Solving Classical and New Single Allocation Hub Location Problems on Euclidean Data

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Abstract

Transport networks with hub structure organise the exchange of shipments between many sources and sinks. All sources and sinks are connected to a small number of hubs which serve as transhipment points, so that only few, strongly consolidated transport relations exist. While hubs and detours lead to additional costs, the savings from bundling shipments, i.e. economies of scale, usually outweigh these costs. Typical applications for hub networks are in cargo, air freight, postal and parcel transport services.

In this paper we consider three classical and two recent formulations of single allocation hub location problems, i.e. hub location problems in which every source and sink is connected to exactly one hub. Solving larger instances of these problems to optimality is difficult because the inherent quadratic structure of the problem has to be linearised: This leads to a sharp rise in variable numbers. Our new approach relies on the fact that many instances—including the famous CAB and AP data sets—have a Euclidean structure: The distances between the possible hub locations are Euclidean distances in the plane. This enables us to construct a new linearisation together with a row generation procedure which solves instances up to 200 nodes to optimality.

Keywords: Quadratic Optimisation, Linearisation, Euclidean Distances, Mixed Integer Program

1. Introduction

Hubs are used to consolidate and disseminate shipments in many-to-many transport networks. Hub Location Problems deal with finding the location of hubs (out of a given list of possible locations) and the allocation of the sources/sinks of shipments to these located hubs. The idea behind non-direct transport is that the economies of scale generated by consolidation often outweigh the additional costs of hubs and detours. These economies of scale are

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typically estimated by using a constant discount factor for transport on the hub-hub connections. Hub location problems arise in various application settings in telecommunication [vdL99] and transportation (air passenger [JSY96], cargo [RAB07], public [Gel08]) network design.

The first mathematical model for the hub location problem was given by O'Kelly [O'K87]. He proposed a quadratic integer programming formulation for the problem of minimizing the total transport cost for a given number of hubs to locate (p-Hub Median Problem). The following hub location literature mainly focused on the transportation cost objective locating both a fixed and a variable number of hubs. Important early studies are [O'K92]; [EK96, EK99] [SKSKO96]. More recent studies are [TCEY05], [MCL06], [CDF09, CCL11, CCL12]. For a more complete and also historical introduction, the reader is referred to [AK08, CO12].

In this paper we concentrate on *single allocation* hub location problems in which each source/sink establishes exactly one connection to a hub. These models are applied to situations in which sorting at the source is not possible (or too costly) so that all shipments are transported as a whole to the allocated hub. Typical examples are postal or parcel networks. To be more precise, the problem can be described as follows:

For a set of sources/sinks V, a matrix of shipments W_{ij}, $i, j \in V$ is given which should be transported from i to j. No direct shipping is applied; instead, every $i \in V$ is allocated to one hub k_i and the shipments are sent $i \to k_i \to k_j \to j$. The aim of Single Allocation Hub Location Problems is to choose a set of hubs out of a set of possible hubs and to allocate each $i \in V$ to a hub so that the overall costs are minimized. A variety of formulations has been discussed and solved in literature. Three "classical" problems are:

1. The Uncapacitated Single Allocation p Hub Median Problem (USApHMP), defined in [O'K87]: The aim is to choose p hubs and assign every $i \in V$ to them to minimize overall transport costs. No hub costs and no capacity constraints are involved.
2. The Uncapacitated Single Allocation Hub Location Problem (USAHLP), defined in [O'K92]: Instead of a fixed number of hubs to choose, a cost factor F_k is given for establishing a hub at location k.
3. The Capacitated Single Allocation Hub Location Problem (CSAHLP), defined in [EK99, Bry98]: In addition to the costs and restrictions of USAHLP, we consider a capacity Γ_k for each hub $k \in V$. This capacity has to be larger than all incoming flow from the sources.

Furthermore, two additional formulations will be discussed:

4. The Capacitated Single Allocation Hub Location Problem with Multiple Levels (CSAHLPM), defined in [CNSdG10]: Each hub has a given list of capacity levels from which exactly one can be installed. These have different capacities and fixed costs.
5. The Single Allocation Hub Location Problem under Congestion (SAHLPC) defined in [DCM12]. Here, capacity is not restricted by hard constraints, but instead, a non-linear cost factor for congestion at hubs is introduced.

The main aim of this paper is to describe a new method of linearising the binary quadratic terms involved in the models and demonstrating the strength of this new approach by solving a variety of large instances to optimality. This method can be applied if the distances present in the objective function are Euclidean distances—a situation which is often encountered and is also true in the usual Benchmark instances.

The next section will describe the five different single allocation hub location models in detail. Section 3 explains the new solution method based on projections in Euclidean space. In Sect. 4, a speed-up for the solution process is discussed. Section 5 gives the results of a large number of numerical experiments, while Sect. 6 contains the conclusions.

2. Single Allocation Hub Location Models

The general situation for Single Allocation Hub Location Problems can be described follows: A set V of locations describes both the sources/sinks and the possible hub locations. Although it is an easy exercise to separate the source/sink set from the hub location set, we follow the conventions of the literature and use just one set. For each $(i, j) \in V \times V$, there is a shipment $W_{ij} \in \mathbb{R}_{\geq 0}$ to be transported from i to j. For that, every source/sink $i \in V$ is allocated to a hub $k_i \in V$ and the resulting path for the shipment is $i \to k_i \to k_j \to j$.

Binary variables x_{ik} are introduced, meaning the allocation of i to k. The variables x_{kk}, $k \in V$, describe the decision of establishing a hub. Therefore, every problem contains the constraints:

$$\sum_k x_{ik} = 1 \qquad\qquad \forall i \in V \qquad\qquad (1)$$

$$x_{ik} \leq x_{kk} \qquad\qquad \forall i, k \in V \qquad\qquad (2)$$

$$x_{ik} \in \{0, 1\} \qquad\qquad \forall i, k \in V \qquad\qquad (3)$$

Sums are always understood to be over V if not stated otherwise. We see that (1) makes sure that every source/sink is allocated while (2) states that one can only allocate i to k if k is chosen as a hub.

Several linearised versions of the following models exist (see e.g. [Cam94, CO12]), mostly with $O(n^3)$ or $O(n^4)$ variables ($n = |V|$). We will use the original quadratic formulation and develop a smaller linearisation in Sect. 3.

2.1. USApHMP

The Uncapacitated Single Allocation p Hub Median Problem consists of finding exactly p hubs so that the overall transport costs are minimized. For that, unit transport costs c_{ij}, $i, j \in V$, are assumed which are multiplied with different factors χ, α and δ for depot-hub, hub-hub and hub-depot connections.

To simulate economies of scale, it is assumed that $\alpha < \chi$ and $\alpha < \delta$. The model is then:

$$\text{Min} \quad \sum_i \sum_j W_{ij}\left(\chi \sum_k c_{ik}x_{ik} + \delta \sum_m c_{jm}x_{jm} + \alpha \sum_k \sum_m c_{km}\, x_{ik}x_{jm}\right)$$

$$\sum_k x_{kk} = p \tag{4}$$

$$(1), (2), (3)$$

If we introduce the constants

$$d_{km} = \alpha c_{km} \qquad\qquad K_{ik} = \chi c_{ik} \sum_{j \in V} W_{ij} + \delta c_{ik} \sum_{j \in V} W_{ji}$$

we can write the objective function as

$$\text{Min} \quad \sum_i \sum_k K_{ik}x_{ik} + \sum_i \sum_j \sum_k \sum_m W_{ij}d_{km}\, x_{ik}x_{jm}$$

We see that the cost structure for pre and onward carriage is of minor importance because it can be preprocessed. The main difficulty lies in the term $\sum_i \sum_j \sum_k \sum_m W_{ij}d_{km}\, x_{ik}x_{jm}$ in the objective function.

2.2. USAHLP

The Uncapacitated Single Allocation Hub Location Problem uses the parameters of the USApHMP, but replaces the fixed number p by a set F_k, $k \in V$, of hub costs. Using the definitions of the preceding subsection, we write

$$\text{Min} \quad \sum_k F_k x_{kk} + \sum_i \sum_k K_{ik}x_{ik} + \sum_i \sum_j \sum_k \sum_m W_{ij}d_{km}\, x_{ik}x_{jm}$$

$$(1), (2), (3)$$

2.3. CSAHLP

The Capacitated Single Allocation Hub Location Problem extends the USAHLP by a capacity Γ_k per hub $k \in V$. This capacity is incorporated in an additional constraint:

$$\text{Min} \quad \sum_k F_k x_{kk} + \sum_i \sum_k K_{ik}x_{ik} + \sum_i \sum_j \sum_k \sum_m W_{ij}d_{km}\, x_{ik}x_{jm}$$

$$\sum_i \sum_j W_{ij}x_{ik} \leq \Gamma_k x_{kk} \qquad\qquad \forall k \in V \tag{5}$$

$$(1), (2), (3)$$

As we see in (5), the capacity restrictions only applies to the flow from sources (which have to be sorted, labelled etc.).

2.4. CSAHLPM

The Capacitated Single Allocation Hub Location Problem with Multiple Capacity Levels was introduced by [CNSdG10]. It replaces the single capacity level of the CSAHLP by a number of different capacity levels Γ_k^q with different costs F_k^q, where $q \in Q$ enumerates the capacity levels. For this purpose, we introduce the additional set of binary variables z_{kq}, $k \in V$, $q \in Q$ which indicate that capacity level q was chosen for hub k. The model can then be written as

$$\text{Min} \quad \sum_k \sum_{q \in Q} F_k^q z_{kq} + \sum_i \sum_k K_{ik} x_{ik} + \sum_i \sum_j \sum_k \sum_m W_{ij} d_{km}\, x_{ik} x_{jm}$$

$$\sum_i \sum_j W_{ij} x_{ik} \leq \sum_{q \in Q} \Gamma_k^q z_{kq} \qquad\qquad \forall k \in V \qquad (6)$$

$$\sum_q z_{kq} = x_{kk} \qquad\qquad \forall k \in V \qquad (7)$$

$$z_{kq} \in \{0,1\} \qquad\qquad \forall k \in V, q \in Q \qquad (8)$$

$$(1), (2), (3)$$

The additional set of variables makes the problem more difficult to solve.

2.5. SAHLPC

The Single Allocation Hub Location Problem under Congestion was introduced in [DCM12]. Instead of restricting the throughput for each hub, a congestion cost function is assigned to each hub. The problem differs in two ways from the previously defined problems:

First of all, the throughput g_k of a hub k is measured differently: In addition to the incoming flow from the sources, flow from other hubs is also taken into account. Secondly, the throughput is not bounded, but for $g_k \geq \Gamma_k$, a cost function $\tau_k(g_k)$ of the form $a(g_k - \Gamma_k)^b$ for given constants a and $b \geq 1$ is applied. We get the non-linear model:

$$\text{Min} \quad \sum_k \tau_k(g_k) + \sum_k F_k x_{kk} + \sum_i \sum_k K_{ik} x_{ik} + \sum_i \sum_j \sum_k \sum_m W_{ij} d_{km}\, x_{ik} x_{jm}$$

$$\sum_i \sum_j W_{ij} x_{ik} + \sum_i \sum_j W_{ji} x_{ik}(1 - x_{jk}) \leq g_k \qquad \forall k \in V \qquad (9)$$

$$\tau_k(g_k) = \begin{cases} 0 & g_k \leq \Gamma_k \\ a(g_k - \Gamma_k)^b & g_k \geq \Gamma_k \end{cases} \qquad\qquad (10)$$

$$g_k \geq 0, \tau_k \geq 0 \qquad\qquad \forall k \in V \qquad (11)$$

$$(1), (2), (3)$$

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We made some minor changes compared to [DCM12]: We did not introduce a capacity reduction factor γ for each hub but integrated it into the given capacity. Furthermore, we wrote (9) as quadratic constraint instead of using 4-index path variables. Additionally, we corrected the formula for $\tau_k(g_k)$: It was originally written as $\max\left\{0, a(g_k - \Gamma_k)^b\right\}$ which leads for even b to the unwanted result of congestion for non-used hubs.

Both (9) and (10) should be linearised for our purposes. For (9) we can use a usual technique (already described by Danzig) to turn a binary quadratic inequality into an exponential set of linear inequalities. We know that

$$x_{ik}(1 - x_{jk}) \geq x_{ik} - x_{jk}$$

$$x_{ik}(1 - x_{jk}) \geq 0,$$

so that for every binary vector $\eta \in \{0, 1\}^{n^2}$ we can introduce the constraints

$$\sum_i \sum_j W_{ij} x_{ik} + \sum_i \sum_j W_{ji} \eta_{ik}(1 - \eta_{jk})(x_{ik} - x_{jk}) \leq g_k \quad \forall k \in V, \forall \eta \in \{0, 1\}^{n^2}$$

$$(12)$$

If we have a solution $\{\hat{x}\}$ with $\hat{x}_{ik} = \eta_{ik}$ for all $i, k \in V$, (12) enforces (9), so that it is a proper linear replacement.

For (10), we use the fact that $\tau_k(g_k)$ is convex; it is also differentiable for $g_k \neq \Gamma_k$. For every fixed value $\bar{g}_k > \Gamma_k$, we can, therefore, introduce the constraint

$$\tau_k \geq ab\left(\bar{g}_k - \Gamma_k\right)^{b-1}(g_k - \bar{g}_k) + a\left(\bar{g}_k - \Gamma_k\right)^b \qquad \forall \bar{g}_k > \Gamma_k \qquad (13)$$

meaning that the value of τ_k is above the tangent to $\tau_k(g_k)$ at $\bar{g}_k$. The constraint (13) enforces (10) if $g_k = \bar{g}_k$. Hence, only a finite number of these constraints is necessary to replace (10).

Both (12) and (13) will be integrated into the Branch-and-Cut-algorithm: Every time an integral solution $\{\hat{x}, \hat{g}, \hat{\tau}\}$ is found, we check (12) for $\eta_{ik} := \hat{x}_{ik}$ and (13) for $\bar{g}_k := \hat{g}_k$. If they are valid, the solution is admissible because these are the most violated inequalities. Otherwise, these constraints are added to the problem and the found integral solution is discarded.

3. The Euclidean Projection Method

The Euclidean projection method relies on d_{km} being a Euclidean distance. This is true for the standard benchmark instances CAB and AP [Bea12] and also for the URAND instances from [IUBM10]. Furthermore, it serves as a good approximation in many cases. The method is based on our previous study [MCRB15].

The aim of this section is to describe a method to replace the term

$$T = \sum_i \sum_j \sum_k \sum_m W_{ij} d_{km}\, x_{ik} x_{jm} \qquad (14)$$

by a linear term and a set of linear constraints. Let us introduce positive real variables y_{ij}, $i, j \in V$. Then, we can replace T by

$$T_1 = \sum_i \sum_j W_{ij} y_{ij} \tag{15}$$

$$y_{ij} \geq \sum_k \sum_m d_{km} x_{ik} x_{jm} \qquad \forall i, j \in V \tag{I_{ij}}$$

$$y_{ij} \geq 0 \qquad \forall i, j \in V \tag{16}$$

Each of the quadratic constraints (I_{ij}) will now be replaced by a set of n^2 linear inequalities. To construct these inequalities, we need a short digression to linear algebra.

Following from the assumption that d_{km} is a Euclidean distance, we can assign a vector e_k in a Euclidean vector space $\mathbb{E}$ (usually the plane) to every possible hub $k \in V$ such that $d_{km} = ||e_k - e_m||$ in the vector space norm. The inner product $\langle \bullet, \bullet \rangle$ of $\mathbb{E}$ can be used to define orthogonal projections $P_w : \mathbb{E} \to \mathbb{E}_w$ from $\mathbb{E}$ to the one-dimensional space $\mathbb{E}_w$ spanned by a vector $w \in \mathbb{E}$. The space $\mathbb{E}_w$ can be canonically identified with $\mathbb{R}$ by the isometric bijection $\lambda w/||w|| \mapsto \lambda$, $\lambda \in \mathbb{R}$, so that we can interpret P_w as map from $\mathbb{E}$ to $\mathbb{R}$, which can be written as

$$P_w(u) = \frac{\langle u, w \rangle}{||w||} \tag{17}$$

Now we know that the distance between orthogonal projections is never greater than the original distance:

$$d_{km} = ||e_k - e_m|| \geq \left| P_w(e_k) - P_w(e_m) \right| \geq P_w(e_k) - P_w(e_m) \tag{18}$$

and that equality holds if $w = e_k - e_m$.

Let us define $\lambda_k^{hl} = P_{e_h - e_l}(e_k)$ for $e_h \neq e_l$. Then, for every $h \neq l$, we can derive the following inequality:

$$\sum_k \sum_m d_{km} x_{ik} x_{jm} \geq \sum_k \sum_m \left(\lambda_k^{hl} - \lambda_m^{hl} \right) x_{ik} x_{jm}$$

$$= \sum_k \sum_m \lambda_k^{hl} x_{ik} x_{jm} - \sum_k \sum_m \lambda_m^{hl} x_{ik} x_{jm}$$

$$= \sum_k \lambda_k^{hl} x_{ik} \sum_m x_{jm} - \sum_m \lambda_m^{hl} x_{jm} \sum_k x_{ik}$$

Using (1):

$$= \sum_k \lambda_k^{hl} x_{ik} - \sum_m \lambda_m^{hl} x_{jm}$$

If, in a solution $\{\hat{x}\}$, we have $\hat{x}_{ih} = \hat{x}_{jl} = 1$, this constraint becomes an equality. From (1) we know that for every $i, j \in V$ there exists $h_i, l_j \in V$ with

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$\hat{x}_{ih_i} = \hat{x}_{jl_j} = 1$. This shows, that the constraint (I_{ij}) is equivalent to the set of constraints

$$y_{ij} \geq \sum_k \lambda_k^{hl} x_{ik} - \sum_m \lambda_m^{hl} x_{jm} \qquad \forall h \neq l \in V \qquad (I_{ij}^{hl})$$

If $h_i \neq l_j$, the constraint (I_{ij}^{hl}) forces (I_{ij}) to be fulfilled. If $h_i = l_j$, the term T_1 is zero and (I_{ij}^{hl}) is trivial.

For computational purposes, one should note that one can compute the values λ_k^{hl} more directly using the Law of Cosines in the triangle formed by k, h and l (illustrated in Fig. 1):

$$\begin{aligned}
\lambda_k^{hl} &= d_{kl} \cos(\angle klh) \\
&= d_{kl} \cdot \frac{d_{kl}^2 + d_{hl}^2 - d_{kh}^2}{2 d_{kl} d_{hl}} \\
&= \frac{1}{2} d_{hl} + \frac{1}{2 d_{hl}} (d_{kl}^2 - d_{hl}^2)
\end{aligned}$$

As the first term is independent of k, it can also be deleted without altering the resulting inequalities (I_{ij}^{hl}).

Summarized, the problem USApHMP can be written as

$$\text{Min} \quad \sum_i \sum_k K_{ik} x_{ik} + \sum_i \sum_j W_{ij} y_{ij} \qquad (19)$$

$$(1), (2), (3), (4), (I_{ij}^{hl})$$

135   and similarly, USAHLP, CSAHLP, CSAHLPM and SAHLPC can be rewritten.

Because (I_{ij}^{hl}) is a set of approximately n^4 constraints, we cannot consider them all in practical problem sizes of $n > 50$. Therefore, we use a row generation scheme for these constraints:

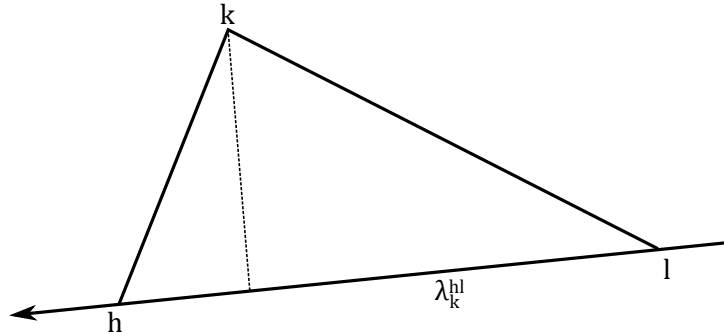Figure 1: Illustration of the orthogonal projection of e_k onto the vector $e_l - e_h$. The length λ_{hl}^k is the cosine of the angle at l multiplied by the length of kl.

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1. We solve the problem with just a small set of (I_{ij}^{hl}) inequalities (this small set is determined by a heuristic procedure described in Sect. 4). We receive binary values $\hat{x}_{ik}$, $i, k \in V$ for the allocation variables.
2. We use the binary values $\hat{x}_{ik}$ to construct a primal solution for the original problem. This can be easily done by setting $\hat{y}_{ij} = \sum_k \sum_m d_{km} \hat{x}_{ik} \hat{x}_{jm}$.
3. For every $i, j \in V$, we determine h, l with $\hat{x}_{ih} = \hat{x}_{jl} = 1$. If $h \neq l$, we check the inequality (I_{ij}^{hl}) on the given solution and add it, if it is not fulfilled.
4. If inequalities were added, we resolve and return to step (2.), otherwise optimality is proven.

4. Accelerate the Solving Procedure

The row generation procedure of Sect. 3 can be accelerated if we start with a good upper bound and a good set of initial constraints. The idea is to solve a relaxed problem to gain a set of "good hub locations". Then we restrict ourselves to these good hub locations and solve a problem with binary x_{ik}. The values of x_{ik} form a valid choice of hubs and allocations and can, therefore, be uniquely extended to a solution of the original problem. Furthermore, by comparing the y and x values, we find violated inequalities that can be added to the problem. For the description of the procedure, we introduce the following notation:

Let HLP be one of the five hub location problems discussed above without the (I_{ij}^{hl}) inequalities, let R-HLP be a relaxation (e.g. the LP relaxation) of the model HLP and $\text{HLP}(\hat{H})$ be the restriction of HLP, where hubs can only be chosen from the set $\hat{H} \subset V$. We follow the following heuristic row generation procedure to generate both an upper bound and a set of starting inequalities:

1. We solve R-HLP with just a small set of (I_{ij}^{hl})-type inequalities, resulting in a solution $\{\hat{x}, \hat{y}\}$. A good choice is to start with all inequalities (I_{ij}^{hl}) for which $i = h$ and $j = l$.
2. We solve $\text{HLP}(\hat{H})$ with all (I_{ij}^{hl})-type inequalities already added to R-HLP, where $\hat{H}$ is the set of $k \in V$ with positive $\hat{x}_{kk}$. The resulting (binary) values of x are called $\overline{x}_{ik}$.
3. For every $i, j \in V$, we determine h, l with $\overline{x}_{ih} = \overline{x}_{jl} = 1$ in $\text{HLP}(\hat{H})$. If $h \neq l$, we check the inequality (I_{ij}^{hl}) on $\text{HLP}(\hat{H})$ and add it to R-HLP, if the test fails.
4. If inequalities were added, we resolve R-HLP and return to step (2.), otherwise we leave the heuristic procedure.

Every solve of $\text{HLP}(\hat{H})$ gives binary allocation variables $\overline{x}_{ik}$ which determine a primal solution that gives an upper bound for the optimal solution of HLP. Furthermore, a set of inequalities is provided.

What is a good choice for the definition of the relaxed R-HLP? For US-ApHMP and USAHLP, we chose the LP relaxation; for CSAHLP, CSAHLPM and SAHLPC we relaxed all variables except for x_{kk}, $k \in V$. These choices lead to good numerical results in Sect. 5.

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For SAHLPC we have additional inequalities to add. We made use of the very efficient Branch-and-Cut scheme of Gurobi and added the constraints (12) and (13) by so-called Lazy Constraints in the callback process. This means that each time a new incumbent is found, it is checked for the most violated inequalities. If it passes the tests, it is accepted as solution; otherwise it is removed and the constraints are added to the problem.

5. Computational Results

Computational experiments for Hub Location Problems are usually performed on the Australian Post (AP) or Civil Aeronautics Board (CAB) data set. We choose to use the AP data set because the CAB data set only consists of instances up to 25 nodes. The AP data set without hub costs and capacities can be found in [Bea12]; the full set can be provided upon request (we thank Mr. Contreras for giving us access to these data sets). For our tests, we used the Gurobi 6.0 solver on a 3.4 GHz computer with 16GB RAM, coding our procedure in $C^\sharp$.

The AP data set consists of instances from 10 nodes to 200 nodes, but we will consider only instances from 50 on because the smaller ones are no challenge for modern solvers. The instances are named 50LL, 50LT, 50TL, 50TT, 60LL, ... The first letter indicates whether loose or tight fixed costs apply, while the second letter indicates if the capacities are loose or tight. As fixed costs and capacities are not used in all of the problems, unnecessary instances are left out.

The instances for the USApHMP are written as nLL:p, where n is the number of nodes and p the number of hubs to choose. We see in Table 1 that all these instances could be solved to optimality within a time of 2 hours. The first paper which gave the optimal results for all these instances was (up to our knowledge) our own work [MCRB15]. Good lower bounds were provided by [RBMC15]. The results coincide with the heuristic results of [IUBM10] which are the best available heuristic values.

The instances for USAHLP only depend on the first L/T. These are all solved to optimality with a maximum time of 25 minutes (see Table 2). These coincide with the best known results of [BOBA13]. The optimal values up to 100 coincide with the optimal values given in [SC09]. For the comparison of the numerical values one should note that there appears to be a second version of the fixed costs for AP data sets originating from [TCEY05]. These lead to much lower numerical values for the 200 nodes instances.

The instances of CSAHLP depend on all available information. They can be solved to optimality except for 150LT, but the longest solves takes 6 hours (see Table 3). It is interesting that the gap fell always below 1 % in less than 20 minutes. [CDF11] solved all instances but 200LL to optimality with maximum computational time of one day on a 2.3 GHz computer. The best known solution of [CDF09] for 200LL turns out to be optimal. [SM15] states that the optimal solution for 200LL with 4 hubs or less would be 241992.97 which we consider to be wrong.

Table 1: AP results for USApHMP. The second and third column give the first iteration and time for reaching a gap below 1%. The following three columns give the iteration, time and added inequalities until optimality is reached. Iterations are given in the form $I(P)$ where I are the MIP iterations and P are the iterations of the initial heuristic solve. Optimal objective values are in bold face.

Name	Gap < 1%		Optimal Solution			Objective	Hubs
	Iter	sec	Iter	sec	+Ineq		
50LL:2	0 (2)	00.79	1 (5)	03.17	5008	**178484.29**	14, 35
50LL:3	0 (2)	00.95	1 (3)	02.46	4204	**158569.93**	14, 28, 35
50LL:4	0 (2)	01.09	1 (3)	03.02	4452	**143378.05**	14, 28, 33, 35
50LL:5	0 (2)	01.23	1 (3)	03.92	4448	**132366.95**	4, 14, 28, 33, 35
60LL:2	0 (2)	01.48	0 (4)	02.36	7172	**179920.21**	16, 42
60LL:3	0 (2)	01.82	1 (5)	08.35	7644	**160338.58**	16, 33, 42
60LL:4	0 (2)	02.35	1 (3)	08.17	6302	**144719.69**	16, 33, 39, 42
60LL:5	0 (2)	02.72	1 (4)	13.62	10562	**132850.29**	4, 17, 34, 39, 42
70LL:2	0 (2)	02.03	0 (3)	02.48	7286	**180093.20**	20, 49
70LL:3	0 (2)	02.57	0 (3)	03.41	8222	**160933.23**	20, 38, 49
70LL:4	0 (2)	02.62	1 (4)	15.76	10444	**145619.65**	20, 38, 46, 49
70LL:5	0 (2)	03.66	1 (6)	55.46	15634	**135835.20**	5, 21, 39, 46, 49
75LL:2	0 (2)	02.44	0 (3)	02.98	8360	**180118.91**	21, 52
75LL:3	0 (2)	03.24	1 (3)	08.67	9438	**161056.74**	21, 40, 52
75LL:4	0 (2)	03.65	1 (3)	17.83	11828	**145734.21**	21, 40, 49, 52
75LL:5	0 (2)	04.82	1 (6)	90.85	17942	**136011.35**	5, 22, 42, 49, 52
90LL:2	0 (2)	04.70	0 (4)	07.06	15942	**179821.64**	24, 63
90LL:3	0 (2)	05.88	1 (5)	29.29	20554	**160437.43**	24, 49, 62
90LL:4	0 (2)	10.51	1 (5)	66.21	20318	**145133.69**	24, 49, 58, 62
90LL:5	0 (2)	08.33	1 (7)	181.40	33828	**135808.25**	5, 24, 49, 58, 62
100LL:2	0 (2)	06.50	0 (4)	09.70	19676	**180223.80**	28, 71
100LL:3	0 (2)	08.67	1 (5)	38.67	24748	**160847.00**	28, 55, 70
100LL:4	0 (2)	10.34	1 (5)	76.72	25568	**145896.58**	28, 55, 64, 70
100LL:5	0 (2)	12.63	2 (5)	363.38	37009	**136929.44**	7, 28, 55, 64, 70
125LL:2	0 (2)	14.18	0 (4)	20.33	30764	**180372.19**	33, 86
125LL:3	0 (2)	18.11	1 (5)	76.46	33792	**161117.17**	33, 65, 85
125LL:4	0 (2)	20.05	1 (5)	211.45	39974	**146173.22**	33, 65, 79, 85
125LL:5	0 (2)	32.17	1 (7)	1153.59	69062	**137175.68**	7, 35, 67, 79, 85
150LL:2	0 (2)	27.47	0 (5)	42.08	44792	**180898.84**	38, 100
150LL:3	0 (2)	41.11	1 (4)	157.95	48646	**161490.48**	38, 80, 100
150LL:4	0 (2)	69.19	1 (5)	477.64	56110	**146521.33**	38, 80, 94, 100
150LL:5	0 (2)	39.37	1 (7)	1082.77	73794	**137425.91**	7, 40, 82, 94, 100
175LL:2	0 (2)	50.35	0 (4)	69.52	60752	**182120.64**	46, 120
175LL:3	0 (2)	86.77	1 (5)	564.02	84986	**162553.71**	43, 92, 120
175LL:4	0 (2)	183.00	1 (7)	1485.20	96910	**147316.45**	46, 95, 108, 120
175LL:5	0 (2)	221.40	1 (7)	3697.66	109362	**139354.52**	9, 51, 98, 108, 120
200LL:2	0 (2)	94.49	0 (4)	129.48	79578	**182459.25**	56, 140
200LL:3	0 (2)	141.51	1 (5)	1005.52	110018	**162887.03**	53, 107, 140
200LL:4	0 (2)	131.86	1 (5)	1644.09	121808	**147767.30**	56, 110, 131, 140
200LL:5	0 (3)	629.78	1 (6)	5144.54	157792	**140062.65**	14, 61, 113, 131, 141

Table 2: AP results for USAHLP. Description as in Table 1.

	Gap < 1%		Optimal Solution				
Name	Iter	sec	Iter	sec	+Ineq	Objective	Hubs
50LT	0 (2)	00.88	0 (3)	01.06	3706	**237421.99**	15, 36
50TT	0 (2)	00.94	1 (2)	02.28	3690	**300420.99**	24
60LT	0 (2)	01.42	1 (6)	07.13	7387	**228855.08**	18, 42
60TT	0 (2)	01.53	1 (4)	05.23	5604	**264742.11**	27, 43
70LT	0 (2)	02.23	0 (4)	03.15	7746	**226188.20**	21, 49
70TT	0 (1)	01.15	0 (1)	01.15	4830	**261294.99**	48
75LT	0 (2)	02.60	1 (4)	12.29	11242	**235847.50**	23, 52
75TT	0 (1)	01.48	0 (1)	01.48	5550	**288778.29**	53
90LT	0 (2)	04.58	0 (5)	07.32	12750	**225475.48**	27, 62
90TT	0 (1)	02.49	0 (4)	06.12	12310	**257415.86**	27, 61
100LT	0 (2)	05.79	0 (3)	06.91	14732	**238016.28**	29, 73
100TT	0 (1)	03.24	0 (1)	03.24	9900	**305097.95**	52
125LT	0 (2)	13.14	1 (4)	43.30	23782	**227949.00**	32, 86
125TT	0 (2)	11.53	0 (4)	16.08	23016	**258839.68**	11, 88
150LT	0 (2)	25.79	1 (5)	107.24	34880	**225450.10**	40, 101
150TT	0 (2)	18.38	0 (4)	26.30	35030	**234778.74**	42, 134
175LT	0 (2)	44.15	1 (4)	188.17	46215	**227655.38**	46, 120
175TT	0 (2)	32.62	0 (4)	44.55	47080	**247876.80**	61, 121
200LT	0 (2)	49.51	0 (4)	68.18	60604	**233802.98**	43, 148
200TT	0 (3)	220.35	1 (6)	1399.53	102980	**272188.11**	54, 122

The instances for CSAHLPM require additional information. We followed the pattern described in [CNSdG10] to generate hub levels: The maximal hub level equals in size and cost the values of CSAHLP. Additional levels are produced recursively:

$$\Gamma_k^q = 0.7 \cdot \Gamma_k^{q+1} \tag{20}$$

$$F_k^q = \rho \cdot 0.7 \cdot F_k^{q+1} \tag{21}$$

Here, ρ is a factor to model economies of scale in hub building. We consider instances with 3 and 5 levels and the ρ values 1.1 and 1.2. [CNSdG10] solved these instances up to 50 nodes mostly to optimality, but gave no objective values. They used only instances with tight capacity constraints. We present optimal results up to 100 nodes in Table 4 and very good results (gap < 2%) in Table 5. We see that the 5-1.1 instances are often difficult and lead to a high number of chosen hubs.

For SAHLPC we used the AP data set together with the values $a = 0.005$ and $b = 2$ that were also used by [DCM12]. Although [DCM12] used the AP data set, they created their own capacities and fixed costs for which we did not find a source. Hence our results cannot be directly compared to theirs. Nevertheless, as Table 6 shows, we can solve all instances to near optimal results.

For every solution, we give the chosen hubs and the objective value to enable anyone to check the results for correctness.

Table 3: AP results for CSAHLP. Description as in Table 1. The time limit was set to 12 hours.

| Name | Gap < 1% | | Optimal Solution | | | | |
	Iter	sec	Iter	sec	+Ineq	Objective	Hubs
50LL	0 (2)	01.15	0 (3)	01.74	3762	**238520.59**	15, 35
50TL	0 (1)	01.33	1 (3)	07.18	3552	**319015.77**	3, 24
50LT	0 (2)	06.11	1 (3)	13.78	4526	**272897.49**	6, 26, 32, 46
50TT	1 (11)	302.84	3 (11)	656.16	8492	**417440.99**	6, 26, 28, 48
60LL	0 (2)	02.03	0 (3)	03.08	5488	**225917.21**	17, 42
60TL	0 (2)	01.97	0 (3)	03.05	5616	**252496.66**	15, 55
60LT	0 (2)	10.21	1 (3)	27.20	6878	**253616.51**	6, 19, 40, 42
60TT	0 (3)	19.47	1 (7)	95.90	10254	**351203.17**	17, 26, 40
70LL	0 (2)	07.52	1 (7)	42.01	7940	**236817.36**	20, 51
70TL	0 (2)	03.44	0 (5)	10.36	10000	**271283.82**	20, 66
70LT	0 (2)	14.46	1 (8)	187.28	13384	**256920.45**	8, 36, 47, 49
70TT	0 (2)	19.22	1 (6)	97.35	9436	**387380.20**	25, 48, 65, 68
75LL	0 (1)	01.73	0 (3)	06.63	8340	**238024.22**	21, 51
75TL	0 (2)	06.10	0 (4)	21.60	12082	**303363.55**	22, 52
75LT	0 (2)	17.66	1 (4)	59.98	9438	**256188.12**	22, 52, 53
75TT	0 (2)	22.70	1 (6)	125.15	13498	**347189.82**	27, 33, 53, 74
90LL	0 (2)	06.81	0 (4)	14.32	12472	**224195.72**	23, 63
90TL	0 (2)	19.97	1 (5)	89.98	14846	**281561.56**	28, 68, 77
90LT	0 (2)	41.83	1 (5)	191.79	14570	**246026.24**	28, 62, 65
90TT	0 (2)	31.99	1 (4)	102.19	15644	**337008.93**	27, 51, 59, 61
100LL	0 (2)	42.13	1 (4)	176.31	20878	**246713.97**	29, 64, 73
100TL	0 (3)	65.80	1 (6)	174.03	16558	**362950.09**	44, 52
100LT	0 (2)	20.61	1 (6)	141.23	17722	**256155.33**	29, 68, 76
100TT	0 (3)	187.16	1 (10)	1122.14	21672	**474068.96**	5, 34, 86, 95
125LL	0 (2)	66.51	1 (5)	415.41	33253	**239889.33**	33, 81, 90
125TL	0 (2)	19.44	0 (5)	53.24	24516	**246486.69**	29, 112
125LT	0 (2)	97.57	1 (4)	279.53	27798	**251259.16**	35, 79, 85, 87
125TT	0 (2)	35.58	1 (4)	109.19	28154	**291807.35**	11, 44, 77, 88
150LL	0 (2)	49.14	0 (4)	153.85	44720	**234765.44**	38, 100
150TL	0 (2)	152.49	1 (7)	1419.80	61060	**262543.08**	37, 134
150LT	0 (2)	225.71	–		249797.493 at gap 00.45 %		37, 73, 95, 104
150TT	1 (5)	1565.99	2 (5)	3243.56	49638	**322976.47**	42, 65, 77, 134, 144
175LL	0 (2)	120.93	1 (8)	1010.55	61660	**227997.58**	44, 121
175TL	0 (2)	66.41	0 (4)	152.07	45902	**244860.41**	43, 129
175LT	0 (2)	423.24	1 (4)	2196.45	72956	**251540.80**	46, 109, 122, 166
175TT	1 (6)	2780.41	4 (6)	17634.54	69007	**312195.54**	57, 90, 132, 151
200LL	0 (2)	302.39	1 (3)	725.40	68358	**231069.50**	53, 126, 184
200TL	0 (2)	269.46	1 (4)	902.96	69250	**273443.81**	54, 95, 186
200LT	0 (2)	1355.87	3 (3)	7460.18	94585	**267218.35**	41, 124, 148, 168
200TT	0 (2)	510.99	1 (9)	4117.31	75950	**290582.04**	54, 113, 168, 186

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Table 4: AP results for CSAHLPM from 50 to 100. The instance names indicate the number of hub levels and the "economies of scale"-factor. Description as in Table 1. Time limit was set to 12 hours.

	Gap < 1%		Optimal Solution				
Name	Iter	sec	Iter	sec	+Ineq	Objective	Hubs
50LT:3-1.2	1 (4)	17.10	1 (4)	17.10	4924	**261943.51**	6, 26, 32, 46
50LT:3-1.1	1 (5)	36.39	5 (5)	190.93	7172	**255649.68**	14, 19, 26, 32, 46
50LT:5-1.2	1 (6)	49.62	1 (6)	49.62	6000	**261943.51**	6, 26, 32, 46
50LT:5-1.1	1 (5)	74.26	1 (5)	74.26	5792	**247930.65**	4, 9, 12, 14, 26, 32, 46
50TT:3-1.2	1 (4)	40.36	2 (4)	62.06	5685	**408522.79**	6, 12, 26, 48
50TT:3-1.1	1 (8)	300.40	1 (8)	300.40	6664	**401829.91**	6, 12, 26, 48
50TT:5-1.2	1 (4)	54.13	2 (4)	82.03	5685	**408522.79**	6, 12, 26, 48
50TT:5-1.1	1 (4)	101.51	1 (4)	101.51	5322	**396053.56**	6, 12, 26, 30, 41, 48
60LT:3-1.2	1 (4)	66.58	2 (4)	120.16	9096	**249636.90**	6, 19, 40, 42
60LT:3-1.1	1 (5)	88.78	2 (5)	140.18	9948	**239622.84**	6, 15, 19, 32, 40, 42
60LT:5-1.2	1 (4)	100.44	2 (4)	158.62	8699	**249636.90**	6, 19, 40, 42
60LT:5-1.1	1 (5)	285.32	2 (5)	487.60	9412	**234745.62**	6, 8, 17, 32, 37, 40, 42, 59
60TT:3-1.2	1 (6)	39.02	1 (6)	39.02	7494	**345332.23**	17, 26, 40, 47
60TT:3-1.1	1 (8)	63.32	1 (8)	63.32	8172	**336066.27**	17, 26, 40, 47
60TT:5-1.2	1 (4)	28.02	1 (4)	28.02	7180	**344672.22**	17, 26, 40, 47
60TT:5-1.1	1 (4)	41.94	1 (4)	41.94	7116	**333884.15**	9, 17, 26, 40, 47
70LT:3-1.2	1 (7)	145.93	4 (7)	331.37	11627	**247895.33**	8, 21, 39, 47, 64
70LT:3-1.1	1 (6)	340.08	2 (6)	460.64	13216	**238756.23**	8, 21, 39, 47, 64
70LT:5-1.2	1 (5)	213.39	1 (5)	213.39	9688	**245394.16**	8, 21, 39, 47, 64
70LT:5-1.1	1 (3)	224.41	2 (3)	1325.08	11031	**229785.52**	8, 11, 16, 21, 36, 39, 47, 49, 64
70TT:3-1.2	1 (6)	103.42	1 (6)	103.42	10328	**384424.12**	25, 48, 65, 68
70TT:3-1.1	1 (6)	113.09	1 (6)	113.09	10328	**382244.31**	25, 48, 65, 68
70TT:5-1.2	1 (6)	128.84	1 (6)	128.84	10328	**384424.12**	25, 48, 65, 68
70TT:5-1.1	1 (8)	266.89	1 (8)	266.89	11440	**381193.56**	25, 39, 48, 65, 68
90LT:3-1.2	0 (3)	102.66	2 (6)	675.49	28340	**243423.99**	27, 38, 62, 65
90LT:3-1.1	1 (3)	225.93	3 (3)	1634.64	17959	**235536.76**	23, 33, 38, 62, 65
90LT:5-1.2	1 (4)	302.90	2 (4)	442.85	17723	**241166.92**	27, 33, 38, 62, 65
90LT:5-1.1	1 (7)	1640.76	2 (7)	2512.28	27223	**227053.10**	9, 12, 28, 38, 62, 65, 70, 76
90TT:3-1.2	0 (2)	30.61	1 (4)	104.93	15644	**337008.93**	27, 51, 59, 61
90TT:3-1.1	0 (2)	42.47	1 (4)	149.70	15644	**337008.93**	27, 51, 59, 61
90TT:5-1.2	0 (2)	36.51	1 (4)	131.99	15644	**337008.93**	27, 51, 59, 61
90TT:5-1.1	1 (5)	324.80	1 (5)	324.80	17148	**337008.93**	27, 51, 59, 61
100LT:3-1.2	1 (5)	496.96	4 (5)	4938.89	25644	**254902.25**	29, 44, 68, 76
100LT:3-1.1	1 (6)	1196.20	3 (6)	4972.68	32144	**247076.09**	29, 34, 64, 76, 96
100LT:5-1.2	1 (6)	612.20	5 (6)	3427.21	26682	**253454.89**	29, 44, 68, 76
100LT:5-1.1	1 (11)	5477.62	3 (11)	19542.61	46874	**239578.15**	6, 18, 24, 29, 34, 58, 68, 76, 86, 95
100TT:3-1.2	1 (9)	1403.85	6 (9)	2926.34	29764	**473974.23**	5, 34, 41, 52, 95
100TT:3-1.1	0 (4)	488.84	1 (7)	1107.66	25318	**467581.96**	5, 34, 52, 86, 95
100TT:5-1.2	1 (9)	1803.82	4 (9)	3070.19	31310	**473974.23**	5, 34, 41, 52, 95
100TT:5-1.1	1 (10)	5789.73	–		467516.418 at gap 00.03 %		5, 34, 41, 52, 86, 95

Table 5: AP results for CSAHLPM from 60. The instance names indicate the number of hub levels and the "economies of scale"-factor. The time limit was set to 12 hours. Description as in Table 1.

	Gap < 1%		Optimal Solution				
Name	Iter	sec	Iter	sec	+Ineq	Objective	Hubs
125LT:3-1.2	1 (5)	492.50	3 (5)	897.74	33670	**243150.50**	6, 42, 78, 85, 87
125LT:3-1.1	0 (2)	119.27	1 (4)	549.63	29896	**236468.56**	6, 42, 78, 85, 87
125LT:5-1.2	1 (4)	713.52	2 (4)	1260.67	36749	**242098.71**	6, 35, 66, 79, 85, 87
125LT:5-1.1	1 (5)	1984.46	3 (5)	4179.90	47270	**228920.56**	6, 35, 42, 66, 79, 85, 87
125TT:3-1.2	1 (5)	277.63	1 (5)	277.63	28890	**285172.38**	11, 44, 77, 88
125TT:3-1.1	1 (5)	315.18	1 (5)	315.18	28890	**281586.48**	11, 44, 77, 88
125TT:5-1.2	1 (5)	342.68	1 (5)	342.68	28890	**285172.38**	11, 44, 77, 88
125TT:5-1.1	1 (6)	694.94	1 (6)	694.94	34532	**280740.72**	11, 44, 77, 88, 124
150LT:3-1.2	0 (2)	219.69	1 (4)	1009.42	41098	**244797.83**	37, 81, 95, 101, 104
150LT:3-1.1	1 (7)	2254.43	4 (7)	6198.60	59400	**238401.09**	7, 40, 65, 81, 97, 101, 104
150LT:5-1.2	1 (4)	3153.25	2 (4)	5223.05	51150	**243153.26**	7, 40, 81, 95, 101, 104
150LT:5-1.1	1 (8)	14205.79	–		229457.392 at gap 00.16 %		7, 17, 40, 65, 73, 85, 92, 95, 101, 104
150TT:3-1.2	1 (6)	1928.99	3 (6)	2926.03	61432	**318225.33**	18, 65, 77, 134, 144
150TT:3-1.1	1 (6)	2622.18	4 (6)	6631.99	61266	**308114.99**	11, 65, 68, 77, 134
150TT:5-1.2	1 (7)	1923.89	2 (7)	2911.17	49314	**316605.53**	18, 65, 77, 134, 144
150TT:5-1.1	1 (6)	3304.52	1 (6)	3304.52	53052	**305783.14**	11, 18, 65, 68, 77, 134
175LT:3-1.2	1 (4)	6478.97	3 (4)	12441.05	75916	**246271.40**	17, 59, 109, 122, 166
175LT:3-1.1	1 (6)	6256.02	4 (6)	26660.46	91178	**238413.43**	17, 51, 98, 109, 121, 166
175LT:5-1.2	1 (6)	13516.10	2 (6)	20104.35	99319	**244237.79**	46, 98, 109, 122, 166
175LT:5-1.1	1 (5)	8192.18	3 (5)	28034.20	80240	**228329.11**	17, 40, 60, 80, 98, 109, 121, 166
175TT:3-1.2	1 (7)	5305.44	3 (7)	14703.41	87183	**312193.78**	57, 90, 132, 151
175TT:3-1.1	1 (7)	5620.08	3 (7)	9094.58	74887	**305710.76**	57, 90, 110, 132, 163
175TT:5-1.2	2 (8)	7778.71	6 (8)	22552.71	83262	**311726.73**	87, 101, 132, 163
175TT:5-1.1	1 (5)	9982.46	–		303477.374 at gap 00.08 %		57, 90, 101, 110, 132, 163
200LT:3-1.2	1 (5)	23076.42	–		259808.254 at gap 00.01 %		41, 96, 139, 171
200LT:3-1.1	–	–	–		252557.711 at gap 01.67 %		41, 96, 133, 168, 171
200LT:5-1.2	1 (4)	16184.00	2 (4)	32963.46	103473	**258691.91**	41, 47, 124, 168, 171
200LT:5-1.1	–	–	–		242239.9 at gap 01.45 %		13, 41, 47, 80, 97, 119, 141, 168, 170
200TT:3-1.2	1 (3)	1719.15	4 (3)	5730.80	101825	**288598.79**	21, 57, 113, 168, 186
200TT:3-1.1	1 (3)	2486.55	4 (3)	8577.22	91980	**282236.13**	21, 57, 113, 168, 186
200TT:5-1.2	1 (3)	1758.20	2 (3)	3100.92	96891	**286165.23**	47, 54, 113, 168, 186
200TT:5-1.1	1 (4)	1781.22	1 (4)	1781.22	77980	**275300.99**	15, 47, 53, 57, 113, 168, 186

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Table 6: AP results for SAHLPC. The instance names indicate the number of hub levels and the "economies of scale"-factor. Description as in Table 1. The time restriction was 12 hours.

Name	Gap < 1%		Optimal Solution			Objective	Hubs
	Iter	sec	Iter	sec	+Ineq		
50LL	0 (2)	02.36	1 (4)	06.21	3762	**238775.94**	15, 35
50TL	0 (1)	01.23	0 (1)	01.23	2450	**303509.52**	24
50LT	1 (3)	163.31	3 (3)	594.02	4407	**269698.61**	6, 26, 35
50TT	0 (1)	03.42	0 (1)	03.42	2450	**337821.04**	26
60LL	0 (2)	01.94	0 (3)	02.94	5488	**225917.21**	17, 42
60TL	0 (2)	05.13	2 (4)	29.40	5678	**254522.61**	15, 55
60LT	0 (3)	48.18	5 (3)	1602.95	7072	**254717.52**	19, 42
60TT	1 (4)	545.75	5 (4)	2892.56	7007	**319766.46**	17, 27, 43
70LL	0 (2)	17.53	2 (4)	93.62	10277	**237876.34**	20, 51
70TL	0 (2)	08.11	3 (4)	301.79	7444	**270918.72**	20, 64
70LT	1 (6)	516.31	3 (6)	1143.27	12633	**252066.38**	8, 39, 44, 49
70TT	0 (1)	06.05	0 (1)	06.05	4830	**309297.71**	48
75LL	0 (1)	05.31	0 (3)	19.61	8340	**238591.99**	21, 51
75TL	2 (3)	59.48	2 (3)	59.48	9719	**298762.22**	22, 48, 75
75LT	0 (3)	155.33	3 (5)	4495.73	12559	**259452.49**	22, 52
75TT	1 (1)	97.78	3 (1)	475.89	8248	**324160.40**	53
90LL	0 (2)	07.56	0 (4)	14.95	12472	**224195.72**	23, 63
90TL	1 (3)	97.96	3 (3)	200.57	13737	**280434.90**	28, 68, 77
90LT	1 (5)	1789.33	3 (5)	10729.63	20148	**249172.45**	27, 62, 66
90TT	1 (4)	434.57	2 (4)	822.20	12550	**303637.04**	27, 61
100LL	0 (2)	52.47	3 (4)	537.49	19719	**245852.11**	29, 73
100TL	0 (1)	11.85	0 (1)	11.85	9900	**322358.68**	52
100LT	1 (4)	1153.89	4 (4)	3709.45	17583	**258573.18**	29, 68, 76
100TT	0 (1)	18.74	2 (1)	116.21	11310	**368396.44**	52
125LL	0 (2)	71.66	2 (5)	662.53	33704	**240274.60**	37, 86
125TL	0 (2)	21.14	3 (5)	262.46	24666	**246666.54**	29, 112
125LT	1 (4)	6472.69	2 (4)	9684.61	36282	**251680.17**	6, 42, 79, 85
125TT	0 (2)	78.08	2 (4)	395.27	26726	**279587.88**	11, 44, 88
150LL	0 (2)	153.44	0 (5)	554.35	45386	**235681.93**	42, 100
150TL	0 (2)	105.47	2 (3)	558.22	34915	**262139.83**	37, 134
150LT	1 (3)	7755.27	–			248698.804 at gap 00.02 %	37, 81, 95, 104
150TT	–	–	–			278438.681 at gap 01.01 %	42, 65, 134
175LL	0 (2)	276.67	0 (5)	960.38	75855	**230793.01**	53, 121
175TL	0 (2)	151.86	1 (5)	732.84	61212	**246090.20**	43, 129
175LT	1 (2)	7257.89	3 (2)	36225.68	57780	**247953.13**	17, 59, 109, 166
175TT	0 (3)	777.38	–			289236.988 at gap 00.98 %	57, 121
200LL	1 (4)	2512.25	4 (4)	12189.69	68804	**234850.34**	53, 126, 184
200TL	0 (2)	563.41	2 (4)	3373.61	70032	**275253.98**	54, 95, 186
200LT	–	–	–			271491.046 at gap 03.98 %	57, 148
200TT	0 (3)	2919.58	–			296214.827 at gap 00.98 %	54, 113, 186

6. Conclusion

The numerical experiments show that the Euclidean projection method can solve even large instances of different Single Allocation Hub Location Problems to optimality or near optimality. Furthermore, it quickly arrives at low gaps and can, therefore, be applied to practical problems. The next aim is to adapt it for problems with additional, complicating constraints, like stochastic or robust versions of the problems above.

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