Cutting planes from extended LP formulations

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Abstract

Given a mixed-integer set defined by linear inequalities and integrality requirements on some of the variables, we consider extended formulations of its continuous (LP) relaxation and study the effect of adding cutting planes in the extended space. In terms of optimization, extended LP formulations do not lead to better bounds as their projection onto the original space is precisely the original LP relaxation. However, adding cutting planes in the extended space can lead to stronger bounds. In this paper we show that for every 0-1 mixed-integer set with n integer and k continuous variables, there is an extended LP formulation with $2n + k - 1$ variables whose elementary split closure is integral. The proof is constructive but it requires an inner description of the LP relaxation. We then extend this idea to general mixed-integer sets and construct the best extended LP formulation for such sets with respect to lattice-free cuts. We also present computational results on the two-row continuous group relaxation showing the strength of cutting planes derived from extended LP formulations.

1 Introduction

Given a mathematical programming formulation of an optimization problem, an extended formulation is one which uses additional variables to represent the same problem, usually to simplify or to strengthen the formulation in some way. In integer programming, extended formulations are primarily used to obtain a new LP relaxation which is stronger than the original LP relaxation of the integer program in the sense that the projection of the new relaxation onto the original variables is strictly contained in the original LP relaxation; thus it is better to optimize the objective function over the new relaxation rather than the original LP relaxation. The RLT-based extended formulations of Sherali and Adams [30] are well-known examples with this property. Conforti, Cornuéjols and Zambelli [12] and Balas [4] survey extended formulations in integer programming.

Extended formulations are also used to represent a polyhedron defined by exponentially many linear inequalities as the projection of a higher dimensional polyhedron defined by only polynomially many linear inequalities. See Barahona and Mahjoub [8] and Balas and Pulleyblank [6] for examples of such compact extended formulations. On the other hand, Fiorini et al. [20] and Rothvoss [29] have recently shown the nonexistence of compact extended formulations for the convex hull of solutions of the clique problem and the perfect matching problem, respectively.

In this paper we take a different approach and study *extended LP formulations* – or extended formulations of the LP relaxations of integer programs. The extended formulations that we study are not stronger than the original LP relaxation but might lead to stronger relaxations after the addition of cutting planes. In [9], we showed (along with Jim Luedtke) that given two or more split disjunctions, split cuts from these disjunctions yield stronger relaxations when applied to an extended LP formulation instead of the original LP formulation. Furthermore, we gave an example where the extended LP formulation is strictly stronger after adding split cuts. Earlier, Modaresi, Kilinç and Vielma [27] observed that split cuts added to a specific extended formulation of a linear relaxation of a conic integer program yielded a stronger relaxation than split cuts added to the original continuous, convex relaxation.

As there are infinitely many possible extended LP formulations for a given mixed-integer program, a natural question is whether there exists a strongest possible extended LP formulation with respect to subsequent strengthening by split cuts or lattice-free cuts. We answer this question in the affirmative, and give an explicit construction of one such extended LP formulation for general mixed-integer programs. In the negative direction, for the well-known Cook-Kannan-Schrijver [13] example with infinite split rank, we show that extended LP formulations do not help in the sense that the split closure of the original set equals the projection of the split closure of any extended LP formulation onto the original space.

For 0-1 mixed-integer programs with n integer variables, we give an extended LP formulation with $n-1$ additional continuous variables such that adding all split cuts derived from 0-1 split disjunctions to it yields an integral polyhedron, and thus this extended LP formulation is strongest possible with respect to addition of split cuts. In addition, we present two families of polytopes, namely the (generalized) cropped cube and the stable set polytope of a clique, for which our extended LP formulation is described by polynomially many inequalities. We also show that no extended LP formulations of the cropped cube with fewer than $n-1$ additional variables yields an integral polyhedra after adding split cuts.

Finally, we perform computational experiments where we numerically compare the effect of split cuts added to an extended LP formulation to the effect of split cuts added to the original LP relaxation. In these experiments, we focus on the two-row continuous group problem with nonnegativity constraints on the integer variables. Throughout the paper we work with pointed polyhedra for convenience.

2 Preliminaries

We study extended formulations of polyhedral sets and the effect of applying split (and more general lattice-free) cuts to these sets. More precisely, let $n = n_1 + n_2 > 0$ and consider two mixed-integer sets $P^{IP} = P \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$ and $Q^{IP} = Q \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2+q})$ such that the continuous relaxations

$$P = \{x \in \mathbb{R}^n \,:\, Ax \leq b\} \text{ and } Q = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^q \,:\, Cx + Gy \leq d\}$$

satisfy

$$P = \operatorname{proj}_x(Q)$$

where $\operatorname{proj}_x(Q)$ is the set obtained by orthogonal projection of points in Q onto the space of x variables. Clearly, any such Q satisfies

$$Q \subseteq (P \times \mathbb{R}^q).$$

We call the set Q an *extended LP formulation* of the set P. Given an objective function that only depends on the x variables, optimizing it over the set P or the set Q yields the same value. Furthermore, the same observation also holds for the sets P^{IP} and Q^{IP} as $P^{IP} = \operatorname{proj}_x(Q^{IP})$.

It is known that applying split cuts to the extended formulation Q and projecting the resulting set to the space of the x variables can sometimes yield a smaller set (and therefore a better approximation of the integer set P^{IP}) than the set obtained by applying split cuts to P, see [9]. We next review some results and concepts that we use in the paper.

2.1 Extended formulations and orthogonal projections

We next formally define orthogonal projections and some basic properties. Given a polyhedron $Q = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^q \,:\, Cx + Gy \leq d\}$, its orthogonal projection onto the space of the x variables is

$$\operatorname{proj}_x(Q) = \{x \in \mathbb{R}^n \,:\, \exists y \in \mathbb{R}^q \; s.t. \; (x,y) \in Q\}.$$

The *projection cone* of Q is the polyhedral set $K = \{u \in \mathbb{R}^m \,:\, u^T G = 0,\ u \geq 0\}$ where m is the number of inequalities in the definition of Q above. If $K = \{0\}$, then $\operatorname{proj}_x(Q) = \mathbb{R}^n$ and therefore in this paper we can assume that $K \neq \{0\}$. Then, an inequality description of the projection is obtained by

$$\operatorname{proj}_x(Q) = \{x \in \mathbb{R}^n \,:\, u^T Cx \leq u^T d \text{ for all } u \in K\}.$$

Let K^r denote the extreme rays of K and note that, in the definition of the set $\operatorname{proj}_x(Q)$ one can use K^r instead of K. The recession cone of Q is

$$\operatorname{rec}(Q) = \{(d,l) \in \mathbb{R}^n \times \mathbb{R}^q \,:\, Cd + Gl \leq 0\},$$

and the recession cone of its projection is

$$\operatorname{rec}(\operatorname{proj}_x(Q)) = \{d \in \mathbb{R}^n \,:\, u^T C d \le 0 \text{ for all } u \in K^r\}.$$

Consequently, we have

$$\operatorname{rec}(\operatorname{proj}_x(Q)) = \operatorname{proj}_d(\operatorname{rec}(Q)).$$

For a given point $\bar{x} \in \operatorname{proj}_x(Q)$, its pre-image is defined to be the set of points $\{(x, y) \in Q \,:\, x = \bar{x}\}$. We define the pre-image of a ray of $\operatorname{proj}_x(Q)$ similarly.

2.2 Lattice-free cuts

Lattice-free cuts can be seen as a generalization of split cuts which form a well-known class of valid inequalities for mixed-integer programming problems. A set $L \subset \mathbb{R}^n$ is called a strictly lattice-free set for the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ if $L \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$, where $n_1 + n_2 = n$. We emphasize that we do not require L to be a convex set. Strictly lattice-free sets are closely related to valid inequalities for mixed-integer sets as the region cut-off by any valid inequality is strictly lattice-free. Consider a mixed-integer set $P^{IP} = P \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$ where $P \subset \mathbb{R}^n$. As $L \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset$,

$$P^{IP} = P \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = (P \setminus L) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}).$$

Consequently, $P^{IP} \subseteq \operatorname{conv}(P \setminus L)$ and therefore, any inequality valid for $P \setminus L$ is also valid for P^{IP}. Here, for a given set $X \subseteq \mathbb{R}^n$, we use $\operatorname{conv}(X)$ to denote its convex hull. Let a collection of strictly lattice-free sets $\mathcal{L} = \{L_1, L_2, \ldots\}$ be given. The lattice-free closure of P with respect to $\mathcal{L}$ is defined as

$$\mathcal{L}(P) = \bigcap_{L \in \mathcal{L}} \operatorname{conv}(P \setminus L). \tag{1}$$

Clearly, $P \supseteq \mathcal{L}(P) \supseteq P^{IP}$. A basic strictly lattice-free set is the *split set:*

$$S(\pi, \gamma) = \{x \in \mathbb{R}^n \,:\, \gamma + 1 > \pi^T x > \gamma\},$$

where $\pi \in \mathbb{Z}^n$, $\gamma \in \mathbb{Z}$, and $\pi_j = 0$ for $j \in \{n_1 + 1, \ldots, n\}$. Let $\mathcal{S}^*$ denote the collection of all such split sets, that is,

$$\mathcal{S}^* = \Big\{ S(\pi, \gamma) : \pi \in \mathbb{Z}^n,\ \gamma \in \mathbb{Z},\ \pi_j = 0 \text{ for } j \in \{n_1 + 1, \ldots, n\} \Big\}$$

and note that every $S \in \mathcal{S}^*$ is strictly lattice-free and therefore $P \supseteq \mathcal{S}^*(P) \supseteq P^{IP}$. Many authors define split cuts in terms of *split disjunctions* instead of split sets. In fact, lattice-free cuts generalize *polyhedral* disjunctive cuts that we define next. Let $D_k = \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \,:\, A^k x \le b^k\}$ be finitely many polyhedral sets indexed by $k \in K$. The set $D = \cup_{k \in K} D_k$ is called a valid *disjunction* for the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ if

$$\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \subseteq D.$$

A linear inequality is called a *disjunctive cut* for P, derived from the valid disjunction D, if it is valid for $\text{conv}(P \cap D)$. As $D^c = \mathbb{R}^n \setminus D$ is a strictly lattice-free set for the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$, disjunctive cuts derived from D are lattice-free cuts derived from D^c.

Given a collection of strictly lattice-free sets $\mathcal{L}$ for the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ and an extended formulation $Q \subseteq \mathbb{R}^{n+q}$ of $P \subseteq \mathbb{R}^n$, we define

$$\mathcal{L}(Q) = \bigcap_{L \in \mathcal{L}} \text{conv}(Q \setminus L^+), \tag{2}$$

where $L^+ = L \times \mathbb{R}^q$. Note that L^+ is a strictly lattice-free set for the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2+q}$ for $L \in \mathcal{L}$.

2.3 Lattice-free cuts for extended LP formulations

In a recent paper [9], we apply split cuts in an extended space instead of the original space for stochastic programming problems. More precisely, let $P \subseteq \mathbb{R}^n$ be a polyhedral set and let $Q \subseteq \mathbb{R}^{n+q}$ be an extended formulation of it, i.e., $P = \text{proj}_{\mathbb{R}^n}(Q)$. Here we assume that the first n variables in the definition of Q are the same as the ones in P and the operator $\text{proj}_{\mathbb{R}^n}(\cdot)$, keeps the first n coordinates and projects out the rest. We will use this convention throughout the paper when it is unambiguous. Bodur et al. [9] show that for any $\mathcal{S} \subseteq \mathcal{S}^*$

$$\mathcal{S}(P) \supseteq \text{proj}_{\mathbb{R}^n}(\mathcal{S}(Q)). \tag{3}$$

In addition, they show that the two sets above are equal when $|\mathcal{S}| = 1$. It is easy to generalize these observations on split cuts applied to polyhedral sets to non-polyhedral sets and more general cutting planes. We next extend this to general lattice-free cuts.

Theorem 2.1. *Let $P \subseteq \mathbb{R}^n$ and an extended formulation $Q \subseteq \mathbb{R}^{n+q}$ be given and let $\mathcal{L}$ be a collection of strictly lattice-free sets in $\mathbb{R}^n$. Then,*

$$\mathcal{L}(P) \supseteq \text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q)). \tag{4}$$

Furthermore, $\mathcal{L}(P) = \text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q))$ if $|\mathcal{L}| = 1$.

Proof. The proof is essentially same as the proof of the same fact in [9] for split sets when P is a polyhedron. If $x \in \text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q))$, then there exists $y \in \mathbb{R}^q$ such that $(x, y) \in \mathcal{L}(Q)$. As $(x, y) \in \text{conv}(Q \setminus L^+)$ for all $L \in \mathcal{L}$, we have $x \in \text{conv}(P \setminus L)$ for all $L \in \mathcal{L}$ and the proof is complete. □

In [9], we also show that the inclusion in inequality (3) (and therefore in inequality (4)) can be strict when $|\mathcal{S}| > 1$. In other words, split cuts on an extended formulation can be strictly better. The proof is based on the following simple example in $\mathbb{R}^2$, where

$$P = \text{conv}((0, 1/2), (1, 1/2), (1/2, 0), (1/2, 1))$$

and the associated extended formulation in $\mathbb{R}^3$ is

$$Q = \operatorname{conv}((0, 1/2, 1), (1, 1/2, 1), (1/2, 0, 0), (1/2, 1, 0)),$$

see Figure 1. It is easy to argue that the split closure of P is the point $(1/2, 1/2)$ whereas using the split sets $S_i = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R} \;:\; 1 > x_i > 0\}$ for $i = 1, 2$ proves that the split closure of Q is empty.

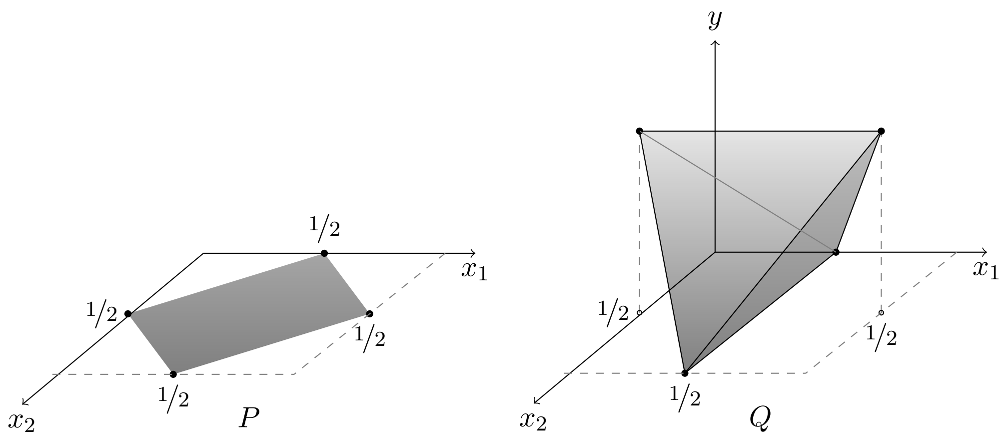

Figure 1: The set P and its extended formulation Q

3 Mixed-integer sets with 0-1 variables

Consider a 0-1 mixed-integer set $P^{IP} = P \cap (\{0, 1\}^{n_1} \times \mathbb{R}^{n_2})$ where $P = \{x \in \mathbb{R}^n \;:\; Ax \leq b\}$ and the 0-1 bounds for the first n_1 variables are included in the constraints. Furthermore, assume that P is a pointed polyhedron defined by

$$P = \operatorname{conv}(\bar{x}^1, \ldots, \bar{x}^m) + \operatorname{cone}(\bar{r}^1, \ldots, \bar{r}^\ell),$$

where $\{\bar{x}^j\}_{j=1,\ldots,m}$ is the set of extreme points of P and $\{\bar{r}^j\}_{j=1,\ldots,\ell}$ is the set of extreme rays of P. Here, for a finite set $X \subseteq \mathbb{R}^n$, we let cone(X) denote the set of nonnegative linear combinations of vectors in X. For $t \in \{1, \ldots, n_1\}$ we define the following extended LP formulation of P in $\mathbb{R}^{n+t}$

$$Q^t = \operatorname{conv}(\hat{x}^1, \ldots, \hat{x}^m) + \operatorname{cone}(\hat{r}^1, \ldots, \hat{r}^\ell),$$

where $\hat{r}^j = (\bar{r}^j, \mathbf{0})$ for $j = 1, \ldots, \ell$ and $\mathbf{0}$ is the vector of zeros in $\mathbb{R}^t$, while $\hat{x}^j = (\bar{x}^j, y^j)$ where

$$y_i^j = \begin{cases} 1, & \text{if } \bar{x}_i^j \text{ fractional} \\ 0, & \text{o.w.} \end{cases}, \; i = 1, \ldots, t,$$

for $j = 1, \ldots, m$. Notice that both sets $\text{cone}(\bar{r}^1, \ldots, \bar{r}^\ell)$ and $\text{cone}(\hat{r}^1, \ldots, \hat{r}^\ell)$ have the same dimension and there is a one-to-one correspondence between their members, that is, $\bar{r} \in \text{cone}(\bar{r}^1, \ldots, \bar{r}^\ell)$ if and only if $(\bar{r}, 0) \in \text{cone}(\hat{r}^1, \ldots, \hat{r}^\ell)$. Also note that if $\bar{r} \in \text{cone}(\bar{r}^1, \ldots, \bar{r}^\ell)$ then $\bar{r} \in \{0\}^{n_1} \times \mathbb{R}^{n_2}$ due to the fact that the first n_1 variables in P have finite upper and lower bounds.

Our main observation is that applying split cuts from 0-1 splits to the extended formulation Q^t yields an integral polyhedron in the first t coordinates and consequently, applying 0-1 split cuts to Q^{n_1} and then projecting it to the original space gives the convex hull of P^{IP}. We next present two technical results that lead to the main observation. For $t \in \{1, \ldots, n_1\}$, let

$$P^t = Q^t \cap \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^t : y_j = 0 \text{ for } j = 1, \ldots, t\}. \tag{5}$$

Lemma 3.1. *Let $t \in \{1, \ldots, n_1\}$ be given. If $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}^t$ is an extreme point of P^t then, $x^*_j \in \{0, 1\}$ for all $j = 1, \ldots, t$.*

Proof. First note that $y_j \geq 0$ is a valid inequality for Q^t for all $j = 1, \ldots, t$ and therefore P^t is a face of Q^t. Consequently, (x^*, y^*) is an extreme point of Q^t as well. Furthermore, as all points in P^t satisfy $y = \mathbf{0}$, we have $y^* = \mathbf{0}$. By definition of Q^t, if $(x^*, \mathbf{0})$ is an extreme point of Q^t, then x^* is an extreme point of P with the property that $x^*_j \in \{0, 1\}$ for all $j = 1, \ldots, t$. □

We define the collection of the 0-1 split sets $\mathcal{S}^{0,1} = \cup_{i=1}^n S_i$ where

$$S_i = \{x \in \mathbb{R}^n \ : \ 1 > x_i > 0\}.$$

Lemma 3.2. *For any $j \in \{1, \ldots, t\}$, the equality $y_j = 0$ is a valid split cut for Q^t that can be derived from the 0-1 split set*

$$S^+_j = \{x \in \mathbb{R}^{n+t} \ : \ 1 > x_j > 0\}.$$

Proof. Let $j \in \{1, \ldots, t\}$ be given and define $W_0 = Q^t \cap \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^t : x_j = 0\}$. Clearly W_0 is a face of Q^t as $x_j \geq 0$ is a valid inequality for Q^t. Consequently, all extreme points of W_0 are extreme points of Q^t that satisfy $x_j = 0$. By definition, all such points have $y_j = 0$. As all rays of Q^t and therefore of W_0 have zero components for the coordinates corresponding to the y variables, we have that $y_j = 0$ holds for all points in W_0. Similarly, all points in $W_1 = Q^t \cap \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^t : x_j = 1\}$ also have $y_j = 0$. As $Q^t \setminus S^+_j = W_0 \cup W_1$, we conclude that $y_j = 0$ is a split cut for Q^t. □

Combining Lemmas 3.1 and 3.2 we obtain the following result.

Theorem 3.3. *The 0-1 split closure of Q^{n_1} using S^+_j for $j = 1, \ldots, n_1$ is integral and therefore*

$$\text{proj}_x \left(\mathcal{S}^{0,1}(Q^{n_1})\right) = \text{conv}(P^{IP}).$$

Proof. Clearly, $\text{proj}_x \left(\mathcal{S}^{0,1}(Q^{n_1})\right) \supseteq \text{conv}(P^{IP})$. To see that the reverse inclusion also holds, note that by Lemma 3.2 for all $j \in \{1, \ldots, n_1\}$, the equality $y_j = 0$ is a valid split cut based on a 0-1 split set. Therefore $\mathcal{S}^{0,1}(Q^{n_1})$ is contained in the set P^{n_1} and $\text{proj}_x \left(\mathcal{S}^{0,1}(Q^{n_1})\right) \subseteq \text{proj}_x \left(P^{n_1}\right)$. As P^{n_1} is integral (by Lemma 3.1), $\text{proj}_x \left(P^{n_1}\right) = \text{conv}(P^{IP})$ and the proof is complete. □

Therefore, we proved that every 0-1 mixed-integer set in $\mathbb{R}^n$ has an extended formulation in $\mathbb{R}^{n+n_1}$ such that the 0-1 split closure of the extended formulation is integral. It is in fact possible to improve this result and show that the 0-1 split closure of $Q^{n_1-1} \subseteq \mathbb{R}^{n+n_1-1}$ is also integral. Even though this result is stronger, we also present Theorem 3.3 as we find Q^{n_1} a more natural extended formulation than Q^{n_1-1}. Later in Section 3.1, we will also show that this improved bound on the dimension of the extended formulation is tight in the sense that for a certain 0-1 mixed-integer set, no extended formulation with fewer than $n_1 - 1$ additional variables yields an integral 0-1 split closure.

Theorem 3.4. *The 0-1 split closure of Q^{n_1-1} using S_j^+ for $j = 1, \dots, n_1$ is integral and therefore*

$$\text{proj}_x\left(\mathcal{S}^{0,1}(Q^{n_1-1})\right) = \text{conv}(P^{IP}).$$

Proof. Clearly,

$$\mathcal{S}^{0,1}(Q^{n_1-1}) = \bigcap_{j=1}^{n_1} \text{conv}(Q^{n_1-1} \setminus S_j^+) \subseteq \text{conv}(Q^{n_1-1} \setminus S_{n_1}^+) \cap P^{n_1-1}, \tag{6}$$

where P^{n_1-1} is defined in (5) and the last inclusion is due to Lemma 3.2. As discussed in the proof of Lemma 3.1, the set P^{n_1-1} is a face of Q^{n_1-1} and consequently

$$\text{conv}(Q^{n_1-1} \setminus S_{n_1}^+) \cap P^{n_1-1} = \text{conv}(P^{n_1-1} \setminus S_{n_1}^+) \tag{7}$$

as for any face F of a polyhedron P, it holds that $\text{conv}(P \setminus S) \cap F = \text{conv}(F \setminus S)$ for a (split) set S, see [13, Equation 9]. In addition, again by Lemma 3.1, we know that P^{n_1-1} is integral in the first $n_1 - 1$ coordinates. Consequently, both $W_0 = P^{n_1-1} \cap \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{n_1-1} : x_{n_1} = 0\}$ and $W_1 = P^{n_1-1} \cap \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{n_1-1} : x_{n_1} = 1\}$ are integral in the first $n_1 - 1$ coordinates as they are both faces of P^{n-1}. Furthermore, as W_0 and W_1 are also integral in the n_1^{th} coordinate, we have that both W_0 and W_1 are in fact integral in the first n_1 coordinates. As $\text{conv}(P^{n_1-1} \setminus S_{n_1}^+) = \text{conv}(W_0 \cup W_1)$, we therefore conclude that $\text{conv}(P^{n_1-1} \setminus S_{n_1}^+)$ is integral. Furthermore, by combining the inclusion in inequality (6) with equation (7) we have $\mathcal{S}^{0,1}(Q^{n_1-1})$ is integral in the first n_1 coordinates and therefore, so is $\text{proj}_x\left(\mathcal{S}^{0,1}(Q^{n_1-1})\right)$. □

Also note that there is a one-to-one correspondence between the extreme points and rays of Q^{n_1-1} and Q^{n_1} and consequently it is easy to see that Q^{n_1} is an extended LP formulation for Q^{n_1-1}:

$$Q^{n_1-1} = \text{proj}_{\mathbb{R}^{n+n_1-1}}\left(Q^{n_1}\right). \tag{8}$$

Constructing the extended formulation of a given polyhedron P seems to be difficult as one needs the explicit knowledge of the extreme points and extreme rays (i.e., the inner description) of P. Furthermore, an exponential number of inequalities might be necessary to describe the extended formulation even when P has only a polynomial number of facets. In the remainder of this section we describe two well-known mixed-integer sets and give their extended LP formulations.

Remember that $\mathcal{S}^*$ denotes the collection of all split disjunctions. For any given $\mathcal{S} \subseteq \mathcal{S}^*$ and integer $k \geq 0$ we define the k^{th} *split closure of* P with respect to $\mathcal{S}$ as $\mathcal{S}^k(P)$. Formally, this closure is defined iteratively as follows: $\mathcal{S}^0(P) = P$ and $\mathcal{S}^k(P) = \mathcal{S}(\mathcal{S}^{k-1}(P))$ for $k \geq 1$. We say that a valid inequality for $P \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$ has split rank k if it is not valid for $(\mathcal{S}^*)^{k-1}(P)$, but is valid for $(\mathcal{S}^*)^k(P)$.

The two sets that we present in the following sections have the property that even though their LP relaxations have an exponential number of extreme points, the extended formulation is defined by a polynomial number of inequalities. Furthermore, for both sets, the convex hull of integer solutions are defined by inequalities that have high split rank and yet the 0-1 split closure of the extended formulation is integral by Theorem 3.3.

3.1 The cropped cube

We next present a set, first described in [11], that actually has a compact extended formulation even though the formulation in the original space has an exponential number of facets.

Let $P^{IP} = P \cap \{0,1\}^n$ where

$$P = \Big\{ x \in \mathbb{R}^n \; : \quad \sum_{i \in I} x_i + \sum_{i \in N \setminus I} (1 - x_i) \geq 1/2, \quad \forall I \subseteq N, \quad 1 \geq x_i \geq 0, \quad \forall i \in N \quad \Big\}$$

and $N = \{1, \ldots, n\}$. It is easy to see that $P^{IP} = \emptyset$ as each vertex of the hypercube $[0,1]^n$ is cut off precisely by one inequality generated by the set I where $i \in I$ if and only if the i^{th} coordinate of the vertex is zero. Consequently, all $2^n + 2n$ inequalities that appear in the description of P are facet defining. Furthermore, it is easy to see that all vertices of P have precisely one coordinate that is equal to $1/2$ and the remaining coordinates are integral. We call P the *cropped cube*, see Figure 2.

In [11], Cook, Chvátal and Hartmann show that the Chvátal rank of P is n and later Cornuéjols and Li [14] extend this result to show that the split rank of P is also n. In other words, to prove that P^{IP} is empty, one needs rank-n split cuts. This is the highest rank split cuts that one needs for a 0-1 mixed-integer set as the n^{th} split closure of such a set is always integral.

In their paper [14], Cornuéjols and Li construct a family of polyhedral sets $P_1, P_2 \ldots, P_n$ in $\mathbb{R}^n$ that satisfy the following properties: (i) P_1 is same as P, (ii) P_n contains only the point $(1/2, \ldots, 1/2)$, and, (iii) the split closure of P_q contains P_{q+1} for $q = 1, \ldots, n-1$. Consequently, they argue that the $(n-1)^{\text{th}}$ split closure of P is not empty and therefore the cropped cube has split rank n.

The sets P_q are defined as the convex hull of their extreme points and a point $x^* \in \mathbb{R}^n$ is an extreme point of P_q if and only if q components of x^* is equal to $1/2$ and the remaining components are either zero or one. Figure 2 shows the cropped cube $P = P_1$ and the *generalized cropped cube* P_2 in $\mathbb{R}^3$. Let Q^n_q denote the extended LP formulation of P_q as defined earlier in Section 3. We

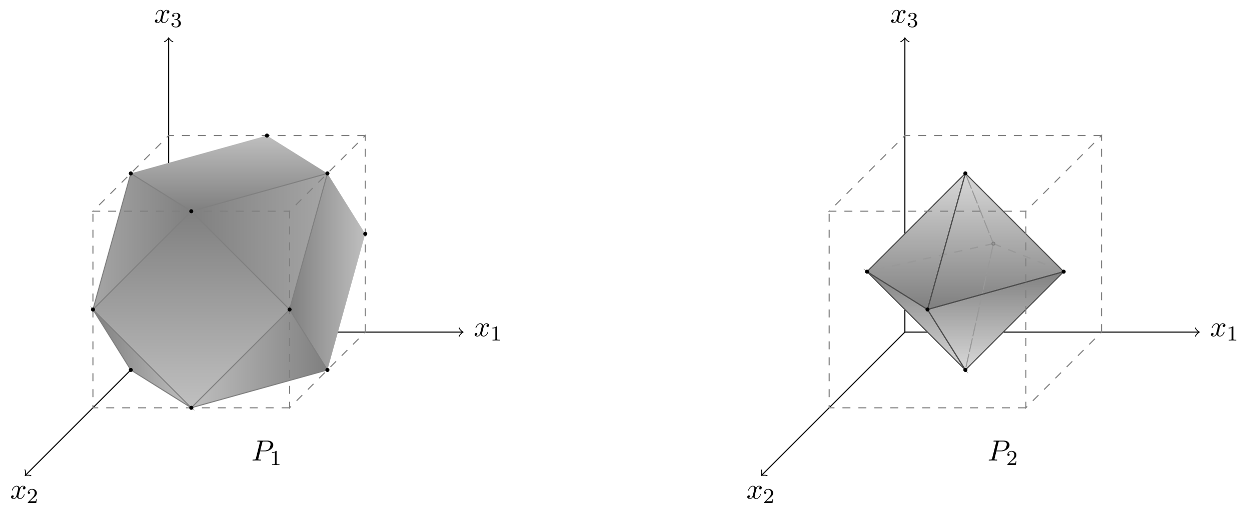

Figure 2: The cropped cube P_1 and the generalized cropped cube P_2 in $\mathbb{R}^3$

next show that for any given $q \in \{1 \dots, n\}$, the formulation Q_q^n is defined by $3n$ inequalities and one equation in $\mathbb{R}^{2n}$.

Lemma 3.5. *$Q_q^n = X_q$, where*

$$X_q = \Big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \; : \qquad y_i \leq 2x_i, \qquad \forall i \in N, \tag{9}$$

$$2x_i + y_i \leq 2, \qquad \forall i \in N, \tag{10}$$

$$y_i \geq 0, \qquad \forall i \in N, \tag{11}$$

$$\sum_{i \in N} y_i = q \qquad \Big\}. \tag{12}$$

Proof. First note that both sets are polytopes and all extreme points of Q_q^n are contained in X_q and therefore $Q_q^n \subseteq X_q$. Next, we show the reverse inclusion by arguing that every extreme point of X_q is an extreme point of Q_q^n. Note that any extreme point of X_q is defined by $2n$ linearly independent equalities obtained from inequalities (9)-(12). More precisely, (x^*, y^*) satisfies $2n - 1$ inequalities from inequalities (9)-(11) as equality together with (12). Notice that for any fixed $i \in N = \{1 \dots, n\}$ at most two of the three inequalities (9)-(11) associated with i can hold as equality (if inequality (11) is tight, then $y_i^* = 0$, in which case, if inequality (9) is also tight, then $x_i^* = 0$ and therefore inequality (10) cannot be tight). Therefore, for all but possibly one $i \in N$, two of the inequalities (9)-(11) must be tight.

Let $t \in N$ be such that each $i \in N \setminus \{t\}$ has two tight inequalities associated with it. For any fixed $i \in N \setminus \{t\}$, if inequalities (9) and (11) are tight, then $y_i^* = 0$ and $x_i^* = 0$ as discussed above. If, on the other hand, inequalities (10) and (11) are tight, then similarly, $y_i^* = 0$ and $x_i^* = 1$. Finally, if inequalities (9) and (10) are tight, then it is easy to see that $y_i^* = 1$ and $x_i^* = 1/2$. Consequently, the claim holds for all $i \in N \setminus \{t\}$.

Notice that by (12), there must be at most q and at least $q-1$ indices $i \in N \setminus \{t\}$ such that $y_i^* = 1$. We will next consider two cases. If there are q such indices, then by (12) we have $y_t^* = 0$. On the other hand, if there are $q-1$ such indices, then $y_t^* = 1$.

If $y_t^* = 1$, inequality (11) associated with t cannot be tight. In addition, if inequality (9) is tight, then $x_t^* = 1/2$, or, if inequality (10) is tight, then $x_t^* = 1/2$. Consequently, in this case, the claim holds for t and therefore for all $i \in N$.

Finally, if $y_t^* = 0$, inequality (11) associated with t is tight. However, one of the inequalities (9) and (10) must also be tight because if that is not the case, two distinct points obtained from (x^*, y^*) by increasing and decreasing x_t^* by a small amount while keeping all other components the same are contained in X_q, implying that (x^*, y^*) is not an extreme point of X_q. Consequently, even though $y_t^* = 0$, one of inequality (9) or (10) must still hold as equality. Therefore, as claimed, x_t^* is either zero or one, depending on which inequality is tight. $\square$

As $P^{IP} = \emptyset$, by Theorem 3.3 we know that the 0-1 split closure of the extended formulation X_1 is empty even though one needs rank n split cuts in the original space to prove that P^{IP} is empty. We note that Pierre Bonami [10] has also independently discovered an extended LP formulation for P_1 that has an empty 0-1 split closure. Note that the extended formulation of the generalized cropped cube $Q_q^{n-1} \subseteq \mathbb{R}^{2n-1}$ described in Theorem 3.4 also has an empty 0-1 split closure. By equation (8), this smaller extended formulation can be obtained for all q by simply projecting out the last variable from X_q.

Next, we show that any extended formulation of the cropped cube with dimension less than $2n-1$ has a nonempty 0-1 split closure. This proves that the bound $n + n_1 - 1$ on the dimension of the extended formulation with integral 0-1 split closure is tight for 0-1 mixed-integer sets.

Lemma 3.6. *Let $Q \subseteq \mathbb{R}^{2n-2}$ be an extended formulation of the cropped cube $P \subseteq \mathbb{R}^n$. Then,* $\text{proj}_x\left(\mathcal{S}^{0,1}(Q)\right) \neq \emptyset$.

Proof. Let $\tilde{x} \in \mathbb{R}^n$ be the vector of halves, i.e., $\tilde{x}_i = 1/2$ for all $i \in N$. We will show that $\tilde{x} \in \text{proj}_x\left(\mathcal{S}^{0,1}(Q)\right)$. Let $V = V_1 \cup \cdots \cup V_n$ be the set of extreme points of P, where $V_i = \{x^{i,1}, \ldots, x^{i,J}\}$ is the set of extreme points whose i^{th} coordinate is equal to $1/2$ and the remaining coordinates are binary for each $i \in N$, and $J = 2^{n-1}$. As Q is an extended formulation of P, each point in P has at least one pre-image in Q. Then, for each $i \in N$ and $j \in \{1, \ldots, J\}$, there exists $y^{i,j} \in \mathbb{R}^{n-2}$ such that $(x^{i,j}, y^{i,j}) \in Q$. For each $i \in N$, define

$$\bar{y}^i = \frac{1}{J} \sum_{j=1}^{J} y^{i,j}.$$

Then, as $\frac{1}{J} \sum_{j=1}^{J} x^{i,j} = \tilde{x}$, we get $(\tilde{x}, \bar{y}^i) \in Q$, for all $i \in N$. As $\{\bar{y}^1, \ldots, \bar{y}^n\}$ constitutes a set of n points in $\mathbb{R}^{n-2}$, from Radon's lemma [25, Theorem 1.3.1], there exist a point and $\tilde{y} \in \mathbb{R}^{n-2}$ and two disjoint subsets $N_1, N_2 \subseteq N$ such that

$$\tilde{y} \in \operatorname{conv}(\{\bar{y}^i\}_{i \in N_1}) \cap \operatorname{conv}(\{\bar{y}^i\}_{i \in N_2}). \tag{13}$$

We next show that $(\tilde{x}, \tilde{y}) \in \mathcal{S}^{0,1}(Q)$.

Let $k \in N$ and consider the 0-1 split set S_k^+. For any $i \in N \setminus \{k\}$, as $(x^{i,j}, y^{i,j}) \in Q \setminus S_k^+$ for all $j = 1, \ldots, J$, we have $(\tilde{x}, \bar{y}^i) \in \operatorname{conv}(Q \setminus S_k^+)$. Moreover, as $N_1 \cap N_2 = \emptyset$, we can assume that $k \notin N_1$ without loss of generality. These observations together with (13) imply that

$$(\tilde{x}, \tilde{y}) \in \operatorname{conv}(\{(\tilde{x}, \bar{y}^i)\}_{i \in N_1}) \subseteq \operatorname{conv}(Q \setminus S_k^+).$$

Hence, $(\tilde{x}, \tilde{y}) \in \mathcal{S}^{0,1}(Q) = \cap_{k=1}^n \operatorname{conv}(Q \setminus S_k^+)$, which proves that $\tilde{x} \in \operatorname{proj}_x\left(\mathcal{S}^{0,1}(Q)\right)$.

□

3.2 The stable set polytope of a clique

Let G be a graph on $n \geq 3$ nodes (denoted by $1, \ldots, n$), and let E stand for the set of its edges. The *stable set polytope* of G is the convex hull of incidence vectors of independent sets in G and is denoted by STAB(G):

$$\text{STAB}(G) = \operatorname{conv}(\{x \in \{0,1\}^n : x_i + x_j \leq 1,\ \forall \{i,j\} \in E\}).$$

The continuous relaxation of STAB(G), denoted by FSTAB(G), is called the *fractional stable set polytope*:

$$\text{FSTAB}(G) = \{x \in [0,1]^n : x_i + x_j \leq 1,\ \forall \{i,j\} \in E\}.$$

If $C \subseteq \{1, \ldots, n\}$ is a clique in G, i.e., each pair of nodes in C is connected by an edge in E, then the *clique inequality* $\sum_{i \in C} x_i \leq 1$ is valid for STAB(G), but not for FSTAB(G) when $|C| > 2$. Balinski [7] showed that every vertex of FSTAB(G) is half-integral, i.e., $x_i = 0, 1$ or $1/2$ for $i = 1, \ldots, n$.

In the remainder of the section, we will assume $G = K_n$, the complete graph on n nodes, and study the clique inequality

$$\sum_{i=1}^n x_i \leq 1, \tag{14}$$

which is valid for STAB(K_n). Let P^n stand for FSTAB(K_n). In fact, STAB(K_n) equals the set of points in P^n that satisfy inequality (14). This inequality is not valid for P^n. Hartmann [22] showed that inequality (14) is a rank-$\lceil \log_2(n-1) \rceil$ Chvátal-Gomory cut for P^n. Therefore, the clique inequality can be obtained as a rank-k split cut for P^n for some $k \leq \lceil \log_2(n-1) \rceil$.

We show that the split rank of the clique inequality (14) essentially matches the Chvátal rank using the fact that P^n is an *anti-blocking polyhedron* (i.e., if $x \in P^n$ and $0 \leq y \leq x$, then $y \in P^n$). Anti-blocking polyhedra have the property that their Chvátal closure is also anti-blocking. A similar property holds for the split closure, and follows from the next lemma.

Lemma 3.7. *Consider $P^{IP} = P \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$ where P is an anti-blocking polyhedron. If $c^T x \le d$ is a valid split cut for P^{IP}, then $(c^+)^T x \le d$ is also a valid split cut where $c_i^+ = \max\{0, c_i\}$ for all $i = 1, \dots, n_1 + n_2$.*

Proof. See appendix. □

Theorem 3.8. *The split rank of the clique inequality (14) with respect to P^n for $n \ge 3$ is at least $\lceil \log_2(n-1) \rceil$.*

Proof. Let $N = \{1, \dots, n\}$ and let $e_1, \dots, e_n$ be the unit vectors in $\mathbb{R}^n$. In addition, let $\mathbf{0}$ and $\mathbf{1}$ be the vectors in $\mathbb{R}^n$ with all components 0 and 1, respectively. Let $P^{n,l}$ stand for $(\mathcal{S}^*)^l(P^n)$ for non-negative integers l. Note that $e_1, \dots, e_n$ and $\mathbf{0}$ belong to $STAB(K_n) \subseteq P^{n,l} \setminus S$ for all $l \ge 0$ and for any split set $S \in \mathcal{S}^*$. We will prove, by induction, that the point $\frac{1}{1+2^l}\mathbf{1}$ belongs to $P^{n,l}$ for all $l \ge 0$. The result will follow as $\frac{1}{1+2^l}\mathbf{1}$ does not satisfy the clique inequality when $1 + 2^l < n$.

It is easy to see that $\frac{1}{2}\mathbf{1}$ belongs to $P^{n,0}= P^n$. Let $l \ge 0$ and assume that $\frac{1}{1+2^l}\mathbf{1} \in P^{n,l}$. To obtain a contradiction, assume that $\frac{1}{1+2^{l+1}}\mathbf{1} \notin P^{n,l+1}$ for some $n \ge 1$. Then for some split set $S \in \mathcal{S}^*$, we have

$$\frac{1}{1+2^{l+1}}\mathbf{1} \notin \mathrm{conv}(P^{n,l} \setminus S) \Rightarrow \frac{1}{1+2^{l+1}}\mathbf{1} \notin P^{n,l} \setminus S. \tag{15}$$

This implies that

$$\frac{1}{1+2^{l}}\mathbf{1} \notin \mathrm{conv}(P^{n,l} \setminus S) \Rightarrow \frac{1}{1+2^{l}}\mathbf{1} \notin P^{n,l} \setminus S. \tag{16}$$

This is because $\frac{1}{1+2^{l+1}}\mathbf{1}$ is a convex combination of $\frac{1}{1+2^l}\mathbf{1}$ and $\mathbf{0}$; if $\frac{1}{1+2^l}\mathbf{1}$ belonged to $\mathrm{conv}(P^{n,l} \setminus S)$, then so would $\frac{1}{1+2^{l+1}}\mathbf{1}$.

Let $S = \{x \in \mathbb{R}^n : b < a^T x < b+1\}$, where $a = (a_1, \dots, a_n)$ is an integral vector and b is an integer. We can assume that $b \ge 0$; if $b < 0$, then we can multiply both a and b by -1 to get the same split set but with the desired representation. Then, from (16) we have $\frac{1}{1+2^l}\mathbf{1} \in S$, i.e.,

$$b < a^T(\frac{1}{1+2^l}\mathbf{1}) < b+1 \Rightarrow \frac{1}{1+2^l}\sum_{i=1}^n a_i = b + \frac{s}{1+2^l} \text{ for some } s \in \{1, \dots, 2^l\}.$$

Now consider the set $NZ = \{i \in N : a_i > 0\}$, i.e., the set of indices for which a_i is positive. If $|NZ| > 2^l$, choose an arbitrary subset NZ' of NZ of size 2^l. If $|NZ| \le 2^l$, then let $NZ' = NZ$. Note that in either case, we have $\sum_{i \in NZ'} a_i \ge s$. Consider the point p derived from $\frac{1}{1+2^l}\mathbf{1}$ by setting to zero all components corresponding to indices in NZ'. Then we have

$$a^T p = \frac{1}{1+2^l}(\sum_{i=1}^n a_i - \sum_{i \in NZ'} a_i) \le b.$$

Furthermore $p \in P^{n,l}$ as Lemma 3.7 implies that $P^{n,l}$ is an anti-blocking polyhedron. Therefore, $p \in P^{n,l} \setminus S$ and it differs from $\frac{1}{1+2^l}\mathbf{1}$ in at most 2^l components (corresponding to indices in NZ'), and in these components it has value zero. Therefore, the point $\frac{1}{1+2^{l+1}}\mathbf{1}$ can be expressed as a convex combination of points in $P^{n,l} \setminus S$ as follows:

$$\frac{1}{1+2^{l+1}}\mathbf{1} = \frac{1+2^l}{1+2^{l+1}}p + \frac{1}{1+2^{l+1}}\sum_{i\in NZ'} e_i + \frac{2^l - |NZ'|}{1+2^{l+1}}\mathbf{0}.$$

We therefore obtain a contradiction to (15) and the result follows. □

Let Q^n be the extended formulation of P^n as defined at the beginning of Section 3. As the 0-1 split closure of Q^n gives STAB(G) by Theorem 3.3, the clique inequality $\sum_{i=1}^n x_i \leq 1$ is valid for the 0-1 split closure of Q^n. We will next show that Q^n has a compact description.

It follows from results of Nemhauser and Trotter [28] that the set of vertices of P^n is given by the set $V = \{\mathbf{0}, e_1, \ldots, e_n\} \cup V_{1/2}$ where

$$V_{1/2} \quad = \bigcup_{I\subseteq N, |I|\geq 3} \{x \in \mathbb{R}^n : x_i = 1/2,\ \forall i \in I,\ x_i = 0,\ \forall i \in N \setminus I\}.$$

In other words, $V_{1/2}$ consists of all vectors in $\mathbb{R}^n$ with three or more nonzero components where all nonzero components have value 1/2. We next show that even though P^n has an exponential number of extreme points, its extended formulation Q^n is defined by $2n$ variables and $4n$ constraints in $\mathbb{R}^{2n}$.

Theorem 3.9. *$Q^n = X$, where*

$$X = \Big\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \quad 2x_i \geq y_i, \qquad \forall i \in N, \tag{17}$$

$$\sum_{j\in N} y_j \geq 3y_i, \qquad \forall i \in N, \tag{18}$$

$$2 \geq \sum_{j\in N}(2x_j - y_j) + 2y_i, \qquad \forall i \in N, \tag{19}$$

$$y_i \geq 0, \qquad \forall i \in N\Big\}. \tag{20}$$

Proof. See appendix. □

Consequently, the clique inequality (14), which has a split rank of at least $\lceil \log(n-1) \rceil$, is implied by rank 1 split cuts of the extended formulation above. Moreover, all split cuts for the extended formulation that are needed to imply the clique inequality (14) can be derived from 0-1 split sets in the extended space.

4 Properties of good extended LP formulations

As we show later in this section, extending Theorem 3.3 to general mixed-integer sets to show that there always exists an extended formulation with an integral split closure is not possible. However, one can still construct extended formulations that would lead to good relaxations after applying split and more general lattice-free cuts. We next discuss properties of good extended formulations for general mixed-integer sets.

Remember that given a collection of strictly lattice-free sets $\mathcal{L}$ for the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$, where $n_1 + n_2 = n$, and an extended formulation $Q \subseteq \mathbb{R}^{n+q}$ of $P \subseteq \mathbb{R}^n$, we define

$$\mathcal{L}(Q) = \bigcap_{L \in \mathcal{L}} \mathrm{conv}(Q \setminus L^+),$$

where $L^+ = L \times \mathbb{R}^q$. Note that L^+ is a strictly lattice-free set for the mixed-integer lattice $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2+q}$ for $L \in \mathcal{L}$. We next discuss properties of good extended formulations in terms of lattice-free cuts.

4.1 Minimality

Consider the following two *trivial* extended formulations of $P \subseteq \mathbb{R}^n$:

$$Q_0 = P \times \{0\}^q, \qquad \text{and} \qquad Q_\infty = P \times \mathbb{R}^q.$$

It is possible to show that both of these extended formulations are useless in terms of lattice-free cuts in the sense that

$$\mathcal{L}(P) = \mathrm{proj}_{\mathbb{R}^n}\left(\mathcal{L}(Q_0)\right) = \mathrm{proj}_{\mathbb{R}^n}\left(\mathcal{L}(Q_\infty)\right)$$

for any collection of strictly lattice-free sets $\mathcal{L}$. More generally, consider two sets $Q_1, Q_2 \subseteq \mathbb{R}^{n+q}$ and let $\mathcal{L}$ be a collection of sets in $\mathbb{R}^n$. Then,

$$Q_1 \subseteq Q_2 \quad \Rightarrow \quad Q_1 \setminus L^+ \subseteq Q_2 \setminus L^+, \ \ \forall L \in \mathcal{L} \quad \Rightarrow \quad \bigcap_{L \in \mathcal{L}} \mathrm{conv}(Q_1 \setminus L^+) \subseteq \bigcap_{L \in \mathcal{L}} \mathrm{conv}(Q_2 \setminus L^+).$$

Therefore, if $Q_1, Q_2 \subseteq \mathbb{R}^{n+q}$ are two different extended LP formulations of $P \subseteq \mathbb{R}^n$, then

$$Q_1 \subseteq Q_2 \quad \Rightarrow \quad \mathcal{L}(Q_1) \subseteq \mathcal{L}(Q_2) \quad \Rightarrow \quad \mathrm{proj}_{\mathbb{R}^n}\left(\mathcal{L}(Q_1)\right) \subseteq \mathrm{proj}_{\mathbb{R}^n}\left(\mathcal{L}(Q_2)\right)$$

for any collection of strictly lattice-free sets $\mathcal{L}$. Consequently, smaller (in terms of containment) extended formulations lead to stronger lattice-free cuts. This observation establishes that $\mathcal{L}(Q_0) \subseteq \mathcal{L}(Q_\infty)$.

We call an extended LP formulation *minimal* if it does not strictly contain any other extended LP formulation. Note that minimal extended formulations are not unique even for fixed q. We next define minimality formally.

Definition 4.1. *Let $Q \subseteq \mathbb{R}^{n+q}$ be an extended formulation of $P \subseteq \mathbb{R}^n$. It is called a minimal extended formulation if for all $Q' \subseteq \mathbb{R}^{n+q}$ such that $Q' \subsetneq Q$*

$$P \not\subseteq \mathrm{proj}_{\mathbb{R}^n}\left(Q'\right).$$

We next make a basic observation about minimal extended formulations.

Lemma 4.2. *Let Q be an extended formulation of a pointed polyhedron P. Q is a minimal extended formulation of P if and only if both of the following two conditions hold:*
(i) The pre-image of an extreme point (ray) of P is an extreme point (ray) of Q, and,
(ii) Q does not have any additional extreme points or extreme rays.

Proof. Let P be defined by its extreme points and extreme rays as follows:

$$P = \text{conv}(\bar{x}^1, \ldots, \bar{x}^m) + \text{cone}(\bar{r}^1, \ldots, \bar{r}^\ell).$$

As Q is an extended formulation of P, each point and each ray of P has at least one pre-image in Q. Let $\hat{x}^i = (\bar{x}^i, y^i) \in Q$ for $i = 1, \ldots, m$ and $\hat{r}^j = (\bar{r}^j, w^j) \in \text{rec}(Q)$ for $j = 1, \ldots, \ell$ and consider

$$Q' = \text{conv}(\hat{x}^1, \ldots, \hat{x}^m) + \text{cone}(\hat{r}^1, \ldots, \hat{r}^\ell).$$

Clearly $Q' \subseteq Q$. Furthermore, notice that any $x \in P$ can be written as $x = \sum_{i=1}^m \lambda_i \bar{x}^i + \sum_{j=1}^\ell \mu_j \bar{r}^j$ for some $\mu \in \mathbb{R}^\ell_+$ and $\lambda \in \mathbb{R}^m_+$ with $\mathbf{1}^T\lambda = 1$. But, as $\sum_{i=1}^m \lambda_i(\bar{x}^i, y^i) + \sum_{j=1}^\ell \mu_j(\bar{r}^j, w^j) \in Q'$, we have $x \in \text{proj}_x(Q')$ establishing that Q' is an extended formulation of P.

We next show that all $\hat{x}^i$ for $i = 1, \ldots, m$ and all $\hat{r}^j$ for $j = 1, \ldots, \ell$ are extreme for Q'. Assume not and let $(\bar{x}^t, y^t)$ be not extreme for some $m \geq t \geq 1$. Then $(\bar{x}^t, y^t) = \sum_{i=1}^m \lambda_i(\bar{x}^i, y^i) + \sum_{j=1}^\ell \mu_j(\bar{r}^j, w^j)$ for some $\mu \in \mathbb{R}^\ell_+$ and $\lambda \in \mathbb{R}^m_+$ with $\lambda_t = 0$ and $\mathbf{1}^T\lambda = 1$. However, in this case, using the same μ and λ it is possible to argue that $\bar{x}^t$ cannot be extreme for P either, proving the claim for extreme points. Using a similar argument, it is possible to show the claim for extreme rays as well. Therefore, Q' satisfies conditions (i) and (ii).

Assume Q is a minimal extended formulation, then Q' cannot be strictly contained in Q and $Q = Q'$. Consequently, Q satisfies conditions (i) and (ii).

Conversely, assume that Q satisfies conditions (i) and (ii). If Q is not a minimal extended formulation of P there exists another extended formulation $\hat{Q}$ strictly contained in Q. Therefore $\hat{Q}$ either does not contain one of the extreme points of Q, or, it has a smaller recession cone than that of Q. In this case, one of the extreme points or rays of P does not have a pre-image in $\hat{Q}$, contradicting the assumption that $\hat{Q}$ is an extended formulation. □

4.2 Increasing dimension

In this section we will show that a necessary condition for an extended LP formulation to yield stronger relaxations is that its dimension is strictly greater than that of the original formulation.

We start with an observation that highlights the difference between the dimension of an extended formulation and the dimension of the space it is defined in. Combining this result with Lemma 4.2, we will later bound the number of new variables necessary to obtain the strongest extended formulation in terms of lattice-free cuts.

Theorem 4.3. *Let $P \subseteq \mathbb{R}^n$ be a polyhedral set, $Q \subseteq \mathbb{R}^{n+q}$ be an extended formulation of it, and let $t = \dim(Q) - \dim(P)$. If $t < q$, then there exists an extended formulation $Q' \subseteq \mathbb{R}^{n+t}$ of P such that $\mathrm{proj}_{\mathbb{R}^n}(\mathcal{L}(Q)) = \mathrm{proj}_{\mathbb{R}^n}(\mathcal{L}(Q'))$ for any collection of strictly lattice-free sets $\mathcal{L}$.*

Proof. Let $d_P = \dim(P)$ and $d_Q = \dim(Q)$. First assume that $n - d_P > 0$, and therefore all $x \in P$ satisfy $(n - d_P)$ linearly independent equations of the form $Ax = b$, where A has full row rank. Similarly, all $(x, y) \in Q$ satisfy $(n + q - d_Q)$ linearly independent equations. Note that if $(x, y) \in Q$ then $x \in P$, and therefore $Ax = b$ holds. Consequently, points $(x, y) \in Q$ satisfy $Ax + 0y = b$ together with $(n + q - d_Q) - (n - d_P) = q - t$ additional linearly independent equalities of the form $Cx + Ey = d$. We then have the following system of equalities

$$\begin{matrix} n - d_P & \{ \\ q - t & \{ \end{matrix} \begin{bmatrix} A & 0 \\ C & E \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix},$$

where the matrix $[A, 0; C, E]$ has full row rank.

We will now argue that E has full row rank. For a contradiction, assume that the rank of E is less than $q - t$. In this case, after some elementary row operations on the last $q - t$ rows, we can obtain a row of the form $c^T x + 0y = g$ that has zero coefficients on y variables. The new the system has the following form:

$$\begin{matrix} n - d_P & \{ \\ 1 & \{ \\ q - t - 1 & \{ \end{matrix} \begin{bmatrix} A & 0 \\ c & 0 \\ C' & E' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ g \\ d' \end{bmatrix},$$

where the matrix $[A, 0; c, 0; C', E']$ has full row rank, and consequently c is linearly independent of the rows of A. Consequently there are $n - d_P + 1$ linearly independent equalities satisfied by all points in P, which contradicts with the fact that the dimension of P is d_P. Hence, E must have full row rank.

As E is a full row rank matrix, after some elementary row operations on the last $q - t$ rows, we can obtain the system

$$\begin{matrix} n - d_P & \{ \\ q - t & \{ \end{matrix} \begin{bmatrix} A & 0 & 0 \\ C'' & I & E'' \end{bmatrix} \begin{bmatrix} x \\ y_B \\ y_N \end{bmatrix} = \begin{bmatrix} b \\ d'' \end{bmatrix},$$

where I is the $(q - t) \times (q - t)$ identity matrix and y_B and y_N form a partition of y for an index set $B \subseteq \{1, \ldots, q\}$ of size $q - t$ and the complement set $N = \{1, \ldots, q\} \setminus B$. Therefore,

$$y_B = d'' - C''x - E''y_N$$

holds for all $(x, y) \in Q$. Projecting out linearly dependent variables from Q does not reduce the dimension. Therefore, if we let $Q' := \mathrm{proj}_{x, y_N}(Q)$, we have $\dim(Q') = d_Q$. Moreover, $Q' \subseteq \mathbb{R}^{n+t}$ is an extended LP formulation of P because

$$\mathrm{proj}_x(Q') = \mathrm{proj}_x\left(\mathrm{proj}_{x, y_N}(Q)\right) = \mathrm{proj}_x(Q) = P.$$

We next argue that $\mathcal{L}(Q') = \text{proj}_{x,y_N}(\mathcal{L}(Q))$. As Q is an extended LP formulation of Q', it holds that $\mathcal{L}(Q') \supseteq \text{proj}_{x,y_N}(\mathcal{L}(Q))$. We show the reverse inclusion by contradiction. Assume that there exists a point

$$(\bar{x}, \bar{y}_N) \in \mathcal{L}(Q') \setminus \text{proj}_{x,y_N}(\mathcal{L}(Q)). \tag{21}$$

Let $\bar{y}_B = d'' - C''\bar{x} - E''\bar{y}_N$. By (21), there is at least one strictly lattice-free set $L \in \mathcal{L}$ such that $(\bar{x}, \bar{y}_B, \bar{y}_N) \notin \text{conv}(Q \setminus (L \times \mathbb{R}^q))$ whereas, $(\bar{x}, \bar{y}_N) \in \text{conv}(Q' \setminus (L \times \mathbb{R}^t))$. Therefore, for some $l \in \mathbb{Z}_+$ we have

$$(\bar{x}, \bar{y}_N) \in \text{conv}\{(x^1, y_N^1), \ldots, (x^l, y_N^l)\},$$

where $(x^i, y_N^i) \in Q' \setminus (L \times \mathbb{R}^t)$ for all $i = 1, \ldots, l$. But in this case, $(x^i, y_B^i, y_N^i) \in Q \setminus (L \times \mathbb{R}^q)$ for all $i = 1, \ldots, l$ where $y_B^i = d'' - C''x^i - E''y_N^i$ and therefore $(\bar{x}, \bar{y}_B, \bar{y}_N) \in \text{conv}(Q \setminus (L \times \mathbb{R}^q))$, a contradiction. Therefore, the claim, $\text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q)) = \text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q'))$, holds when $d_P < n$.

We next consider the case when $d_P = n$. In this case, there are no equations satisfied by all points in P and $[C, E]$ has full row rank. Furthermore, by following the same steps, we see that E has full row rank and the rest follows similarly. Hence, $\text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q)) = \text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q'))$ holds when $d_P = n$ as well, which completes the proof. □

Remark 4.4. *We note that the proof of Theorem 4.3 does not use the fact that the sets in $\mathcal{L}$ are strictly lattice-free. Consequently, the result in fact holds for any collection of sets $\mathcal{L}$ where $\mathcal{L}(P)$ and $\mathcal{L}(Q)$ are defined for arbitrary sets as in equations (1) and (2).*

Remember the trivial extended formulations $Q_0 = P \times \{0\}^q$ and $Q_\infty = P \times \mathbb{R}^q$ defined earlier. Q_0 has the same dimension as P and by Theorem 4.3, applying lattice-free cuts to Q_0 is same as applying them directly to P. We next state this observation formally.

Corollary 4.5. *Let $P \subseteq \mathbb{R}^n$ be a polyhedral set, Q be an extended formulation of it. If $\dim(Q) = \dim(P)$, then $\text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q)) = \mathcal{L}(P)$ for any collection of strictly lattice-free sets $\mathcal{L}$.*

Therefore, $\text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q_0)) = \mathcal{L}(P)$, and consequently, $\text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q_\infty)) = \mathcal{L}(P)$ as $\mathcal{L}(Q_0) \subseteq \mathcal{L}(Q_\infty)$. Note that the construction presented in Section 3 has the property that the dimension of the resulting extended LP formulation Q^t is larger than that of the original formulation for all $t \geq 1$.

We next combine Theorem 4.3 and 4.2 to bound the number of new variables necessary to obtain the strongest extended formulation in terms of lattice-free cuts.

Corollary 4.6. *Let $P \subseteq \mathbb{R}^n$ be a polyhedral set and $Q \subseteq \mathbb{R}^{n+q}$ be a minimal extended formulation of it. Then $\dim(Q) \leq k - 1$ where k denotes the total number of extreme points and extreme rays of P.*

Furthermore, letting $t = k - 1 - \dim(P)$, if $q > t$, then there exists an extended formulation $Q' \subseteq \mathbb{R}^{n+t}$ of P such that $\text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q)) = \text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q'))$ for any collection of strictly lattice-free sets $\mathcal{L}$.

4.3 Limitations of extended LP formulations for general mixed-integer sets

Note that any minimal extended formulation of P has the same number of extreme points and extreme rays as P and therefore its dimension is bounded by the total number of extreme points and extreme rays of P. Therefore using Corollary 4.5 and 4.6 we observe that:

Corollary 4.7. *Let* $P \subseteq \mathbb{R}^n$ *be a pointed polyhedral set with* $\dim(P) = k - 1$ *where* k *denotes the total number of extreme points and extreme rays of* P*. Then,* $\text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q)) = \mathcal{L}(P)$ *for any extended formulation* Q *of* P *and any collection of strictly lattice-free sets* $\mathcal{L}$*.*

Now, remember the Cook, Kannan and Schrijver's example [13] $P^{IP}_{\text{CKS}} = P_{\text{CKS}} \cap (\mathbb{Z}^2 \times \mathbb{R})$ where

$$P_{\text{CKS}} = \text{conv}((0, 0, 0), (2, 0, 0), (0, 2, 0), (1/2, 1/2, 1)).$$

This mixed-integer set has unbounded split rank in the sense that one does not obtain the integer hull by applying split cuts repeatedly. Notice that P_{CKS} has exactly $\dim(P_{\text{CKS}}) + 1$ extreme points and therefore by Corollary 4.7 it does not have an extended formulation with a smaller split closure (after projection) than that of the original set P_{CKS}. Therefore unlike mixed-integer sets with 0-1 variables, it is not possible to show a result similar to Theorem 3.3 for mixed-integer sets with general integer variables. We note that the main theorem of [17] states that a polyhedron has finite split (lattice-free) rank if and only if its projection onto the space of integer variables satisfies certain properties. As an extended LP formulation has the same projection onto the space of integer variables as the original set, the result of Del Pia [17] also implies that there exists no extended formulation of P_{CKS} with finite split rank.

5 The strongest extended formulation for general mixed-integer sets

In this section we will give an extended formulation Q of a polyhedral set $P \subseteq \mathbb{R}^n$ which is strongest possible with respect to lattice-free cuts in the sense that for any other extended formulation Q' of P, $\text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q)) \subseteq \text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q'))$ for any collection of strictly lattice-free sets $\mathcal{L}$. As discussed in Section 4.3, it is not always possible to find an extended formulation of a general mixed-integer set that would lead to better lattice-free cuts. In such a case Q will not yield better lattice-free cuts than P. Moreover, if Q does not yield any better lattice-free cuts, that will imply that no extended LP formulation of P can yield better lattice-free cuts. For convenience, if A is a matrix, we will abuse notation and let conv(A) stand for the convex hull of the columns of A, and let cone(A) be the set of nonnegative linear combinations of the columns of A.

Let $P \subseteq R^n$ be a polyhedral set with $m > 0$ extreme points and $\ell \geq 0$ extreme rays,

$$P = \text{conv}(V) + \text{cone}(R), \tag{22}$$

where the columns of $V \in \mathbb{R}^{n\times m}$ and $R \in \mathbb{R}^{n\times \ell}$ correspond to extreme points and extreme rays of P, respectively. Define

$$X(P) = \text{conv}\begin{pmatrix} V \\ I_m \\ \mathbf{0} \end{pmatrix} + \text{cone}\begin{pmatrix} R \\ \mathbf{0} \\ I_\ell \end{pmatrix} \tag{23}$$

to be the extended formulation of P obtained by appending unit vectors to the extreme points and rays. Here we use I_s to denote the $s \times s$ identity matrix and $\mathbf{0}$ to denote a matrix of zeroes of appropriate dimension. It is easy to see that $P = \text{proj}_{\mathbb{R}^n}(X(P))$ and therefore $X(P)$ is indeed an extended formulation of P. In addition, by Lemma 4.2, $X(P)$ is minimal as there is a one-to-one correspondence between the extreme points and extreme rays of the two sets. Finally, note that the extreme points of $X(P)$ are affinely independent. In addition, points obtained by adding any one of the extreme rays of $X(P)$ to a fixed extreme point of $X(P)$ gives additional affinely independent points, establishing that the dimension of $X(P)$ is at least $m + \ell - 1$. Combined with Corollary 4.6, this means that $\dim(X(P)) = m + \ell - 1$. We next show that $X(P)$ is the strongest possible extended formulation of P with respect to lattice-free cuts.

Theorem 5.1. *Let Q be an extended formulation of $P \subseteq \mathbb{R}^n$. Then*

$$\text{proj}_{\mathbb{R}^n}(\mathcal{L}(X(P))) \subseteq \text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q))$$

for any collection $\mathcal{L}$ of strictly lattice-free sets in $\mathbb{R}^n$.

Proof. Without loss of generality, we can assume that Q is a minimal extended formulation of P. By Lemma 4.2, there is a one-to-one correspondence between the extreme points and extreme rays of P and Q, respectively. In other words, for some integer $k > 0$ there is a $k \times m$ matrix V' and another $k \times \ell$ matrix R' such that

$$Q = \text{conv}\begin{pmatrix} V \\ V' \end{pmatrix} + \text{cone}\begin{pmatrix} R \\ R' \end{pmatrix}.$$

We next define a polyhedral set Q^+ as follows:

$$Q^+ = \text{conv}\begin{pmatrix} V \\ V' \\ I_m \\ \mathbf{0} \end{pmatrix} + \text{cone}\begin{pmatrix} R \\ R' \\ \mathbf{0} \\ I_\ell \end{pmatrix}.$$

As Q^+ is an extended formulation of Q and also an extended formulation of $X(P)$, we have

$$\text{proj}_{\mathbb{R}^n}\left(\mathcal{L}(Q^+)\right) \subseteq \text{proj}_{\mathbb{R}^n}(\mathcal{L}(Q)) \quad \text{and} \quad \text{proj}_{\mathbb{R}^n}\left(\mathcal{L}(Q^+)\right) \subseteq \text{proj}_{\mathbb{R}^n}(\mathcal{L}(X(P))) \tag{24}$$

by Theorem 2.1. On the other hand, $\dim(Q^+) = m + \ell - 1$ and therefore $\dim(Q^+) = \dim(X(P))$ and consequently

$$\text{proj}_{\mathbb{R}^n}(\mathcal{L}(X(P))) = \text{proj}_{\mathbb{R}^n}\left(\mathcal{L}(Q^+)\right) \tag{25}$$

by Corollary 4.5. Combining inequality (24) and (25) completes the proof. $\square$

Note that, we can write $X(P)$ explicitly as follows:

$$X(P) = \left\{ \begin{pmatrix} x \\ \lambda \\ \mu \end{pmatrix} \in \mathbb{R}^{n+m+\ell} \;:\; \exists \bar{\lambda}, \bar{\mu} \;\text{ s.t. } \begin{pmatrix} x \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} V \\ I_m \\ \mathbf{0} \end{pmatrix} \bar{\lambda} + \begin{pmatrix} R \\ \mathbf{0} \\ I_\ell \end{pmatrix} \bar{\mu}, \quad \mathbf{1}^T\bar{\lambda} = 1, \quad \bar{\lambda}, \bar{\mu} \geq 0 \right\}.$$

Clearly, the equations defining $X(P)$ imply that $\lambda = \bar{\lambda}$ and $\mu = \bar{\mu}$ and therefore we can rewrite this set as follows:

$$X(P) = \{(x, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell \;:\; x = V\lambda + R\mu,\; \mathbf{1}^T\lambda = 1,\; \lambda, \mu \geq 0\}. \tag{26}$$

Writing P explicitly, we have:

$$P = \{x \in \mathbb{R}^n : \exists \lambda, \mu \text{ s.t. } x = V\lambda + R\mu,\; \mathbf{1}^T\lambda = 1,\; \lambda, \mu \geq 0\}. \tag{27}$$

Therefore the only difference between P and $X(P)$ is that the multipliers for the extreme points and extreme rays in the definition of P are explicit variables in $X(P)$. It is somewhat surprising that this seemingly trivial change can result in stronger lattice-free cuts for $X(P)$ compared to the lattice-free cuts for P.

We next study the extended formulation $X(P)$ in more detail for the special case of split sets. Our results can be extended to general strictly lattice-free sets but it requires more notation. In [3, Thm 3.3] Balas proved a result on convex hulls of unions of polyhedra which, in case of a split set $S^k = \{x \in \mathbb{R}^n : \gamma^k + 1 > (\pi^k)^T x > \gamma^k\}$ and polyhedron $\hat{P} = \{x \in \mathbb{R}^n : Ax \leq b\}$, implies that

$$\begin{aligned}
\text{conv}(\hat{P} \setminus S^k) = \{x \in \mathbb{R}^n \;:\; \exists \bar{x}\;, \bar{\bar{x}} \in \mathbb{R}^n, \nu \in \mathbb{R} \;\text{ s.t.} & \qquad (28)\\
x = \bar{x} + \bar{\bar{x}}, \qquad & 0 \leq \nu \leq 1, \qquad (29)\\
A\bar{x} \leq \nu b, \qquad & A\bar{\bar{x}} \leq (1-\nu)b, \qquad (30)\\
(\pi^k)^T x \leq \nu\gamma^k, \qquad & (\pi^k)^T x \geq (1-\nu)(\gamma^k + 1)\}. \qquad (31)
\end{aligned}$$

Let P be defined as in equation (27) and consider a collection of split sets $\mathcal{S} = \{S^k : k \in K\}$. Using the extended formulation of the split closure with respect to a single split set in (28)-(31), we can write $\mathcal{S}(P)$ via an extended formulation as

$$\begin{aligned}
\mathcal{S}(P) = \Big\{ x \in \mathbb{R}^n : \exists \bar{x}^k, \bar{\bar{x}}^k, \bar{\lambda}^k, \bar{\bar{\lambda}}^k, \bar{\mu}^k, \bar{\bar{\mu}}^k, \nu^k \text{ s.t.} & \\
x = \bar{x}^k + \bar{\bar{x}}^k, \qquad 0 \leq \nu^k \leq 1, \qquad & k \in K,\\
\bar{x}^k = V\bar{\lambda}^k + R\bar{\mu}^k, \qquad \bar{\bar{x}}^k = V\bar{\bar{\lambda}}^k + R\bar{\bar{\mu}}^k, \qquad & k \in K,\\
\mathbf{1}^T\bar{\lambda}^k = \nu^k, \qquad \mathbf{1}^T\bar{\bar{\lambda}}^k = 1 - \nu^k, \qquad & k \in K,\\
(\pi^k)^T\bar{x}^k \leq \nu^k\gamma^k, \qquad (\pi^k)^T\bar{\bar{x}}^k \geq (1-\nu^k)(\gamma^k + 1), \qquad & k \in K,\\
\bar{\lambda}^k, \bar{\mu}^k \geq 0, \qquad \bar{\bar{\lambda}}^k, \bar{\bar{\mu}}^k \geq 0, \qquad & k \in K \Big\}. \qquad (32)
\end{aligned}$$

Similarly, $\mathcal{S}(X(P))$ can be written as the set of points (x, λ, μ) such that x satisfies the inequalities in (32), and, in addition, λ and μ satisfy $\lambda = \bar{\lambda}^k + \bar{\bar{\lambda}}^k$ and $\mu = \bar{\mu}^k + \bar{\bar{\mu}}^k$ for all $k \in K$. But these conditions on λ and μ imply the following result.

Corollary 5.2. *Let P and $X(P)$ be defined as in equations (27) and (26), and let $\mathcal{S} = \{S^k : k \in K\}$ be a given collection of split sets. Then,*

$$\mathrm{proj}_x\left(\mathcal{S}(X(P))\right) = \big\{x \in \mathbb{R}^n \ : \ \exists \bar{x}^k, \bar{\bar{x}}^k, \bar{\lambda}^k, \bar{\bar{\lambda}}^k, \bar{\mu}^k, \bar{\bar{\mu}}^k, \nu^k \quad \text{s.t.} \quad (32) \text{ holds}, \\ \bar{\lambda}^k + \bar{\bar{\lambda}}^k = \bar{\lambda}^{k'} + \bar{\bar{\lambda}}^{k'}, \ \ \bar{\mu}^k + \bar{\bar{\mu}}^k = \bar{\mu}^{k'} + \bar{\bar{\mu}}^{k'} \text{ for all } k \neq k' \in K\big\}. \tag{33}$$

To understand the implications of Corollary 5.2, consider the case that P is a polytope, thus R and $\mu, \bar{\mu}^k, \bar{\bar{\mu}}^k$ do not exist in (32) and in Corollary 5.2. If $x \in \mathcal{S}(P)$, then for some $k \in K$, x must equal the convex combination of two points $\bar{x}^k$ and $\bar{\bar{x}}^k$ (call these points *friends* of x with respect to the split S^k), where $\bar{x}^k$ is a convex combination of the columns of V where the multiplier for the j^{th} column of V is $\bar{\lambda}^k_j$, and $\bar{\bar{x}}^k$ is defined similarly in terms of $\bar{\bar{\lambda}}^k$. If $x \in \mathrm{proj}_x\left(\mathcal{S}(X(P))\right)$, then for the j^{th} column of V, we have $\bar{\lambda}^k_j + \bar{\bar{\lambda}}^k_j = \bar{\lambda}^{k'}_j + \bar{\bar{\lambda}}^{k'}_j$ for $k \neq k' \in K$. In other words, for each $k \in K$, the two friends of x with respect to the split S^k are dependent on the two friends of x for $k' \neq k$ in the sense that the sum of the multipliers of the j^{th} column of V for the two friends for k is exactly equal to the same sum for $k' \neq k$.

Lastly, we note that the upper bound of $m + \ell$ on the number additional continuous variables used to obtain the strongest extended formulation with respect to lattice-free cuts is not tight. We next provide a special case when we can tighten the bound. In particular, we show that if some of the extreme rays are unit vectors, then the additional variables associated with them can be easily projected out without changing the strength of the extended formulation.

Lemma 5.3. *Let P and $X(P)$ be defined as in equations (22) and (23). Let $R_U \subseteq R$ be the set of extreme rays of P which are unit vectors. Then, there exists an extended formulation $\tilde{X} \subseteq \mathbb{R}^{n+m+\ell-|R_U|}$ of P such that $\mathrm{proj}_{\mathbb{R}^n}\left(\mathcal{L}(\tilde{X})\right) = \mathrm{proj}_{\mathbb{R}^n}\left(\mathcal{L}(X(P))\right)$ for any collection $\mathcal{L}$ of strictly lattice-free sets in $\mathbb{R}^n$.*

Proof. Let $k = |R_U| \leq n$ and without loss of generality assume that the vectors in R are ordered in such a way that the first k vectors form R_U. Define

$$\tilde{X} = \mathrm{conv}\begin{pmatrix} V \\ I_m \\ \mathbf{0} \end{pmatrix} + \mathrm{cone}\begin{pmatrix} R_U & R \setminus R_U \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{\ell-k} \end{pmatrix}.$$

$X(P)$ is clearly an extended formulation of $\tilde{X}$. As noted before, we know that $\dim(X(P)) = m + \ell - 1$. Similarly, by Corollary 4.6 we have $\dim(\tilde{X}) \leq m + \ell - 1$. Also, it is easy to see that the extreme points of $\tilde{X}$ together with points obtained by adding any one of the extreme rays of $\tilde{X}$ to a fixed extreme point of $\tilde{X}$ give $m+\ell$ affinely independent points, establishing that $\dim(\tilde{X}) \geq m+\ell-1$. As $\dim(\tilde{X}) = \dim(X(P))$, from Corollary 4.5, we obtain $\text{proj}_{\mathbb{R}^{n+m+\ell-k}} (\mathcal{L}(X(P))) = \mathcal{L}(\tilde{X})$, which implies the desired result. □

5.1 Mixing set

We next consider the *mixing set*, introduced by Günlük and Pochet [21]:

$$P^{MIX} = P^{LP} \cap (\mathbb{R}_+ \times \mathbb{Z}^n), \text{ where } P^{LP} = \{(s,x) \in \mathbb{R}_+ \times \mathbb{R}^n : s + x_i \geq b_i,\ i = 1, \dots, n\}.$$

Without loss of generality, assume that $b_1 < b_2 < \cdots < b_n$. For convenience, define $b_0 = 0$. P^{LP} has one extreme point, which is $(0,b)$, and $n+1$ extreme rays, namely $(1, -\mathbf{1})$ and $\{(0, e_i)\}_{i=1,\dots,n}$. From Corollay 4.7, we know that it is not possible to obtain extended formulations of it that would yield stronger split cuts.

Now, we focus on a restriction of P^{MIX} obtained by enforcing non-negativity constraints on the integer variables. Let $P_+^{MIX} = P^{MIX} \cap \mathbb{R}_+^{n+1}$ and let $P_+^{LP} = P^{LP} \cap \mathbb{R}_+^{n+1}$. Note that P_+^{LP} has $n+1$ extreme points and $n+1$ extreme rays, namely

$$V_+^{LP} = \{(\hat{s}^j, \hat{x}^j)\}_{j=0,\dots,n} \text{ and } R_+^{LP} = \{(0, e_i)\}_{i=1,\dots,n} \cup (1, \mathbf{0}),$$

where for $j \in \{0, \dots, n\}$, the vertex $(\hat{s}^j, \hat{x}^j)$ is defined as

$$\hat{s}^j = b_j,\ \hat{x}_i^j = 0 \text{ for } i = 1, \dots, j \text{ and } \hat{x}_i^j = b_i - b_j \text{ for } i = j+1, \dots, n.$$

Let X_+^{LP} denote the extended formulation of P_+^{LP} as defined in (26), that is,

$$X_+^{LP} = \{(s,x,\lambda,\mu) \in \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1} : \\ (s,x) = \sum_{j=1}^{n+1} \lambda_j (b_{j-1}, \hat{x}^{j-1}) + \sum_{j=1}^{n} \mu_j (0, e_i) + \mu_{n+1}(1, \mathbf{0}),\ \sum_{j=1}^{n+1} \lambda_j = 1\}.$$

We next give a smaller extended formulation of P_+^{LP} defined by $2n+2$ inequalities in $\mathbb{R}^{2n+1}$ by projecting out the last $n+2$ variables from X_+^{LP}. Projecting out μ variables from X_+^{LP} gives :

$$\text{proj}_{(s,x,\lambda)} \left(X_+^{LP} \right) = \{(s,x,\lambda) \in \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R}_+^{n+1} : (s,x) \geq \sum_{j=1}^{n+1} \lambda_j (b_{j-1}, \hat{x}^{j-1}),\ \mathbf{1}^T \lambda = 1\}.$$

Then, it is easy to see that if we further project out the variable λ_{n+1} using $\lambda_{n+1} = 1 - \sum_{j=1}^{n} \lambda_j$

and $\lambda_{n+1} \geq 0$, we obtain the set

$$\mathcal{X} = \Big\{(s, x, \lambda) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i \leq 1, \tag{34}$$

$$s + \sum_{i=1}^{n} (b_n - b_{i-1})\lambda_i \geq b_n, \tag{35}$$

$$x_i \geq \sum_{j=1}^{i} (b_i - b_{j-1})\lambda_j, \qquad i = 1, \dots, n, \tag{36}$$

$$\lambda_i \geq 0, \qquad i = 1, \dots, n\Big\}. \tag{37}$$

Therefore, $\mathcal{X} = \text{proj}_{(s,x,\lambda_1,\dots,\lambda_n)}\left(X_+^{LP}\right)$. Furthermore, as $\text{proj}_{(s,x)}(\mathcal{X}) = P_+^{LP}$, $\mathcal{X}$ is indeed an extended formulation of P_+^{LP}.

In the remainder of this section, we assume that $0 = b_0 < b_1 < b_2 < \cdots < b_n \leq 1$ and study the so-called *mixing inequalities*, which are introduced by Günlük and Pochet [21]. For all $I' = \{i_1, \dots, i_t\} \subseteq \{1, \dots, n\}$, the $|I'|$*-term mixing inequality of type I*

$$s + b_{i_1} x_{i_1} + \sum_{k=2}^{|I'|} (b_{i_k} - b_{i_{k-1}}) x_{i_k} \geq b_{i_{|I'|}} \tag{38}$$

is known to be facet-defining for the convex hull of the mixing set P^{MIX} [21].

Dash and Günlük [16] show that the split rank of the $|I'|$-term mixing inequality of type I is at most $|I'|$ for the set P^{LP} and therefore for P_+^{LP}. In addition, Dey [18] presents a general technique to obtain lower bounds on the split rank of intersection cuts that gives a lower bound of $\lceil \log_2(|I'| + 1) \rceil$ on the split rank of inequality (38) for P^{LP}. Proposition 3 and Corollary 1 in [18] easily extend to P_+^{LP} (by treating non-negativity constraints as additional constraints) and Dey's results imply that the split rank of inequality (38) is at least $\lceil \log_2(|I'| + 1) \rceil$ for P_+^{LP}. We next show that the mixing inequality (38) is implied by 0-1 split cuts for the extended formulation $\mathcal{X}$.

Lemma 5.4. *Suppose that* $0 = b_0 < b_1 < b_2 < \cdots < b_n \leq 1$ *and let* $I' = \{i_1, \dots, i_t\} \subseteq \{1, \dots, n\}$ *be a subset of indices given in increasing order. Then, the* $|I'|$*-term mixing inequality of type I given in* (38) *is valid for* $\mathcal{S}^{0,1}(\mathcal{X})$*, where* $\mathcal{X}$ *is defined by* (34)-(37).

Proof. We first show that for any $i \in \{1, \dots, n\}$, the inequality

$$x_i \geq \sum_{j=1}^{i} \lambda_j \tag{39}$$

is valid for $\text{conv}(\mathcal{X} \setminus S_i^+)$, where $S_i^+ = \{(s, x, \lambda) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n : 0 < x_i < 1\}$. If $x_i \leq 0$, then $x_i = 0$ and so, from (36), $\lambda_j = 0$ for all $j = 1, \dots, i$. Thus, (39) is valid for $\mathcal{X} \cap \{x_i \leq 0\}$. On the other hand, if $x_i \geq 1$, then (34) implies that $\sum_{j=1}^{n} \lambda_j \leq 1 \leq x_i$. Therefore, (39) is valid for $\mathcal{X} \cap \{x_i \geq 1\}$ as well.

Secondly, we show that for any $k \in \{1, \ldots, n\}$, the inequality

$$s + \sum_{i=1}^{k} (b_k - b_{i-1})\lambda_i \geq b_k \tag{40}$$

is valid for $\mathcal{X}$. Rewriting inequality (35) we obtain

$$s + \sum_{i=1}^{k} (b_n - b_{i-1})\lambda_i \geq b_n - \sum_{i=k+1}^{n} (b_n - b_{i-1})\lambda_i,$$

which, after subtracting $\sum_{i=1}^{k} (b_n - b_k)\lambda_i$ from both sides of the inequality, can be equivalently written as

$$s + \sum_{i=1}^{k} (b_k - b_{i-1})\lambda_i \geq b_n - \Big(\sum_{i=k+1}^{n} (b_n - b_{i-1})\lambda_i + \sum_{i=1}^{k} (b_n - b_k)\lambda_i \Big). \tag{41}$$

Due to (34) and (37), we also know that

$$\sum_{i=k+1}^{n} (b_n - b_{i-1})\lambda_i + \sum_{i=1}^{k} (b_n - b_k)\lambda_i \leq b_n - b_k.$$

Consequently, the right hand side of inequality (41) can be weakened to obtain (40).

If we multiply the inequalities (39) corresponding to the indices $i_k \in I'$ by $(b_{i_k} - b_{i_{k-1}})$ and add those to the inequality (40) corresponding to $k = i_{|I'|}$, we get

$$b_{i_1} x_{i_1} + \sum_{k=2}^{|I'|} (b_{i_k} - b_{i_{k-1}}) x_{i_k} + s + \sum_{j=1}^{i_{|I'|}} (b_{i_{|I'|}} - b_{j-1})\lambda_j \geq b_{i_{|I'|}} + b_{i_1} \sum_{j=1}^{i_1} \lambda_j + \sum_{k=2}^{|I'|} (b_{i_k} - b_{i_{k-1}}) \sum_{j=1}^{i_k} \lambda_j,$$

whose right hand side can be written as

$$b_{i_{|I'|}} + \sum_{j=1}^{i_1} \lambda_j \underbrace{\Big(\sum_{k=1}^{|I'|} (b_{i_k} - b_{i_{k-1}}) \Big)}_{b_{i_{|I'|}}} + \sum_{j=i_1+1}^{i_2} \lambda_j \underbrace{\Big(\sum_{k=2}^{|I'|} (b_{i_k} - b_{i_{k-1}}) \Big)}_{b_{i_{|I'|}} - b_{i_1}} + \cdots + \sum_{j=i_{|I'|-1}+1}^{i_{|I'|}} \lambda_j (b_{i_{|I'|}} - b_{i_{|I'|-1}}).$$

Therefore, we obtain

$$s + b_{i_1} x_{i_1} + \sum_{k=2}^{|I'|} (b_{i_k} - b_{i_{k-1}}) x_{i_k} \geq b_{i_{|I'|}} + \sum_{j=1}^{i_1} b_{j-1}\lambda_j + \sum_{j=i_1+1}^{i_2} \underbrace{(b_{j-1} - b_{i_1})}_{\geq 0} \lambda_j + \sum_{j=i_2+1}^{i_3} \underbrace{(b_{j-1} - b_{i_2})}_{\geq 0} \lambda_j + \cdots,$$

whose right hand side can be weakened to get the desired valid inequality. □

5.2 Two-row continuous group relaxation

In this section, we consider the so-called *two-row continuous group relaxation* first studied by Andersen, Louveaux, Weismantel and Wolsey [2]:

$$W = \Big\{(x, s) \in \mathbb{Z}^2 \times \mathbb{R}^n \ : \ x = \hat{f} + \sum_{j \in J} \hat{r}^j s_j, \ s \geq 0\Big\},$$

where $\hat{f} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ and $\hat{r}^j \in \mathbb{Q}^2 \setminus \{0\}$, for all $j \in J = \{1, \ldots, n\}$. This set arises as a relaxation of a mixed-integer set obtained by selecting two equations valid for the set (e.g., by using an associated simplex tableau), and then relaxing all other inequalities together with the integrality constraint on all but two of the variables and non-negativity constraints on the two remaining integer variables. This set has attracted some attention in recent years, see [1, 19] and references therein.

Let W^{LP} denote the continuous relaxation of W and notice that as W^{LP} has $n+2$ variables and is defined by two (linearly independent) equations and n inequalities, it has dimension n and has exactly one extreme point $(\hat{f}, \mathbf{0})$. Furthermore, it is known that W^{LP} has exactly n extreme rays $(\hat{r}^j, e^j)$ where $e^j \in \mathbb{R}^n$ denotes the unit vector where the j^{th} component is 1. As W^{LP} has dimension n and is defined by $n+1$ extreme points and extreme rays, by Corollary 4.7 we know that it is not possible to obtain extended formulations of it that would lead to stronger split cuts.

Now consider a restriction of W obtained by imposing non-negativity constraints on the integer variables, i.e., $W_+ = W \cap \mathbb{R}^{n+2}_+$, and let $W^{LP}_+ = W^{LP} \cap \mathbb{R}^{n+2}_+$. Notice that W^{LP}_+ is defined by two equations and $n+2$ inequalities and potentially has $\binom{n+2}{n}$ extreme points and $\binom{n+2}{n-1}$ extreme rays. Therefore, Corollary 4.7 does not apply and we can construct an extended formulation it of the form (23) that has higher dimension than W^{LP}_+. Consequently, the extended formulation may provide a better approximation of the integer set after being strengthened with lattice-free cuts and in particular, split cuts.

We perform computational experiments to compare the strength of split cuts derived from the original formulation and its extended formulation. To construct the extended formulation $X(W^{LP}_+)$, we generate all extreme points and extreme rays of W^{LP}_+ as follows: We first enumerate all possible candidate $\binom{n+2}{2}$ extreme points and $\binom{n+2}{3}$ extreme rays, and then simply choose the non-negative ones among these to build the extended formulation (26). Furthermore, for a given list of splits, we can optimize over their split closure by solving either (32) or (33) for the formulations W^{LP}, W^{LP}_+ and $X(W^{LP}_+)$.

To compare these formulations, we generated a number of test instances as follows:

1. We generated instances with $n \in \{10, 20, \ldots, 100\}$ and for each n, we generated 100 instances.
2. For each instance we generated $\hat{f} \in (0, 1)^2$ uniformly at random.
3. We generated all $\hat{r}^j \in (-1, 1)^2$, for all $j \in J$, again uniformly at random.

4. We also generated an objective function $c^T x + d^T s$ for each instance such that $c = \mathbf{0}$ and $d \in (0,1)^n$ is generated uniformly at random.

To generate cuts, we use 16 split sets in total that include the two elementary split sets $S_i = \{1 > x_i > 0\}$ for $i = 1,2$ together with the 14 non-elementary split sets used in [19], namely, $S(\pi,\gamma) = \{\gamma + 1 > \pi^T x > \gamma\}$ for all $\pi \in [-3,3]^2$ with co-prime elements and $\gamma = \lfloor \pi^T \hat{f} \rfloor$. We denote this collection of split sets by $\mathcal{S}_{16}$.

For a given instance, let z_{IP} and z_{LP} denote the optimal objective function value obtained by minimizing $d^T s$ over the sets W_+ and W_+^{LP}, respectively. Note that for our test instances z_{LP} value is always zero as $(x,s) = (\hat{f}, \mathbf{0})$ is a feasible solution and $d \geq 0$. For any relaxation R of W_+, we measure the strength of the split cuts applied to the relaxation by the percentage optimality gap closed, calculated as

$$\text{Gap closed} = 100 * (z_R - z_{LP})/(z_{IP} - z_{LP}),$$

where $z_R = \{\min\ d^T s : (x,s) \in \mathcal{S}_{16}(R)\}$. We used IBM ILOG Cplex 12.4 as the LP/MILP solver.

In Table 1, we report the average percentage optimality gap closed by three relaxations strengthened by split cuts in an incremental fashion. We denote the gap closed by the split cuts applied to formulations W^{LP}, W_+^{LP} and $X(W_+^{LP})$ by R_0, R_1 and R_2, respectively. Clearly, $100 \geq R_2 \geq R_1 \geq R_0 \geq 0$. The averages are taken over the instances where there is an improvement and the numbers in parentheses show the number of such instances. For example, the second column of the first row means that $R_1 > R_0$ in 48 instances and the average difference taken over these instances is 14.74.

$\lvert J\rvert$	R_0	$R_1 - R_0$	$R_2 - R_1$	Remaining
10	87.93 (100)	14.74 (48)	6.88 (17)	18.23 (21)
20	89.97 (100)	15.32 (38)	5.32 (21)	14.05 (22)
30	89.34 (100)	15.76 (40)	4.99 (26)	11.76 (26)
40	91.08 (100)	14.57 (30)	6.14 (17)	14.02 (25)
50	89.67 (100)	12.95 (40)	6.50 (22)	14.89 (25)
60	86.94 (100)	13.98 (53)	5.80 (26)	14.27 (29)
70	91.49 (100)	10.64 (40)	5.61 (19)	16.80 (19)
80	92.20 (100)	9.21 (43)	3.56 (27)	10.66 (27)
90	91.04 (100)	10.81 (40)	6.61 (28)	8.97 (31)
100	88.54 (99)	21.19 (32)	5.15 (16)	18.24 (25)

Table 1: Incremental Average Gap Closed Percentages over nonzeros and nonzero counts

We first observe that split cuts are computationally very effective when applied to the original relaxation W^{LP} which does not have non-negativity constraints on the integer variables. This was also observed in [19]. When non-negativity constraints on the integer variables are present, we see a difference in 40% of the instances, and the improvement on the gap closed is significant. When we go one step further and apply split cuts to the extended formulation $X(W_+^{LP})$, in 22% of the instances we close an extra gap of 5.7%. Therefore, the effect of applying split cuts to the extended formulation is quite noticeable. We also note that in 75% of the instances split cuts close the optimality gap completely but in the rest there is still a noticeable gap remaining. In 32% of the instances split cuts applied to the original formulation W_+^{LP} does not close all of the optimality gap, but this number decreases by 7% when the extended formulation $X(W_+^{LP})$ is used instead. Finally, we also experimented with using many more split sets but did not observe significantly different behaviour.

6 Concluding remarks

In this paper we described how to construct the best possible extended LP formulation of a given mixed-integer set in terms of lattice-free cuts. Our construction, however, uses an inner description of the original LP relaxation which is usually not given explicitly. In terms of practical applications, computing the inner description might be prohibitively time consuming except for very special sets. However, for practical applications, one does not need the best possible extended LP formulation. In practice, we need extended formulations that are easy to construct and yet give better bounds after adding cuts that are already implemented in current software. The remaining practical question is how to construct such extended LP formulations for general mixed-integer sets.

Throughout the paper we work with pointed polyhedra for convenience but our results hold for non-pointed polyhedra as well. To adapt the proofs of our results, one should represent P uniquely as $P = P^B + P^R + P^{LS}$ where P^{LS} is the lineality space of P and $P^B + P^R$ is contained in the orthogonal complement of P^{LS}. Furthermore P^B is a polytope and P^R is a pointed cone. In this setting all references to "extreme points" of P have to be replaced by "minimal faces" of P which are extreme points of P^B plus P^{LS}. For example, in Lemma 4.2, there will be a one-to-one correspondence between the minimal faces of P and its extended formulation.

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Appendix

1. Proof of Lemma 3.7

Let $n = n_1 + n_2$. As P is anti-blocking, it is defined by nonredundant system of inequalities $P = \{x \in \mathbb{R}^n : Ax \leq b;\ x \geq 0\}$ where $A, b \geq 0$. Assume that the cut $c^T x \leq d$ can be derived using the split set $S(\pi, \gamma)$ and let $c_k < 0$ for some index k.

As $c^T x \leq d$ is valid for $P^1 = \{x \in \mathbb{R}^n : Ax \leq b,\ \pi^T x \leq \gamma,\ x \geq 0\}$, there exists multipliers $\mu_1, \mu_2 \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}^n_+$, such that

$$c = \mu_1 a + \mu_2 \pi - \lambda, \qquad \text{and} \qquad d \geq \mu_1 f + \mu_2 \gamma,$$

where $a^T x \leq f$ is a non-negative linear combination of the rows of $Ax \leq b$ (and therefore $a \geq 0$). Now consider the split set $S(\pi^+, \gamma)$ where π^+ is identical to π except $\pi^+_k = 0$. Further let λ^+ be identical to λ except $\lambda^+_k = \mu_1 a_k \geq 0$. Now, observe that the inequality $\bar{c}^T x \leq d$ is valid for $\bar{P}^1 = \{x \in \mathbb{R}^n : Ax \leq b,\ (\pi^+)^T x \leq \gamma,\ x \geq 0\}$ where $\bar{c} = \mu_1 a + \mu_2 \pi^+ - \lambda^+$ and note that $\bar{c}$ is identical to c except $\bar{c}_k = 0$.

Using the same argument for $P^2 = \{x \in \mathbb{R}^n : Ax \leq b,\ \pi^T x \geq \gamma + 1,\ x \geq 0\}$ establishes that $\bar{c}^T x \leq d$ is valid for P^2 and therefore is a valid split cut for P^{IP}.

Repeating this process for all negative components of c completes the proof. □

2. Proof of Theorem 3.9

Given a point $(x, y) \in X$, let $T_{ny} = \{j \in N : y_j > 0\}$, i.e., T_{ny} stands for the indices of nonzero components of y. Let y_{max} be the maximum value among all components of y, and let $T_{max} = \{j \in N : y_j = y_{max}\}$. Let

$$\begin{aligned} T_2 &= \{i \in N : (18) \text{ for index } i \text{ is tight for } (x,y)\} \text{ and} \\ T_3 &= \{i \in N : (19) \text{ for index } i \text{ is tight for } (x,y)\}. \end{aligned}$$

If (x, y) is an extreme point of X, it must be the unique solution of a list L of some $2n$ linearly independent constraints from (17) - (20). From among these $2n$ constraints, let L_1, L_2, L_3, L_y stand for, respectively, the sets of indices of constraints of types (17), (18), (19) and (20) that are contained in L. By definition, we have

$$L_2 \subseteq T_2,\ L_3 \subseteq T_3,\ T_{ny} \neq \emptyset \Longrightarrow T_{max} \subseteq T_{ny}, \tag{42}$$

$$L_y \subseteq N \setminus T_{ny} \text{ and } |L_1| + |L_2| + |L_3| + |L_y| = 2n. \tag{43}$$

Using these definitions, we first show some properties which we need in the proof.

Lemma 6.1. *Let (x, y) be an extreme point of X with $y \neq 0$. Then*
(i) $T_2, T_3 \subseteq T_{max}$ and $|L_2 \cup L_3| + |L_y| \leq n$.
(ii) $|T_{ny}| \geq 3$.
(iii) $|L_2 \cap L_3| \leq 1$ and $|L_1| \geq n - 1$.
(iv) $T_{max} = T_{ny}$.

Proof. (i) Let $k \in T_{max}$. If the inequality (18) is tight for some index $i \in N \setminus T_{max}$ then $y_i < y_k$, and inequality (18) for index k is violated, contradicting the fact that $(x, y) \in X$. Therefore $T_2 \subseteq T_{max}$. The proof of the fact that $T_3 \subseteq T_{max}$ is very similar. To see the last inequality, notice that by inequality (42) we have $L_2, L_3 \subset T_{max} \subset T_{ny}$, and therefore $|L_2 \cup L_3| + |L_y| \leq n$.

(ii) Let $y_{i_1} > 0$ for some $i_1 \in N$ and let $i_2 \in N \setminus \{i_1\}$. Adding the constraints in (18) corresponding to the indices i_1 and i_2, we obtain

$$2 \sum_{j \in N} y_j \geq 3(y_{i_1} + y_{i_2}) \Rightarrow 2 \sum_{j \in N \setminus \{i_1, i_2\}} y_j \geq y_{i_1} + y_{i_2},$$

which implies that at least one of the components of y different from i_1, i_2 is nonzero. Repeating the same argument with this new component and y_{i_1} shows that there are at least 3 such entries.

(iii) Suppose that $|L_2 \cap L_3| \geq 2$. Then there are two indices $i_1 \neq i_2$ such that the corresponding constraints from (18) and (19) are contained in L. But subtracting $\sum_{j \in N} y_j = 3y_{i_2}$ from $\sum_{j \in N} y_j = 3y_{i_1}$, we get $y_{i_1} - y_{i_2} = 0$. Similarly, subtracting $2 = \sum_{j \in N}(2x_j - y_j) + 2y_{i_2}$ from $2 = \sum_{j \in N}(2x_j - y_j) + 2y_{i_1}$, we also get $y_{i_1} - y_{i_2} = 0$, which means that the above four constraints from L are not linearly independent, a contradiction. Therefore $|L_2 \cap L_3| \leq 1$. But this fact, combined with (i) yields $|L_2| + |L_3| \leq n - |L_y| + 1$. Then equation (43) implies that $|L_1| \geq n - 1$.

(iv) Note that $T_{max} \subseteq T_{ny}$ by (42). For contradiction, assume $|T_{max}| \leq |T_{ny}| - 1$ and therefore $0 < y_\ell < y_{max}$ for some index $\ell \in N$. If $|L_2| = 0$, then (i) and equation (43) imply that

$$|L_3| \leq |T_3| \leq |T_{max}| \leq |T_{ny}| - 1 \Rightarrow |L_2| + |L_3| \leq |T_{ny}| - 1 \Rightarrow |L_2| + |L_3| + |L_y| \leq n - 1.$$

As $|L_1| \leq n$, this means that L cannot have $2n$ constraints, a contradiction. Therefore, we can assume that $|L_2| \neq 0$. Let $k \in L_2$, i.e., $\sum_{j \in N} y_j = 3y_k$. Then, (18) implies that $y_k \geq y_i$ for all $i \in N$. Thus, $y_k \in T_{max}$ and $y_k = y_{max}$. As $0 < y_\ell < y_k$, there has to be at least another index of y different from ℓ such that the corresponding component is positive and strictly less than y_k in order for the constraint $\sum_{j \in N} y_j = 3y_k$ to hold. Then we have $|T_{max}| \leq |T_{ny}| - 2$. This fact, combined with (i) and (iii) and equation (42) implies that $|L_2| + |L_3| \leq |T_{ny}| - 1$. As $|L_1| \leq n$, using equation (43), we get $|L| \leq 2n - 1$, a contradiction. Therefore $T_{max} = T_{ny}$. □

Proof of Theorem 3.9. We will first prove that $\{q^1, \ldots, q^m\} \subseteq X$; as X is convex this will imply that $Q^n \subseteq X$. Let $q^j = (x^j, y^j)$ for some $1 \leq j \leq m$. If $x^j = \mathbf{0}$ or if x^j is a unit vector in $\mathbb{R}^n$, then $y^j = \mathbf{0}$, and the constraints (17)-(20) are trivially satisfied. Let $x^j \in V_{1/2}$. The i^{th} component of y^j is 1 if and only if the i^{th} component of x^j is $1/2$. Therefore (17) holds with equality for all $i = 1, \ldots, n$; in this case the constraints (19) simply become $y_i \leq 1$ for $i \in N$, which are all satisfied by y^j. Finally, at least three components of x^j are $1/2$, therefore at least three components of y^j are 1. This, combined with the fact that all components of y^j are at most 1, implies that all constraints in (18) are satisfied by q^j. Therefore $Q^n \subseteq X$.

For the reverse inclusion, we will show that each extreme point of X belongs to V'. Note that X is a pointed polyhedron as equations (20) and (17) imply that it is contained in $\mathbb{R}^n_+ \times \mathbb{R}^n_+$. Furthermore, inequalities (19) together with inequalities (17) imply that $2y_i \leq 2$ and therefore $y_i \leq 1$ for all $i \in N$. In addition, when all y_i are bounded, (19) also implies that all x_i are bounded as well. Therefore, X is in fact a polytope and is equal to the convex hull of its extreme points.

Let $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ be an extreme point of X. If $x_i = 1$ for some index $i \in N$, rewriting (19) as

$$2 \geq \sum_{j \in N \setminus \{i\}} (2x_j - y_j) + (2x_i + y_i), \ \forall i \in N, \tag{44}$$

it is clear that $x_i = 1 \Rightarrow y_i = 0$ and $2x_j = y_j$, for all $j \in N \setminus \{i\}$. Moreover, for any index $k \in N \setminus \{i\}$, equation (44) with i replaced by k becomes

$$2 \geq \sum_{j \in N \setminus \{k\}} (2x_j - y_j) + (2x_k + y_k) = (2x_i - y_i) + (2x_k + y_k).$$

Together with $x_i = 1$ and $y_i = 0$, the above inequality implies that $2x_k + y_k = 0 \Rightarrow x_k = y_k = 0$ for $k \in N \setminus \{i\}$. Therefore $(x, y) = (e_i, \mathbf{0})$ which shows that $(x, y) \in V' \subseteq Q^n$.

From now on, we assume that $x_i < 1$, for all $i \in N$. We first assume $y = \mathbf{0}$. By Lemma 6.1(iii), we have $|L_1| \geq n-1$, and therefore at most one of the inequalities of type (17) is not tight for (x, y). Therefore at most one component of x is nonzero. This implies that (x, y) is a convex combination of $(e_k, \mathbf{0})$ and $(\mathbf{0}, \mathbf{0})$ for some $k \in N$, and therefore (x, y) is equal to one of these two points and is contained in V'.

Henceforth assume that $y \neq \mathbf{0}$. If $L_3 = \emptyset$, then L does not contain any constraints of type (19), and therefore $x = \mathbf{0}, y = \mathbf{0}$ is a solution of the constraints in L. As these constraints have a unique solution, we have $(x, y) = (\mathbf{0}, \mathbf{0})$ which contradicts the assumption that $y \neq \mathbf{0}$. Therefore, $L_3 \neq \emptyset$.

We will now consider two cases.

Case 1: All constraints of type (17) are tight for (x, y). Let $k \in L_3$. As $2x_i - y_i = 0$ for all $i \in N$, the tight inequality (19) for index k implies that $2 = 2y_k \Rightarrow y_k = 1$. But this means $y_{max} = 1$, and, by Lemma 6.1(iv), all nonzero components of y have value 1. Furthermore, Lemma 6.1(ii) implies that at least three components of y have value 1. This combined with the fact that $2x_i = y_i$ for all $i \in N$ means that $x \in V_{1/2}$ and therefore $(x, y) \in V'$.

Case 2: At least one constraint of type (17) is not tight for (x, y). Therefore, $|L_1| \leq n - 1$ and by Lemma 6.1(iii) $|L_1| = n - 1$. Let $\{\ell\} = N \setminus L_1$, therefore $2x_\ell > y_\ell$. It also follows that $y_{max} < 1$ as $y_k = 1$ for any $k \in N$ would imply that $2x_i - y_i = 0$ for all $i \in N$, a contradiction.

Further, if $|L_2 \cap L_3| = 0$, then $|L_2 \cup L_3| = |L_2| + |L_3|$ and Lemma 6.1(i) would imply that $|L_2| + |L_3| + |L_y| \leq n$. This, in turn, implies that $|L_1| = n$, a contradiction. Therefore $|L_2 \cap L_3| \geq 1$, and $L_2 \neq \emptyset$. Consider an index $k \in L_2$, then $y_k = y_{max}$ because $L_2 \subseteq T_{max}$. As all nonzero components of y have value y_{max} by Lemma 6.1(iv), and inequality (18) is tight for index k, we conclude that $|T_{ny}| = 3$.

Case 2a: Let $\ell \notin T_{ny}$. Let $\delta = \min\{y_{max}/3, x_\ell, 1 - x_\ell\} > 0$. Define (x^1, y^1) by

$$x_i^1 = \begin{cases} x_i + \delta/2, & \text{if } i \in T_{ny} \\ x_i - \delta, & \text{if } i = \ell \\ x_i, & \text{o.w.} \end{cases}, \quad y_i^1 = \begin{cases} y_i + \delta, & \text{if } i \in T_{ny} \\ y_i, & \text{o.w.} \end{cases}$$

and $(x^2, y^2) = 2(x, y) - (x^1, y^1)$. Then both (x^1, y^1) and (x^2, y^2) belong to X and (x, y) is a convex combination of these points, contradicting the fact that it is an extreme point of X.

Case 2b: Let $\ell \in T_{ny}$. Let $\delta = \min\{1 - x_\ell + y_\ell/2, y_{max}/3, (2x_\ell - y_\ell)/2\} > 0$. Define (x^1, y^1) by

$$x_i^1 = \begin{cases} x_i + \delta/2, & \text{if } i \in T_{ny} \setminus \{\ell\} \\ x_i - \delta/2, & \text{if } i = \ell \\ x_i & \text{o.w.} \end{cases}, \quad y_i^1 = \begin{cases} y_i + \delta, & \text{if } i \in T_{ny} \\ y_i, & \text{o.w.} \end{cases}$$

and $(x^2, y^2) = 2(x, y) - (x^1, y^1)$. Then both (x^1, y^1) and (x^2, y^2) belong to X and (x, y) is a convex combination of these points, contradicting the fact that it is an extreme point of X.

As the claim holds for both cases, the proof is complete. □