + \alpha\|\mathcal{H}^{-1}(\widetilde{w}^k - w^{k+1})\|_{\mathcal{G}}^2 \end{aligned} \tag{3.12}$$

where the second equality is using the following identity relation

$$2\langle u - v, \mathcal{T}(u - w)\rangle = \|u - v\|_{\mathcal{T}}^2 + \|u - w\|_{\mathcal{T}}^2 - \|v - w\|_{\mathcal{T}}^2. \tag{3.13}$$

for a positive semidefinite linear operator $\mathcal{T}$. Using the identity (3.13), we also have

$$\begin{aligned}
&\frac{2}{\tau_k\sigma^2}\langle x^k - x^{k+1}, x^k - x^*\rangle + 2\langle z_1^k - z_1^*,\, \mathcal{T}_1(z_1^k - z_1^{k+1})\rangle \\
&= \frac{1}{\tau_k\sigma^2}\left(\|x^{k+1}-x^k\|^2 + \|x^k - x^*\|^2 - \|x^{k+1}-x^*\|^2\right) \\
&\quad + \|z_1^k - z_1^*\|_{\mathcal{T}_1}^2 - \|z_1^{k+1} - z_1^*\|_{\mathcal{T}_1}^2 + \|z_1^{k+1} - z_1^k\|_{\mathcal{T}_1}^2.
\end{aligned} \tag{3.14}$$

By combining equations (3.10), (3.12) and (3.14) with inequality (3.3), it follows that

$$\begin{aligned}
&\frac{1}{\alpha}\left(\|\widetilde{w}^k - w^*\|_{\mathcal{G}}^2 - \|\widetilde{w}^{k+1} - w^*\|_{\mathcal{G}}^2\right) + \alpha\|\mathcal{H}^{-1}(\widetilde{w}^k - w^{k+1})\|_{\mathcal{G}}^2 + \|z_1^k - z_1^*\|_{\mathcal{T}_1}^2 \\
&\quad - \|z_1^{k+1} - z_1^*\|_{\mathcal{T}_1}^2 + \frac{1}{\tau_k\sigma^2}\left(\|x^{k+1}-x^k\|^2 + \|x^k - x^*\|^2 - \|x^{k+1}-x^*\|^2\right) \\
&\geq \Big\|\sum_{i=2}^{p}\mathcal{A}_i^*(z_i^{k+1} - \widetilde{z}_i^k) - \frac{1}{\tau_k\sigma}(x^{k+1}-x^k)\Big\|^2 + \frac{1}{(\tau_k\sigma)^2}\|x^{k+1}-x^k\|^2 \\
&\quad + \sum_{i=2}^{p}\|z_i^{k+1} - \widetilde{z}_i^k\|_{\mathcal{A}_i\mathcal{A}_i^* + 2\mathcal{T}_i}^2 + \|z_1^{k+1} - z_1^k\|_{\mathcal{T}_1}^2 + \frac{2}{\sigma}\sum_{i=1}^{p}\|z_i^{k+1} - z_i^*\|_{\Sigma_i}^2.
\end{aligned} \tag{3.15}$$

By the expressions of the operators $\mathcal{H}$ and $\mathcal{M}$, an elementary computation yields that

$$\mathcal{M}^*\mathcal{H}^{-1} = \begin{bmatrix}
\mathcal{A}_2\mathcal{A}_2^* & 0 & \cdots & 0 & 0 \\
0 & \mathcal{A}_3\mathcal{A}_3^* & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \mathcal{A}_{p-1}\mathcal{A}_{p-1}^* & 0 \\
0 & 0 & \cdots & 0 & \frac{1}{\alpha}\mathcal{A}_p\mathcal{A}_p^*
\end{bmatrix},$$

and consequently

$$\begin{aligned}
\alpha\|\mathcal{H}^{-1}(\widetilde{w}^k - w^{k+1})\|_{\mathcal{G}}^2 &= \alpha\langle(\widetilde{w}^k - w^{k+1}), (\mathcal{H}^{-1})^*\mathcal{G}\mathcal{H}^{-1}(\widetilde{w}^k - w^{k+1})\rangle \\
&= \alpha\langle\mathcal{M}^*\mathcal{H}^{-1}(\widetilde{w}^k - w^{k+1}), (\widetilde{w}^k - w^{k+1})\rangle \\
&= \alpha\textstyle\sum_{i=2}^{p-1}\|\mathcal{A}_i^*(z_i^{k+1} - \widetilde{z}_i^k)\|^2 + \|\mathcal{A}_p^*(z_p^{k+1} - z_p^k)\|^2.
\end{aligned}$$

Substituting this equality into inequality (3.15), we immediately obtain that

$$\begin{aligned}
&\frac{1}{\alpha}\left(\|\widetilde{w}^k - w^*\|_{\mathcal{G}}^2 - \|\widetilde{w}^{k+1} - w^*\|_{\mathcal{G}}^2\right) + \|z_1^k - z_1^*\|_{\mathcal{T}_1}^2 \\
&\quad - \|z_1^{k+1} - z_1^*\|_{\mathcal{T}_1}^2 + \frac{1}{\tau_k\sigma^2}\left(\|x^k - x^*\|^2 - \|x^{k+1}-x^*\|^2\right) \\
&\geq \sum_{i=2}^{p-1}\|z_i^{k+1} - \widetilde{z}_i^k\|_{(1-\alpha)\mathcal{A}_i\mathcal{A}_i^* + 2\mathcal{T}_i}^2 + \|z_1^{k+1} - z_1^k\|_{\mathcal{T}_1}^2 + 2\|z_p^{k+1} - z_p^k\|_{\mathcal{T}_p}^2 \\
&\quad + \Big\|\sum_{i=2}^{p}\mathcal{A}_i^*(z_i^{k+1} - \widetilde{z}_i^k) - \frac{x^{k+1} - x^k}{\tau_k\sigma}\Big\|^2 + \frac{1-\tau_k}{(\tau_k\sigma)^2}\|x^{k+1}-x^k\|^2 + \frac{2}{\sigma}\sum_{i=1}^{p}\|z_i^{k+1} - z_i^*\|_{\Sigma_i}^2.
\end{aligned}$$

Let $\widetilde{\mathcal{G}}$ be an operator with the same form as $\mathcal{G}$ except that the pth diagonal element is replaced by $\mathcal{A}_p(\alpha\mathcal{I}+\sum_{j=2}^{p-1}\mathcal{B}_j)\mathcal{A}_p^*+\alpha\mathcal{T}_p+\frac{2\alpha}{\sigma}\Sigma_p$. Then the last inequality is equivalent to

$$\begin{aligned}
&\frac{1}{\alpha}\left(\|\widetilde{w}^k-w^*\|_{\widetilde{\mathcal{G}}}^2-\|\widetilde{w}^{k+1}-w^*\|_{\widetilde{\mathcal{G}}}^2\right)+\frac{1}{\tau_k\sigma^2}\left(\|x^k-x^*\|^2-\|x^{k+1}-x^*\|^2\right)\\
&\quad+\|z_1^k-z_1^*\|_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1}^2-\|z_1^{k+1}-z_1^*\|_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1}^2\\
&\geq\sum_{i=2}^{p-1}\|z_i^{k+1}-\widetilde{z}_i^k\|_{(1-\alpha)\mathcal{A}_i\mathcal{A}_i^*+2\mathcal{T}_i}^2+\|z_1^{k+1}-z_1^k\|_{\mathcal{T}_1}^2+2\|z_p^{k+1}-z_p^k\|_{\mathcal{T}_p}^2\\
&\quad+\Big\|\sum_{i=2}^{p}\mathcal{A}_i^*(z_i^{k+1}-\widetilde{z}_i^k)-\frac{1}{\tau_k\sigma}(x^{k+1}-x^k)\Big\|^2+\frac{1-\tau_k}{(\tau_k\sigma)^2}\|x^{k+1}-x^k\|^2\\
&\qquad+\frac{2}{\sigma}\|z_1^k-z_1^*\|_{\Sigma_1}^2+\frac{2}{\sigma}\|z_p^k-z_p^*\|_{\Sigma_p}^2+\frac{2}{\sigma}\sum_{i=2}^{p-1}\|z_i^{k+1}-z_i^*\|_{\Sigma_i}^2. \qquad (3.16)
\end{aligned}$$

Recall that $F(z_1,z_2,\ldots,z_p)=\sum_{i=1}^p\mathcal{A}_i^*z_i-c$. Therefore, we have that

$$\sum_{i=2}^{p}\mathcal{A}_i^*(z_i^{k+1}-\widetilde{z}_i^k)=F(z_1^{k+1},z_2^{k+1},\ldots,z_p^{k+1})-F(z_1^{k+1},\widetilde{z}_2^k,\ldots,\widetilde{z}_p^k).$$

In addition, from equation (2.3) it follows that $x^{k+1}-x^k=\sigma\tau_kF(z_1^{k+1},z_2^{k+1},\ldots,z_p^{k+1})$. Substituting the two equalities into (3.16), we obtain the desired inequality

$$\begin{aligned}
&\frac{1}{\alpha}\left(\|\widetilde{w}^k-w^*\|_{\widetilde{\mathcal{G}}}^2-\|\widetilde{w}^{k+1}-w^*\|_{\widetilde{\mathcal{G}}}^2\right)+\frac{1}{\tau_k\sigma^2}\left(\|x^k-x^*\|^2-\|x^{k+1}-x^*\|^2\right)\\
&\quad+\|z_1^k-z_1^*\|_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1}^2-\|z_1^{k+1}-z_1^*\|_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1}^2\\
&\geq\sum_{i=2}^{p-1}\|z_i^{k+1}-\widetilde{z}_i^k\|_{(1-\alpha)\mathcal{A}_i\mathcal{A}_i^*+2\mathcal{T}_i}^2+\|z_1^{k+1}-z_1^k\|_{\mathcal{T}_1}^2+2\|z_p^{k+1}-z_p^k\|_{\mathcal{T}_p}^2\\
&\quad+\|F(z_1^{k+1},\widetilde{z}_2^k,\ldots,\widetilde{z}_p^k)\|^2+(1-\tau_k)\|F(z_1^{k+1},z_2^{k+1},\ldots,z_p^{k+1})\|^2\\
&\qquad+\frac{2}{\sigma}\|z_1^k-z_1^*\|_{\Sigma_1}^2+\frac{2}{\sigma}\|z_p^k-z_p^*\|_{\Sigma_p}^2+\frac{2}{\sigma}\sum_{i=2}^{p-1}\|z_i^{k+1}-z_i^*\|_{\Sigma_i}^2. \qquad (3.17)
\end{aligned}$$

By the definition of τ_k in (2.4), we have that the sequence $\{\tau_k\}$ is nonincreasing and $\tau_k=\overline{\tau}$ for all $k\geq\overline{k}$ once $\tau_{\overline{k}}=\overline{\tau}$. In view of this, we next prove the results of part (a) and part (b) by the case where $\tau_k>\overline{\tau}$ for all k or $\tau_k=\overline{\tau}$ for all $k\geq\overline{k}$.

Case 1: $\tau_k>\overline{\tau}$ for all k. In this case, the definition of τ_k implies $\delta_k\geq 1-\tau_k$, and then

$$\begin{aligned}
&\|F(z_1^{k+1},\widetilde{z}_2^k,\ldots,\widetilde{z}_p^k)\|^2+(1-\tau_k)\|F(z_1^{k+1},z_2^{k+1},\ldots,z_p^{k+1})\|^2\\
&\geq\varepsilon\|F(z_1^{k+1},z_2^{k+1},\ldots,z_p^{k+1})\|^2+\varepsilon\|\mathcal{A}_p^*(z_p^{k+1}-z_p^k)\|^2.
\end{aligned}$$

Together with the above inequality (3.17), we immediately obtain that

$$\begin{aligned}
&\frac{\tau_k}{\alpha}\left(\|\widetilde{w}^k-w^*\|_{\widetilde{\mathcal{G}}}^2-\|\widetilde{w}^{k+1}-w^*\|_{\widetilde{\mathcal{G}}}^2\right)+\frac{1}{\sigma^2}\left(\|x^k-x^*\|^2-\|x^{k+1}-x^*\|^2\right)\\
&\quad+\tau_k\|z_1^k-z_1^*\|_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1}^2-\tau_k\|z_1^{k+1}-z_1^*\|_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1}^2\\
&\quad\geq \tau_k\sum_{i=2}^{p-1}\|z_i^{k+1}-\widetilde{z}_i^k\|_{(1-\alpha)\mathcal{A}_i\mathcal{A}_i^*+2\mathcal{T}_i}^2+\tau_k\|z_1^{k+1}-z_1^k\|_{\mathcal{T}_1}^2\\
&\quad+\tau_k\varepsilon\|F(z_1^{k+1},z_2^{k+1},\ldots,z_p^{k+1})\|^2+\tau_k\|z_p^{k+1}-z_p^k\|_{2\mathcal{T}_p+\varepsilon\mathcal{A}_p\mathcal{A}_p^*}^2\\
&\quad+\frac{2\tau_k}{\sigma}\|z_1^k-z_1^*\|_{\Sigma_1}^2+\frac{2\tau_k}{\sigma}\|z_p^k-z_p^*\|_{\Sigma_p}^2+\frac{2\tau_k}{\sigma}\sum_{i=2}^{p-1}\|z_i^{k+1}-z_i^*\|_{\Sigma_i}^2.
\end{aligned}\tag{3.18}$$

Notice that $\tau_{k+1}\leq\tau_k$ for all $k\geq 1$ in this case. Therefore, we have that

$$\begin{aligned}
&\sum_{k=0}^{\infty}\tau_k\left\{\sum_{i=2}^{p-1}\|z_i^{k+1}-\widetilde{z}_i^k\|_{(1-\alpha)\mathcal{A}_i\mathcal{A}_i^*+2\mathcal{T}_i}^2+\|z_1^{k+1}-z_1^k\|_{\mathcal{T}_1}^2+\varepsilon\|F(z_1^{k+1},\ldots,z_p^{k+1})\|^2\right.\\
&\quad\left.+\|z_p^{k+1}-z_p^k\|_{2\mathcal{T}_p+\varepsilon\mathcal{A}_p\mathcal{A}_p^*}^2+\frac{2}{\sigma}\|z_1^k-z_1^*\|_{\Sigma_1}^2+\frac{2}{\sigma}\|z_p^k-z_p^*\|_{\Sigma_p}^2+\frac{2}{\sigma}\sum_{i=2}^{p-1}\|z_i^{k+1}-z_i^*\|_{\Sigma_i}^2\right\}\\
&\leq\frac{1}{\alpha}\sum_{k=0}^{\infty}\left[\tau_k\|\widetilde{w}^k-w^*\|^2-\tau_{k+1}\|\widetilde{w}^{k+1}-w^*\|^2\right]+\frac{1}{\sigma^2}\sum_{k=0}^{\infty}\left[\|x^k-x^*\|^2-\|x^{k+1}-x^*\|^2\right]\\
&\quad+\sum_{k=0}^{\infty}\left[\tau_k\|z_1^k-z_1^*\|_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1}^2-\tau_{k+1}\|z_1^{k+1}-z_1^*\|_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1}^2\right]\\
&\leq\frac{\tau_0}{\alpha}\|\widetilde{w}^0-w^*\|^2+\frac{1}{\sigma^2}\|x^0-x^*\|^2+\tau_0\|z_1^0-z_1^*\|_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1}^2.
\end{aligned}\tag{3.19}$$

Since $\alpha\in(0,1)$ and $\tau_k>\overline{\tau}>0$, from inequality (3.19) it follows that

$$\lim_{k\to+\infty}\left[\sum_{i=2}^{p-1}\|z_i^{k+1}-\widetilde{z}_i^k\|_{(1-\alpha)\mathcal{A}_i\mathcal{A}_i^*+2\mathcal{T}_i}^2+\|z_p^{k+1}-z_p^k\|_{2\mathcal{T}_p+\varepsilon\mathcal{A}_p\mathcal{A}_p^*}^2\right]=0,\tag{3.20}$$

which, together with the choice of $\mathcal{T}_i$ for $i=2,3,\ldots,p$, implies that

$$\lim_{k\to+\infty}\|z_i^{k+1}-\widetilde{z}_i^k\|=0,\quad i=2,3,\ldots,p.\tag{3.21}$$

Notice that inequality (3.19) also implies that $\lim_{k\to\infty}\|F(z_1^{k+1},\ldots,z_p^{k+1})\|^2=0$. This, together with equation (2.3) and $\tau_k\leq\tau_0$, yields that

$$\lim_{k\to+\infty}\|x^{k+1}-x^k\|=\lim_{k\to+\infty}\|F(z_1^{k+1},z_2^{k+1},\ldots,z_p^{k+1})\|=0.\tag{3.22}$$

The last two equations show that the results of part (a) hold. We next prove that the conclusions of part (b) hold. Notice that equation (3.18) and $\tau_{k+1}\leq\tau_k$ imply that

$$\begin{aligned}
&\frac{\tau_k}{\alpha}\|\widetilde{w}^k-w^*\|_{\widetilde{\mathcal{G}}}^2+\frac{1}{\sigma^2}\|x^k-x^*\|^2+\tau_k\|z_1^k-z_1^*\|_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1}^2\\
&\geq\frac{\tau_{k+1}}{\alpha}\|\widetilde{w}^{k+1}-w^*\|_{\widetilde{\mathcal{G}}}^2+\frac{1}{\sigma^2}\|x^{k+1}-x^*\|^2+\tau_{k+1}\|z_1^{k+1}-z_1^*\|_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1}^2.
\end{aligned}$$

Hence, the sequence $\{\frac{\tau_k}{\alpha}\|\widetilde{w}^k - w^*\|^2_{\widetilde{\mathcal{G}}} + \frac{1}{\sigma^2}\|x^k - x^*\|^2 + \tau_k\|z_1^k - z_1^*\|^2_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1}\}$ is convergent, which implies that the sequences $\{\widetilde{w}^k\}$ and $\{x^k\}$ are bounded, and $\{\|z_1^k - z_1^*\|^2_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1}\}$ is bounded. Together with $\lim_{k\to+\infty}\|w^{k+1} - \widetilde{w}^k\| = 0$ in part (a), it follows that $\{w^k\}$ is bounded. From the boundedness of $\{w^k\}$ and $\{F(z_1^{k+1}, \ldots, z_p^{k+1})\}$ we deduce that the sequence $\{\mathcal{A}_1^* z_1^k\}$ is also bounded, which implies that the boundedness of $\{\|z_1^k - z_1^*\|^2_{\mathcal{A}_1\mathcal{A}_1^*+\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1}\}$. Thus, there exists a subsequence $\{(z_1^k, \ldots, z_p^k, x^k)\}_{k\in K}$ that converges to a limit point, to say $(z_1^\infty, \ldots, z_p^\infty, x^\infty)$. By part (a), $\{(\widetilde{z}_1^k, \ldots, \widetilde{z}_p^k)\}_{k\in K}$ also converges to $(z_1^\infty, \ldots, z_p^\infty)$.

Next we argue that $(z_1^\infty, \ldots, z_p^\infty)$ is an optimal solution to problem (1.1) and x^∞ is the associated Lagrange multiplier. Since $\lim_{k\to+\infty}\|F(z_1^{k+1}, \ldots, z_p^{k+1})\| = 0$, we have $\mathcal{A}_1^* z_1^\infty + \mathcal{A}_2^* z_2^\infty + \cdots + \mathcal{A}_p^* z_p^\infty - c = 0$. In addition, taking the limit $k \to \infty$ with $k \in K$ on the both sides of (3.4) and using the closedness of the graphs of $\partial\theta_i$ (see [24]), we have $-\mathcal{A}_i x^\infty \in \partial\theta_i(z_i^\infty)$ for $i = 1, \ldots, p$. The two sides and equation (3.1) imply that $(z_1^\infty, \ldots, z_p^\infty)$ is an optimal solution of (1.1) and x^∞ is the associated Lagrange multiplier.

To complete the proof of part (b), we only need to show that $(z_1^\infty, \ldots, z_p^\infty, x^\infty)$ is the unique limit point of $\{(z_1^k, \ldots, z_p^k, x^k)\}$. Recall that $(z_1^\infty, \ldots, z_p^\infty)$ is an optimal solution to (1.1) and x^∞ is the associated Lagrange multiplier. So, we could replace $(z_1^*, \ldots, z_p^*, x^*)$ with $(z_1^\infty, \ldots, z_p^\infty, x^\infty)$ in the previous arguments, starting from (3.4). Thus, inequalities (3.18)-(3.19) still hold with $(z_1^*, \ldots, z_p^*, x^*)$ replaced by $(z_1^\infty, \ldots, z_p^\infty, x^\infty)$, and then $\{\frac{\tau_k}{\alpha}\|\widetilde{w}^k - w^\infty\|^2_{\widetilde{\mathcal{G}}} + \frac{1}{\sigma^2}\|x^k - x^\infty\|^2 + \tau_k\|z_1^k - z_1^\infty\|^2_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1}\}$ is convergent. Since this sequence is nonnegative and has a limit point 0 for the subsequence $\{(\widetilde{z}_1^k, \ldots, \widetilde{z}_p^k, x^k)\}_{k\in K}$, we have

$$\lim_{k\to\infty}\frac{\tau_k}{\alpha}\|\widetilde{w}^k - w^\infty\|^2_{\widetilde{\mathcal{G}}} + \frac{1}{\sigma^2}\|x^k - x^\infty\|^2 + \tau_k\|z_1^k - z_1^\infty\|^2_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1} = 0.$$

Moreover, $\lim_{k\to\infty}\|\mathcal{A}_1^* z_1^k\| = \|\mathcal{A}_1^* z_1^\infty\|$ since $\lim_{k\to\infty} z_1^k = z_1^\infty$. By the results of part (a), $\tau_k > \overline{\tau}$ and the positive definiteness of $\mathcal{A}_1\mathcal{A}_1^* + \mathcal{T}_1 + \frac{2}{\sigma}\Sigma_1$, it follows that

$$\lim_{k\to\infty} z_i^k = z_i^\infty \quad \text{for} \quad i = 1, 2, \ldots, p \quad \text{and} \quad \lim_{k\to\infty} x^k = x^\infty. \tag{3.23}$$

Thus, we show that $(z_1^\infty, \ldots, z_p^\infty, x^\infty)$ is the unique limit point of $\{(z_1^k, \ldots, z_p^k, x^k)\}$.

Case 2: $\tau_k = \overline{\tau}$ **for all** $k \geq \overline{k}$ with $\overline{k} \in \mathbb{N}$. Now inequality (3.17) is specialized as

$$\begin{aligned}
&\frac{1}{\alpha}\left(\|\widetilde{w}^k - w^*\|^2_{\widetilde{\mathcal{G}}} - \|\widetilde{w}^{k+1} - w^*\|^2_{\widetilde{\mathcal{G}}}\right) + \frac{1}{\overline{\tau}\sigma^2}\left(\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2\right) \\
&\quad + \|z_1^k - z_1^*\|^2_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1} - \|z_1^{k+1} - z_1^*\|^2_{\mathcal{T}_1+\frac{2}{\sigma}\Sigma_1} \\
&\geq \sum_{i=2}^{p-1}\|z_i^{k+1} - \widetilde{z}_i^k\|^2_{(1-\alpha)\mathcal{A}_i\mathcal{A}_i^*+2\mathcal{T}_i} + \|z_1^{k+1} - z_1^k\|^2_{\mathcal{T}_1} + 2\|z_p^{k+1} - z_p^k\|^2_{\mathcal{T}_p} \\
&\quad + \|F(z_1^{k+1}, \widetilde{z}_2^k, \ldots, \widetilde{z}_p^k)\|^2 + (1 - \overline{\tau})\|F(z_1^{k+1}, z_2^{k+1}, \ldots, z_p^{k+1})\|^2 \\
&\qquad + \frac{2}{\sigma}\|z_1^k - z_1^*\|^2_{\Sigma_1} + \frac{2}{\sigma}\|z_p^k - z_p^*\|^2_{\Sigma_p} + \frac{2}{\sigma}\sum_{i=2}^{p-1}\|z_i^{k+1} - z_i^*\|^2_{\Sigma_i}.
\end{aligned}$$

Since $\overline{\tau} \in (0,1)$, using the same arguments as those for Case 1, we have

$$\lim_{k\to+\infty} \textstyle\sum_{i=2}^{p-1} \|z_i^{k+1} - \tilde{z}_i^k\|^2_{(1-\alpha)\mathcal{A}_i\mathcal{A}_i^*+2\mathcal{T}_i} = 0, \quad \lim_{k\to\infty} \|z_p^{k+1} - z_p^k\|^2_{\mathcal{T}_p} = 0, \tag{3.24}$$

$$\lim_{k\to\infty} \|F(z_1^{k+1}, \tilde{z}_2^k, \ldots, \tilde{z}_p^k)\|^2 = 0, \quad \lim_{k\to\infty} \|F(z_1^{k+1}, z_2^{k+1}, \ldots, z_p^{k+1})\|^2 = 0. \tag{3.25}$$

Combining the first limit in (3.24) with the assumption of $\mathcal{T}_i$ for $i = 2, \ldots, p-1$, we have

$$\lim_{k\to+\infty} \|z_i^{k+1} - \tilde{z}_i^k\| = 0, \quad i = 2, 3, \ldots, p-1. \tag{3.26}$$

From equations (3.25) and (3.26) and the second limit in (3.24), we may deduce that

$$\lim_{k\to\infty} \|z_p^{k+1} - z_p^k\|_{2\mathcal{T}_p + \mathcal{A}_p\mathcal{A}_p^*} = 0.$$

This, along with the assumption of $\mathcal{T}_p$, implies that $\lim_{k\to+\infty} \|z_p^{k+1} - z_p^k\| = 0$, while the second limit in (3.25) implies that $\lim_{k\to+\infty} \|x^{k+1} - x^k\| = 0$. Thus, we complete the proof of part (a). Using the same arguments as those for Case 1 yields part (b). □

If all the linear operators $\mathcal{A}_i$ are surjective, then one can also obtain the conclusion of Theorem 3.1 by setting all $\mathcal{T}_i$ to be the zero operator in the proof of Theorem 3.1.

Corollary 3.1 *Suppose that Assumption 3.1 holds and the linear operators $\mathcal{A}_i$ for $i = 1, 2, \ldots, p$ are all surjective. Then, we have the following conclusions:*

- *(a)* $\lim_{k\to+\infty} \|z_i^{k+1} - \tilde{z}_i^k\| = 0$ *for* $i = 2, 3, \ldots, p$ *and* $\lim_{k\to+\infty} \|x^{k+1} - x^k\| = 0$.
- *(b) The sequences* $\{(z_1^k, \ldots, z_p^k)\}$ *and* $\{(\tilde{z}_1^k, \ldots, \tilde{z}_p^k)\}$ *converge to an optimal solution to (1.1), and* $\{x^k\}$ *converges to an optimal solution to the dual problem of (1.1).*

4 Applications to doubly nonnegative SDPs

Let $\mathcal{S}_+^n$ be the cone of $n \times n$ symmetric and positive semidefinite matrices in the space $\mathbb{S}^n$ of $n \times n$ symmetric matrices, which is endowed with the Frobenius inner product and its induced norm $\|\cdot\|$. The doubly nonnegative SDP problem takes the form of

$$\max\Big\{ -\langle C, X\rangle \mid \mathcal{A}_E X = b_E,\ \mathcal{A}_I X \geq b_I,\ X \in \mathcal{S}_+^n,\ X - M \in \mathcal{K}\Big\}, \tag{4.1}$$

where $b_E \in \mathbb{R}^{m_E}$, $b_I \in \mathbb{R}^{m_I}$, and X–$M \in \mathcal{K}$ means that every entry of X–M is nonnegative (of course, one can only require a subset of the entries of $X - M$ to be nonnegative or non-positive or free). An elementary calculation yields the dual of problem (4.1) as

$$\begin{aligned} &\min\ \big(\delta_{\mathbb{R}_+^{m_I}}(y_I) - \langle b_I, y_I\rangle\big) + \big(\delta_{\mathcal{K}^*}(Z) - \langle M, Z\rangle\big) - \langle b_E, y_E\rangle + \delta_{\mathcal{S}_+^n}(S) \\ &\ \text{s.t.}\ \ \mathcal{A}_I^* y_I + Z + \mathcal{A}_E^* y_E + S = C, \end{aligned} \tag{4.2}$$

where $\mathcal{K}^*$ is the positive dual cone of $\mathcal{K}$. Here we always assume that $\mathcal{A}_E$ is surjective. Clearly, problem (4.2) takes the form of (1.1) with $p = 4$, and takes the form of (1.1) with $p = 3$ if the inequality constraint $\mathcal{A}_I X \geq b_I$ is removed. Hence, we can apply the proposed corrected ADMM with adaptive step-size for solving problem (4.2).

For problem (4.2), instead of using the constraint qualification (CQ) in Assumption 3.1, we use the following more familiar Slater's CQ in the field of conic optimization.

Assumption 4.1 *(a) For problem (4.1), there exists a point $\widehat{X} \in \mathbb{S}^n$ such that*

$$\mathcal{A}_E \widehat{X} = b_E, \ \mathcal{A}_I \widehat{X} \geq b_I, \ \widehat{X} \in \operatorname{int}(\mathcal{S}^n_+), \ \widehat{X} \in \mathcal{K}.$$

(b) For problem (4.2), there exists a point $(\widehat{y}_I, \widehat{Z}, \widehat{y}_E, \widehat{S}) \in \mathbb{R}^{m_I} \times \mathbb{S}^n \times \mathbb{R}^{m_E} \times \mathbb{S}^n$ such that

$$\mathcal{A}_I^* \widehat{y}_I + \widehat{Z} + \mathcal{A}_E^* \widehat{y}_E + \widehat{S} = c, \ \widehat{S} \in \operatorname{int}(\mathcal{S}^n_+), \ \widehat{Z} \in \mathcal{K}^*, \ \widehat{y}_I \in \mathbb{R}^{m_I}_+.$$

From [1, Corollary 5.3.6], under Assumption 4.1, the strong duality for (4.1) and (4.2) holds, and the following Karush-Kuhn-Tucker (KKT) condition has nonempty solutions:

$$\begin{cases} \mathcal{A}_E X - b_E = 0, \\ \mathcal{A}_I^* y_I + Z + \mathcal{A}_E^* y_E + S - C = 0, \\ \langle X, S \rangle = 0, \ X \in \mathcal{S}^n_+, \ S \in \mathcal{S}^n_+, \\ \langle X, Z \rangle = 0, \ X \in \mathcal{K}, \ Z \in \mathcal{K}^*, \\ \langle y_I, \mathcal{A}_I X - b_I \rangle = 0, \ \mathcal{A}_I X - b_I \geq 0, \ y_I \in \mathbb{R}^{m_I}_+. \end{cases} \tag{4.3}$$

Let $\sigma > 0$ be given. The augmented Lagrange function for (4.2) is defined as follows

$$\begin{aligned} L_\sigma(y_I, Z, y_E, S; X) := \ & \delta_{\mathbb{R}^{m_I}_+}(y_I) - \langle b_I, y_I \rangle + (\delta_{\mathcal{K}^*}(Z) - \langle M, Z \rangle) - \langle b_E, y_E \rangle \\ & + \delta_{\mathcal{S}^n_+}(S) + \langle X, \mathcal{A}_I^* y_I + Z + \mathcal{A}_E^* y_E + S - C \rangle \\ & + \frac{\sigma}{2} \left\| \mathcal{A}_I^* y_I + Z + \mathcal{A}_E^* y_E + S - C \right\|^2 \\ & \forall (y_I, Z, y_E, S, X) \in \mathbb{R}^{m_I} \times \mathbb{S}^n \times \mathbb{R}^{m_E} \times \mathbb{S}^n \times \mathbb{S}^n. \end{aligned}$$

Notice that the minimization of $L_\sigma(y_I, Z, y_E, S; X)$ with respect to variables Z and S, respectively, have a closed form solution, while the minimization of $L_\sigma(y_I, Z, y_E, S; X)$ with respect to variable y_E is solvable since the operator $\mathcal{A}_E$ is assumed to be surjective. Hence, when applying the corrected semi-proximal ADMM for solving (4.2), we do not introduce any proximal term to the three minimization problems. In addition, we adopt the solution order $y_I \to Z \to y_E \to S$ for the subproblems involved in (S.1). By Remark 2.1, such a solution order can guarantee that the hard constraints $y_I \in \mathbb{R}^{m_I}_+$ and $S \in \mathcal{S}^n_+$ are satisfied, and when the inequality constraint $\mathcal{A}_I X \geq b_I$ is removed, the hard constraints $Z \in \mathcal{K}^*$ and $S \in \mathcal{S}^n_+$ are satisfied. Extensive numerical tests indicate that such a solution order is the best one. Thus, we obtain the following algorithm.

Algorithm CADMM: A corrected 4-block ADMM for solving (4.2)

Given parameters $\sigma > 0, \alpha = 0.999, \overline{\tau} = 0.1$ and $\varepsilon = 0.1$. Choose $\tau_0 = 1.95$ and a starting point $(y_I^0, Z^0, y_E^0, S^0, X^0) = (\widetilde{y}_I^0, \widetilde{Z}^0, \widetilde{y}_E^0, \widetilde{S}^0, X^0) \in \mathbb{R}_+^{m_I} \times \mathcal{K}^* \times \mathbb{R}^{m_E} \times \mathcal{S}_+^n \times \mathcal{S}_+^n$. Let $\mathcal{T} = \lambda_{\max}(\mathcal{A}_I\mathcal{A}_I^*)\mathcal{I} - \mathcal{A}_I\mathcal{A}_I^*$. For $k = 0, 1, \ldots$, perform the kth iteration as follows.

Step 1. Compute the following minimization problems

$$
\left\{
\begin{aligned}
y_I^{k+1} &= \mathop{\arg\min}_{y_I \in \mathbb{R}_+^{m_I}} \; L_\sigma(y_I, \widetilde{Z}^k, \widetilde{y}_E^k, \widetilde{S}^k; X^k) + \frac{\sigma}{2}\|y_I - \widetilde{y}_I^k\|_{\mathcal{T}}^2, && \text{(4.4a)}\\
Z^{k+1} &= \mathop{\arg\min}_{Z \in \mathcal{K}^*} \; L_\sigma(y_I^{k+1}, Z, \widetilde{y}_E^k, \widetilde{S}^k; X^k), && \text{(4.4b)}\\
y_E^{k+1} &= \mathop{\arg\min}_{y_E \in \mathbb{R}^{m_E}} \; L_\sigma(y_I^{k+1}, Z^{k+1}, y_E, \widetilde{S}^k; X^k), && \text{(4.4c)}\\
S^{k+1} &= \mathop{\arg\min}_{S \in \mathcal{S}_+^n} \; L_\sigma(y_I^{k+1}, Z^{k+1}, y_E^{k+1}, S; X^k). && \text{(4.4d)}
\end{aligned}
\right.
$$

Step 2. Let $X^{k+1} = X^k + \tau_k\sigma\big(\mathcal{A}_I^* y_I^{k+1} + Z^{k+1} + \mathcal{A}_E^* y_E^{k+1} + S^{k+1} - C\big)$ where

$$
\tau_k := \begin{cases} \min(1+\delta_k, \tau_{k-1}) & \text{if } 1+\delta_k > \overline{\tau} \\ \overline{\tau} & \text{otherwise} \end{cases} \quad \text{for } k \geq 1
$$

with

$$
\delta_k = \frac{\big\|\mathcal{A}_I^* y_I^{k+1} + \widetilde{Z}^k + \mathcal{A}_E^* \widetilde{y}_E^k + \widetilde{S}^k - C\big\|^2 - \varepsilon\big\|S^{k+1} - S^k\big\|^2}{\|\mathcal{A}_I^* y_I^{k+1} + Z^{k+1} + \mathcal{A}_E^* y_E^{k+1} + S^{k+1} - C\|^2} - \varepsilon.
$$

Step 3. Let $\widetilde{S}^{k+1} = S^{k+1}, \widetilde{y}_I^{k+1} = y_I^{k+1}$, and

$$
\left\{
\begin{aligned}
\widetilde{y}_E^{k+1} &= \widetilde{y}_E^k + \alpha(y_E^{k+1} - \widetilde{y}_E^k) - \big(\mathcal{A}_E\mathcal{A}_E^*\big)^{-1}(S^{k+1} - S^k),\\
\widetilde{Z}^{k+1} &= \widetilde{Z}^k + \alpha(Z^{k+1} - \widetilde{Z}^k) - (S^{k+1} - S^k) - \mathcal{A}_E^*(\widetilde{y}_E^{k+1} - \widetilde{y}_E^k).
\end{aligned}
\right. \tag{4.5}
$$

4.1 Doubly nonnegative SDP problem sets

In our numerical experiments, we test the following five classes of doubly nonnegative SDP (DNN-SDP) problems, which can also be found in the literature [33, 32, 27].

(i) SDP relaxation of BIQ problems. It has been shown in [2] that under some mild assumptions, the following binary integer quadratic programming (BIQ) problem

$$
\min\left\{\frac{1}{2}x^{\mathbb{T}}Qx + \langle c, x\rangle \mid x \in \{0,1\}^n\right\}
$$

is equivalent to the completely positive programming (CPP) problem given by

$$
\min\left\{\frac{1}{2}\langle Q, Y\rangle + \langle c, x\rangle \mid \mathrm{diag}(Y) - x = 0,\; X = [Y\;\, x; x^{\mathbb{T}}\;\, 1] \in \mathcal{C}_{n+1}\right\},
$$

where $\mathcal{C}_{n+1} := \{0\} \cup \left\{ X \in \mathbb{S}^{n+1} \mid X = \sum_{k\in K} z^k (z^k)^{\mathbb{T}} \text{ for some } \{z^k\}_{k\in K} \subset \mathbb{R}^{n+1}_+ \backslash \{0\} \right\}$ is the $(n+1)$-dimensional completely positive cone. It is well known that the CPP problem is intractable although $\mathcal{C}_{n+1}$ is convex. To solve the CPP problem, one would typically relax $\mathcal{C}_{n+1}$ to $\mathcal{S}^{n+1}_+ \cap \mathcal{K}$, and obtain the following SDP relaxation problem

$$\begin{aligned} \min \ & \frac{1}{2}\langle Q, Y\rangle + \langle c, x\rangle \\ \text{s.t.} \ & \operatorname{diag}(Y) - x = 0, \quad \alpha = 1, \\ & X = \begin{bmatrix} Y & x \\ x^{\mathbb{T}} & \alpha \end{bmatrix} \in \mathcal{S}^{n+1}_+, \quad X \in \mathcal{K} \end{aligned} \tag{4.6}$$

where $\mathcal{K} = \left\{ X \in \mathbb{S}^{n+1} \mid X \geq 0 \right\}$ is the polyhedral cone. In our numerical experiments, the test data for the matrix Q and the vector c are taken from Biq Mac Library maintained by Wiegele, which is available at http://biqmac.uni-klu.ac.at/biqmaclib.html

(ii) θ_+ problems. This class of DNN-SDP problems arises from the relaxation of maximum stable set problems. Given a graph G with edge set E, the SDP relaxation of the maximum stable set problem for the graph G is given by

$$\theta_+(G) := \max \left\{ \langle ee^{\mathbb{T}}, X\rangle \mid \langle \Xi_{ij}, X\rangle = 0 \ \ (i,j) \in E, \ \langle I, X\rangle = 1, \ X \in \mathcal{S}^n_+, \ X \in \mathcal{K} \right\},$$

where e is the vector of ones with dimension known from the context, $\Xi_{ij} = e_i e_j^{\mathbb{T}} + e_j e_i^{\mathbb{T}}$ with e_i denoting the ith column of the $n \times n$ identity matrix, and $\mathcal{K} = \left\{ X \in \mathbb{S}^n \mid X \geq 0 \right\}$. In our numerical experiments, we test the graph instances G considered in [28, 31, 32].

(iii) SDP relaxation of QAP problems. Let $\mathbb{P}^n$ be the set of $n \times n$ permutation matrices. Given matrices $A, B \in \mathbb{S}^n$, the quadratic assignment problem is defined as

$$\overline{v}_{\mathrm{QAP}} := \min \left\{ \langle X, AXB\rangle \mid X \in \mathbb{P}^n \right\}.$$

We identify a matrix $X = [x_1 \ x_2 \ \ldots \ x_n] \in \mathbb{R}^{n\times n}$ with the n^2-vector $x = [x_1; \ldots; ; x_n]$, and let Y^{ij} be the $n \times n$ block corresponding to $x_i x_j^{\mathbb{T}}$ in the $n^2 \times n^2$ matrix $xx^{\mathbb{T}}$. It has been shown in [22] that $\overline{v}_{\mathrm{QAP}}$ is bounded below by the number yielded by

$$\begin{aligned} v := \min \ & \langle B \otimes A, Y\rangle \\ \text{s.t.} \ & \textstyle\sum_{i=1}^n Y^{ii} = I, \\ & \langle I, Y^{ij}\rangle = \delta_{ij} \quad \forall 1 \leq i \leq j \leq n, \\ & \langle \Gamma, Y^{ij}\rangle = 1 \quad \forall 1 \leq i \leq j \leq n, \\ & Y \in \mathcal{S}^{n^2}_+, \quad Y \in \mathcal{K}, \end{aligned} \tag{4.7}$$

where Γ is the matrix of ones, $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ and $\mathcal{K} = \left\{ X \in \mathbb{S}^{n^2} \mid X \geq 0 \right\}$. In our numerical experiments, the test instances (A, B) are taken from the QAP Library [10].

(iv) RCP problems. This class of DNN-SDP problems arises from the SDP relaxation of clustering problems described in [23, Eq(13)] and takes the following form

$$\min\Big\{\big\langle W, I-X\big\rangle \mid Xe = e,\ \langle I, X\rangle = \kappa,\ X \in \mathcal{S}^n_+,\ X \in \mathcal{K}\Big\}, \tag{4.8}$$

where W is the so-called affinity matrix whose entries represent the similarities of the objects in the dataset, e is the vector of ones, κ is the number of clusters, and $\mathcal{K}$ is the cone $\{X \in \mathbb{S}^n \mid X \geq 0\}$. All the data sets we test are from the UCI Machine Learning Repository (available at http://archive.ics.uci.edu/ml/datasets.html). For some large size data sets, we only select the first n rows. For example, the original data set "spambase" has 4061 rows and we select the first 1500 rows to obtain the test problem "spambase-large.2" for which the number "2" means that there are $\kappa = 2$ clusters.

(v) SDP relaxation of FAP problems. Let $G = (V, E)$ be an undirected graph with vertex set V and edge set $E \in V \times V$, and W be a weight matrix for G such that $W_{ij} = W_{ji}$ is the weight associated with $(i, j) \in E$. For those edges $(i, j) \notin E$, we assume $W_{ij} = W_{ji} = 0$. Let $U \subseteq E$ be a given edge subset. This class of problems has the form

$$\begin{aligned}\max\ & \Big\langle \frac{\kappa-1}{2\kappa} L(G,W) - \frac{1}{2}\mathrm{Diag}(We), X\Big\rangle \\ \text{s.t.}\ & \mathrm{diag}(X) = e, \quad X \in \mathcal{S}^n_+, \\ & \langle -\Xi_{ij}, X\rangle = 2/(\kappa-1) \quad \forall (i,j) \in U \subseteq E, \\ & \langle -\Xi_{ij}, X\rangle \leq 2/(\kappa-1) \quad \forall (i,j) \in E\backslash U,\end{aligned} \tag{4.9}$$

where $\kappa > 1$ is an integer, $L(G, W) := \mathrm{Diag}(We) - W$ is the Laplacian matrix, Ξ_{ij} and e are same as above. Let $M_{ij} = -\frac{1}{\kappa-1}$ if $(i, j) \in E$ and otherwise $M_{ij} = 0$. Then (4.9) is also equivalent to

$$\begin{aligned}\max\ & \Big\langle \frac{\kappa-1}{2\kappa} L(G,W) - \frac{1}{2}\mathrm{Diag}(We), X\Big\rangle \\ \text{s.t.}\ & \mathrm{diag}(X) = e, \quad X \in \mathcal{S}^n_+,\ X - M \in \mathcal{K},\end{aligned} \tag{4.10}$$

where $\mathcal{K} = \{X \in \mathbb{S}^n \mid X_{ij} = 0\ \ \forall (i,j) \in U,\ X_{ij} \geq 0\ \ \forall (i,j) \in E\backslash U\}$ (see [27]).

The above five classes of DNN-SDP problems come from the SDP relaxation for some difficult combinatorial optimization problems. For these problems, one usually adds some additional valid inequalities so as to obtain tighter bound for the original combinatorial optimization problems. For example, to obtain a tighter bound for the BIQ problems, one can add four classes of valid inequalities to (4.6) and get the following problems:

$$\begin{aligned}\min\ & \frac{1}{2}\langle Q, Y\rangle + \langle c, x\rangle \\ \text{s.t.}\ & \mathrm{diag}(Y) - x = 0, \quad \alpha = 1, \\ & X = \begin{bmatrix} Y & x \\ x^{\mathbb{T}} & \alpha \end{bmatrix} \in \mathcal{S}^{n+1}_+, \quad X \in \mathcal{K}, \\ & -Y_{ij} + x_i \geq 0,\ -Y_{ij} + x_j \geq 0,\ Y_{ij} - x_i - x_j \geq -1,\ \forall i < j,\ j = 2, \ldots, n-1, \\ & \quad Y_{ij} + Y_{ik} + Y_{jk} - x_i - x_j - x_k \geq -1, \quad j \neq i,\ k \neq i,\ k \neq j\end{aligned} \tag{4.11}$$

where $\mathcal{K} = \{X \in \mathbb{S}^{n+1} \mid X \geq 0\}$, and the set of the first three inequalities are obtained from the valid inequalities $x_i(1-x_j) \geq 0, x_j(1-x_i) \geq 0, (1-x_i)(1-x_j) \geq 0$ when x_i, x_j are binary variables. In the sequel, we call (4.11) the extended BIQ problem.

4.2 Numerical results for DNN-SDPs without $\mathcal{A}_I X \geq b_I$ constraints

In this subsection, we apply CADMM for solving the doubly nonnegative SDP problems without inequality constraints $\mathcal{A}_I X \geq b_I$ described in last subsection, and compare its performance with that of the 3-block ADMM with step-size $\tau = 1.618$ and the 3-block ADMM with Guassian back substitution proposed in [12]. We call the last two methods ADMM3d and ADMM3g, respectively. We have implemented CADMM, ADMM3d and ADMM3g in MATLAB, where the correction step-size of ADMM3g was set to be 0.999 instead of 1 as in [12] for the convergence guarantee. Among others, the solution order of subproblems involved in the 3-block ADMM and the prediction step of ADMM3g is same as that of subproblems in (S.1) of CADMM. Extensive numerical tests show that this order is also the best for ADMM3d and ADMM3g. Notice that ADMM3d here is different from the ADMM developed by Wen et al. [33] since the latter uses the solution order of $y \to Z \to S$. The computational results for all the DNN-SDP problems are obtained on a Windows system with Intel(R) Core(TM) i3-2120 CPU@3.30GHz.

We measure the accuracy of an approximate optimal solution (X, y_E, S, Z) for (4.1) and (4.2) by using the relative residual $\eta = \max\{\eta_P, \eta_D, \eta_{\mathcal{S}}, \eta_{\mathcal{K}}, \eta_{\mathcal{S}^*}, \eta_{\mathcal{K}^*}, \eta_{C_1}, \eta_{C_2}\}$ where

$$\eta_P = \frac{\|\mathcal{A}_E X - b_E\|}{1+\|b_E\|},\ \eta_D = \frac{\|\mathcal{A}_E^* y_E + S + Z - C\|}{1+\|C\|},\ \eta_{\mathcal{S}} = \frac{\|\Pi_{\mathcal{S}_+^n}(-X)\|}{1+\|X\|},\ \eta_{\mathcal{K}} = \frac{\|\Pi_{\mathcal{K}^*}(-X)\|}{1+\|X\|},$$
$$\eta_{\mathcal{S}^*} = \frac{\|\Pi_{\mathcal{S}_+^n}(-S)\|}{1+\|S\|},\ \eta_{\mathcal{K}^*} = \frac{\|\Pi_{\mathcal{K}^*}(-Z)\|}{1+\|Z\|},\ \eta_{C_1} = \frac{\langle X, S\rangle}{1+\|X\|+\|S\|},\ \eta_{C_2} = \frac{\langle X, Z\rangle}{1+\|X\|+\|Z\|}.$$

In addition, we also compute the relative gap by $\eta_g = \frac{\langle C,X\rangle - \langle b_E, y_E\rangle}{1+|\langle C,X\rangle| + |\langle b_E, y_E\rangle|}$. We terminated the solvers CADMM, ADMM3g and ADMM3d whenever $\eta < 10^{-6}$ or the number of iteration is over the maximum number of iterations $k_{\max} = 20000$.

In the implementation of all the solvers, the penalty parameter σ is dynamically adjusted according to the progress of the algorithms. The exact details on the adjustment strategies are too tedious to be presented here but it suffices to mention that the key idea to adjust σ is to balance the progress of primal feasibilities $(\eta_P, \eta_{\mathcal{S}}, \eta_{\mathcal{K}})$ and dual feasibilities $(\eta_D, \eta_{\mathcal{S}^*}, \eta_{\mathcal{K}^*})$. In addition, all the solvers also adopt some kind of restart strategies to ameliorate slow convergence. During the numerical tests, we use the same adjustment strategy of σ and restart strategy for all the solvers.

Table 1 reports the number of problems that are successfully solved to the accuracy of 10^{-6} in η by each of the three solvers within the maximum number of iterations. We see that CADMM and ADMM3d solved successfully all instances from **BIQ**, **RCP**, **FAP** and θ_+, and for **QAP** problems CADMM and ADMM3d solved successfully **39** and **35**, respectively; while ADMM3g solved successfully all instances from **RCP** and **FAP**, but failed to **1** tested problem from **BIQ**, **5** tested problems from θ_+ and **58** tested problems from **QAP**. That is, CADMM solved the most number of instances to the required accuracy, with ADMM3d in the second place, followed by ADMM3g.

Table 1: Numbers of problems that are solved to the accuracy of 10^{-6} in η

Solvers / Problem set	CADMM	ADMM3d	ADMM3g
BIQ(165)	165	165	164
θ_+(113)	113	113	108
QAP(95)	39	35	37
RCP(120)	120	120	120
FAP(13)	13	13	13
Total(506)	450	446	443

Table 2 reports the detailed numerical results of CADMM, ADMM3d and ADMM3g in solving all test instances. From this table, one can learn that CADMM requires the fewest iterations for about 69% test problems though the computing time is comparable even a little more than that of the ADMM3d due to some additional computation cost in the correction step, while ADMM3g requires the most iterations for most of problems and at least 1.5 times as many iterations as CADMM does for about 30% test problems.

Figure 1 (respectively, Figure 2) shows the performance profiles of CADMM, ADM-M3d and ADMM3g in terms of number of iterations and computing time, respectively, for the total 165 **BIQ** (respectively, 120 **RCP**) tested problems. We recall that a point (x, y) is in the performance profiles curve of a method if and only if it can solve $(100y)\%$ of all tested problems no slower than x times of any other methods. It can be seen that CADMM requires the least number of iterations for at least 90% **BIQ** tested problems and 75% **RCP** tested problems, and its computing time is at most 1.3 times as many as that of the fastest solver for 90% instances; while ADMM3g requires the most number of iterations for almost all test instances, and for about 20% BIQ tested problems, its number of iterations is at least twice as many as that of the best solver.

Figure 3 (respectively, Figure 4) shows the performance profiles of CADMM, ADM-M3d and ADMM3g in terms of number of iterations and computing time, respectively, for the total 113 θ_+ and 13 **FAP** (respectively, 95 **QAP**) tested problems. One can see from Figure 3 that CADMM requires the comparable iterations as ADMM3d does for the θ_+ and **FAP** tested problems, for which the former requires the least number of iterations for about 60% problems and the latter requires the least number of iterations for at most 40% problems, while ADMM3g requires at least 1.5 times as many iterations as the best solvers for about 30% problems. Figure 4 indicates that for the **QAP** tested problems, which are the most difficult among the five classes, CADMM has remarkable superiority to ADMM3d and ADMM3g in terms of iterations and computing time, and CADMM requires the least number of iterations for more than 30% problems, while ADMM3d and ADMM3g need the least number of iterations only for 5% problems.

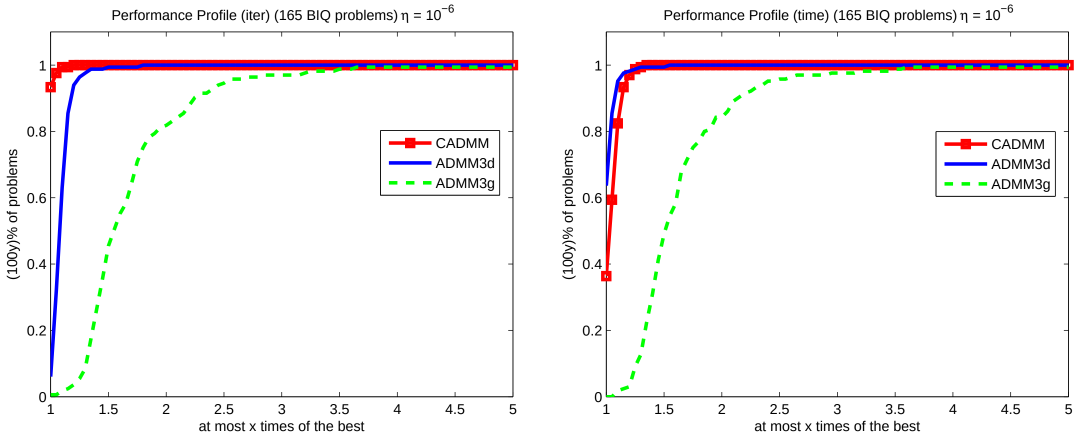

Figure 1: Performance profiles of the number of iterations and computing time for BIQ

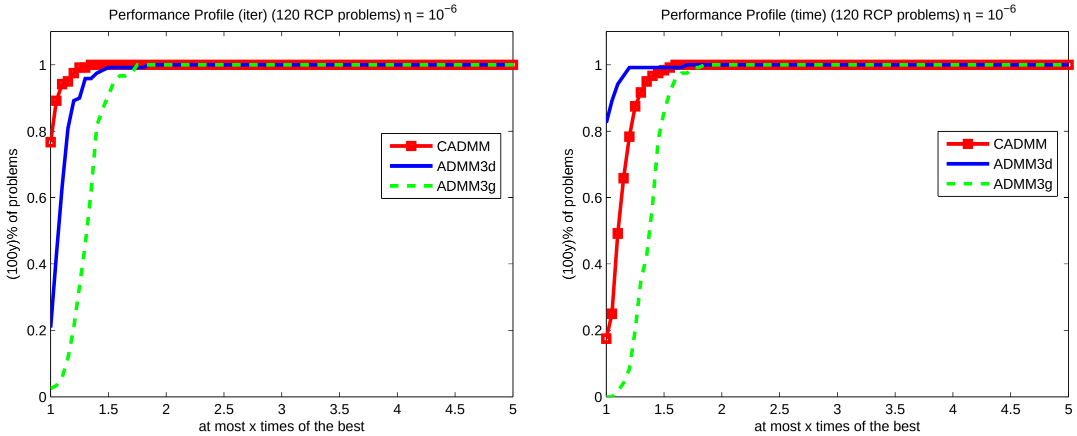

Figure 2: Performance profiles of the number of iterations and computing time for RCP

4.3 Numerical results for DNN-SDPs with $\mathcal{A}_I X \geq b_I$ constraints

We apply CADMM for solving the extended BIQ problems described in (4.11), and compare its performance with the linearized ADMMG in [14] (we call the method LADMM4g and use the parameter $\alpha = 0.999$ in the Gaussian back substitution step). Notice that one may apply the directly extended ADMM with 4 blocks (although without convergent guarantee) for solving (4.11) by adding a proximal term $\frac{\sigma}{2}\|y_I - y_I^k\|^2_{\mathcal{T}}$ for the y_I part, where $\mathcal{T} = \|\mathcal{A}_I\mathcal{A}_I^*\|\mathcal{I} - \mathcal{A}_I\mathcal{A}_I^*$. We call this method ADMM4d, and compare the performance of CADMM with that of ADMM4d with $\tau = 1.618$. The computational results for all the extended BIQ problems are obtained on the same desktop computer as before.

We measure the accuracy of an approximate optimal solution (X, y_I, Z, y_E, S) for (4.1) and (4.2) by the relative residual $\eta = \max\{\eta_P, \eta_D, \eta_{\mathcal{S}}, \eta_{\mathcal{K}}, \eta_{\mathcal{S}^*}, \eta_{\mathcal{K}^*}, \eta_{C_1}, \eta_{C_2}, \eta_I, \eta_{I^*}\}$,

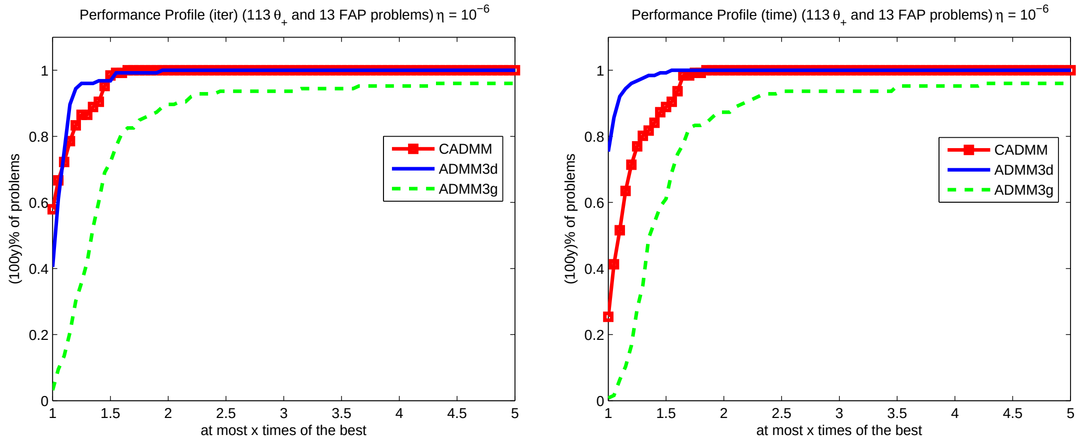

Figure 3: Performance profiles of the number of iterations and computing time for θ_+ and FAP

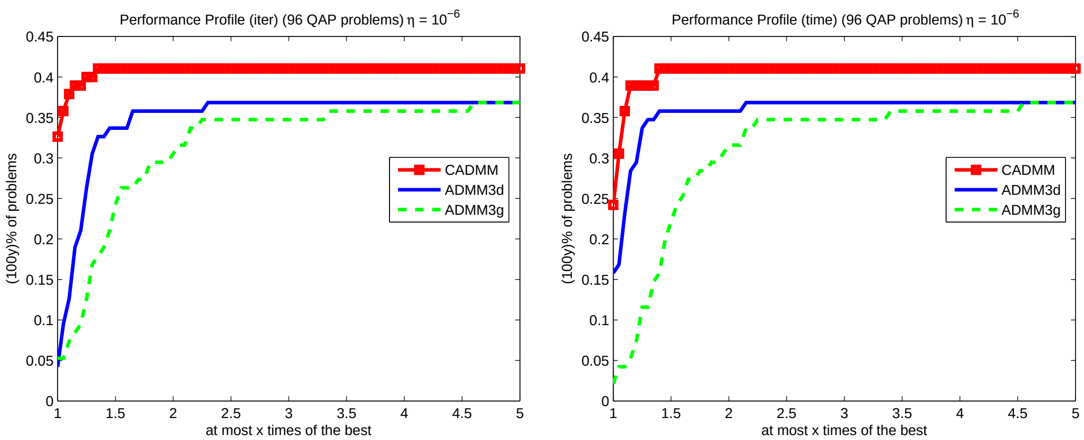

Figure 4: Performance profiles of the number of iterations and computing time for QAP

where $\eta_P, \eta_{\mathcal{S}}, \eta_{\mathcal{K}}, \eta_{\mathcal{S}^*}, \eta_{\mathcal{K}^*}, \eta_{C_1}, \eta_{C_2}$ are defined as before, and $\eta_D, \eta_I, \eta_{I^*}$ are given by

$$\eta_D = \frac{\|\mathcal{A}_I^* y_I + Z + \mathcal{A}_E^* y_E + S - C\|}{1 + \|C\|}, \ \ \eta_I = \frac{\|\max(0, b_I - \mathcal{A}_I X)\|}{1 + \|b_I\|}, \ \ \eta_{I^*} = \frac{\|\max(0, -y_I)\|}{1 + \|y_I\|}.$$

We also compute the relative gap by $\eta_g = \frac{\langle C, X\rangle - (\langle b_E, y_E\rangle + \langle b_I, y_I\rangle)}{1 + |\langle C, X\rangle| + |\langle b_E, y_E\rangle + \langle b_I, y_I\rangle|}$. The solvers CADMM, ADMM4d and LADMM4g were terminated whenever $\eta < 10^{-6}$ or the number of iteration is over the maximum number of iterations $k_{\max} = 40000$.

Table 3 reports the detailed numerical results for the solvers CADMM, ADMM4d and LADMM4g in solving a collection of **165** extended BIQ problems. Figure 5 shows the performance profiles of CADMM, ADMM4d and LADMM4g in terms of the number of iterations and the computing time, respectively, for the total **165** extended BIQ tested problems. It can be seen that CADMM requires the least number of iterations for 80% tested problems although its computing time is comparable with that of ADMM4d, which requires the least computing time for 90% tested problems, while ADMM4g requires 1.5 times as many as iterations as CADMM does for 73% tested problems.

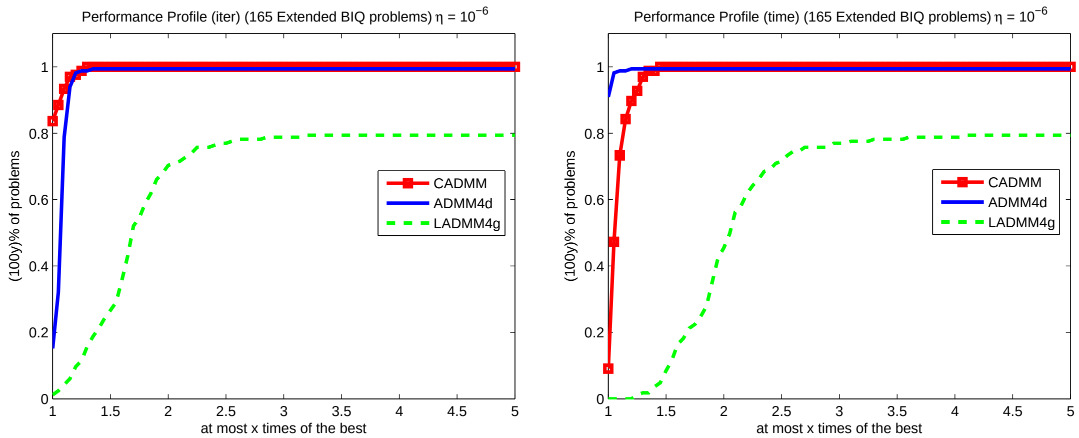

Figure 5: Performance profiles of the number of iterations and computing time for EBIQ

5 Conclusions

We have proposed a corrected semi-proximal ADMM by making suitable correction for the directly extended semi-proximal ADMM with a large step-size, which does not only have convergent guarantee but also enjoys good numerical performance for the general p-block ($p \geq 3$) convex optimization problems with linear equality constraints. Extensive numerical tests for the doubly nonnegative SDP problems with many linear equality and/or inequality constraints show that the corrected semi-proximal ADMM is superior to the directly extended ADMM with step-size $\tau = 1.618$ in terms of the number of iterations, and it requires fewer iterations than the latter for 70% test problems within the comparable computing time. In particular, for 40% tested problems, its number of iterations is only 67% that of the multi-block ADMM with Gaussian back substitution. Thus, the proposed corrected semi-proximal ADMM to a certain extent resolves the dilemma facing all the existing modified versions of the directly extended ADMM. To the best of our knowledge, this is also the first convergent semi-proximal ADMM for the general multi-block convex optimization problem (1.1).

We see from the τ column of Table 2-3 that for most of test instances, the step-size τ_k of the prediction step computed by our proposed formula lies in the interval $[1.8, 1.95]$, while for some test instances (for example, QAP test problems) it will reduce to be strictly less than 1 when the relative residual η is less than a certain threshold. It is interesting that the corrected semi-proximal ADMM still yields the desired result, provided that the small step-size appears after the relative residual η is less than some threshold. This phenomenon seems to match well with the linear convergence rate analysis of the multi-block ADMM in [16]. In our future research work, we will focus on the convergence rate analysis of the corrected semi-proximal ADMM. Another future research work is to explore the effective convergent algorithms for general p-block ($p \geq 3$) separable convex optimization based on the directly extended ADMM with large step-size.

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Table 2: Performance of CADMM, ADMM3d and ADMM3g on θ_+, FAP, QAP, BIQ and RCP problems with accuray=10^{-6}. In this table, τ column reports the final value of τ_k of CADMM, and the computing time is in the format of "hours:minutes:seconds'. In the name of the last 120 problems, "-s, -m, -l" means "-small, -medium, -large", respectively.

problem	m	n	iteration cadm	iteration adm3d	iteration adm3g	η cadm	η adm3d	η adm3g	gap cadm	gap adm3d	gap adm3g	τ cadm	time cadm	time adm3d	time adm3g
theta4	1949	200	311	319	485	9.9-7	9.7-7	3.3-7	3.6-7	3.6-7	-4.9-8	1.84	04	04	07
theta42	5986	200	186	193	222	9.9-7	9.9-7	5.2-7	6.1-8	4.6-8	9.9-9	1.64	03	03	03
theta6	4375	300	308	325	401	8.6-7	8.9-7	4.8-7	-1.7-6	-1.5-6	-7.2-7	1.85	11	10	14
theta62	13390	300	175	168	192	9.9-7	9.6-7	7.9-7	7.8-8	1.4-7	6.5-8	1.59	06	06	07
theta8	7905	400	306	334	404	9.9-7	8.7-7	8.8-7	1.6-6	-1.3-6	-2.3-7	1.84	19	18	24
theta82	23872	400	176	160	189	9.9-7	9.2-7	4.6-7	-1.8-8	-2.0-7	9.9-8	1.65	11	09	13
theta83	39862	400	187	152	184	9.9-7	9.7-7	7.4-7	1.5-9	7.5-8	-4.4-9	1.92	13	09	12
theta10	12470	500	361	379	432	7.4-7	8.9-7	9.1-7	1.4-6	1.6-6	4.0-7	1.84	35	34	42
theta102	37467	500	170	150	191	9.2-7	9.7-7	5.7-7	1.3-7	3.0-7	2.3-7	1.66	18	14	20
theta103	62516	500	191	162	182	9.8-7	9.4-7	4.0-7	-6.4-9	5.3-8	-3.6-8	1.93	20	16	19
theta104	87245	500	209	169	198	9.9-7	9.6-7	4.8-7	-4.4-8	9.0-8	-4.8-8	1.94	23	17	21
theta12	17979	600	374	381	442	9.9-7	8.9-7	9.3-7	-1.8-6	1.5-6	7.5-7	1.85	54	51	1:03
theta123	90020	600	187	167	192	9.4-7	9.5-7	5.7-7	2.0-8	5.3-8	-1.3-8	1.92	30	25	30
theta162	127600	800	179	150	176	9.4-7	9.1-7	4.7-7	-2.2-9	4.2-8	-3.7-8	1.91	55	42	53
theta32	2286	150	235	219	310	9.9-7	9.8-7	4.5-7	1.1-7	9.7-8	3.9-8	1.89	02	02	03
MANN-a45	533116	1035	113	74	90	9.9-7	7.1-7	9.9-7	9.6-7	-4.4-6	-8.3-6	1.90	45	25	35
hamming6-4	705	64	226	224	231	7.6-7	8.3-7	5.7-7	-2.3-6	-2.1-6	1.8-6	1.62	01	00	01
hamming8-4	20865	256	257	262	293	9.4-7	7.9-7	1.0-6	-3.0-6	3.6-6	-1.3-6	1.64	06	06	07
MANN-a9	919	45	90	81	76	8.6-7	9.8-7	8.1-7	3.7-7	-1.3-6	-1.8-6	1.76	00	00	00
MANN-a27	70552	378	76	63	71	9.2-7	8.0-7	7.7-7	-1.2-6	-3.4-6	3.9-6	1.76	04	03	04
johnson8-2-4	211	28	104	104	107	9.4-7	7.8-7	9.0-7	9.4-7	1.2-6	8.2-7	1.95	00	00	00
johnson8-4-4	1856	70	106	109	117	7.4-7	6.6-7	7.1-7	-2.6-6	-2.5-6	2.1-6	1.38	00	00	00
johnson16-2-4	5461	120	210	196	207	8.4-7	9.7-7	9.7-7	2.8-7	1.5-6	1.3-6	1.40	01	01	01
johnson32-2-4	107881	496	409	398	409	9.4-7	9.5-7	9.4-7	-3.2-8	3.7-7	-1.2-6	1.89	36	29	35
sanr200-0.7	13869	200	249	234	401	9.9-7	9.8-7	1.1-7	-2.8-8	1.6-7	-1.5-8	1.95	04	03	06
san200-0.7-1	13931	200	3029	2926	3583	9.9-7	9.9-7	9.4-7	-1.4-6	-1.4-6	-1.4-6	1.52	44	38	51
san200-0.7-2	13931	200	1210	831	755	9.1-7	9.3-7	9.3-7	2.3-6	2.2-6	4.1-6	1.77	14	09	09
sanr200-0.9	17864	200	463	575	1001	9.9-7	9.9-7	8.5-8	-3.0-7	-3.9-7	-4.7-8	1.95	07	08	15
san200-0.9-1	17911	200	9455	10549	14801	9.9-7	9.9-7	9.2-7	-5.2-6	-5.2-6	-5.0-6	1.95	2:17	2:14	3:33
san200-0.9-2	17911	200	1313	1330	1788	9.9-7	9.9-7	8.1-7	-1.0-6	-1.3-6	-1.2-6	1.95	19	17	26
san200-0.9-3	17911	200	802	816	2501	9.9-7	9.9-7	1.9-7	-8.4-7	-6.9-7	-4.8-7	1.95	12	10	36
sanr400-0.5	39985	400	210	144	228	9.8-7	9.9-7	3.1-7	-2.4-8	4.4-8	-7.8-8	1.95	14	09	15
san400-0.5-1	39901	400	523	575	807	9.9-7	9.9-7	6.2-7	-2.0-7	-1.9-7	-1.6-7	1.71	34	34	52
sanr400-0.7	55870	400	228	159	243	9.6-7	9.3-7	2.7-7	-4.2-8	1.4-7	-9.8-8	1.95	16	10	16
san400-0.7-1	55861	400	450	432	823	9.7-7	9.9-7	1.4-7	4.8-8	-2.2-6	-9.7-7	1.41	30	25	53
san400-0.7-2	55861	400	5125	9787	8921	9.9-7	9.7-7	9.9-7	-1.4-5	-1.1-5	-9.3-6	0.11	4:01	6:05	6:36
san400-0.7-3	55861	400	2860	1999	2037	9.8-7	9.9-7	9.8-7	-5.8-6	-3.7-6	5.6-6	0.49	2:12	1:17	1:32
san400-0.9-1	71821	400	960	1048	1442	9.9-7	9.9-7	8.3-7	-1.1-6	-1.1-6	-2.1-6	1.95	1:06	1:01	1:36
c-fat200-1	1535	200	1418	1417	20000	8.6-7	9.5-7	1.5-5	1.8-8	-1.7-8	4.0-8	1.80	14	12	3:09
c-fat200-2	3236	200	182	205	226	7.6-7	8.6-7	1.0-6	7.6-6	-7.0-6	-1.0-6	1.92	02	02	02
c-fat200-5	8474	200	196	170	296	9.9-7	7.4-7	6.4-7	3.0-8	-3.0-8	-4.6-9	1.90	02	01	03
c-fat500-1	4460	500	1034	1185	1376	9.5-7	8.7-7	9.8-7	6.2-6	5.8-6	8.2-7	1.84	1:15	1:18	1:38
c-fat500-2	9140	500	428	431	532	8.4-7	8.6-7	9.2-7	1.0-5	-1.1-5	3.9-7	1.90	30	28	37
c-fat500-5	23192	500	201	190	221	2.9-7	7.1-7	7.9-7	8.5-6	1.7-5	2.9-6	1.92	13	11	14
c-fat500-10	46628	500	155	164	219	9.6-7	7.5-7	2.3-7	-1.5-7	-3.9-7	-5.6-8	1.91	10	09	14
DSJC500-5	62625	500	202	143	201	9.4-7	9.9-7	2.8-7	-7.8-9	5.3-8	-8.6-8	1.94	22	14	22
DSJC1000-5	249827	1000	214	162	224	9.6-7	9.2-7	3.1-7	-2.6-8	-6.4-9	-6.8-8	1.94	1:58	1:21	2:02
hamming6-2	1825	64	218	194	261	4.9-7	8.6-7	9.7-7	-1.1-5	-1.7-5	-7.6-7	1.76	00	00	01
hamming-6-4	1313	64	68	69	73	9.9-7	9.8-7	8.4-7	4.3-6	-3.3-6	2.2-6	1.95	00	00	00
hamming8-2	31617	256	564	491	668	7.2-7	8.5-7	9.7-7	4.6-5	-6.8-5	-1.4-6	1.09	11	08	13
hamming-8-4	11777	256	133	134	137	8.3-7	9.0-7	9.7-7	-1.7-6	-2.1-6	-1.4-6	1.87	03	02	03
hamming-9-8	2305	512	2978	2824	3082	9.7-7	9.9-7	9.8-7	-1.5-6	7.1-7	3.8-7	0.67	3:27	2:53	3:30
hamming-10-2	23041	1024	701	718	752	9.4-7	9.2-7	7.9-7	3.7-6	3.6-6	1.3-6	1.89	4:59	4:42	5:15
hamming-7-5-6	1793	128	592	580	585	8.9-7	9.7-7	9.5-7	2.0-6	1.1-6	1.2-6	1.67	03	03	03
hamming-8-3-4	16129	256	224	244	282	6.8-7	4.3-7	7.5-7	3.2-7	1.5-7	-2.0-8	1.54	05	05	07
hamming-9-5-6	53761	512	537	537	533	9.0-7	9.9-7	9.6-7	-1.6-6	-1.9-6	-1.3-6	1.47	47	43	46
brock200-1	14835	200	457	485	1101	9.8-7	9.9-7	9.8-8	-1.0-7	-8.8-8	-5.7-8	1.95	07	07	17
brock200-2	9877	200	208	157	218	9.6-7	9.6-7	3.8-7	-2.1-8	4.9-8	-9.4-8	1.95	03	02	03
brock200-3	12049	200	225	181	244	9.5-7	9.7-7	3.1-7	-4.9-8	-1.7-7	-1.2-7	1.95	03	03	04
brock200-4	13090	200	226	192	263	8.6-7	9.9-7	3.6-7	-4.8-8	-2.1-7	-1.5-7	1.95	03	03	04
brock400-1	59724	400	230	167	247	9.9-7	9.6-7	2.8-7	-2.5-8	-1.3-8	-9.7-8	1.95	16	10	17
brock400-2	59787	400	230	164	251	9.8-7	9.7-7	3.1-7	-9.0-8	-9.1-8	-1.1-7	1.95	16	10	17
brock400-3	59682	400	230	164	251	9.5-7	9.5-7	2.8-7	-1.0-7	-3.3-8	-1.1-7	1.95	16	10	17
brock400-4	59766	400	230	170	250	9.9-7	9.5-7	2.8-7	-5.3-8	1.5-8	-9.2-8	1.95	16	10	17
brock800-1	207506	800	224	155	218	9.9-7	9.2-7	2.5-7	-5.0-8	3.5-8	-10.0-8	1.95	1:12	45	1:10
brock800-2	208167	800	225	154	218	9.8-7	9.2-7	2.4-7	-8.3-8	1.2-8	-1.0-7	1.95	1:12	45	1:09
brock800-3	207334	800	225	154	220	9.9-7	9.5-7	2.4-7	-7.6-8	1.7-8	-10.0-8	1.95	1:12	45	1:10
brock800-4	207644	800	225	154	216	9.7-7	9.1-7	2.5-7	-6.2-8	1.9-8	-9.8-8	1.95	1:12	45	1:09
keller4	5101	171	386	332	688	9.9-7	9.9-7	7.0-8	3.5-7	-4.4-7	-5.1-8	1.95	03	03	06
keller5	225991	776	698	749	725	9.9-7	9.9-7	9.9-7	1.7-6	1.5-6	-1.6-6	1.40	2:37	2:27	2:41
p-hat300-1	10934	300	1262	1445	5401	9.9-7	9.9-7	2.9-7	-4.2-7	-4.6-7	-3.3-7	1.88	43	44	3:01
p-hat300-2	21929	300	2939	3428	20000	9.9-7	9.9-7	4.6-7	-1.6-6	-1.5-6	-1.9-6	1.95	1:43	1:47	11:35
p-hat300-3	33391	300	638	729	2301	9.9-7	9.9-7	1.2-7	2.7-7	3.7-7	-1.2-7	1.95	23	23	1:21
C125.9	6964	125	13690	14480	20000	9.9-7	9.9-7	1.4-6	-1.8-6	-2.1-6	-3.6-6	1.93	1:28	1:21	2:07
C250.9	27985	250	968	1462	1348	9.7-7	9.7-7	9.4-7	1.2-5	1.1-5	9.2-6	0.32	19	23	25
C500.9	112333	500	243	214	347	9.9-7	9.8-7	2.5-7	-4.1-8	-1.4-7	-1.7-7	1.95	27	21	38
G43	9991	1000	1141	1198	1296	9.9-7	9.9-7	5.4-7	-1.8-6	-1.8-6	3.4-8	1.82	6:46	6:34	7:37

Table 2: (continued) Performance of CADMM, ADMM3d and ADMM3g on θ_+, FAP, QAP, BIQ and RCP problems with accuray=10^{-6}. In this table, τ column reports the final value of τ_k of CADMM, and the computing time is in the format of "hours:minutes:seconds'. In the name of the last 120 problems, "-s, -m, -l" means "-small, -medium, -large", respectively.

problem	m	n	iteration cadm	iteration adm3d	iteration adm3g	η cadm	η adm3d	η adm3g	gap cadm	gap adm3d	gap adm3g	τ cadm	time cadm	time adm3d	time adm3g
G44	9991	1000	1135	1203	1340	9.1-7	9.5-7	3.8-7	-1.7-6	-9.6-7	-4.0-7	1.83	6:43	6:34	7:52
G45	9991	1000	1163	1198	1341	9.6-7	9.8-7	4.2-7	1.7-6	-1.7-6	-6.4-7	1.82	6:54	6:34	7:53
G46	9991	1000	1187	1274	1331	8.9-7	9.7-7	4.3-7	-1.6-6	1.8-6	-2.6-7	1.81	7:01	6:57	7:49
G47	9991	1000	1135	1181	1353	9.0-7	9.9-7	2.4-7	-1.7-6	1.0-6	-3.9-7	1.85	6:42	6:27	7:55
G51	5910	1000	3133	3201	6901	9.9-7	9.9-7	5.4-7	-6.9-7	-1.4-6	-3.9-8	1.89	24:36	23:50	54:32
G52	5917	1000	9038	10629	9301	9.9-7	9.9-7	9.9-7	-1.3-7	-3.5-7	2.3-7	1.89	1:11:49	1:17:37	1:12:32
G53	5915	1000	13309	14509	20000	9.9-7	9.9-7	1.6-6	-1.9-6	-1.6-6	-2.4-6	1.89	1:45:10	1:48:25	2:36:28
G54	5917	1000	3289	4088	4801	9.9-7	9.9-7	6.0-7	-2.5-6	2.3-6	-5.3-7	1.89	25:43	30:11	37:14
1dc.64	544	64	366	366	406	9.0-7	9.4-7	9.8-7	-2.3-7	5.3-7	2.1-6	1.95	01	01	01
1et.64	265	64	214	245	301	9.4-7	9.8-7	9.2-7	1.8-6	-2.4-6	6.8-7	1.87	00	00	01
1tc.64	193	64	329	343	419	8.5-7	9.2-7	9.2-7	2.7-6	-2.8-6	8.1-7	1.95	01	01	01
1dc.128	1472	128	2613	2661	3484	9.9-7	9.9-7	9.9-7	-1.7-6	-1.8-6	-1.9-6	1.69	15	14	19
1et.128	673	128	359	363	481	9.2-7	8.6-7	2.5-7	-2.5-6	-2.6-6	-3.9-8	1.91	02	02	02
1tc.128	513	128	1186	1063	1144	9.0-7	9.6-7	9.9-7	2.2-7	-2.3-6	2.6-7	1.91	05	04	05
1zc.128	1121	128	179	175	228	7.8-7	9.3-7	7.5-7	2.2-6	2.5-6	-3.9-8	1.68	01	01	01
2dc.128	5174	128	727	827	980	9.5-7	9.9-7	5.5-7	7.6-8	-1.1-7	-7.4-7	1.92	04	04	06
1dc.256	3840	256	5915	8069	5335	9.6-7	9.6-7	9.7-7	-4.8-6	4.6-6	4.7-6	0.90	1:38	1:56	1:29
1et.256	1665	256	791	825	1144	9.8-7	9.9-7	2.2-7	3.8-7	2.3-7	6.6-8	1.82	16	15	22
1tc.256	1313	256	1815	1393	3001	9.9-7	9.9-7	2.6-7	-1.9-7	1.1-7	-3.2-7	1.89	38	27	1:02
1zc.256	2817	256	249	277	289	9.0-7	7.9-7	9.7-7	-2.3-6	-1.9-6	9.1-7	1.73	05	05	05
2dc.256	17184	256	9572	9817	17845	9.9-7	9.9-7	9.8-7	-1.1-5	-1.2-5	-8.7-6	1.95	3:35	3:25	6:32
1dc.512	9728	512	2672	2856	3519	9.9-7	9.9-7	9.3-7	-1.7-7	-1.7-7	-2.0-7	1.82	4:17	4:11	5:32
1et.512	4033	512	946	1067	1553	9.9-7	9.9-7	1.5-7	-9.8-8	3.2-7	-3.5-7	1.90	1:22	1:23	2:12
1tc.512	3265	512	2852	4282	4301	9.9-7	9.9-7	3.8-7	-6.6-7	-1.1-7	-2.9-7	1.88	4:22	6:20	6:28
2dc.512	54896	512	3712	4240	5301	9.9-7	9.9-7	4.0-7	-3.9-6	-2.5-6	-2.1-6	1.95	5:41	5:53	7:59
1zc.512	6913	512	486	534	657	9.3-7	8.8-7	6.3-7	-2.8-6	2.7-6	-2.2-8	1.76	40	40	53
1dc.1024	24064	1024	3058	3300	3942	9.9-7	9.9-7	8.4-7	-5.5-7	-7.4-7	-7.9-7	1.84	24:35	24:59	31:20
1et.1024	9601	1024	1689	1580	2314	9.9-7	9.9-7	4.5-7	4.3-8	9.0-8	-1.7-7	1.89	12:42	11:00	17:09
1tc.1024	7937	1024	3545	3947	6801	9.9-7	9.9-7	3.3-7	-4.4-7	-1.1-6	-5.5-7	1.89	27:35	28:32	52:12
1zc.1024	16641	1024	749	781	879	8.7-7	8.6-7	9.6-7	2.7-6	2.7-6	8.9-7	1.74	5:29	6:34	6:20
2dc.1024	169163	1024	5809	6441	20000	9.9-7	9.9-7	1.0-6	-7.9-6	-7.9-6	9.1-6	1.95	46:06	47:35	2:39:50
1dc.2048	58368	2048	5445	5736	6647	9.9-7	9.9-7	9.3-7	-7.8-7	-6.2-7	-8.7-7	1.87	3:02:23	3:02:40	3:47:51
1et.2048	22529	2048	2766	3118	4301	9.9-7	9.9-7	5.0-7	-1.5-6	-1.4-6	-1.3-6	1.89	1:26:11	1:32:34	2:17:35
1tc.2048	18945	2048	3460	4766	6101	9.9-7	9.9-7	5.2-7	-4.8-7	-8.4-7	-8.1-7	1.89	2:05:05	2:45:08	3:28:32
1zc.2048	39425	2048	1302	1337	2377	9.5-7	9.1-7	7.2-7	-3.0-6	2.4-6	-1.2-6	1.82	40:55	38:46	1:20:24
2dc.2048	504452	2048	4177	4323	5191	9.9-7	9.9-7	9.0-7	-4.5-6	-4.2-6	-7.5-6	1.84	2:16:34	2:12:21	2:48:57
fap01	52	52	639	557	715	9.5-7	9.5-7	9.7-7	9.2-7	-7.4-6	2.0-6	1.15	01	01	01
fap02	61	61	882	1022	1130	9.8-7	9.2-7	9.0-7	7.1-6	2.5-6	1.7-5	1.90	01	02	02
fap03	65	65	565	642	904	9.9-7	9.9-7	9.9-7	-5.7-6	-6.5-6	-6.4-6	1.95	01	01	02
fap04	81	81	499	569	712	9.9-7	9.8-7	9.9-7	1.6-5	1.6-5	1.6-5	1.95	01	02	02
fap05	84	84	1665	1903	2187	9.9-7	9.9-7	9.9-7	-2.2-6	-1.7-6	-1.3-6	1.95	06	06	08
fap06	93	93	633	624	1232	9.9-7	9.9-7	9.9-7	-2.3-6	-2.9-6	-1.4-6	1.95	03	02	05
fap07	98	98	698	690	889	9.9-7	9.9-7	9.9-7	-1.4-6	-1.4-6	-1.3-6	1.95	03	02	03
fap08	120	120	433	515	580	9.9-7	9.9-7	9.9-7	-5.0-6	-3.8-6	-5.1-6	1.95	02	02	03
fap09	174	174	412	482	555	9.9-7	9.9-7	9.9-7	9.8-7	1.1-6	9.1-7	1.95	04	04	05
fap10	183	183	3706	4010	5099	9.9-7	9.9-7	9.9-7	-3.9-5	-3.9-5	-4.0-5	1.90	53	54	1:11
fap11	252	252	3324	3730	4568	9.9-7	9.9-7	9.9-7	-5.0-5	-4.9-5	-5.0-5	1.90	1:25	1:31	1:55
fap12	369	369	5468	6561	7467	9.9-7	9.9-7	9.9-7	-4.6-5	-4.4-5	-4.6-5	1.90	4:41	5:18	6:20
fap25	2118	2118	8047	8235	10314	9.8-7	9.9-7	9.9-7	-1.2-5	-1.2-5	-1.2-5	1.90	5:24:18	5:31:01	7:16:36
bur26a	1051	676	20000	20000	20000	1.4-5	1.3-5	1.7-5	-2.5-5	-2.4-5	-2.6-5	1.95	52:59	52:47	57:27
bur26b	1051	676	20000	20000	20000	1.4-5	1.5-5	1.1-5	-2.8-5	-3.0-5	-2.4-5	1.95	53:29	50:13	58:02
bur26c	1051	676	20000	20000	20000	1.8-5	2.1-5	2.0-5	-3.1-5	-3.5-5	-2.8-5	1.94	53:02	48:48	55:29
bur26d	1051	676	20000	20000	20000	1.9-5	1.2-5	2.0-5	-3.9-5	-3.3-5	-4.0-5	1.95	52:50	53:45	58:10
bur26e	1051	676	20000	20000	20000	1.1-5	2.7-6	1.4-5	-1.0-5	-2.8-6	-1.2-5	1.90	52:56	51:55	57:19
bur26f	1051	676	20000	20000	19058	2.6-6	6.2-6	9.9-7	-4.5-6	-5.8-6	-4.3-6	1.90	52:08	48:11	53:28
bur26g	1051	676	9133	11020	19199	9.9-7	9.6-7	9.9-7	-4.5-7	4.4-8	-5.0-6	1.86	23:31	28:22	52:34
bur26h	1051	676	10785	20000	16070	9.9-7	1.3-6	9.8-7	-2.4-6	-1.9-6	-2.5-6	1.88	27:21	48:57	44:12
chr12a	232	144	4472	4499	4078	2.5-7	3.9-7	6.7-7	-3.7-6	3.7-6	-4.1-6	1.06	22	20	21
chr12b	232	144	4761	3682	3642	4.4-7	6.6-7	9.0-7	-9.4-6	1.0-5	-8.2-6	1.07	21	16	17
chr12c	232	144	19941	20000	20000	8.9-7	1.0-5	1.8-5	-1.9-5	-4.1-5	-4.1-5	1.09	2:04	1:56	2:06
chr15a	358	225	20000	20000	20000	2.6-5	2.8-5	4.6-5	**-1.5-4**	**-1.6-4**	**-1.8-4**	1.92	4:48	4:22	4:49
chr15b	358	225	3909	4872	5231	8.9-7	9.9-7	9.7-7	-2.5-5	-2.8-5	-2.7-5	0.39	53	58	1:11
chr15c	358	225	3735	4279	4685	9.6-7	9.1-7	9.8-7	2.7-5	2.5-5	-2.6-5	0.29	50	50	1:01
chr18a	511	324	20000	20000	20000	1.4-5	1.4-5	2.0-5	-3.2-5	-3.1-5	-3.1-5	1.91	11:47	10:49	11:56
chr18b	511	324	1196	1563	1653	9.9-7	9.9-7	9.9-7	-5.4-7	-1.1-6	-9.0-7	1.92	43	52	1:01
chr20a	628	400	12994	16606	19613	9.3-7	9.9-7	9.1-7	-1.9-5	-1.1-5	-1.2-5	0.29	11:12	13:07	17:27
chr20b	628	400	8987	9665	13145	8.4-7	9.1-7	9.9-7	3.1-5	-3.3-5	-3.5-5	0.27	8:00	7:39	11:38
chr20c	628	400	15319	15687	16231	8.5-7	9.1-7	9.7-7	-1.5-5	-1.1-5	-1.3-5	1.86	10:56	10:10	12:01
chr22a	757	484	7153	9283	10896	9.7-7	9.8-7	9.6-7	3.7-5	3.7-5	3.6-5	0.19	9:26	10:40	14:08
chr22b	757	484	9545	9325	11473	9.9-7	9.8-7	9.5-7	3.6-5	-3.5-5	-3.4-5	0.34	12:31	10:58	15:32
chr25a	973	625	8402	10379	12436	9.7-7	9.9-7	9.9-7	4.1-5	4.2-5	4.2-5	0.47	19:35	21:31	29:28
els19	568	361	4751	5008	5446	9.7-7	7.9-7	9.6-7	2.8-6	-7.0-8	-4.2-6	1.85	3:03	3:03	3:42
esc16a	406	256	20000	20000	20000	1.2-5	1.3-5	1.0-6	-4.4-5	-5.0-5	-2.0-5	1.95	6:21	5:16	6:56
esc16b	406	256	20000	20000	20000	1.7-6	2.5-6	4.1-5	-9.9-5	-9.8-5	**-4.8-4**	1.89	5:44	5:01	7:33
esc16c	406	256	20000	20000	20000	1.5-5	3.9-5	6.2-5	**-1.5-4**	**-1.4-4**	**-3.0-4**	1.94	6:08	5:20	7:01
esc16d	406	256	247	352	551	9.4-7	9.7-7	9.9-7	-5.7-7	-8.3-7	-1.8-6	1.95	05	06	10
esc16e	406	256	337	441	714	9.9-7	9.9-7	6.9-7	-7.5-7	-7.7-8	-3.0-6	1.95	06	07	12

Table 2: (continued) Performance of CADMM, ADMM3d and ADMM3g on θ_+, FAP, QAP, BIQ and RCP problems with accuray=10^{-6}. In this table, τ column reports the final value of τ_k of CADMM, and the computing time is in the format of "hours:minutes:seconds'. In the name of the last 120 problems, "-s, -m, -l" means "-small, -medium, -large", respectively.

problem	m	n	iteration			η			gap			τ	time		
			cadm	adm3d	adm3g	cadm	adm3d	adm3g	cadm	adm3d	adm3g	cadm	cadm	adm3d	adm3g
esc16g	406	256	393	444	791	9.9-7	9.9-7	8.7-7	9.5-8	-1.2-7	-1.6-6	1.95	07	07	14
esc16h	406	256	20000	20000	20000	1.3-6	1.5-6	1.7-5	-1.1-5	-1.2-5	-4.6-5	1.95	5:41	4:59	7:48
esc16i	406	256	1496	1584	1876	9.9-7	9.9-7	9.9-7	-6.0-7	-4.5-7	-5.0-7	1.91	26	24	33
esc16j	406	256	409	521	1361	9.8-7	9.9-7	9.2-7	-5.6-6	-4.0-6	-5.0-6	1.95	07	08	24
esc32a	1582	1024	2881	3211	3408	9.9-7	9.9-7	9.9-7	-2.3-7	-1.3-7	-2.7-6	1.90	23:54	27:02	28:44
esc32b	1582	1024	20000	20000	20000	1.7-6	5.3-6	7.4-6	-3.2-5	-4.2-5	-6.6-5	1.95	2:32:39	2:28:27	2:51:39
esc32c	1582	1024	20000	20000	20000	3.7-6	4.2-6	1.3-5	-1.5-5	-1.6-5	-2.9-5	1.73	2:20:45	2:16:15	2:27:11
esc32d	1582	1024	689	1105	1228	9.9-7	9.9-7	9.9-7	-1.6-6	-2.1-7	-1.6-7	1.92	5:07	7:04	9:14
esc32e	1582	1024	977	983	6399	9.9-7	9.9-7	8.4-7	1.9-9	1.4-9	-2.9-6	1.93	6:50	6:12	43:50
esc32f	1582	1024	977	983	6399	9.9-7	9.9-7	8.4-7	1.9-9	1.4-9	-2.9-6	1.93	6:47	6:16	44:07
esc32g	1582	1024	461	546	2113	7.1-7	9.1-7	9.9-7	3.7-8	-3.3-8	-3.0-6	1.95	3:14	3:28	14:40
esc32h	1582	1024	20000	20000	20000	3.3-5	3.2-5	3.3-5	**-1.3-4**	**-1.3-4**	**-2.7-4**	1.95	2:31:52	2:21:28	2:33:37
had12	232	144	20000	20000	20000	1.9-6	2.1-6	8.3-6	-1.0-5	-1.1-5	-6.0-6	1.90	2:03	1:53	2:04
had14	313	196	20000	20000	20000	5.9-6	7.2-6	1.5-5	-1.9-5	-2.3-5	-1.5-5	1.91	3:36	3:16	3:36
had16	406	256	12877	14321	18380	9.9-7	7.7-7	9.6-7	1.2-5	-9.5-6	-1.2-5	0.43	4:45	4:39	6:52
had18	511	324	20000	20000	20000	2.8-5	3.2-5	4.7-5	**-1.4-4**	**-1.7-4**	**-2.0-4**	1.91	11:42	10:30	11:50
had20	628	400	20000	20000	20000	3.0-5	3.1-5	4.9-5	**-1.3-4**	**-1.3-4**	**-1.9-4**	1.91	18:29	16:39	18:37
kra30a	1393	900	20000	20000	20000	3.3-5	3.6-5	4.3-5	**-2.2-4**	**-2.3-4**	**-3.9-4**	1.95	1:50:27	1:42:43	1:50:12
kra30b	1393	900	20000	20000	20000	2.9-5	3.0-5	4.1-5	**-1.7-4**	**-1.8-4**	**-3.7-4**	1.95	1:57:12	1:49:16	1:57:05
kra32	1582	1024	20000	20000	20000	2.9-5	3.1-5	3.3-5	**-1.3-4**	**-1.5-4**	**-2.6-4**	1.95	2:46:55	2:28:44	2:40:10
lipa20a	628	400	3308	3900	4233	9.9-7	9.9-7	9.8-7	1.5-5	1.5-5	-1.5-5	0.22	2:34	2:44	3:07
lipa20b	628	400	2360	3827	3978	8.3-7	9.3-7	9.3-7	-1.5-5	1.7-5	-1.7-5	0.25	1:35	2:02	2:32
lipa30a	1393	900	6090	7543	10923	9.9-7	9.6-7	9.7-7	-2.0-5	-2.0-5	2.0-5	0.16	27:26	33:08	47:51
lipa30b	1393	900	7178	8755	14132	9.8-7	9.9-7	9.2-7	-2.6-5	2.6-5	2.5-5	0.37	28:01	30:13	54:16
lipa40a	2458	1600	13263	20000	20000	9.9-7	2.2-5	1.3-6	-2.7-5	**-6.1-4**	3.6-5	0.26	4:34:35	6:01:41	6:35:39
lipa40b	2458	1600	19245	20000	20000	9.9-7	**4.7-4**	2.3-6	-3.5-5	**1.7-2**	-8.4-5	0.32	5:53:15	5:32:30	5:57:53
nug12	232	144	20000	20000	20000	1.6-5	1.7-5	4.1-5	-7.7-5	-8.0-5	**-2.2-4**	1.95	2:12	2:02	2:11
nug14	313	196	20000	20000	20000	2.6-5	2.7-5	5.2-5	**-1.6-4**	**-1.7-4**	**-2.5-4**	1.95	3:43	3:22	3:38
nug15	358	225	20000	20000	20000	2.5-5	2.5-5	4.3-5	**-1.3-4**	**-1.4-4**	**-2.4-4**	1.95	4:43	4:25	4:50
nug16a	406	256	20000	20000	20000	3.0-5	3.2-5	4.6-5	**-2.0-4**	**-2.2-4**	**-2.6-4**	1.95	7:32	6:41	7:33
nug16b	406	256	20000	20000	20000	2.0-5	2.2-5	4.0-5	**-1.1-4**	**-1.3-4**	**-2.5-4**	1.95	6:46	6:02	6:50
nug17	457	289	20000	20000	20000	2.6-5	2.8-5	4.5-5	**-1.4-4**	**-1.6-4**	**-2.5-4**	1.95	9:27	8:35	9:33
nug18	511	324	20000	20000	20000	2.3-5	2.4-5	4.3-5	**-1.2-4**	**-1.2-4**	**-2.3-4**	1.95	11:38	10:35	11:48
nug20	628	400	20000	20000	20000	1.8-5	2.5-5	4.0-5	-9.8-5	**-1.3-4**	**-2.0-4**	1.95	18:26	16:45	18:24
nug21	691	441	20000	20000	20000	2.2-5	2.2-5	4.5-5	**-1.3-4**	**-1.4-4**	**-2.3-4**	1.95	23:04	21:21	23:22
nug22	757	484	20000	20000	20000	2.5-5	2.6-5	4.6-5	**-1.7-4**	**-1.7-4**	**-2.6-4**	1.95	27:30	25:16	27:46
nug24	898	576	20000	20000	20000	2.0-5	2.1-5	4.4-5	**-1.1-4**	**-1.2-4**	**-2.1-4**	1.95	41:02	37:58	41:15
nug25	973	625	20000	20000	20000	1.6-5	1.7-5	4.0-5	-9.3-5	**-1.0-4**	**-1.8-4**	1.95	46:22	42:58	46:27
nug27	1132	729	20000	20000	20000	2.1-5	2.2-5	3.5-5	**-1.2-4**	**-1.4-4**	**-1.9-4**	1.95	1:09:59	1:04:49	1:09:59
nug28	1216	784	20000	20000	20000	1.8-5	2.0-5	3.3-5	**-1.0-4**	**-1.2-4**	**-1.7-4**	1.95	1:21:07	1:14:53	1:21:28
nug30	1393	900	20000	20000	20000	1.7-5	1.8-5	3.9-5	-9.6-5	-9.9-5	**-1.7-4**	1.95	1:53:30	1:46:02	1:54:51
rou12	232	144	20000	20000	20000	3.5-5	3.7-5	5.1-5	**-3.0-4**	**-3.3-4**	**-3.8-4**	1.95	2:21	2:11	2:21
rou15	358	225	20000	20000	20000	2.6-5	2.7-5	3.7-5	**-1.1-4**	**-1.2-4**	**-2.1-4**	1.95	5:24	5:02	5:23
rou20	628	400	20000	20000	20000	1.7-5	1.8-5	3.6-5	-6.3-5	-6.7-5	**-1.6-4**	1.95	19:13	17:36	19:21
scr12	232	144	1333	1516	1880	9.0-7	7.9-7	9.8-7	5.2-6	1.5-6	-2.3-6	1.86	09	10	13
scr15	358	225	3291	3073	2990	9.7-7	9.8-7	9.4-7	1.5-5	1.4-5	1.5-5	1.58	43	41	43
scr20	628	400	20000	20000	20000	1.7-5	1.8-5	3.2-5	**-1.2-4**	**-1.3-4**	**-1.9-4**	1.95	18:39	17:08	18:46
ste36a	1996	1296	20000	20000	20000	1.4-5	1.5-5	4.0-5	**-1.0-4**	**-1.1-4**	**-1.3-4**	1.95	3:23:41	3:10:15	3:26:53
ste36b	1996	1296	20000	20000	20000	2.0-5	2.7-5	4.3-5	**-1.0-4**	**-1.3-4**	**-1.2-4**	1.95	3:18:59	3:03:46	3:20:45
ste36c	1996	1296	20000	20000	20000	1.8-5	2.1-5	4.3-5	**-1.1-4**	**-1.2-4**	**-1.3-4**	1.95	3:26:01	3:14:00	3:27:33
tai12a	232	144	2419	1948	2411	9.3-7	9.7-7	8.7-7	-1.1-5	1.2-5	-1.0-5	1.09	15	11	15
tai12b	232	144	7363	6781	8255	7.5-7	6.2-7	6.5-7	4.7-6	1.7-6	1.8-6	1.10	42	37	50
tai15a	358	225	20000	20000	20000	2.0-5	2.1-5	3.4-5	-7.6-5	-8.4-5	**-1.8-4**	1.95	5:30	5:08	5:24
tai15b	358	225	7815	7715	8470	9.9-7	9.9-7	9.9-7	-4.9-6	-4.9-6	-4.9-6	1.51	2:00	1:51	2:11
tai17a	457	289	20000	20000	20000	1.8-5	1.9-5	3.4-5	-5.9-5	-6.4-5	**-1.7-4**	1.95	10:06	9:13	10:05
tai20a	628	400	20000	20000	20000	1.6-5	1.6-5	3.3-5	-5.0-5	-5.5-5	**-1.5-4**	1.95	19:26	17:48	19:26
tai20b	628	400	12864	16162	16185	9.9-7	7.9-7	8.0-7	3.0-5	1.1-5	-8.9-6	1.10	10:15	11:29	13:25
tai25a	973	625	2031	4503	1992	9.9-7	9.9-7	9.9-7	-1.1-6	-6.5-7	-1.2-6	1.82	5:19	11:01	5:14
tai25b	973	625	20000	20000	20000	3.4-5	3.6-5	6.5-5	**-3.7-4**	**-3.9-4**	**-4.9-4**	1.95	49:39	46:32	51:18
tai30a	1393	900	20000	20000	20000	1.1-5	1.2-5	3.0-5	-3.2-5	-3.6-5	**-1.0-4**	1.95	2:04:18	1:53:31	2:01:21
tai30b	1393	900	20000	20000	20000	2.3-5	2.4-5	4.7-5	**-2.1-4**	**-2.1-4**	**-2.8-4**	1.95	1:53:24	1:45:16	1:52:49
tai35a	1888	1225	20000	20000	20000	8.9-6	9.5-6	2.5-5	-2.7-5	-2.9-5	-7.6-5	1.95	4:33:08	4:29:44	4:36:20
tai35b	1888	1225	20000	20000	20000	2.5-5	2.6-5	4.5-5	**-2.1-4**	**-2.3-4**	**-2.8-4**	1.95	4:14:07	4:00:53	4:16:31
tai40a	2458	1600	20000	20000	20000	8.1-6	8.7-6	2.5-5	-2.3-5	-2.6-5	-7.3-5	1.95	6:17:40	5:58:21	6:37:50
tai40b	2458	1600	20000	20000	20000	2.3-5	2.4-5	4.8-5	**-1.7-4**	**-1.8-4**	**-2.3-4**	1.95	5:55:42	5:56:03	6:34:46
tho30	1393	900	20000	20000	20000	2.3-5	2.6-5	4.2-5	**-1.4-4**	**-1.5-4**	**-2.3-4**	1.95	1:26:26	1:21:04	1:28:26
tho40	2458	1600	20000	20000	20000	1.8-5	2.0-5	3.8-5	**-1.0-4**	**-1.1-4**	**-1.8-4**	1.95	6:06:20	5:37:56	6:18:31
be100.1	101	101	1670	1956	3058	9.7-7	9.7-7	9.8-7	2.2-7	1.8-6	-7.8-7	1.94	06	07	11
be100.2	101	101	1800	1827	2454	9.9-7	9.9-7	9.9-7	3.0-7	-3.2-7	-1.1-6	1.92	07	06	09
be100.3	101	101	2518	2448	3178	9.9-7	9.9-7	9.5-7	8.1-8	2.1-7	2.8-7	1.92	09	09	12
be100.4	101	101	1999	2036	2876	9.9-7	9.9-7	9.9-7	-5.2-7	-3.6-7	-4.2-7	1.94	08	07	11
be100.5	101	101	1653	1786	2399	9.9-7	9.9-7	9.9-7	-1.4-7	-2.3-7	-1.1-7	1.94	06	06	08
be100.6	101	101	1919	2047	2495	9.9-7	9.9-7	9.9-7	-2.4-7	-4.3-7	2.5-7	1.92	07	07	09
be100.7	101	101	1772	1835	2655	9.9-7	9.9-7	9.9-7	-1.4-9	-8.8-7	2.7-7	1.92	07	06	09
be100.8	101	101	1440	1622	2206	9.6-7	9.8-7	9.9-7	-3.1-7	-9.0-7	-3.8-7	1.92	06	05	08
be100.9	101	101	1484	1749	1943	9.9-7	9.9-7	9.9-7	1.5-7	1.2-7	2.2-7	1.95	06	06	07
be100.10	101	101	1134	1332	1558	9.9-7	9.8-7	9.6-7	-1.4-6	-3.9-7	2.1-6	1.94	05	05	06

Table 2: (continued) Performance of CADMM, ADMM3d and ADMM3g on θ_+, FAP, QAP, BIQ and RCP problems with accuray=10^{-6}. In this table, τ column reports the final value of τ_k of CADMM, and the computing time is in the format of "hours:minutes:seconds'. In the name of the last 120 problems, "-s, -m, -l" means "-small, -medium, -large", respectively.

			iteration			η			gap			τ	time		
problem	m	n	cadm	adm3d	adm3g	cadm	adm3d	adm3g	cadm	adm3d	adm3g	cadm	cadm	adm3d	adm3g
be120.3.1	121	121	2003	2448	3101	9.9-7	9.7-7	8.6-7	-4.9-7	-9.3-8	-1.1-7	1.94	10	11	14
be120.3.2	121	121	2267	2503	3298	9.9-7	9.9-7	9.9-7	-1.3-7	-1.7-7	1.5-8	1.93	11	12	15
be120.3.3	121	121	1781	1965	2407	9.9-7	9.8-7	9.9-7	3.2-8	4.9-7	-9.3-7	1.92	09	09	11
be120.3.4	121	121	1948	2164	4384	9.7-7	9.9-7	9.9-7	1.5-6	-3.3-7	4.0-6	1.91	09	10	19
be120.3.5	121	121	2532	2809	4501	9.9-7	9.9-7	9.7-7	8.6-8	8.2-8	-3.9-8	1.93	12	13	20
be120.3.6	121	121	2475	2695	3975	9.9-7	9.9-7	9.9-7	-1.4-7	-5.3-8	2.6-8	1.93	12	13	18
be120.3.7	121	121	3843	3991	8281	9.9-7	9.9-7	9.9-7	-1.6-7	-9.4-8	-2.9-8	1.92	19	19	37
be120.3.8	121	121	2998	3375	4695	9.9-7	9.9-7	8.0-7	5.0-7	5.0-7	2.8-7	1.93	14	15	20
be120.3.9	121	121	3313	3496	10501	9.9-7	9.9-7	9.8-7	-2.4-7	-2.8-7	-2.1-8	1.94	16	17	48
be120.3.10	121	121	1637	1823	2208	9.9-7	9.9-7	9.9-7	-8.5-7	2.4-7	-1.6-7	1.93	08	08	10
be120.8.1	121	121	1775	1818	2338	9.9-7	9.9-7	9.9-7	2.5-7	-1.2-6	5.4-7	1.92	08	08	10
be120.8.2	121	121	3008	3372	4701	9.9-7	9.9-7	9.9-7	-2.5-7	-9.8-8	-9.5-8	1.92	15	16	21
be120.8.3	121	121	1805	2134	2933	9.9-7	9.9-7	9.0-7	-3.7-9	4.0-8	1.2-7	1.93	09	10	13
be120.8.4	121	121	1963	2210	2763	9.9-7	9.9-7	9.9-7	-3.6-7	-8.1-8	-9.5-8	1.93	10	10	12
be120.8.5	121	121	2080	2338	5240	9.9-7	9.9-7	9.9-7	4.0-7	-5.7-8	1.2-6	1.92	11	11	24
be120.8.6	121	121	2023	2080	2797	9.9-7	9.9-7	9.8-7	-2.8-8	-7.5-8	-8.7-8	1.93	10	09	12
be120.8.7	121	121	1694	1775	4058	9.8-7	9.7-7	9.8-7	-1.4-7	-3.0-8	1.2-6	1.92	08	08	17
be120.8.8	121	121	1366	1595	1979	9.7-7	9.7-7	9.6-7	-2.9-6	-2.6-7	-5.6-7	1.93	07	07	09
be120.8.9	121	121	1602	1562	2143	9.9-7	9.7-7	9.9-7	-8.6-7	1.8-6	2.4-7	1.92	08	07	10
be120.8.10	121	121	2160	2759	3755	9.9-7	9.9-7	9.9-7	2.3-7	5.3-8	1.6-7	1.93	11	13	17
be150.3.1	151	151	2217	2318	3697	9.9-7	9.9-7	9.6-7	6.9-7	-1.3-6	1.6-6	1.93	15	15	22
be150.3.2	151	151	2035	2692	3656	9.9-7	9.9-7	9.9-7	5.5-7	-1.2-7	-1.2-7	1.92	14	17	22
be150.3.3	151	151	2114	2382	3803	9.9-7	9.9-7	9.8-7	-4.8-8	-1.8-6	5.4-6	1.92	14	15	23
be150.3.4	151	151	2273	2687	3964	9.9-7	9.9-7	9.8-7	9.2-9	-1.9-6	3.4-8	1.92	15	17	24
be150.3.5	151	151	2147	2285	2886	9.9-7	9.9-7	9.9-7	-4.2-7	-2.7-7	-2.3-6	1.92	14	15	18
be150.3.6	151	151	2296	2396	3477	9.9-7	9.9-7	9.2-7	-1.4-7	-1.9-7	-8.9-8	1.93	15	15	21
be150.3.7	151	151	2210	2386	3160	9.9-7	9.9-7	9.9-7	-1.9-7	-4.5-8	-3.6-7	1.92	14	15	19
be150.3.8	151	151	2879	3033	4901	9.9-7	9.9-7	9.9-7	-2.9-7	-3.5-7	-5.7-8	1.93	19	19	29
be150.3.9	151	151	1362	1489	1834	9.4-7	9.6-7	9.6-7	1.4-6	1.3-6	1.0-7	1.92	09	10	11
be150.3.10	151	151	3120	3429	5720	9.9-7	9.9-7	9.9-7	-3.5-7	-2.8-7	-1.3-7	1.93	20	21	35
be150.8.1	151	151	1861	1882	2646	9.9-7	9.9-7	9.9-7	-1.1-6	-2.1-6	-3.5-7	1.91	12	12	16
be150.8.2	151	151	1992	2082	2806	9.9-7	9.9-7	9.9-7	2.0-7	-3.0-7	9.0-7	1.91	13	13	17
be150.8.3	151	151	2072	2491	3713	9.9-7	9.6-7	9.4-7	3.5-6	1.8-6	2.4-6	1.92	14	16	23
be150.8.4	151	151	2201	2416	3201	9.9-7	9.9-7	9.1-7	-1.8-7	7.9-8	-1.7-7	1.93	15	15	20
be150.8.5	151	151	2285	2528	3736	9.9-7	9.9-7	9.9-7	-2.5-7	-4.3-7	-4.2-7	1.92	15	16	23
be150.8.6	151	151	2153	2303	3349	9.9-7	9.9-7	8.8-7	-2.5-7	-5.4-7	-2.3-7	1.92	14	14	19
be150.8.7	151	151	3119	2957	4409	9.9-7	9.9-7	9.9-7	4.8-7	-3.4-7	3.4-7	1.93	20	18	26
be150.8.8	151	151	3275	3485	5420	9.9-7	9.9-7	9.9-7	-6.2-7	-5.0-7	-2.0-7	1.91	22	22	33
be150.8.9	151	151	2814	2812	3810	9.9-7	9.9-7	9.1-7	-4.1-7	-2.3-7	-4.0-7	1.93	19	19	24
be150.8.10	151	151	2315	2525	3422	9.9-7	9.9-7	9.9-7	3.0-7	-2.4-7	-9.5-7	1.92	15	15	21
be200.3.1	201	201	2250	2353	3223	9.9-7	9.8-7	9.9-7	-1.9-6	-6.2-7	3.2-6	1.92	25	22	32
be200.3.2	201	201	2571	2659	3587	9.9-7	9.9-7	9.8-7	-1.4-6	2.2-7	-1.5-6	1.92	30	25	37
be200.3.3	201	201	3836	4080	6444	9.9-7	9.9-7	9.9-7	-3.0-7	-5.5-7	-5.2-8	1.91	43	39	1:07
be200.3.4	201	201	2911	3199	4666	9.9-7	9.9-7	9.9-7	-2.9-7	-3.5-7	-9.9-7	1.91	32	30	48
be200.3.5	201	201	3346	3379	4659	9.9-7	9.9-7	9.9-7	-1.7-7	4.1-8	2.7-7	1.92	38	32	48
be200.3.6	201	201	2557	2483	3467	9.9-7	9.9-7	9.8-7	6.4-7	-3.0-6	1.0-6	1.92	27	23	34
be200.3.7	201	201	3240	3601	5769	9.9-7	9.9-7	9.9-7	-7.8-7	3.0-8	3.0-7	1.91	34	33	57
be200.3.8	201	201	2586	2817	4271	9.8-7	9.9-7	9.9-7	-1.4-6	-5.5-7	1.1-6	1.92	29	26	43
be200.3.9	201	201	3618	3754	6201	9.9-7	9.9-7	9.5-7	-3.3-7	-2.4-7	-6.8-7	1.92	40	35	1:03
be200.3.10	201	201	2682	2864	3900	9.9-7	9.9-7	9.9-7	-3.3-7	-1.2-7	-1.5-7	1.92	29	26	39
be200.8.1	201	201	3220	3413	5810	9.9-7	9.9-7	9.9-7	6.4-7	-6.7-8	2.5-8	1.92	34	32	59
be200.8.2	201	201	2428	2527	3288	9.9-7	9.9-7	9.5-7	-3.0-7	-1.1-6	-4.6-7	1.91	24	22	31
be200.8.3	201	201	2985	3029	4376	9.9-7	9.9-7	9.9-7	-4.0-7	-4.3-7	5.1-7	1.93	32	29	45
be200.8.4	201	201	2630	2928	4579	9.9-7	9.9-7	9.9-7	-5.4-7	1.0-6	-3.7-6	1.92	28	27	46
be200.8.5	201	201	2395	2724	3629	9.9-7	9.9-7	9.9-7	5.6-7	-6.8-7	9.6-7	1.92	26	26	37
be200.8.6	201	201	3081	3290	5874	9.9-7	9.9-7	9.4-7	-7.0-7	-6.0-7	1.1-6	1.91	34	31	59
be200.8.7	201	201	2665	2908	5492	9.9-7	9.9-7	9.9-7	1.4-6	3.3-6	1.7-6	1.91	27	26	54
be200.8.8	201	201	2554	2825	3984	9.9-7	9.9-7	9.9-7	-7.7-7	-6.2-7	1.3-6	1.92	26	26	39
be200.8.9	201	201	2460	2741	4110	9.9-7	9.9-7	9.9-7	-3.4-6	-6.4-7	-5.3-8	1.92	28	26	41
be200.8.10	201	201	2386	2601	3547	9.9-7	9.9-7	9.9-7	7.8-7	-3.7-7	-1.0-6	1.92	25	24	36
be250.1	251	251	3952	4362	7414	9.9-7	9.9-7	9.9-7	-7.1-7	-4.3-8	-9.1-7	1.91	1:04	59	1:47
be250.2	251	251	3619	3816	5779	9.9-7	9.9-7	9.9-7	-1.5-6	-8.2-7	-6.0-7	1.91	58	51	1:23
be250.3	251	251	3448	3951	6498	9.9-7	9.9-7	9.9-7	-2.2-7	-1.2-6	2.3-8	1.91	54	53	1:34
be250.4	251	251	6030	7622	12801	9.9-7	9.9-7	9.8-7	-1.2-6	-1.1-6	-1.0-7	1.91	1:37	1:42	3:05
be250.5	251	251	3772	4218	5082	9.9-7	9.9-7	9.7-7	-7.3-7	-7.4-7	-7.9-7	1.92	1:01	57	1:15
be250.6	251	251	4011	4234	5454	9.9-7	9.9-7	9.9-7	-7.0-7	-4.1-7	-1.0-6	1.91	1:00	55	1:17
be250.7	251	251	3996	4317	7744	9.9-7	9.9-7	9.5-7	-6.7-7	-1.3-6	6.1-7	1.91	1:02	58	1:50
be250.8	251	251	3691	3979	5757	9.9-7	9.9-7	9.9-7	-2.6-7	-1.1-6	-1.4-6	1.91	56	52	1:20
be250.9	251	251	4589	4934	6543	9.9-7	9.9-7	9.9-7	-1.2-6	-1.2-6	2.0-7	1.92	1:15	1:08	1:37
be250.10	251	251	5039	5513	8436	9.9-7	9.9-7	9.9-7	-7.3-7	-7.3-7	-8.5-8	1.91	1:18	1:12	1:59
bqp50-1	51	51	2965	3485	8391	9.9-7	9.9-7	8.7-7	-1.7-6	-7.4-7	-1.9-7	1.95	06	07	17
bqp50-2	51	51	3759	4403	4531	9.9-7	9.9-7	9.5-7	-6.6-8	-1.3-7	-4.8-7	1.91	07	08	08
bqp50-3	51	51	2960	3032	4218	9.8-7	9.9-7	8.0-7	9.9-7	-4.1-6	-1.5-6	1.92	06	05	07
bqp50-4	51	51	7472	8810	12613	9.9-7	9.9-7	9.9-7	-5.2-7	-5.0-7	-5.1-7	1.90	15	17	25
bqp50-5	51	51	2200	2382	5411	9.9-7	9.9-7	9.9-7	-2.2-7	-1.6-7	-6.2-8	1.94	04	04	11
bqp50-6	51	51	2537	2583	4403	9.9-7	9.9-7	9.9-7	-5.1-9	-2.3-7	-8.2-8	1.92	05	05	09
bqp50-7	51	51	1324	1496	1883	9.7-7	9.8-7	9.4-7	7.3-8	3.1-7	1.1-6	1.93	03	03	04

Table 2: (continued) Performance of CADMM, ADMM3d and ADMM3g on θ_+, FAP, QAP, BIQ and RCP problems with accuray=10^{-6}. In this table, τ column reports the final value of τ_k of CADMM, and the computing time is in the format of "hours:minutes:seconds'. In the name of the last 120 problems, "-s, -m, -l" means "-small, -medium, -large", respectively.

			iteration			η			gap			τ	time		
problem	m	n	cadm	adm3d	adm3g	cadm	adm3d	adm3g	cadm	adm3d	adm3g	cadm	cadm	adm3d	adm3g
bqp50-8	51	51	1587	1623	1936	9.9-7	9.9-7	9.9-7	-1.7-8	2.5-7	-2.6-7	1.94	03	03	04
bqp50-9	51	51	1394	1426	1868	9.3-7	9.7-7	9.9-7	1.5-6	-1.3-6	-3.2-6	1.94	03	03	04
bqp50-10	51	51	1306	1371	1733	9.7-7	9.8-7	9.9-7	2.6-7	-1.1-6	3.7-7	1.93	03	03	03
bqp100-1	101	101	1511	1560	2073	9.9-7	9.9-7	9.9-7	-3.8-7	9.4-7	3.7-7	1.93	06	05	07
bqp100-2	101	101	2869	3269	4201	9.9-7	9.9-7	9.7-7	3.6-7	2.1-7	2.5-7	1.91	10	11	14
bqp100-3	101	101	4664	4839	16850	9.9-7	9.9-7	9.9-7	-5.6-7	-5.4-7	-2.2-8	1.93	16	16	56
bqp100-4	101	101	3100	2644	3704	9.9-7	9.9-7	9.9-7	1.8-8	-1.5-7	-1.6-7	1.93	12	09	13
bqp100-5	101	101	3486	3676	7801	9.9-7	9.9-7	9.8-7	-5.4-7	-6.7-7	-1.1-7	1.93	13	13	27
bqp100-6	101	101	1738	1860	2366	9.9-7	9.9-7	9.9-7	3.1-8	-5.4-7	-1.4-7	1.93	06	06	08
bqp100-7	101	101	1890	2028	2520	9.9-7	9.8-7	9.8-7	-5.8-7	-3.5-7	-4.1-7	1.92	07	07	09
bqp100-8	101	101	3230	3512	7001	9.9-7	9.9-7	9.8-7	-2.8-7	-3.2-7	-1.6-8	1.93	12	12	25
bqp100-9	101	101	3081	3125	5901	9.9-7	9.9-7	9.3-7	1.9-8	1.2-7	1.7-7	1.91	11	10	20
bqp100-10	101	101	3268	3454	8796	9.9-7	9.9-7	9.9-7	-1.0-6	-9.2-7	-2.1-7	1.92	12	11	29
bqp250-1	251	251	4170	4409	7691	9.9-7	9.9-7	9.9-7	-1.1-6	-7.1-7	1.0-7	1.91	1:06	59	1:51
bqp250-2	251	251	4437	4471	6712	9.9-7	9.9-7	9.9-7	-1.0-6	3.3-8	9.4-8	1.91	1:08	59	1:35
bqp250-3	251	251	3995	4708	8906	9.9-7	9.8-7	9.9-7	-4.2-8	-6.5-7	3.4-8	1.91	1:00	1:00	2:02
bqp250-4	251	251	3605	3919	4991	9.9-7	9.9-7	9.9-7	-6.2-7	-1.3-6	-2.9-8	1.91	57	53	1:13
bqp250-5	251	251	5322	5555	12893	9.9-7	9.9-7	9.9-7	-8.3-8	1.0-8	3.6-8	1.91	1:22	1:14	3:04
bqp250-6	251	251	4135	4234	6101	9.9-7	9.9-7	9.8-7	-1.2-6	-9.3-7	-8.9-8	1.92	1:04	57	1:29
bqp250-7	251	251	4272	4655	7420	9.9-7	9.9-7	8.5-7	-2.2-6	-1.2-6	-5.5-7	1.91	1:04	1:00	1:43
bqp250-8	251	251	2887	3181	4090	9.9-7	9.9-7	9.9-7	-6.3-7	-1.0-6	1.3-6	1.91	45	42	58
bqp250-9	251	251	4336	4839	8709	9.9-7	9.9-7	9.9-7	-3.4-6	-2.7-7	3.5-6	1.91	1:05	1:02	2:00
bqp250-10	251	251	3053	3448	4592	9.9-7	9.9-7	9.8-7	-1.5-6	-1.2-6	-2.4-6	1.91	46	45	1:03
bqp500-1	501	501	5911	6702	10101	9.9-7	9.9-7	9.1-7	-1.5-6	-9.1-7	2.9-6	1.90	7:13	7:04	11:35
bqp500-2	501	501	6897	8377	20000	9.9-7	9.9-7	1.8-5	-1.4-6	1.9-8	**-1.6-4**	1.91	8:43	8:55	23:03
bqp500-3	501	501	6566	9548	16429	9.9-7	9.9-7	9.5-7	-1.9-6	-2.7-6	-3.1-6	1.90	7:55	9:54	18:25
bqp500-4	501	501	6264	7322	11100	9.9-7	9.9-7	9.8-7	-1.0-6	6.9-7	-4.8-6	1.90	7:28	7:32	12:21
bqp500-5	501	501	6435	8408	13841	9.9-7	9.9-7	9.9-7	5.3-6	3.8-6	2.5-6	1.90	7:56	8:55	15:56
bqp500-6	501	501	6234	6822	10195	9.9-7	9.9-7	8.9-7	-4.6-6	5.6-6	6.7-8	1.90	7:44	7:19	11:49
bqp500-7	501	501	6472	7107	10821	9.9-7	9.9-7	9.6-7	-1.0-6	-1.7-6	-1.0-6	1.90	8:05	7:31	12:21
bqp500-8	501	501	6169	7376	10591	9.9-7	9.9-7	9.9-7	-3.2-7	-1.3-6	-4.5-6	1.90	7:57	7:40	11:49
bqp500-9	501	501	6246	6617	11108	9.9-7	9.9-7	9.9-7	-1.0-6	-1.1-6	-3.1-6	1.90	7:40	6:59	12:37
bqp500-10	501	501	6849	8493	15222	9.9-7	9.9-7	9.0-7	2.8-6	4.8-6	3.6-7	1.90	8:21	8:56	17:16
gka1a	51	51	3402	3649	3393	9.9-7	9.9-7	9.6-7	2.7-8	-1.1-7	1.6-6	1.90	06	06	06
gka2a	61	61	2382	2514	3757	9.8-7	9.9-7	9.9-7	1.1-6	-2.0-6	-1.5-7	1.91	04	04	07
gka3a	71	71	1076	1156	1562	9.2-7	9.8-7	9.6-7	-6.4-7	-6.3-7	5.0-7	1.94	03	03	04
gka4a	81	81	2362	2432	3697	9.9-7	9.6-7	9.1-7	-2.6-6	2.5-6	-2.5-6	1.92	06	06	09
gka5a	51	51	1161	1308	1640	9.9-7	9.7-7	9.8-7	1.3-6	-6.8-7	6.5-7	1.94	02	02	03
gka6a	31	31	641	1128	1542	9.4-7	9.9-7	9.3-7	-5.5-7	2.8-7	1.0-7	1.95	01	01	02
gka7a	31	31	1915	2118	3939	9.9-7	9.9-7	9.9-7	-4.6-7	-4.5-7	-9.7-8	1.95	03	03	06
gka8a	101	101	4704	5144	10582	9.9-7	9.8-7	9.9-7	1.7-6	1.5-6	3.8-6	1.91	16	15	32
gka1b	21	21	462	513	556	9.7-7	9.7-7	9.5-7	-1.2-5	-1.7-5	1.7-5	1.90	01	01	01
gka2b	31	31	1528	1645	1961	9.9-7	9.9-7	8.5-7	-1.4-6	-2.7-6	-4.4-6	1.91	02	02	03
gka3b	41	41	601	598	657	9.0-7	8.6-7	7.9-7	-3.0-5	-3.0-5	-2.9-5	1.90	01	01	01
gka4b	51	51	571	590	606	9.3-7	9.3-7	8.3-7	-4.1-5	-4.0-5	3.7-5	1.90	01	01	01
gka5b	61	61	567	642	764	8.9-7	9.8-7	9.9-7	-4.0-5	4.3-5	4.5-5	1.90	01	01	01
gka6b	71	71	544	565	623	8.2-7	8.8-7	8.9-7	-4.3-5	-4.7-5	-4.8-5	1.90	01	01	01
gka7b	81	81	803	814	1038	9.9-7	9.9-7	9.9-7	-5.5-6	-6.9-6	-7.6-6	1.91	02	02	03
gka8b	91	91	709	674	883	8.2-7	8.9-7	8.5-7	5.3-5	-5.4-5	-5.4-5	1.90	02	02	02
gka9b	101	101	1260	1234	1434	7.5-7	6.8-7	9.2-7	2.0-5	-2.7-5	-1.7-5	1.90	04	03	04
gka10b	126	126	1681	1741	1949	9.9-7	9.9-7	9.9-7	-2.1-5	-2.0-5	-1.7-5	1.91	09	09	10
gka1c	41	41	2902	3380	9401	9.9-7	9.9-7	9.6-7	-4.2-7	-3.4-7	-6.7-9	1.92	04	04	14
gka2c	51	51	1473	1602	2147	9.6-7	9.9-7	9.4-7	-2.6-8	-3.5-6	9.8-7	1.93	03	03	04
gka3c	61	61	1421	1506	2378	9.7-7	9.8-7	9.9-7	-7.2-7	-2.3-8	2.3-7	1.92	03	03	05
gka4c	71	71	2278	2123	3007	9.9-7	9.9-7	9.9-7	-3.2-8	1.6-7	9.9-8	1.93	05	05	07
gka5c	81	81	3807	4086	13201	9.9-7	9.9-7	9.9-7	-2.1-7	-1.7-7	-1.8-8	1.93	10	11	36
gka6c	91	91	3638	3941	8293	9.9-7	9.9-7	7.2-7	3.4-7	3.4-7	9.8-9	1.92	12	12	25
gka7c	101	101	3534	3889	7444	9.9-7	9.9-7	9.9-7	1.3-7	-1.8-7	3.6-8	1.92	13	13	25
gka1d	101	101	3127	3209	4413	9.9-7	9.9-7	9.9-7	-2.7-7	-2.9-7	-7.6-8	1.92	12	11	15
gka2d	101	101	2160	2241	3249	9.9-7	9.9-7	9.8-7	9.7-8	1.6-7	1.4-7	1.94	09	09	13
gka3d	101	101	3145	3228	6801	9.9-7	9.9-7	9.8-7	1.5-7	9.8-8	8.9-9	1.93	13	12	24
gka4d	101	101	1995	2294	2978	9.9-7	9.9-7	9.9-7	1.3-7	7.6-8	-2.4-8	1.95	08	08	11
gka5d	101	101	1542	1614	1971	9.9-7	9.9-7	9.9-7	3.5-7	-1.1-6	-3.5-7	1.93	06	06	07
gka6d	101	101	1686	1738	2296	9.8-7	9.4-7	9.9-7	-7.2-8	1.3-6	1.2-7	1.92	07	06	08
gka7d	101	101	1540	1587	2047	9.6-7	9.8-7	9.7-7	-1.2-7	-1.9-6	5.1-8	1.93	06	06	07
gka8d	101	101	3085	3563	6301	9.9-7	9.9-7	9.9-7	-3.6-8	-3.4-9	1.3-9	1.93	13	13	23
gka9d	101	101	1355	1512	2084	9.7-7	9.9-7	9.9-7	4.6-7	-3.0-7	4.0-7	1.93	05	05	07
gka10d	101	101	1600	1677	2339	9.8-7	9.9-7	9.9-7	-2.8-7	-1.6-6	-3.4-6	1.92	06	06	08
gka1e	201	201	4048	4229	5401	9.9-7	9.9-7	8.6-7	-4.4-7	-4.8-7	7.3-9	1.91	48	41	54
gka2e	201	201	3362	3501	4285	9.9-7	9.9-7	9.9-7	-7.0-7	-6.9-7	-5.4-7	1.92	40	33	42
gka3e	201	201	3128	3155	4572	9.9-7	9.9-7	9.9-7	-2.3-8	-2.9-7	8.2-9	1.92	38	31	47
gka4e	201	201	3389	3805	6201	9.9-7	9.9-7	9.7-7	-4.7-7	-7.7-8	-2.3-7	1.92	41	36	1:02
gka5e	201	201	3397	3646	4526	9.9-7	9.9-7	9.9-7	3.3-7	4.0-7	-9.4-8	1.93	41	35	45
gka1f	501	501	5992	6881	11771	9.9-7	9.9-7	9.9-7	-1.1-6	-6.8-7	4.2-6	1.91	7:28	7:26	13:43
gka2f	501	501	6715	7364	11343	9.9-7	9.9-7	9.9-7	-1.5-6	-2.1-6	1.6-6	1.91	8:25	7:59	13:14
gka3f	501	501	5715	6462	9885	9.9-7	9.9-7	9.5-7	-1.1-6	-8.8-7	-8.1-7	1.91	7:19	7:06	11:42
gka4f	501	501	6027	6695	14281	9.9-7	9.9-7	9.9-7	-2.4-6	-9.0-7	-2.8-6	1.91	7:37	7:17	16:44

Table 2: (continued) Performance of CADMM, ADMM3d and ADMM3g on θ_+, FAP, QAP, BIQ and RCP problems with accuray=10^{-6}. In this table, τ column reports the final value of τ_k of CADMM, and the computing time is in the format of "hours:minutes:seconds'. In the name of the last 120 problems, "-s, -m, -l" means "-small, -medium, -large", respectively.

problem	m	n	iteration cadm	iteration adm3d	iteration adm3g	η cadm	η adm3d	η adm3g	gap cadm	gap adm3d	gap adm3g	τ cadm	time cadm	time adm3d	time adm3g
gka5f	501	501	6104	6649	10537	9.9-7	9.9-7	9.9-7	-8.4-7	-8.8-7	2.8-6	1.90	7:22	6:59	11:58
soybean-s.2	48	47	441	484	605	9.8-7	9.2-7	8.3-7	-7.9-8	-5.0-8	-1.7-7	1.95	01	01	01
soybean-s.3	48	47	119	118	115	7.0-7	4.9-7	6.1-7	-2.1-7	-2.4-7	1.9-7	1.69	00	00	00
soybean-s.4	48	47	456	510	610	9.9-7	9.1-7	9.3-7	-5.4-8	-1.0-7	-9.1-8	1.87	01	01	01
soybean-s.5	48	47	208	236	284	9.9-7	9.9-7	9.9-7	-2.3-8	-2.0-8	-1.9-8	1.89	00	00	01
soybean-s.6	48	47	316	318	387	9.8-7	9.8-7	9.6-7	-2.0-8	-3.8-8	4.9-8	1.90	01	01	01
soybean-s.7	48	47	411	431	509	9.9-7	9.9-7	9.9-7	-2.8-8	-2.4-8	-4.9-8	1.84	01	01	01
soybean-s.8	48	47	1202	1306	1624	9.7-7	9.9-7	9.9-7	-2.6-8	-6.3-8	-5.7-8	1.91	02	02	03
soybean-s.9	48	47	754	691	908	9.9-7	9.9-7	9.9-7	-4.6-8	-1.3-7	-1.0-7	1.89	01	01	02
soybean-s.10	48	47	309	320	407	9.9-7	9.8-7	9.9-7	-2.3-7	-2.2-7	-2.4-7	1.89	01	01	01
soybean-s.11	48	47	699	609	574	9.9-7	9.9-7	9.9-7	-2.0-8	-2.0-8	-1.3-7	1.92	01	01	01
soybean-l.2	308	307	1105	1513	1709	9.9-7	9.9-7	9.9-7	-5.0-8	-5.3-8	-6.1-8	1.95	24	28	35
soybean-l.3	308	307	816	893	1000	9.8-7	7.6-7	9.0-7	-1.2-8	-7.6-8	-1.0-7	1.86	18	17	21
soybean-l.4	308	307	1366	1478	1849	9.9-7	9.9-7	9.9-7	-9.8-8	-8.9-8	-1.0-7	1.95	32	31	41
soybean-l.5	308	307	806	836	1104	9.9-7	9.9-7	9.8-7	-3.8-8	-1.8-8	-4.3-8	1.92	19	17	25
soybean-l.6	308	307	312	393	435	9.5-7	9.9-7	8.8-7	-5.5-8	-3.2-8	-7.0-8	1.89	08	08	10
soybean-l.7	308	307	1005	924	1235	9.9-7	9.9-7	9.9-7	-10.0-9	-4.0-8	-4.1-8	1.95	25	20	29
soybean-l.8	308	307	695	778	1058	9.9-7	9.9-7	9.9-7	-6.6-8	-4.8-8	-5.9-8	1.91	17	18	24
soybean-l.9	308	307	848	854	1146	9.9-7	9.9-7	9.7-7	5.5-9	2.0-8	1.2-8	1.92	21	18	26
soybean-l.10	308	307	364	373	450	9.8-7	9.9-7	9.6-7	-2.6-9	-1.6-8	-5.1-8	1.93	09	08	11
soybean-l.11	308	307	956	1107	1275	8.9-7	9.5-7	9.2-7	1.8-7	2.4-7	1.4-7	1.89	24	26	30
spambase-s.2	301	300	471	462	608	9.9-7	9.9-7	9.7-7	-1.2-7	-8.8-8	-1.8-7	1.95	10	08	12
spambase-s.3	301	300	675	656	714	9.7-7	9.9-7	9.9-7	-4.7-8	-7.5-8	-7.0-8	1.94	14	11	14
spambase-s.4	301	300	1267	1381	1450	9.9-7	9.9-7	9.9-7	-2.0-9	-8.8-9	-2.5-8	1.94	28	26	30
spambase-s.5	301	300	636	607	673	9.9-7	9.8-7	9.8-7	3.4-7	-8.6-7	6.8-7	1.93	14	12	15
spambase-s.6	301	300	828	916	1027	9.9-7	9.9-7	9.9-7	-2.4-7	-5.3-7	-1.1-7	1.92	19	20	22
spambase-s.7	301	300	869	927	1011	9.9-7	9.9-7	9.9-7	1.1-7	4.3-7	-5.3-8	1.93	20	19	23
spambase-s.8	301	300	1026	1117	1200	9.9-7	9.9-7	9.9-7	1.1-7	9.0-8	7.6-8	1.94	25	22	28
spambase-s.9	301	300	1116	1015	1248	9.9-7	9.9-7	9.3-7	-6.3-7	4.5-8	-4.4-8	1.93	27	21	29
spambase-s.10	301	300	991	1276	1143	9.9-7	9.9-7	9.9-7	-1.3-7	-2.0-7	8.6-8	1.94	25	27	27
spambase-s.11	301	300	1114	1164	1433	9.9-7	9.9-7	9.9-7	-4.0-7	-5.7-7	-5.1-7	1.92	28	26	35
spambase-m.2	901	900	612	520	721	9.8-7	9.8-7	9.3-7	1.9-7	7.5-7	-4.4-7	1.95	2:41	1:47	2:45
spambase-m.3	901	900	1351	1110	1501	9.9-7	9.8-7	9.9-7	-1.4-7	-3.7-7	-3.0-7	1.94	5:36	3:48	5:45
spambase-m.4	901	900	2751	2860	3134	9.1-7	9.8-7	9.6-7	1.3-6	1.1-6	1.4-6	1.91	11:32	9:50	12:07
spambase-m.5	901	900	1739	1698	2159	9.9-7	9.9-7	9.9-7	-1.2-7	-1.9-7	-1.0-7	1.93	7:21	5:59	8:34
spambase-m.6	901	900	1356	1489	1775	9.9-7	9.9-7	9.9-7	-6.6-8	-1.3-8	-6.4-8	1.93	5:45	5:20	7:08
spambase-m.7	901	900	1464	1728	2171	9.9-7	9.9-7	9.9-7	9.2-9	-4.5-8	6.8-8	1.95	6:18	6:19	8:54
spambase-m.8	901	900	1591	1695	1830	9.9-7	9.9-7	9.9-7	-7.1-8	-1.3-7	-7.4-8	1.93	6:50	6:06	7:26
spambase-m.9	901	900	1292	1464	1954	9.9-7	9.9-7	9.9-7	-1.2-6	-1.1-6	-4.0-7	1.92	5:35	5:21	8:00
spambase-m.10	901	900	1361	1503	1798	9.3-7	9.9-7	9.9-7	1.5-6	-1.4-6	1.8-6	1.93	5:59	5:31	7:26
spambase-m.11	901	900	1276	1377	1907	9.6-7	9.2-7	9.9-7	1.6-6	1.4-6	9.9-7	1.93	5:31	5:07	7:49
spambase-l.2	1501	1500	596	692	1027	9.7-7	9.9-7	4.6-7	-1.9-6	-1.4-6	-1.9-6	1.92	8:59	9:30	15:24
spambase-l.3	1501	1500	686	1262	1189	9.7-7	9.9-7	9.9-7	1.6-6	-6.6-7	2.3-6	1.94	10:23	17:25	17:52
spambase-l.4	1501	1500	3114	3193	3389	9.9-7	9.9-7	9.8-7	4.2-7	1.9-7	3.3-7	1.92	47:07	44:23	51:12
spambase-l.5	1501	1500	8684	11278	9355	9.9-7	9.7-7	9.8-7	-2.2-6	-1.9-6	-2.2-6	1.90	2:12:43	2:37:35	2:22:24
spambase-l.6	1501	1500	1823	1973	2632	9.9-7	9.9-7	9.7-7	2.8-6	-2.2-6	-2.5-6	1.94	28:07	27:57	40:28
spambase-l.7	1501	1500	1391	1620	2007	9.9-7	9.9-7	9.9-7	4.3-7	2.8-7	2.0-6	1.94	21:44	23:15	31:04
spambase-l.8	1501	1500	1578	1724	2125	9.9-7	9.3-7	9.8-7	1.7-6	-2.2-6	2.4-6	1.95	24:52	25:06	33:03
spambase-l.9	1501	1500	1984	2176	2752	9.9-7	9.9-7	9.8-7	-2.5-6	-2.0-6	-2.3-6	1.94	32:05	31:48	44:10
spambase-l.10	1501	1500	2864	2783	3219	9.9-7	9.9-7	9.9-7	8.1-7	-1.3-6	-1.1-6	1.92	46:49	40:48	52:14
spambase-l.11	1501	1500	2606	2650	3176	9.6-7	9.9-7	9.2-7	-2.4-6	2.5-6	2.2-6	1.95	42:02	38:38	51:07
abalone-s.2	201	200	402	404	508	9.9-7	9.7-7	9.9-7	-1.3-8	4.5-8	-9.1-9	1.95	03	03	04
abalone-s.3	201	200	300	270	401	9.6-7	9.9-7	9.7-7	-3.7-8	8.5-8	-3.1-8	1.93	03	02	03
abalone-s.4	201	200	512	492	642	9.9-7	9.9-7	9.9-7	-3.3-9	-1.1-8	-1.4-8	1.95	05	04	05
abalone-s.5	201	200	559	684	867	9.9-7	9.9-7	9.9-7	-2.6-8	-3.4-8	-3.4-8	1.95	05	05	08
abalone-s.6	201	200	669	512	736	9.9-7	9.8-7	9.8-7	-1.6-8	-7.2-8	-2.7-8	1.93	06	04	06
abalone-s.7	201	200	1054	1020	1140	9.9-7	9.9-7	9.9-7	-3.2-8	-3.4-8	-4.9-8	1.91	10	08	10
abalone-s.8	201	200	1035	1120	1420	9.9-7	9.9-7	9.8-7	-1.4-7	-1.5-7	-1.1-7	1.91	10	09	13
abalone-s.9	201	200	1197	1195	1391	9.9-7	9.8-7	9.9-7	-1.9-7	-1.3-7	-2.1-7	1.90	12	10	13
abalone-s.10	201	200	1645	1691	1650	9.9-7	9.9-7	9.9-7	-8.0-8	-9.2-8	-1.1-7	1.90	16	14	15
abalone-s.11	201	200	1301	1267	1592	9.9-7	9.9-7	9.8-7	-1.4-7	-2.2-7	-9.5-8	1.90	13	11	15
abalone-m.2	401	400	479	533	621	9.9-7	9.9-7	9.5-7	2.7-8	5.8-8	-2.7-8	1.95	19	17	23
abalone-m.3	401	400	653	742	901	9.9-7	9.9-7	9.9-7	3.1-9	3.9-9	7.8-9	1.95	26	25	34
abalone-m.4	401	400	379	429	521	9.9-7	9.3-7	9.9-7	-9.5-9	-3.2-8	-4.5-9	1.93	16	15	21
abalone-m.5	401	400	643	753	788	9.9-7	9.9-7	9.9-7	-1.6-7	-1.6-7	-1.7-7	1.91	28	26	32
abalone-m.6	401	400	502	639	788	9.9-7	9.9-7	9.9-7	-1.4-7	-1.4-7	-1.4-7	1.89	22	23	32
abalone-m.7	401	400	1169	1249	1592	9.9-7	9.9-7	9.8-7	-4.8-8	-6.8-8	-4.4-8	1.92	50	44	1:05
abalone-m.8	401	400	928	805	926	9.6-7	9.9-7	9.1-7	-1.1-7	-8.0-8	-6.4-8	1.91	41	29	39
abalone-m.9	401	400	1212	1116	1517	9.9-7	9.9-7	9.9-7	-2.7-8	-8.0-8	-5.9-8	1.90	54	42	1:06
abalone-m.10	401	400	1464	1477	1792	9.9-7	9.9-7	9.9-7	-1.6-7	-1.5-7	-1.6-7	1.90	1:05	55	1:18
abalone-m.11	401	400	1501	1403	1771	9.7-7	9.9-7	9.9-7	-1.5-7	-1.9-7	-1.6-7	1.90	1:08	53	1:18
abalone-l.2	1001	1000	475	615	626	9.9-7	9.8-7	7.2-7	4.3-7	4.8-7	1.5-7	1.95	2:35	2:51	3:12
abalone-l.3	1001	1000	742	842	884	9.9-7	9.9-7	9.9-7	-7.3-8	-6.8-8	-7.1-8	1.95	4:08	3:55	4:35
abalone-l.4	1001	1000	550	609	704	9.7-7	9.9-7	9.9-7	2.2-8	2.3-8	-2.7-8	1.94	3:08	2:54	3:42
abalone-l.5	1001	1000	830	923	1137	9.9-7	9.9-7	9.9-7	-8.0-8	-8.2-8	-8.7-8	1.94	4:43	4:24	6:01
abalone-l.6	1001	1000	962	1332	1332	9.9-7	9.9-7	9.9-7	-9.2-8	-9.6-8	-9.4-8	1.91	5:30	6:23	7:04
abalone-l.7	1001	1000	1215	1266	1437	9.8-7	9.9-7	9.9-7	-6.4-8	-1.1-7	-10.0-8	1.90	6:54	6:03	7:36

Table 2: (continued) Performance of CADMM, ADMM3d and ADMM3g on θ_+, FAP, QAP, BIQ and RCP problems with accuray=10^{-6}. In this table, τ column reports the final value of τ_k of CADMM, and the computing time is in the format of "hours:minutes:seconds'. In the name of the last 120 problems, "-s, -m, -l" means "-small, -medium, -large", respectively.

problem	m	n	iteration cadm	iteration adm3d	iteration adm3g	η cadm	η adm3d	η adm3g	gap cadm	gap adm3d	gap adm3g	τ cadm	time cadm	time adm3d	time adm3g
abalone-l.8	1001	1000	1193	1018	1446	9.9-7	9.9-7	9.9-7	-3.2-7	-3.1-7	-2.3-7	1.90	6:45	4:52	7:41
abalone-l.9	1001	1000	1611	2033	2029	9.9-7	9.9-7	9.9-7	-1.4-7	-2.2-7	-2.1-7	1.90	9:09	9:46	10:47
abalone-l.10	1001	1000	1754	1951	2184	8.6-7	9.0-7	9.4-7	-5.5-8	-5.9-8	-6.8-8	1.90	10:05	9:25	11:41
abalone-l.11	1001	1000	2070	2079	2280	9.9-7	9.9-7	9.9-7	-1.4-7	-1.2-7	-1.4-7	1.90	11:53	10:01	12:10
segment-s.2	401	400	2199	2048	2267	9.9-7	9.8-7	9.0-7	-1.1-7	2.0-7	1.1-7	1.95	1:23	1:02	1:24
segment-s.3	401	400	1471	1553	1845	9.9-7	9.9-7	9.9-7	-5.5-8	-5.1-8	-5.6-8	1.84	58	51	1:12
segment-s.4	401	400	1113	1216	1449	9.9-7	9.9-7	9.9-7	-6.6-8	-6.2-8	-9.9-8	1.89	45	41	58
segment-s.5	401	400	2308	2482	3194	9.9-7	9.9-7	9.9-7	-1.5-7	-1.7-7	-1.6-7	1.95	1:46	1:36	2:24
segment-s.6	401	400	1813	1953	2513	9.9-7	9.9-7	9.9-7	-7.8-8	-8.3-8	-7.7-8	1.95	1:23	1:13	1:51
segment-s.7	401	400	912	1043	1374	9.9-7	9.9-7	9.9-7	-1.7-8	-2.8-9	4.1-9	1.93	41	39	59
segment-s.8	401	400	1307	1485	1801	9.9-7	9.9-7	9.9-7	-1.3-7	-1.0-7	-1.0-7	1.91	1:00	57	1:21
segment-s.9	401	400	805	844	1152	9.9-7	9.9-7	9.9-7	-1.6-7	-1.2-7	-8.6-8	1.93	37	33	52
segment-s.10	401	400	1110	1301	1498	9.9-7	9.9-7	9.9-7	-5.9-8	-7.5-8	-8.1-8	1.91	54	54	1:10
segment-s.11	401	400	1229	1456	1802	9.9-7	9.9-7	9.9-7	-5.6-8	-3.7-8	-7.7-8	1.90	1:02	1:03	1:28
segment-m.2	701	700	1131	1101	1093	9.5-7	9.8-7	9.9-7	1.1-6	1.4-6	-1.4-7	1.93	2:29	1:57	2:16
segment-m.3	701	700	501	745	875	9.8-7	9.9-7	8.5-7	-4.6-7	-7.5-7	-2.6-7	1.95	1:08	1:20	1:50
segment-m.4	701	700	2201	2178	2524	9.9-7	9.9-7	9.9-7	-7.2-8	-6.6-8	-6.6-8	1.95	5:06	4:05	5:29
segment-m.5	701	700	2173	2203	2811	9.9-7	9.9-7	9.9-7	-1.1-7	-1.3-7	-1.3-7	1.95	5:07	4:15	6:12
segment-m.6	701	700	2722	3109	3809	9.9-7	9.9-7	9.9-7	-1.6-7	-1.8-7	-1.7-7	1.95	6:47	6:20	8:49
segment-m.7	701	700	3179	3445	4300	9.9-7	9.9-7	9.9-7	-1.5-7	-1.6-7	-1.6-7	1.95	8:16	7:26	10:26
segment-m.8	701	700	3227	3479	3750	9.9-7	9.9-7	9.9-7	-1.8-7	-1.7-7	-1.6-7	1.95	8:16	7:23	9:01
segment-m.9	701	700	2199	2148	2578	9.9-7	9.9-7	9.9-7	-1.5-7	-1.3-7	-1.0-7	1.95	5:30	4:27	6:02
segment-m.10	701	700	1697	1733	2140	9.9-7	9.9-7	9.9-7	-3.4-8	3.2-8	3.1-8	1.95	4:10	3:33	4:57
segment-m.11	701	700	1420	1552	2148	9.9-7	9.9-7	9.9-7	1.4-8	-1.2-7	-4.2-7	1.95	3:29	3:08	4:56
segment-l.2	1001	1000	1248	1396	1570	9.8-7	9.9-7	9.9-7	-1.1-6	1.6-6	-1.6-6	1.89	6:47	6:15	7:52
segment-l.3	1001	1000	489	493	812	9.9-7	9.9-7	9.9-7	-4.8-7	-9.5-7	-1.1-6	1.95	2:43	2:15	4:09
segment-l.4	1001	1000	1946	1973	2434	9.9-7	9.9-7	9.9-7	-8.0-8	-8.9-8	-9.3-8	1.95	10:55	9:18	12:43
segment-l.5	1001	1000	2570	2648	3139	9.9-7	9.9-7	9.9-7	-1.0-7	-1.1-7	-1.0-7	1.95	14:32	12:35	16:32
segment-l.6	1001	1000	3020	3391	3938	9.9-7	9.9-7	9.9-7	-1.7-7	-1.7-7	-1.6-7	1.95	17:29	16:35	21:15
segment-l.7	1001	1000	3208	4135	4423	9.9-7	9.9-7	9.9-7	-1.7-7	-1.8-7	-1.6-7	1.95	19:04	20:46	24:27
segment-l.8	1001	1000	2776	3211	3710	9.9-7	9.9-7	9.9-7	-9.3-8	-8.8-8	-9.8-8	1.95	16:27	16:07	20:30
segment-l.9	1001	1000	2271	2615	3047	9.9-7	9.9-7	9.9-7	-1.3-7	-1.3-7	-1.3-7	1.95	13:19	13:03	16:42
segment-l.10	1001	1000	1742	1827	2106	9.9-7	9.9-7	9.9-7	-2.0-8	3.0-9	-1.3-8	1.95	10:05	8:55	11:19
segment-l.11	1001	1000	1531	1619	1987	9.9-7	9.9-7	9.9-7	3.6-7	-2.3-7	-6.6-8	1.95	8:54	7:59	10:45
housing.2	507	506	3398	3426	4089	9.9-7	9.9-7	9.7-7	-1.9-7	2.0-7	-5.8-8	1.72	3:38	2:59	4:15
housing.3	507	506	1239	1460	1511	9.5-7	9.8-7	9.4-7	-3.5-8	1.5-7	3.8-8	1.92	1:21	1:18	1:35
housing.4	507	506	1253	1222	1601	9.9-7	9.9-7	9.9-7	-4.3-8	-3.6-8	-3.2-8	1.91	1:24	1:06	1:42
housing.5	507	506	1003	1417	1456	9.9-7	9.8-7	9.3-7	-2.7-7	2.3-7	2.2-7	1.92	1:08	1:17	1:34
housing.6	507	506	554	610	745	9.9-7	9.9-7	9.9-7	-7.2-8	-3.0-8	-7.7-8	1.93	38	35	50
housing.7	507	506	623	648	708	9.9-7	9.9-7	9.9-7	-1.6-7	-1.5-7	-1.6-7	1.92	44	38	48
housing.8	507	506	615	691	850	9.9-7	9.9-7	9.9-7	-1.1-7	-1.1-7	-1.2-7	1.93	45	42	59
housing.9	507	506	815	786	1058	9.9-7	9.2-7	8.6-7	-1.7-7	4.9-8	2.4-8	1.91	1:00	49	1:15
housing.10	507	506	902	1022	1303	9.9-7	9.9-7	9.9-7	-1.6-7	-2.0-7	-1.5-7	1.91	1:07	1:03	1:33
housing.11	507	506	823	884	1048	9.9-7	9.9-7	9.9-7	-1.0-7	-1.1-7	-1.0-7	1.92	1:03	57	1:17

Table 3: Performance of CADMM, ADMM4d and LADMM4g on the 165 extended BIQ problems with accuray=10^{-6}. The computing time is in the format of "h:m:s'.

problem	m	n	iteration cadm	iteration adm4d	iteration Ladm4g	η cadm	η adm4d	η Ladm4g	gap cadm	gap adm4d	gap Ladm4g	τ cadm	time cadm	time adm4d	time Ladm4g
be100.1	101	101	15986	17066	28101	9.9-7	9.9-7	9.9-7	6.4-8	2.5-7	-4.6-7	1.83	1:38	1:31	3:09
be100.2	101	101	9452	10803	21101	9.9-7	9.9-7	9.9-7	-6.1-7	-4.6-7	-4.5-7	1.95	58	57	2:21
be100.3	101	101	12768	13842	36001	9.9-7	9.9-7	9.9-7	-3.0-9	-1.4-7	-6.0-7	1.95	1:15	1:14	4:22
be100.4	101	101	14181	15549	26641	9.9-7	9.9-7	9.9-7	-2.6-8	-3.5-8	-1.2-7	1.95	1:24	1:23	2:59
be100.5	101	101	8480	9899	14302	9.9-7	9.9-7	9.6-7	-5.0-7	-3.0-7	-4.7-7	1.95	50	52	1:36
be100.6	101	101	10803	12239	20202	9.9-7	9.9-7	9.9-7	-5.0-7	-7.4-7	7.0-7	1.95	1:03	1:05	2:19
be100.7	101	101	13926	15177	26801	9.9-7	9.9-7	9.9-7	-3.4-7	-5.1-7	-2.5-7	1.90	1:22	1:20	3:00
be100.8	101	101	12099	12640	22301	9.9-7	9.9-7	9.2-7	-4.7-7	-4.5-7	-2.2-7	1.85	1:11	1:07	2:31
be100.9	101	101	11863	12703	30651	9.9-7	9.9-7	9.9-7	-6.5-7	-5.2-7	-3.7-7	1.95	1:10	1:06	3:24
be100.10	101	101	7695	8722	12701	9.9-7	9.9-7	9.7-7	-8.5-7	-8.4-7	-3.4-7	1.95	45	46	1:25
be120.3.1	121	121	18078	20034	32301	9.9-7	9.9-7	9.9-7	1.3-8	-5.3-7	-4.5-7	1.93	2:14	2:12	4:33
be120.3.2	121	121	14797	16023	23401	9.9-7	9.9-7	9.8-7	-1.6-7	3.3-8	-6.8-7	1.95	1:50	1:46	3:21
be120.3.3	121	121	10022	10899	19801	9.9-7	9.9-7	9.9-7	-6.2-7	-5.6-7	-2.4-7	1.95	1:17	1:14	2:49
be120.3.4	121	121	22004	24197	40000	9.9-7	9.9-7	3.1-6	-1.1-7	-1.5-7	-1.1-6	1.95	2:43	2:40	5:40
be120.3.5	121	121	14910	16361	36001	9.9-7	9.9-7	9.9-7	-2.0-8	-2.7-7	-6.6-7	1.95	1:50	1:48	5:17
be120.3.6	121	121	16153	17669	30351	9.9-7	9.9-7	9.9-7	1.4-7	1.7-7	-3.7-7	1.95	2:06	2:02	4:26
be120.3.7	121	121	21032	23244	34543	9.9-7	9.9-7	9.9-7	-2.2-7	-1.2-7	-5.1-7	1.94	2:44	2:42	5:06
be120.3.8	121	121	20004	21389	37101	9.9-7	9.9-7	9.9-7	-4.4-7	-5.4-7	-4.5-7	1.93	2:36	2:28	5:17
be120.3.9	121	121	12477	13338	27766	9.9-7	9.9-7	9.9-7	2.2-7	7.0-8	-4.1-7	1.95	1:32	1:28	3:55
be120.3.10	121	121	13003	14307	21472	9.9-7	9.9-7	9.9-7	3.7-7	-2.1-7	4.0-7	1.94	1:36	1:35	3:03
be120.8.1	121	121	8214	9000	12962	9.9-7	9.9-7	9.9-7	-2.0-7	-3.7-7	-1.4-7	1.95	1:00	59	1:52
be120.8.2	121	121	11804	12407	37201	9.9-7	9.9-7	8.8-7	-1.9-8	-5.6-7	-1.9-7	1.88	1:30	1:25	5:49
be120.8.3	121	121	10499	11278	16841	9.9-7	9.9-7	9.9-7	-3.2-7	-3.3-7	-1.5-7	1.95	1:21	1:18	2:30

Table 3: (continued) Performance of CADMM, ADMM4d and LADMM4g on the 165 extended BIQ problems with accuray=10^{-6}. The computing time is in the format of "h:m:s".

problem	m	n	iteration cadm	iteration adm4d	iteration Ladm4g	η cadm	η adm4d	η Ladm4g	gap cadm	gap adm4d	gap Ladm4g	τ cadm	time cadm	time adm4d	time Ladm4g
be120.8.4	121	121	14513	15505	27001	9.9-7	9.9-7	9.9-7	6.3-7	7.6-7	9.4-8	1.82	1:53	1:49	4:00
be120.8.5	121	121	15238	16661	25002	9.9-7	9.9-7	9.9-7	-4.4-7	-5.3-7	-5.0-7	1.93	2:00	1:58	3:42
be120.8.6	121	121	9737	11345	20101	9.9-7	9.9-7	9.9-7	-6.1-7	-8.2-8	-6.6-8	1.95	1:15	1:18	3:02
be120.8.7	121	121	14726	15535	27201	9.9-7	9.9-7	9.9-7	-4.0-8	1.2-7	-1.3-7	1.84	1:55	1:47	4:07
be120.8.8	121	121	13251	14387	26302	9.9-7	9.9-7	9.9-7	-2.8-8	5.2-9	-1.9-7	1.95	1:42	1:40	3:51
be120.8.9	121	121	11291	12609	17801	9.9-7	9.9-7	9.9-7	1.1-8	1.4-7	-2.8-7	1.94	1:28	1:27	2:37
be120.8.10	121	121	11625	12689	20401	9.9-7	9.9-7	9.7-7	-5.4-7	-8.8-7	-8.4-7	1.95	1:31	1:28	3:05
be150.3.1	151	151	17965	17067	28190	9.9-7	9.9-7	9.9-7	8.7-7	8.3-7	-1.8-7	1.89	3:13	2:43	5:36
be150.3.2	151	151	14490	15005	27001	9.9-7	9.9-7	9.9-7	-4.5-7	-2.9-7	-4.1-7	1.83	2:30	2:22	5:32
be150.3.3	151	151	12420	13709	20601	9.9-7	9.9-7	9.8-7	8.4-7	1.0-6	4.0-7	1.93	2:06	2:05	4:03
be150.3.4	151	151	18830	20617	31126	9.9-7	9.9-7	9.9-7	7.1-8	-9.0-8	-5.4-7	1.87	3:13	3:08	6:08
be150.3.5	151	151	15460	17007	24001	9.9-7	9.9-7	9.9-7	-7.5-7	-6.1-7	-5.8-7	1.90	2:40	2:35	4:44
be150.3.6	151	151	11499	12784	19101	9.9-7	9.9-7	9.9-7	-4.0-7	-3.0-7	8.1-8	1.94	2:02	2:01	3:53
be150.3.7	151	151	19704	20805	40000	9.9-7	9.9-7	2.7-6	-4.5-7	-7.1-7	-6.3-7	1.86	3:31	3:18	8:16
be150.3.8	151	151	17062	18653	30001	9.9-7	9.9-7	9.5-7	-6.7-7	-1.0-6	-6.1-7	1.81	3:03	2:54	6:03
be150.3.9	151	151	9145	10574	17701	9.9-7	9.9-7	9.8-7	-3.0-8	-2.7-7	-5.0-7	1.94	1:34	1:37	3:30
be150.3.10	151	151	16922	18753	30001	9.9-7	9.9-7	9.8-7	-4.9-7	-4.2-7	-6.5-7	1.91	2:54	2:51	5:58
be150.8.1	151	151	11242	12104	17801	9.9-7	9.9-7	9.9-7	-1.6-7	-6.2-7	-9.4-7	1.93	1:58	1:56	3:34
be150.8.2	151	151	9526	10738	16051	9.9-7	9.9-7	9.9-7	-9.3-8	6.0-8	-5.8-7	1.93	1:38	1:39	3:09
be150.8.3	151	151	16109	16406	40000	9.9-7	9.9-7	1.5-6	-3.8-7	3.2-7	-3.5-7	1.82	2:47	2:31	8:25
be150.8.4	151	151	13714	16191	29001	9.9-7	9.9-7	9.8-7	-1.9-7	-1.4-7	-2.8-7	1.92	2:28	2:35	6:11
be150.8.5	151	151	13887	13188	23402	9.9-7	9.9-7	9.9-7	1.3-6	7.9-7	-5.3-7	1.80	2:30	2:08	4:47
be150.8.6	151	151	13227	14739	22101	9.9-7	9.9-7	9.8-7	8.0-7	8.2-7	3.0-7	1.93	2:23	2:20	4:29
be150.8.7	151	151	16646	16922	27801	9.9-7	9.9-7	8.7-7	-2.0-7	-3.2-7	-1.4-7	1.92	2:54	2:35	5:32
be150.8.8	151	151	14280	15379	25801	9.9-7	9.9-7	9.7-7	-3.8-7	-2.9-8	-3.6-7	1.93	2:36	2:30	5:29
be150.8.9	151	151	14811	16483	33301	9.9-7	9.9-7	9.3-7	4.0-7	3.3-7	-3.0-7	1.94	2:40	2:39	6:51
be150.8.10	151	151	13035	14000	22601	9.9-7	9.9-7	9.6-7	-3.6-7	-3.1-7	-3.5-7	1.85	2:19	2:09	4:34
be200.3.1	201	201	15205	15554	24701	9.9-7	9.9-7	9.9-7	-3.5-7	-1.7-7	-3.7-7	1.91	4:46	4:08	8:30
be200.3.2	201	201	15883	17413	26901	9.9-7	9.9-7	9.9-7	9.1-8	9.3-8	1.3-7	1.92	4:38	4:26	9:02
be200.3.3	201	201	20116	21047	32501	9.9-7	9.9-7	9.9-7	3.7-7	6.7-7	2.0-7	1.85	6:03	5:42	11:43
be200.3.4	201	201	19904	20762	31501	9.9-7	9.9-7	9.9-7	-4.9-7	-3.7-7	-4.1-7	1.89	6:11	5:39	11:17
be200.3.5	201	201	16051	17217	25498	9.9-7	9.9-7	9.9-7	-6.8-8	2.7-7	6.6-8	1.91	4:59	4:41	9:07
be200.3.6	201	201	16920	18809	26419	9.9-7	9.9-7	9.9-7	-1.9-9	-9.9-8	3.1-8	1.86	5:14	5:05	9:21
be200.3.7	201	201	21678	23906	35012	9.9-7	9.9-7	9.9-7	-6.7-8	-1.1-7	-6.5-7	1.84	6:45	6:33	12:35
be200.3.8	201	201	20154	21008	36701	9.9-7	9.9-7	9.9-7	-1.6-7	-3.0-7	-4.3-7	1.86	6:16	5:31	13:26
be200.3.9	201	201	18818	20123	31201	9.9-7	9.9-7	9.6-7	-1.1-7	-1.8-8	-4.1-7	1.91	5:55	5:31	11:09
be200.3.10	201	201	12770	14171	20201	9.9-7	9.9-7	9.9-7	-2.2-7	-1.2-6	-8.1-7	1.80	3:41	3:36	6:47
be200.8.1	201	201	19657	20162	30501	9.9-7	9.9-7	9.9-7	9.3-8	2.1-7	-2.2-7	1.73	5:56	5:27	10:47
be200.8.2	201	201	14223	15407	23401	9.9-7	9.9-7	9.4-7	-3.7-7	-4.4-7	-3.0-7	1.90	4:14	4:00	8:03
be200.8.3	201	201	17852	19030	35101	9.9-7	9.9-7	9.7-7	-5.1-7	-3.8-7	-1.3-7	1.82	5:38	5:12	13:05
be200.8.4	201	201	16855	17952	31201	9.9-7	9.9-7	9.8-7	-4.2-7	-4.9-7	-2.8-7	1.79	5:08	4:44	11:11
be200.8.5	201	201	18302	19762	36701	9.9-7	9.9-7	9.8-7	1.9-7	1.3-8	-2.7-8	1.87	5:38	5:16	13:37
be200.8.6	201	201	22596	22712	40000	9.9-7	9.9-7	4.0-6	1.9-7	2.2-7	-8.2-7	1.69	7:10	6:06	14:07
be200.8.7	201	201	18457	19238	30201	9.9-7	9.9-7	9.4-7	6.2-8	1.9-7	2.7-7	1.78	5:50	5:15	10:52
be200.8.8	201	201	23659	24633	40000	9.9-7	9.9-7	3.1-6	9.0-8	7.1-8	-7.6-7	1.75	7:28	6:41	14:09
be200.8.9	201	201	16146	16911	25001	9.9-7	9.9-7	9.9-7	-2.3-7	-1.9-7	-4.5-7	1.83	5:05	4:35	8:48
be200.8.10	201	201	17067	18001	26063	9.9-7	9.9-7	9.9-7	5.1-8	2.5-7	5.2-7	1.80	5:21	4:52	9:08
be250.1	251	251	32610	34592	40000	9.9-7	9.9-7	1.9-6	-5.2-7	-5.6-7	-1.0-6	1.88	15:43	15:06	22:31
be250.2	251	251	24830	26303	35130	9.9-7	9.9-7	9.9-7	-3.6-7	-3.4-7	-1.1-7	1.82	11:43	11:30	19:33
be250.3	251	251	32775	36441	40000	9.9-7	9.9-7	2.2-6	5.5-7	5.8-7	-5.0-7	1.91	16:12	16:22	22:23
be250.4	251	251	34320	37312	40000	9.9-7	9.9-7	3.0-6	-1.1-7	-1.1-7	-8.9-7	1.91	16:52	16:51	23:05
be250.5	251	251	20583	22831	31184	9.9-7	9.9-7	9.9-7	3.4-8	1.3-7	4.1-8	1.91	9:39	9:34	16:54
be250.6	251	251	29456	30930	40000	9.9-7	9.9-7	3.7-6	2.7-7	2.7-7	-8.6-7	1.83	13:52	13:04	22:07
be250.7	251	251	29789	31931	40000	9.9-7	9.9-7	2.4-6	-3.9-8	3.2-9	-8.1-7	1.87	14:35	13:50	22:36
be250.8	251	251	27478	28540	40000	9.9-7	9.9-7	2.4-6	1.2-7	2.5-7	-6.4-7	1.86	13:20	12:17	22:18
be250.9	251	251	19585	21746	28889	9.9-7	9.9-7	9.9-7	4.1-7	3.3-7	3.4-7	1.91	9:14	9:06	15:46
be250.10	251	251	21890	25420	35636	9.9-7	9.9-7	9.9-7	-4.1-7	-1.5-7	-7.9-8	1.87	10:17	10:45	19:22
bqp50-1	51	51	2302	2101	3060	9.1-7	8.9-7	9.9-7	-8.1-7	-2.5-7	-3.4-7	1.90	07	06	10
bqp50-2	51	51	1609	1706	2258	9.5-7	9.9-7	9.9-7	-2.2-6	-4.5-7	-2.0-7	1.91	05	05	08
bqp50-3	51	51	2096	1650	2220	9.6-7	9.9-7	9.7-7	-2.6-6	-2.3-6	3.7-6	1.86	05	04	07
bqp50-4	51	51	2813	2475	2744	9.9-7	9.9-7	9.9-7	1.2-6	4.2-7	2.2-6	1.90	08	06	09
bqp50-5	51	51	3019	3106	7642	9.9-7	9.9-7	9.9-7	2.7-7	-5.1-8	-2.2-8	1.91	08	07	24
bqp50-6	51	51	2616	2372	3301	9.9-7	9.9-7	9.2-7	1.8-6	3.3-6	3.5-7	1.90	07	05	10
bqp50-7	51	51	4132	3721	5274	9.9-7	9.8-7	9.9-7	2.9-6	-3.4-6	-1.3-6	1.86	12	09	17
bqp50-8	51	51	3346	3045	3609	9.9-7	9.6-7	9.9-7	4.8-6	-3.0-6	-4.4-6	1.90	09	07	11
bqp50-9	51	51	3332	3112	3930	9.9-7	9.9-7	9.9-7	-3.6-6	4.1-6	-5.0-6	1.89	09	08	13
bqp50-10	51	51	3285	3993	4583	9.8-7	9.9-7	9.9-7	-7.6-7	-4.1-8	-8.1-9	1.90	10	10	16
bqp100-1	101	101	21532	23675	31372	9.9-7	9.9-7	9.9-7	-7.3-7	-8.2-7	-7.1-7	1.93	2:14	2:11	3:41
bqp100-2	101	101	27469	29024	36064	9.9-7	9.7-7	9.8-7	-4.8-6	-4.9-6	5.6-6	1.88	2:51	2:42	4:14
bqp100-3	101	101	4773	4819	4899	9.8-7	9.5-7	9.9-7	-4.3-6	-5.2-6	4.9-6	1.86	25	22	32
bqp100-4	101	101	4420	4196	5434	9.6-7	9.9-7	9.7-7	5.6-6	6.0-6	3.7-6	1.90	24	20	33
bqp100-5	101	101	7277	7736	9342	9.9-7	9.8-7	9.8-7	6.6-6	-6.5-6	5.5-6	1.89	43	41	1:03
bqp100-6	101	101	19710	22907	31101	9.9-7	9.9-7	9.9-7	-9.2-7	-8.7-7	-7.8-7	1.92	2:04	2:09	3:40
bqp100-7	101	101	28744	31222	40000	9.9-7	9.9-7	1.3-6	-7.9-7	-8.0-7	-1.2-6	1.93	2:59	2:55	4:42
bqp100-8	101	101	3021	3199	4064	9.0-7	8.2-7	9.6-7	1.8-6	1.5-6	-4.4-6	1.85	18	18	28
bqp100-9	101	101	4250	3920	5683	9.8-7	9.7-7	9.6-7	-4.3-6	-5.3-6	4.2-6	1.84	23	19	35
bqp100-10	101	101	6923	7323	8704	9.5-7	9.9-7	9.9-7	4.8-6	5.1-6	5.8-6	1.88	38	36	56
bqp250-1	251	251	28907	30414	40000	9.9-7	9.9-7	4.0-6	3.5-7	4.1-7	-1.2-6	1.82	14:00	13:06	22:28
bqp250-2	251	251	24207	26303	39901	9.9-7	9.9-7	9.9-7	9.3-8	-2.8-7	-4.7-7	1.86	11:16	11:01	21:42

Table 3: (continued) Performance of CADMM, ADMM4d and LADMM4g on the 165 extended BIQ problems with accuray=10^{-6}. The computing time is in the format of "h:m:s'.

			iteration			η			gap			τ	time		
problem	m	n	cadm	adm4d	Ladm4g	cadm	adm4d	Ladm4g	cadm	adm4d	Ladm4g	cadm	cadm	adm4d	Ladm4g
bqp250-3	251	251	28573	31033	40000	9.9-7	9.9-7	1.7-6	8.0-7	9.3-7	5.7-7	1.85	13:26	13:07	21:46
bqp250-4	251	251	21055	22328	34701	9.9-7	9.9-7	9.6-7	4.1-7	5.1-7	7.8-8	1.88	9:53	9:19	19:23
bqp250-5	251	251	29102	30938	40000	9.9-7	9.9-7	1.9-6	-6.2-7	-7.1-7	-7.3-7	1.85	13:57	13:09	22:06
bqp250-6	251	251	19897	20804	33701	9.9-7	9.9-7	9.9-7	3.1-7	5.1-7	-4.0-7	1.91	9:15	8:32	18:43
bqp250-7	251	251	25726	28181	40000	9.9-7	9.9-7	2.2-6	4.6-7	5.1-7	-6.0-8	1.86	11:58	11:40	21:46
bqp250-8	251	251	20746	22407	34590	9.9-7	9.9-7	9.9-7	1.5-7	-3.5-8	-3.1-7	1.91	9:27	9:04	18:33
bqp250-9	251	251	29475	31324	40000	9.9-7	9.9-7	2.3-6	-2.2-7	-2.3-7	-1.0-6	1.86	13:53	13:05	21:54
bqp250-10	251	251	18577	19686	32101	9.9-7	9.9-7	9.9-7	-2.0-7	2.1-7	-4.6-7	1.87	8:33	7:58	17:15
bqp500-1	501	501	29614	31824	40000	9.9-7	9.9-7	2.7-6	-2.2-7	-3.2-7	-5.5-7	1.86	58:11	53:46	1:30:16
bqp500-2	501	501	36701	39134	40000	9.9-7	9.9-7	1.0-5	-5.3-7	-5.5-7	-2.7-6	1.83	1:13:31	1:09:43	1:35:48
bqp500-3	501	501	32335	34701	40000	9.9-7	9.9-7	6.4-6	2.6-7	3.0-7	-1.2-7	1.81	1:04:42	1:01:35	1:33:30
bqp500-4	501	501	34981	36015	40000	9.9-7	9.9-7	1.1-5	2.4-7	2.7-7	-8.9-7	1.80	1:08:24	1:02:33	1:31:25
bqp500-5	501	501	32691	35062	40000	9.9-7	9.9-7	4.1-6	-4.3-7	-4.5-7	-7.2-7	1.84	1:03:19	59:59	1:31:10
bqp500-6	501	501	35186	38019	40000	9.9-7	9.9-7	9.0-6	-3.4-7	-2.4-7	-9.0-7	1.84	1:08:32	1:05:22	1:33:43
bqp500-7	501	501	29781	31503	40000	9.9-7	9.9-7	7.7-6	-5.0-7	-4.9-7	-1.4-6	1.84	58:32	54:15	1:31:14
bqp500-8	501	501	37524	40000	40000	9.9-7	1.0-6	5.7-6	2.7-7	2.9-7	-1.1-7	1.84	1:12:37	1:07:57	1:30:39
bqp500-9	501	501	29410	31211	40000	9.9-7	9.9-7	3.4-6	-8.0-8	2.6-8	-3.6-7	1.83	57:02	53:13	1:30:53
bqp500-10	501	501	34087	35536	40000	9.9-7	9.9-7	8.1-6	-1.2-7	3.0-8	-1.4-6	1.83	1:06:54	1:01:14	1:33:10
fgka1a	51	51	3608	3624	4369	9.9-7	9.9-7	9.9-7	-5.0-7	-4.1-7	4.1-7	1.90	09	08	13
gka2a	61	61	2916	2307	3165	9.9-7	9.9-7	9.7-7	-3.9-6	-2.8-6	4.3-6	1.90	08	06	10
gka3a	71	71	29627	34950	40000	9.9-7	9.9-7	2.6-6	-1.3-6	-1.5-6	-3.6-6	1.93	1:51	1:56	2:49
gka4a	81	81	5711	6227	7271	9.4-7	9.9-7	9.4-7	9.0-8	5.4-7	3.1-7	1.89	24	23	34
gka5a	51	51	3977	3942	4918	9.9-7	9.8-7	9.9-7	-3.2-6	4.4-6	-2.8-6	1.90	11	10	15
gka6a	31	31	2243	1943	2397	9.8-7	9.2-7	9.6-7	-2.6-6	2.2-6	2.0-6	1.90	05	03	05
gka7a	31	31	1567	1592	1831	9.6-7	9.8-7	9.9-7	-2.5-6	4.1-6	3.8-6	1.93	03	03	04
gka8a	101	101	4112	3619	5151	9.5-7	9.5-7	9.9-7	3.2-6	3.7-6	4.8-6	1.89	20	16	28
gka1b	21	21	204	184	193	9.3-7	7.6-7	8.9-7	-1.3-5	-9.9-6	-1.3-5	1.72	00	00	00
gka2b	31	31	827	903	1081	8.5-7	9.9-7	6.7-7	2.2-5	-2.9-5	-1.8-5	1.83	02	02	03
gka3b	41	41	276	270	266	9.1-7	8.9-7	8.9-7	2.8-5	-3.3-5	3.1-5	1.90	01	01	01
gka4b	51	51	310	304	332	8.4-7	9.8-7	8.6-7	3.6-5	-4.3-5	3.5-5	1.79	01	01	01
gka5b	61	61	305	273	253	8.5-7	6.3-7	9.6-7	-3.8-5	2.9-5	4.1-5	1.85	01	01	01
gka6b	71	71	347	346	406	8.5-7	7.4-7	9.1-7	4.6-5	-4.0-5	4.7-5	1.87	01	01	02
gka7b	81	81	631	629	685	9.8-7	6.7-7	9.0-7	-5.3-5	3.8-5	4.9-5	1.85	03	02	03
gka8b	91	91	426	406	459	9.2-7	5.2-7	4.5-7	5.4-5	-3.8-5	2.7-5	1.83	02	02	02
gka9b	101	101	1064	1057	1236	8.9-7	7.4-7	9.2-7	-2.3-5	-1.7-5	2.2-5	1.88	05	05	07
gka10b	126	126	2600	2705	5001	9.9-7	9.9-7	9.9-7	-7.3-5	-7.2-5	-4.3-6	1.90	21	20	45
gka1c	41	41	1204	1362	1627	9.9-7	9.9-7	9.9-7	2.1-6	2.8-6	1.8-6	1.90	03	03	04
gka2c	51	51	3011	3031	3430	9.8-7	9.5-7	9.9-7	4.4-6	-4.2-6	-2.7-6	1.91	08	07	10
gka3c	61	61	4324	3810	4877	9.6-7	9.5-7	9.8-7	3.5-6	-4.6-6	3.0-6	1.86	13	10	16
gka4c	71	71	3210	3403	4235	9.8-7	8.6-7	9.7-7	4.2-6	-1.4-6	-3.1-6	1.90	12	11	18
gka5c	81	81	4202	4444	5294	9.6-7	9.9-7	9.9-7	3.9-6	-3.9-6	1.3-6	1.91	16	15	23
gka6c	91	91	5851	5543	6624	9.8-7	9.8-7	9.7-7	4.1-6	5.2-6	-4.2-6	1.90	27	22	34
gka7c	101	101	4239	5688	5044	9.9-7	9.8-7	9.9-7	-5.5-6	-5.0-6	-5.2-6	1.83	21	25	29
gka1d	101	101	4862	3970	4241	9.9-7	9.8-7	9.9-7	4.8-6	5.8-6	-5.1-6	1.90	25	20	27
gka2d	101	101	7832	8632	13637	9.9-7	9.9-7	9.9-7	-2.5-7	-3.8-7	-2.8-7	1.95	47	46	1:32
gka3d	101	101	14404	16133	26601	9.9-7	9.9-7	9.9-7	-2.5-7	-2.3-7	-2.1-7	1.95	1:27	1:26	2:57
gka4d	101	101	11088	12164	21801	9.9-7	9.9-7	9.8-7	-6.3-7	-6.3-7	-5.1-7	1.95	1:06	1:05	2:32
gka5d	101	101	14074	15565	21522	9.9-7	9.9-7	9.9-7	-7.1-7	-6.2-7	-6.7-7	1.95	1:28	1:24	2:23
gka6d	101	101	18001	19699	38001	9.9-7	9.9-7	9.9-7	2.2-7	1.2-7	-4.9-7	1.95	1:46	1:45	4:13
gka7d	101	101	12419	13600	27201	9.9-7	9.9-7	9.8-7	-5.3-7	-6.8-7	-4.1-7	1.95	1:14	1:12	3:12
gka8d	101	101	13497	14514	23401	9.9-7	9.9-7	9.9-7	5.8-7	3.7-7	9.8-8	1.92	1:20	1:17	2:35
gka9d	101	101	11250	12280	24501	9.9-7	9.9-7	9.9-7	-3.7-7	-5.2-7	-4.7-7	1.95	1:06	1:05	2:46
gka10d	101	101	12161	12945	29002	9.9-7	9.9-7	9.9-7	-1.3-7	-10.0-8	-2.1-7	1.91	1:12	1:09	3:21
gka1e	201	201	36678	39257	40000	9.9-7	9.9-7	5.8-6	-5.8-7	-5.2-7	-3.6-6	1.89	12:57	11:44	15:03
gka2e	201	201	16136	17669	26801	9.9-7	9.9-7	9.9-7	-8.9-8	1.7-7	2.4-7	1.85	5:22	5:17	10:04
gka3e	201	201	17603	18336	28501	9.9-7	9.9-7	9.0-7	1.3-7	1.9-7	-3.3-7	1.82	5:51	5:27	10:46
gka4e	201	201	23408	23594	34942	9.9-7	9.9-7	9.9-7	1.5-7	3.8-7	-3.5-7	1.71	7:47	7:01	13:09
gka5e	201	201	20382	22533	35901	9.9-7	9.9-7	9.9-7	1.2-7	-9.5-8	-2.4-7	1.77	6:46	6:41	13:33
gka1f	501	501	28554	30561	40000	9.9-7	9.9-7	2.0-6	-5.1-7	-5.7-7	4.4-7	1.88	55:01	52:09	1:30:24
gka2f	501	501	32824	34038	40000	9.9-7	9.9-7	9.2-6	-3.9-7	-3.9-7	-9.4-7	1.74	1:04:31	58:51	1:32:59
gka3f	501	501	33675	33852	40000	9.9-7	9.9-7	1.3-5	-3.5-7	-1.9-7	-1.1-6	1.68	1:06:35	57:33	1:31:40
gka4f	501	501	32337	31282	40000	9.9-7	9.9-7	6.4-6	7.3-8	1.2-7	2.8-7	1.61	1:04:33	53:53	1:31:58
gka5f	501	501	31829	34330	40000	9.9-7	9.9-7	9.3-6	4.9-8	9.9-8	-9.3-7	1.88	1:01:44	58:33	1:30:28