

Perspective Envelopes for Bilinear Functions

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Abstract We characterize the convex hull of the set

$$\mathcal{S} \subset \mathbb{R}^3 = \{(x, y, z) \in [\mathbf{x}^l, \mathbf{x}^u] \times [\mathbf{y}^l, \mathbf{y}^u] \times \mathbb{R} \mid x \leq y, z = xy\}.$$

The new characterization, based on perspective functions, dominates the standard McCormick convexification approach. In practice, this result is useful in the presence of linear constraints linking variables x and y , but can also be of great value in global optimization frameworks, suggesting a branching strategy based on dominance, i.e., $x \leq y \vee x \geq y$. The new relaxation yields tight lower bounds, and has the potential to improve the pruning process in spatial branch and bound schemes and consequently reduce the search space effort.

Keywords Bilinear Programming · Convex Relaxation · Perspective Function · McCormick Envelopes · Global Optimization

1 Introduction

Bilinear expressions are the most common nonconvex components in mathematical formulations modeling problems arising in chemical engineering [20, 19, 26, 8, 38, 12, 37, 10], pooling and blending [13, 5], supply chain and transportation science [16, 35, 34, 36], and energy systems [22, 15, 23, 11], to name

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a few. Many convexification approaches [27, 40, 31, 43] are based on the McCormick envelopes [32, 2], these include bound contraction [9], and piecewise McCormick envelopes [10]. Global optimization solvers [6, 1, 39, 33] also heavily rely on these envelopes. Given the set $\mathcal{S}_0 = \{(x, y, z) \in (\mathcal{B} \times \mathbb{R}) \mid z = xy\}$, where $\mathcal{B} = \{[x^l, x^u] \times [y^l, y^u]\}$, Al-Khayyal and Falk [2] were able to define its convex hull in the space of original variables,

$$\text{conv}(\mathcal{S}_0) = \left\{ (x, y, z) \in ([x^l, x^u] \times [y^l, y^u] \times \mathbb{R}) \mid \begin{cases} z \geq x^l y + y^l x - x^l y^l \\ z \geq x^u y + y^u x - x^u y^u \\ z \leq x^l y + y^u x - x^l y^u \\ z \leq x^u y + y^l x - x^u y^l \end{cases} \right\}$$

This set of constraints, known as the McCormick envelopes, defines the convex and the concave envelopes of the bilinear function $f(x, y) = xy$ on the rectangular domain $[x^l, x^u] \times [y^l, y^u]$. The quality of this polyhedral relaxation highly depends on the initial bounds on variables x and y . State-of-the-art global optimization solvers implement bound contraction techniques in order to improve this bounding procedure. Once bound propagation is completed, domain partitioning becomes necessary. Spatial branch and bound schemes [3, 42] are among the most effective partitioning methods in global optimization. By splitting the domain of a given variable, the solver is able to divide the original domain into two smaller regions, further tightening the convex relaxations of each partition. In general, the variables involved in bilinear expressions are also linked through other constraints in the problem formulation. It is thus possible to tighten the convex relaxation of the feasible region by combining the bilinear term with other constraints.

Related Work. In [41], the convex hull of the bilinear function over D-polytopes is derived in the space of original variables. Thereafter, Linderoth [28] produces analytical characterizations on triangular domains. Concurrently, Benson [7] derives the convex hull on parallelograms and trapezoids. More recently, Anstreicher and Burer [4] study higher dimension characterizations. Locatelli and Schoen [30] propose a different approach for computing convex envelopes, based on solving convex programs, Locatelli [29] then uses this result to derive closed-form solutions for specific domains.

In this work, we consider the bilinear term xy in conjunction with a dominance constraint linking variables x and y , i.e., $x \leq y$, and leading to polyhedral domains subsuming right triangles, right trapezoids, and rectangles. To the best of our knowledge, there are no analytical characterizations of this convex hull in the space of original variables. The proof, based on perspective functions, offers a new angle on deriving such convex hulls, and can be easily extended to handle arbitrary linear constraints. The main result is presented in the next section.

2 The New Convex Envelope

2.1 Background on perspective functions

Perspective formulations have been successfully used to model disjunctive constraints in Mixed-Integer Nonlinear Programming (MINLP) [14, 21, 17], dominating standard big-M formulations. Given a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and a real number $u > 0$, the function,

$$f_u(\mathbf{x}) = uf(\mathbf{x}/u)$$

is convex, and represents a *dilated* version of f [24]. The perspective of f , denoted $\tilde{f} : (\mathbb{R}^n \times \mathbb{R}) \rightarrow (\mathbb{R} \cup \{+\infty\})$, is defined as the operator considering all dilations of f , i.e.,

$$\tilde{f}(\mathbf{x}, u) = \begin{cases} uf(\mathbf{x}/u) & \text{if } u > 0 \\ +\infty & \text{otherwise.} \end{cases} \quad (1)$$

Figure 1 illustrates the dilation property of the perspective operator on the square function.

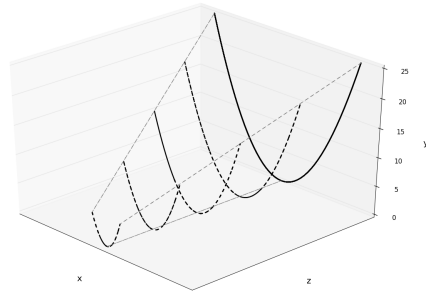


Fig. 1 Several dilations of the square function using the perspective operator.

2.2 Notations

\mathbf{x} denotes a vector variable in \mathbb{R}^n , x a variable in \mathbb{R} , and \mathbf{x} a constant in \mathbb{R} . Given a convex domain $\mathcal{D} \subseteq \mathbb{R}^n$, the epigraph of a continuous function f over \mathcal{D} , denoted $\text{epi}_{\mathcal{D}} f$, is defined as,

$$\text{epi}_{\mathcal{D}} f = \{(\mathbf{x}, z) \in \mathcal{D} \times \mathbb{R} \mid f(\mathbf{x}) \leq z\}.$$

The *convex envelope* of f over \mathcal{D} , denoted $\overline{\text{conv}}_{\mathcal{D}}(f)$, represents the convex hull of its epigraph,

$$\overline{\text{conv}}_{\mathcal{D}}(f) = \text{conv}(\{(\mathbf{x}, z) \in \mathcal{D} \times \mathbb{R} \mid f(\mathbf{x}) \leq z\}).$$

The hypograph of f over \mathcal{D} , denoted $\text{hypo}_{\mathcal{D}} f$, is defined as,

$$\text{hypo}_{\mathcal{D}} f = \{(\mathbf{x}, z) \in \mathcal{D} \times \mathbb{R} \mid f(\mathbf{x}) \geq z\}.$$

The *concave envelope* of f over \mathcal{D} , denoted $\widehat{\text{conv}}_{\mathcal{D}}(f)$, represents the convex hull of its hypograph,

$$\widehat{\text{conv}}_{\mathcal{D}}(f) = \text{conv}(\{(\mathbf{x}, z) \in \mathcal{D} \times \mathbb{R} \mid f(\mathbf{x}) \geq z\}).$$

The projection of a set \mathcal{S} on the vector space \mathbf{v} is denoted $\text{proj}_{\mathbf{v}} \mathcal{S}$, and its closure is written $\text{cl}(\mathcal{S})$.

2.3 The Convex Envelope

We first start by stating the following result from [21].

Lemma 1 [21] *Let $\mathcal{D}_0 \subset \mathbb{R}^n = \{\mathbf{x} \in [\mathbf{x}^{l_0}, \mathbf{x}^{u_0}]\}$, and $\mathcal{D}_1 \subset \mathbb{R}^n = \{\mathbf{x} \in [\mathbf{x}^{l_1}, \mathbf{x}^{u_1}]\}$
 $\Gamma_0 = \{(\mathbf{x}, u) \in \mathcal{D}_0 \times [0, 1] \mid u = 0\}$, and
 $\Gamma_1 = \{(\mathbf{x}, u) \in \mathcal{D}_1 \times [0, 1] \mid f(\mathbf{x}) \leq 0, u = 1\}$,*

$$\text{then } \text{conv}(\Gamma_0 \cup \Gamma_1) = \text{proj}_{(\mathbf{x}, u)} \text{cl}(\Gamma^c),$$

$$\text{where } \Gamma^c = \left\{ (\mathbf{x}, \mathbf{y}, u) \in \mathbb{R}^{2n} \times [0, 1] \left| \begin{array}{l} uf(\mathbf{y}/u) \leq 0 \\ \mathbf{x} - (1-u)\mathbf{x}^{u_0} \leq \mathbf{y} \leq \mathbf{x} - (1-u)\mathbf{x}^{l_0} \\ u\mathbf{x}^{l_1} \leq \mathbf{y} \leq u\mathbf{x}^{u_1}, 0 < u \leq 1 \end{array} \right. \right\}$$

Given a point $\mathbf{x}^* \in \mathbb{R}^n$, and a convex set, based on Lemma 1, we characterize the convex hull of their union.

Corollary 1 *Let $\mathcal{D} \subset \mathbb{R}^n = \{\mathbf{x} \in [\mathbf{x}^l, \mathbf{x}^u]\}$,
 $\Gamma_0 = \{(\mathbf{x}, u) \in \mathbb{R}^n \times [0, 1] \mid \mathbf{x} = \mathbf{x}^*, u = 0\}$, and
 $\Gamma_1 = \{(\mathbf{x}, u) \in \mathcal{D} \times [0, 1] \mid f(\mathbf{x}) \leq 0, u = 1\}$,*

$$\text{then } \text{conv}(\Gamma_0 \cup \Gamma_1) = \text{cl}(\Gamma^c),$$

$$\text{where } \Gamma^c = \left\{ (\mathbf{x}, u) \in \mathbb{R}^n \times [0, 1] \left| \begin{array}{l} uf((\mathbf{x} - (1-u)\mathbf{x}^*)/u) \leq 0 \\ u(\mathbf{x}^l - \mathbf{x}^*) + \mathbf{x}^* \leq \mathbf{x} \leq u(\mathbf{x}^u - \mathbf{x}^*) + \mathbf{x}^* \\ 0 < u \leq 1 \end{array} \right. \right\}$$

Proof Set $\mathbf{x}^{u_0} = \mathbf{x}^{l_0} = \mathbf{x}^*$ in Lemma 1. □

We now apply this result for the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, z) = x^2 - z$.

Lemma 2 Let $\mathcal{D} \subset \mathbb{R}^2 = \{(x, z) \in [\mathbf{x}^l, \mathbf{x}^u] \times [\mathbf{z}^l, \mathbf{z}^u]\}$,
 $\Gamma_0 = \{(x, z, u) \in \mathcal{D} \times [0, 1] \mid x = \mathbf{x}^*, z = \mathbf{z}^*, u = 0\}$, and
 $\Gamma_1 = \{(x, z, u) \in \mathcal{D} \times [0, 1] \mid z \geq x^2, u = 1\}$,

then $\text{conv}(\Gamma_0 \cup \Gamma_1) = \Gamma^p$, where

$$\Gamma^p = \left\{ (x, z, u) \in \mathbb{R}^2 \times [0, 1] \mid \begin{cases} (x - (1-u)\mathbf{x}^*)^2 \leq u(z - (1-u)\mathbf{z}^*) \\ u(\mathbf{x}^l - \mathbf{x}^*) + \mathbf{x}^* \leq x \leq u(\mathbf{x}^u - \mathbf{x}^*) + \mathbf{x}^* \\ u(\mathbf{z}^l - \mathbf{z}^*) + \mathbf{z}^* \leq z \leq u(\mathbf{z}^u - \mathbf{z}^*) + \mathbf{z}^* \end{cases} \right\}$$

Proof By replacing $f(\mathbf{x})$ with $f(x, z) = x^2 - z$ in Corollary 1, we have that

$$\text{conv}(\Gamma_0 \cup \Gamma_1) = \text{cl}(\Gamma^c),$$

where

$$\begin{aligned} \Gamma^c &= \left\{ (x, z, u) \in \mathbb{R} \times [0, 1] \mid \begin{cases} u \left(\left(\frac{x - (1-u)\mathbf{x}^*}{u} \right)^2 - \frac{z - (1-u)\mathbf{z}^*}{u} \right) \leq 0 \\ u(\mathbf{x}^l - \mathbf{x}^*) + \mathbf{x}^* \leq x \leq u(\mathbf{x}^u - \mathbf{x}^*) + \mathbf{x}^* \\ u(\mathbf{z}^l - \mathbf{z}^*) + \mathbf{z}^* \leq z \leq u(\mathbf{z}^u - \mathbf{z}^*) + \mathbf{z}^* \\ 0 < u \leq 1 \end{cases} \right\} \\ &= \left\{ (x, z, u) \in \mathbb{R} \times [0, 1] \mid \begin{cases} (x - (1-u)\mathbf{x}^*)^2 \leq u(z - (1-u)\mathbf{z}^*) \\ u(\mathbf{x}^l - \mathbf{x}^*) + \mathbf{x}^* \leq x \leq u(\mathbf{x}^u - \mathbf{x}^*) + \mathbf{x}^* \\ u(\mathbf{z}^l - \mathbf{z}^*) + \mathbf{z}^* \leq z \leq u(\mathbf{z}^u - \mathbf{z}^*) + \mathbf{z}^* \\ 0 < u \leq 1 \end{cases} \right\} \end{aligned}$$

Finally, observe that $\text{cl}(\Gamma^c) = \Gamma^c \cup \Gamma_0 = \Gamma^p$. □

This leads to the main lemma,

Lemma 3 Consider the bilinear function $f(x, y) = xy$ on the domain

$$\mathcal{D}^c = \left\{ (x, y) \in [\mathbf{x}^l, \mathbf{x}^u] \times [\mathbf{y}^l, \mathbf{y}^u] \mid \begin{cases} x \leq y \\ \frac{\mathbf{x}^u - \mathbf{y}^u}{\mathbf{x}^l - \mathbf{y}^u} x \leq \frac{\mathbf{x}^u - \mathbf{x}^l}{\mathbf{x}^l - \mathbf{y}^u} y + \mathbf{x}^u \\ \frac{\mathbf{y}^l - \mathbf{y}^u}{\mathbf{x}^l - \mathbf{y}^u} x \geq \frac{\mathbf{y}^l - \mathbf{x}^l}{\mathbf{x}^l - \mathbf{y}^u} y + \mathbf{y}^l \end{cases} \right\}$$

s.t. $\mathbf{x}^l < \mathbf{y}^u$,

then $\widetilde{\text{conv}}_{\mathcal{D}^c}(f) = \mathcal{P}^c$,

where

$$\mathcal{P}^c = \left\{ (x, y, z) \in \mathcal{D}^c \times \mathbb{R} \mid \left(x - \mathbf{x}^l \frac{x - y}{\delta} \right)^2 \leq \left(1 - \frac{x - y}{\delta} \right) \left(z - \mathbf{x}^l \mathbf{y}^u \frac{x - y}{\delta} \right) \right\},$$

and $\delta = (\mathbf{x}^l - \mathbf{y}^u)$.

Proof Let $u = 1 - \frac{(x-y)}{\delta}$, we have that

$$\begin{aligned} u = 0 &\Leftrightarrow x - y = \mathbf{x}^l - \mathbf{y}^u \Leftrightarrow x = \mathbf{x}^l \text{ and } y = \mathbf{y}^u \\ u = 1 &\Leftrightarrow x = y \end{aligned}$$

Since $x \leq y$ on \mathcal{D}^c , we can assume w.l.o.g. that $\mathbf{x}^l \leq \mathbf{y}^l$ and $\mathbf{x}^u \leq \mathbf{y}^u$, if $\mathbf{y}^l < \mathbf{x}^l$, we can safely update the lower bound on y , i.e., set $\mathbf{y}^l = \mathbf{x}^l$. A similar reasoning applies for the upper bound. Consequently, $\mathbf{z}^l = \min\{\mathbf{x}^l \mathbf{y}^u, \mathbf{x}^l \mathbf{y}^l\}$ and $\mathbf{z}^u = \max\{\mathbf{x}^u \mathbf{y}^u, \mathbf{x}^l \mathbf{y}^u\}$.

Let $\Gamma_0^c = \{(x, z, u) \in [\mathbf{x}^l, \mathbf{x}^u] \times [\mathbf{z}^l, \mathbf{z}^u] \times [0, 1] \mid x = \mathbf{x}^l, z = \mathbf{x}^l \mathbf{y}^u, u = 0\}$, and $\Gamma_1^c = \{(x, z, u) \in [\mathbf{y}^l, \mathbf{x}^u] \times [\mathbf{z}^l, \mathbf{z}^u] \times [0, 1] \mid x^2 \leq z, u = 1\}$.

Observe that the lower bound on x is \mathbf{y}^l for $u = 1$ ($x = y$ and $\mathbf{x}^l \leq \mathbf{y}^l$).

Based on Lemma 2, we have that $\text{conv}(\Gamma_0^c \cup \Gamma_1^c) = \Gamma^p$, where

$$\Gamma^p = \left\{ (x, y, z) \in \mathbb{R}^3 \left| \begin{array}{l} (x - (1-u)\mathbf{x}^l)^2 \leq u(z - (1-u)\mathbf{x}^l \mathbf{y}^u) \\ u(\mathbf{y}^l - \mathbf{x}^l) + \mathbf{x}^l \leq x \leq u(\mathbf{x}^u - \mathbf{x}^l) + \mathbf{x}^l \\ u(\mathbf{z}^l - \mathbf{x}^l \mathbf{y}^u) + \mathbf{x}^l \mathbf{y}^u \leq z \leq u(\mathbf{z}^u - \mathbf{x}^l \mathbf{y}^u) + \mathbf{x}^l \mathbf{y}^u \end{array} \right. \right\}$$

Using the variable substitution $u = 1 - (x - y)/\delta$, we have that

$$\begin{aligned} (x - (1-u)\mathbf{x}^l)^2 &\leq u(z - (1-u)\mathbf{x}^l \mathbf{y}^u) \\ &\Downarrow \\ \left(x - \mathbf{x}^l \frac{x-y}{\delta}\right)^2 &\leq \left(1 - \frac{x-y}{\delta}\right) \left(z - \mathbf{x}^l \mathbf{y}^u \frac{x-y}{\delta}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} u(\mathbf{x}^u - \mathbf{x}^l) + \mathbf{x}^l \geq x &\Leftrightarrow \frac{\mathbf{x}^u - \mathbf{y}^u}{\mathbf{x}^l - \mathbf{y}^u} x \leq \frac{\mathbf{x}^u - \mathbf{x}^l}{\mathbf{x}^l - \mathbf{y}^u} y + \mathbf{x}^u, \\ u(\mathbf{y}^l - \mathbf{x}^l) + \mathbf{x}^l \leq x &\Leftrightarrow \frac{\mathbf{y}^l - \mathbf{y}^u}{\mathbf{x}^l - \mathbf{y}^u} x \geq \frac{\mathbf{y}^l - \mathbf{x}^l}{\mathbf{x}^l - \mathbf{y}^u} y + \mathbf{y}^l, \\ u(\mathbf{z}^l - \mathbf{x}^l \mathbf{y}^u) + \mathbf{x}^l \mathbf{y}^u \leq z &\Leftrightarrow (\mathbf{z}^l - \mathbf{x}^l \mathbf{y}^u)(y - x) \geq \delta(z - \mathbf{z}^l), \\ u(\mathbf{z}^u - \mathbf{x}^l \mathbf{y}^u) + \mathbf{x}^l \mathbf{y}^u \geq z &\Leftrightarrow (\mathbf{z}^u - \mathbf{x}^l \mathbf{y}^u)(y - x) \leq \delta(z - \mathbf{z}^u). \end{aligned}$$

It is easy to check that the constraint $u(\mathbf{z}^l - \mathbf{x}^l \mathbf{y}^u) + \mathbf{x}^l \mathbf{y}^u \leq z$ (respectively $x \leq y$) is strictly redundant in Γ_0^c (resp. Γ_1^c), and weakly redundant in Γ_1^c

(resp. Γ_0^c), thus it is not facet defining in $\text{conv}(\Gamma_0^c \cup \Gamma_1^c)$. Therefore,

$$\Gamma_p = \left\{ (x, y, z) \in \mathcal{D}^c \times \mathbb{R} \mid \begin{array}{l} \left(x - \mathbf{x}^l \frac{x-y}{\delta} \right)^2 \leq \left(1 - \frac{x-y}{\delta} \right) \left(z - \mathbf{x}^l \mathbf{y}^u \frac{x-y}{\delta} \right) \\ (\mathbf{z}^u - \mathbf{x}^l \mathbf{y}^u)(y-x) \leq \delta(z - \mathbf{z}^u) \end{array} \right\}.$$

To complete the proof, we will split the domain \mathcal{D}^c into,

$\mathcal{D}_1^c = \{(x, y, z) \in \mathcal{D}^c \mid (\mathbf{z}^u - \mathbf{x}^l \mathbf{y}^u)(y-x) \leq \delta(z - \mathbf{z}^u)\}$, and

$\mathcal{D}_2^c = \{(x, y, z) \in \mathcal{D}^c \mid (\mathbf{z}^u - \mathbf{x}^l \mathbf{y}^u)(y-x) \geq \delta(z - \mathbf{z}^u)\}$.

Since $\mathcal{D}^c = \mathcal{D}_1^c \cup \mathcal{D}_2^c$, we have that

$$\text{conv}(\text{epi}_{\mathcal{D}^c} f) = \text{conv}(\text{epi}_{\mathcal{D}_1^c} f) \cup \text{conv}(\text{epi}_{\mathcal{D}_2^c} f)$$

We will next show that $\text{conv}(\text{epi}_{\mathcal{D}_1^c} f) = \Gamma^p$.

1. $\text{conv}(\text{epi}_{\mathcal{D}_1^c} f) \subseteq \Gamma^p$

Since Γ^p is convex, and $\text{epi}_{\mathcal{D}_1^c} f \subseteq \Gamma^p$, based on the definition of the convex hull, we have that $\text{conv}(\text{epi}_{\mathcal{D}_1^c} f) \subseteq \Gamma^p$.

2. $\Gamma^p \subseteq \text{conv}(\text{epi}_{\mathcal{D}_1^c} f)$

Since $\text{epi}_{\mathcal{D}_1^c} f = \{(x, y, z) \in \mathcal{D}_1^c \times \mathbb{R} \mid xy \leq z\}$, it is easy to check that

$\text{epi}_{\mathcal{D}_1^c} f \cap \{u = 0\} \equiv \Gamma_0^c$ and $\text{epi}_{\mathcal{D}_1^c} f \cap \{u = 1\} \equiv \Gamma_1^c$.

Since $(\text{epi}_{\mathcal{D}_1^c} f \cap \{u = 0\}) \cup (\text{epi}_{\mathcal{D}_1^c} f \cap \{u = 1\}) \subseteq \text{epi}_{\mathcal{D}_1^c} f$,

we have that $\text{conv}((\text{epi}_{\mathcal{D}_1^c} f \cap \{u = 0\}) \cup (\text{epi}_{\mathcal{D}_1^c} f \cap \{u = 1\})) \subseteq \text{conv}(\text{epi}_{\mathcal{D}_1^c} f)$,

thus $\text{conv}(\Gamma_0^c \cup \Gamma_1^c) = \Gamma^p \subseteq \text{conv}(\text{epi}_{\mathcal{D}_1^c} f)$

We consequently have

$$\text{conv}(\text{epi}_{\mathcal{D}_1^c} f) = \Gamma^p$$

Based on the definition of \mathcal{D}_2^c , the constraint $xy \leq z$ is redundant in this set, therefore $\text{conv}(\text{epi}_{\mathcal{D}_2^c} f) = \mathcal{D}_2^c$. Combining the previous results, we get that

$$\text{conv}(\text{epi}_{\mathcal{D}^c} f) = \Gamma^p \cup \mathcal{D}_2^c = \mathcal{P}^c$$

□

In order to cover the hole domain, we need to show the following results,

Lemma 4 Consider the bilinear function $f(x, y) = xy$ on the domain

$$\mathcal{D}^l = \left\{ (x, y) \in [\mathbf{x}^l, \mathbf{x}^u] \times [\mathbf{y}^l, \mathbf{y}^u] \mid \begin{array}{l} x \leq y \\ (\mathbf{x}^u - \mathbf{y}^u)x \leq (\mathbf{x}^u - \mathbf{x}^l)y + \mathbf{x}^u(\mathbf{x}^l - \mathbf{y}^u) \end{array} \right\}$$

$$\text{then } \overline{\text{conv}}_{\mathcal{D}^l}(f) = \mathcal{P}^l,$$

where

$$\mathcal{P}^l = \{(x, y, z) \in \mathcal{D}^l \times \mathbb{R} \mid z \geq \mathbf{x}^u y + \mathbf{y}^u x - \mathbf{x}^u \mathbf{y}^u\}.$$

Proof Recall that the convex hull of $\text{epi}_{\mathcal{D}^l} f$ is the smallest convex set containing it. Based on this definition, since \mathcal{P}^l is convex, and $\text{epi}_{\mathcal{D}^l} f \subset \mathcal{P}^l$, we immediately have the first inclusion result, $\text{conv}(\text{epi}_{\mathcal{D}^l} f) \subseteq \mathcal{P}^l$.

In the other direction, observe that the equation $z = \mathbf{x}^u y + \mathbf{y}^u x - \mathbf{x}^u \mathbf{y}^u$ corresponds to the plane passing through the points $(\mathbf{x}^l, \mathbf{y}^u, \mathbf{x}^l \mathbf{y}^u)$, $(\mathbf{x}^u, \mathbf{x}^u, (\mathbf{x}^u)^2)$, and $(\mathbf{x}^u, \mathbf{y}^u, \mathbf{x}^u \mathbf{y}^u)$, thus defining their convex hull. Consider the vertical half-lines passing through these points,

$$\mathcal{L}^1 = \{(x, y, z) \in \mathcal{D}^l \times \mathbb{R} \mid x = \mathbf{x}^l, y = \mathbf{y}^u, z \geq \mathbf{x}^l \mathbf{y}^u\},$$

$$\mathcal{L}^2 = \{(x, y, z) \in \mathcal{D}^l \times \mathbb{R} \mid x = \mathbf{x}^u, y = \mathbf{x}^u, z \geq (\mathbf{x}^u)^2\},$$

$$\text{and } \mathcal{L}^3 = \{(x, y, z) \in \mathcal{D}^l \times \mathbb{R} \mid x = \mathbf{x}^u, y = \mathbf{y}^u, z \geq \mathbf{x}^u \mathbf{y}^u\}.$$

It is easy to see that,

$$\text{conv}(\mathcal{L}^1 \cup \mathcal{L}^2 \cup \mathcal{L}^3) = \{(x, y, z) \in \mathcal{D}^l \times \mathbb{R} \mid z \geq \mathbf{x}^u y + \mathbf{y}^u x - \mathbf{x}^u \mathbf{y}^u\} = \mathcal{P}^l.$$

Since $\mathcal{L}^1 \cup \mathcal{L}^2 \cup \mathcal{L}^3 \subseteq \text{epi}_{\mathcal{D}^l} f$, we have that

$$\text{conv}(\mathcal{L}^1 \cup \mathcal{L}^2 \cup \mathcal{L}^3) \subseteq \text{conv}(\text{epi}_{\mathcal{D}^l} f), \text{ and consequently, } \mathcal{P}^l \subseteq \text{conv}(\text{epi}_{\mathcal{D}^l} f).$$

□

Lemma 5 Consider the bilinear function $f(x, y) = xy$ on the domain

$$\mathcal{D}^r = \left\{ (x, y) \in [\mathbf{x}^l, \mathbf{x}^u] \times [\mathbf{y}^l, \mathbf{y}^u] \mid \begin{array}{l} x \leq y \\ (\mathbf{y}^l - \mathbf{y}^u)x \geq (\mathbf{y}^l - \mathbf{x}^l)y + \mathbf{y}^l(\mathbf{x}^l - \mathbf{y}^u) \end{array} \right\}$$

$$\text{then } \overline{\text{conv}}_{\mathcal{D}^r}(f) = \mathcal{P}^r,$$

where

$$\mathcal{P}^r = \{(x, y, z) \in \mathcal{D}^r \times \mathbb{R} \mid z \geq \mathbf{x}^l y + \mathbf{y}^l x - \mathbf{x}^l \mathbf{y}^l\}.$$

Proof In the proof of Lemma 4 replace the three points $(\mathbf{x}^l, \mathbf{y}^u, \mathbf{x}^l \mathbf{y}^u)$, $(\mathbf{x}^u, \mathbf{x}^u, (\mathbf{x}^u)^2)$, and $(\mathbf{x}^u, \mathbf{y}^u, \mathbf{x}^u \mathbf{y}^u)$ by $(\mathbf{x}^l, \mathbf{y}^u, \mathbf{x}^l \mathbf{y}^u)$, $(\mathbf{y}^l, \mathbf{y}^l, (\mathbf{y}^l)^2)$, and $(\mathbf{x}^l, \mathbf{y}^l, \mathbf{x}^l \mathbf{y}^l)$.

□

We can now state our main result,

Theorem 1 Consider the bilinear function $f(x, y) = xy$ on the domain

$$\mathcal{D} \subset \mathbb{R}^2 = \{(x, y) \in [\mathbf{x}^l, \mathbf{x}^u] \times [\mathbf{y}^l, \mathbf{y}^u] \mid x \leq y\}, \text{ s.t. } \mathbf{x}^l < \mathbf{y}^u,$$

$$\text{then } \overline{\text{conv}}_{\mathcal{D}}(f) = \mathcal{P},$$

where

$$\mathcal{P} = \left\{ (x, y, z) \in \mathcal{D} \times \mathbb{R} \left| \begin{array}{l} \left(x - \mathbf{x}^l \frac{x-y}{\delta} \right)^2 \leq \left(1 - \frac{x-y}{\delta} \right) \left(z - \mathbf{x}^l \mathbf{y}^u \frac{x-y}{\delta} \right) \\ z \geq \mathbf{x}^u y + \mathbf{y}^u x - \mathbf{x}^u \mathbf{y}^u \\ z \geq \mathbf{x}^l y + \mathbf{y}^l x - \mathbf{x}^l \mathbf{y}^l \end{array} \right. \right\},$$

and $\delta = (\mathbf{x}^l - \mathbf{y}^u)$.

Proof We will split \mathcal{D} into three subdomains \mathcal{D}_1 , \mathcal{D}_2 , and \mathcal{D}_3 , such that,

$$\mathcal{D}_1 = \{ (x, y) \in \mathcal{D} \mid (\mathbf{x}^u - \mathbf{y}^u)x \geq (\mathbf{x}^u - \mathbf{x}^l)y + \mathbf{x}^u(\mathbf{x}^l - \mathbf{y}^u) \},$$

$$\mathcal{D}_2 = \left\{ (x, y) \in \mathcal{D} \left| \begin{array}{l} (\mathbf{x}^u - \mathbf{y}^u)x \leq (\mathbf{x}^u - \mathbf{x}^l)y + \mathbf{x}^u(\mathbf{x}^l - \mathbf{y}^u) \\ (\mathbf{y}^l - \mathbf{y}^u)x \leq (\mathbf{y}^l - \mathbf{x}^l)y + \mathbf{y}^l(\mathbf{x}^l - \mathbf{y}^u) \end{array} \right. \right\},$$

$$\mathcal{D}_3 = \{ (x, y) \in \mathcal{D} \mid (\mathbf{y}^l - \mathbf{y}^u)x \geq (\mathbf{y}^l - \mathbf{x}^l)y + \mathbf{y}^l(\mathbf{x}^l - \mathbf{y}^u) \}.$$

Note that $\mathcal{D}_1 \cap \mathcal{D}_2 = \{ (x, y) \in \mathcal{D} \mid (\mathbf{x}^u - \mathbf{y}^u)x = (\mathbf{x}^u - \mathbf{x}^l)y + \mathbf{x}^u(\mathbf{x}^l - \mathbf{y}^u) \}$,

$\mathcal{D}_2 \cap \mathcal{D}_3 = \{ (x, y) \in \mathcal{D} \mid (\mathbf{y}^l - \mathbf{y}^u)x = (\mathbf{y}^l - \mathbf{x}^l)y + \mathbf{y}^l(\mathbf{x}^l - \mathbf{y}^u) \}$,

and $\mathcal{D}_1 \cap \mathcal{D}_3 = \emptyset$. Since $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$, we have that

$$\text{conv}(\text{epi}_{\mathcal{D}} f) = \text{conv}(\text{epi}_{\mathcal{D}_1} f) \cup \text{conv}(\text{epi}_{\mathcal{D}_2} f) \cup \text{conv}(\text{epi}_{\mathcal{D}_3} f).$$

Based on the definition of \mathcal{P}_c in Lemma 3, \mathcal{P}_l in Lemma 4, and \mathcal{P}_r in Lemma 5,

we have that $\text{conv}(\text{epi}_{\mathcal{D}_1} f) = \mathcal{P}^l$, $\text{conv}(\text{epi}_{\mathcal{D}_2} f) = \mathcal{P}^c$, and

$\text{conv}(\text{epi}_{\mathcal{D}_3} f) = \mathcal{P}^r$. Thus, $\widehat{\text{conv}}_{\mathcal{D}}(f) = \text{conv}(\text{epi}_{\mathcal{D}} f) = \mathcal{P}^l \cup \mathcal{P}^c \cup \mathcal{P}^r = \mathcal{P}$.

□

Figures 2-3 compare the feasible region defined by the standard McCormick envelopes, and the new perspective hull.

Remark 1 Observe that for $\mathbf{x}_l = \mathbf{y}_l = 0$ and $\mathbf{y}_u = \mathbf{y}_u = 1$, we get the triangular domains studied in [41, 28], and their convex envelope characterization coincide with this result.

2.4 The Concave Envelope

The following result shows that the concave envelope of the bilinear function is equivalent to the McCormick envelope on the domain of interest.

Theorem 2 Consider the bilinear function $f(x, y) = xy$ on the domain $\mathcal{D} = \{ (x, y) \in [\mathbf{x}^l, \mathbf{x}^u] \times [\mathbf{y}^l, \mathbf{y}^u] \mid x \leq y \}$

$$\text{then } \widehat{\text{conv}}_{\mathcal{D}}(f) = \mathcal{P},$$

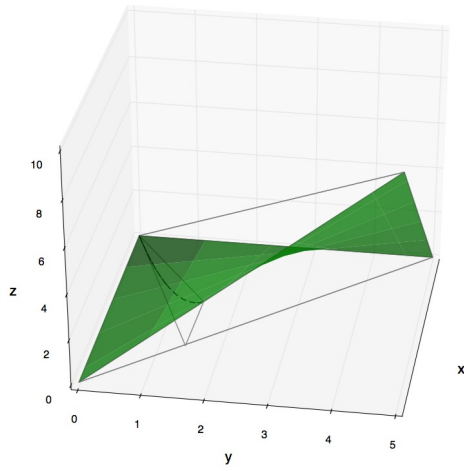


Fig. 2 Original polyhedral McCormick relaxation for $f(x, y) = xy$. The intersection between f and the constraint $x = y$ is represented in dashed lines.

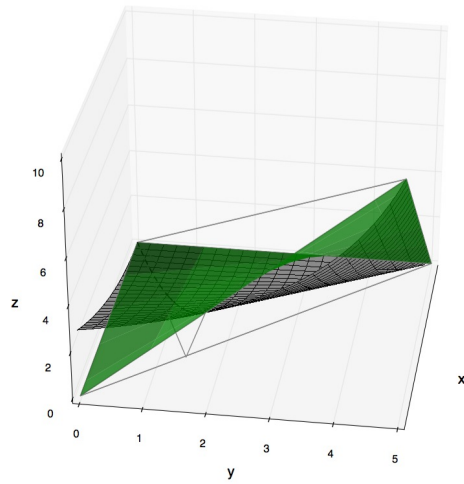


Fig. 3 The new perspective hull, a cone pointed at $(x^l, y^u, x^l y^u)$.

$$\text{where } \mathcal{P} = \left\{ (x, y, z) \in \mathcal{D} \times \mathbb{R} \mid \begin{array}{l} z \leq \mathbf{x}^u y + \mathbf{y}^l x - \mathbf{x}^u \mathbf{y}^l \\ z \leq \mathbf{x}^l y + \mathbf{y}^u x - \mathbf{x}^l \mathbf{y}^u \end{array} \right\}$$

Proof Recall that $\widehat{\text{conv}}_{\mathcal{D}}(f) = \text{conv}(\text{hypo}_{\mathcal{D}} f)$, and that the convex hull of $\text{hypo}_{\mathcal{D}} f$ is the smallest convex set containing it. Based on this definition, since \mathcal{P} is convex, and $\text{hypo}_{\mathcal{D}} f \subset \mathcal{P}$, we immediately have the first inclusion result, $\text{conv}(\text{hypo}_{\mathcal{D}} f) \subseteq \mathcal{P}$.

In the other direction, observe that the equation $z = \mathbf{x}^u y + \mathbf{y}^l x - \mathbf{x}^u \mathbf{y}^l$ corresponds to the plane passing through the points $(\mathbf{x}^l, \mathbf{y}^l, \mathbf{x}^l \mathbf{y}^l)$, $(\mathbf{x}^u, \mathbf{x}^u, (\mathbf{x}^u)^2)$, and $(\mathbf{x}^u, \mathbf{y}^u, \mathbf{x}^u \mathbf{y}^u)$, thus defining their convex hull. Note also that the equation $z = \mathbf{x}^l y + \mathbf{y}^u x - \mathbf{x}^l \mathbf{y}^u$ corresponds to the plane passing through the points $(\mathbf{x}^l, \mathbf{y}^l, \mathbf{x}^l \mathbf{y}^l)$, $(\mathbf{x}^l, \mathbf{y}^u, \mathbf{x}^l \mathbf{y}^u)$, and $(\mathbf{x}^u, \mathbf{y}^u, \mathbf{x}^u \mathbf{y}^u)$, thus defining their convex hull. Consider the vertical half-lines passing through these points,

$$\begin{aligned} \mathcal{L}^1 &= \{(x, y, z) \in \mathcal{D} \times \mathbb{R} \mid x = \mathbf{x}^l, y = \mathbf{y}^l, z \leq \mathbf{x}^l \mathbf{y}^l\}, \\ \mathcal{L}^2 &= \{(x, y, z) \in \mathcal{D} \times \mathbb{R} \mid x = \mathbf{x}^l, y = \mathbf{y}^u, z \leq \mathbf{x}^l \mathbf{y}^u\}, \\ \mathcal{L}^3 &= \{(x, y, z) \in \mathcal{D} \times \mathbb{R} \mid x = \mathbf{x}^u, y = \mathbf{x}^u, z \leq (\mathbf{x}^u)^2\}, \\ \text{and } \mathcal{L}^4 &= \{(x, y, z) \in \mathcal{D} \times \mathbb{R} \mid x = \mathbf{x}^u, y = \mathbf{y}^u, z \leq \mathbf{x}^u \mathbf{y}^u\}. \end{aligned}$$

It is easy to see that,

$$\text{conv}(\mathcal{L}^1 \cup \mathcal{L}^2 \cup \mathcal{L}^3 \cup \mathcal{L}^4) = \mathcal{P}$$

Since $\mathcal{L}^1 \cup \mathcal{L}^2 \cup \mathcal{L}^3 \cup \mathcal{L}^4 \subseteq \text{hypo}_{\mathcal{D}} f$, we have that $\text{conv}(\mathcal{L}^1 \cup \mathcal{L}^2 \cup \mathcal{L}^3 \cup \mathcal{L}^4) \subseteq \text{conv}(\text{hypo}_{\mathcal{D}} f)$, and consequently, $\mathcal{P} \subseteq \text{conv}(\text{hypo}_{\mathcal{D}} f)$.

□

2.5 The Convex Hull

Theorem 3 Consider the set

$$\mathcal{S} \subset \mathbb{R}^3 = \{(x, y, z) \in [\mathbf{x}^l, \mathbf{x}^u] \times [\mathbf{y}^l, \mathbf{y}^u] \times \mathbb{R} \mid x \leq y, z = xy\},$$

s.t. $\mathbf{x}^l < \mathbf{y}^u$, then

$$\text{conv}(\mathcal{S}) = \mathcal{P},$$

where

$$\mathcal{P} = \left\{ (x, y, z) \in \mathbb{R}^3 \left| \begin{array}{l} \left(x - \mathbf{x}^l \frac{x-y}{\delta} \right)^2 \leq \left(1 - \frac{x-y}{\delta} \right) \left(z - \mathbf{x}^l \mathbf{y}^u \frac{x-y}{\delta} \right) \\ z \geq \mathbf{x}^u \mathbf{y} + \mathbf{y}^u x - \mathbf{x}^u \mathbf{y}^u \\ z \geq \mathbf{x}^l \mathbf{y} + \mathbf{y}^l x - \mathbf{x}^l \mathbf{y}^l \\ z \leq \mathbf{x}^u \mathbf{y} + \mathbf{y}^l x - \mathbf{x}^u \mathbf{y}^l \\ z \leq \mathbf{x}^l \mathbf{y} + \mathbf{y}^u x - \mathbf{x}^l \mathbf{y}^u \end{array} \right. \right\},$$

and $\delta = (\mathbf{x}^l - \mathbf{y}^u)$.

Proof This is a consequence of Theorem 1 and 2, and the fact that

$$\text{conv}(\mathcal{S}) = \widetilde{\text{conv}}_{\mathcal{D}}(f) \cap \widehat{\text{conv}}_{\mathcal{D}}(f), \text{ where } f(x, y) = xy.$$

□

3 Computational Impact

To illustrate the potential impact of this result in practice, we design the following computational experiment, based on the nonlinear program,

$$\begin{aligned} \min \sum_{i=1}^2 2z_i^2 + \left(x_i - \frac{\mathbf{x}_i^u - \mathbf{x}_i^l}{2} \right)^2 + \left(y_i - \frac{\mathbf{y}_i^u - \mathbf{y}_i^l}{2} \right)^2 \\ \text{s.t. } \begin{cases} z_i = x_i y_i, x_i \leq y_i, i \in \{1, 2\} \\ \mathbf{x}_i^l \leq x_i \leq \mathbf{x}_i^u, \mathbf{y}_i^l \leq y_i \leq \mathbf{y}_i^u, i \in \{1, 2\} \end{cases} \end{aligned} \quad (\text{NLP})$$

The variable bounds are randomly generated based on the following two schemes,

1. $\mathbf{x}_i^l = \mathbf{y}_i^l = \text{uniform}_{[-2,0]}$, $\mathbf{x}_i^u = \mathbf{x}_i^l + \text{uniform}_{[0,5]}$, and $\mathbf{y}_i^u = \max(\mathbf{x}_i^u, \mathbf{y}_i^l + \text{uniform}_{[0,5]})$.
2. $\mathbf{x}_i^l = \text{uniform}_{[-10,10]}$, $\mathbf{x}_i^u = \mathbf{x}_i^l + \text{uniform}_{[0,10]}$, $\mathbf{y}_i^l = \mathbf{x}_i^l + \text{uniform}_{[0,2]}$, and $\mathbf{y}_i^u = \max(\mathbf{x}_i^u, \mathbf{y}_i^l + \text{uniform}_{[0,10]})$.

$\text{uniform}_{[1, \mathbf{u}]}$ returns a random number following the uniform distribution on the interval $[1, \mathbf{u}]$.

Observe that the objective function is designed to drive the optimal solution away from the boundaries of the variables' domain, where both McCormick envelopes and the new perspective hull are tight.

(NLP) is solved using the nonlinear solver `Ipopt` [44] as a heuristic, the resulting primal solution is then evaluated using the standard McCormick relaxation, and the new perspective envelopes. Observe that the nonlinear component of the perspective formulation is a rotated second-order cone constraint ($uv \geq w^2$), and thus can be handled by commercial solvers such as Cplex [25] or Gurobi [18].

Table 1 Optimality gap reduction

	McCormick av.	Perspective av.	Reduction max
Set 1	59%	29%	71%
Set 2	6%	3%	55%

Table 1 reports the average and the maximum gap reduction comparing between the standard and the new approach, on 200 randomly generated instances for each scenario. The first two columns report the average optimality gap produced by, respectively, the McCormick envelopes, and the perspective hull. The last column reports the maximum gap reduction obtained by using the new envelope.

4 Conclusion

Given the bilinear function $f(x, y) = xy$, and the constraint $x \leq y$, we characterize the convex and the concave envelopes of f in the space of original variables. The new characterization, based on perspective functions, dominates the standard McCormick approach, with promising optimality gap reductions. This result can have a strong impact in global optimization frameworks, potentially improving the pruning process by providing better lower bounds in spatial branch and bound algorithms.

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