

EXTENDED FORMULATIONS FOR INDEPENDENCE POLYTOPES OF REGULAR MATROIDS

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Abstract. We show that the independence polytope of every regular matroid has an extended formulation of size quadratic in the size of its ground set. This generalizes a similar statement for (co-)graphic matroids, which is a simple consequence of Martin’s [9] extended formulation for the spanning-tree polytope. In our construction, we make use of Seymour’s decomposition theorem for regular matroids. As a consequence, the extended formulations can be computed in polynomial time.

Key words. extended formulation, independence polytope, regular matroid, decomposition

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1. Introduction. The theory of extended formulations deals with the concept of representing polytopes as linear projections of other polytopes. An *extension* of a polytope P is some polytope Q together with a linear map π such that $\pi(Q) = P$. Given an outer description of Q by means of linear inequalities and equations, this is called an *extended formulation* for P . The *size* of an extension (Q, π) is defined as the number of facets of Q . The smallest size of any extension of a polytope P (i.e., the smallest number of inequalities in any extended formulation for P) is called the *extension complexity* of P and is denoted by $\text{xc}(P)$.

The construction of extended formulations has played an important role in the design of many algorithms solving combinatorial-optimization problems. It turns out that polytopes associated to tractable combinatorial-optimization problems often admit polynomial size (in their dimension) extensions. However, the area of extended formulation has received renewed attraction due to recent results establishing exponential lower bounds on the extension complexities of certain polytopes including the TSP polytope [4] and the perfect-matching polytope [12]. While most recent research has focused on improving and extending these results, this paper aims at contributing new positive results pertaining to a well-known class of combinatorial polytopes, namely independence polytopes of regular matroids.

Given a matroid $\mathcal{M} = (E, \mathcal{I})$ with ground set E and independent sets \mathcal{I} , the *independence polytope* of \mathcal{M} is defined as

$$P(\mathcal{M}) := \text{conv} \{ \chi(I) \mid I \in \mathcal{I} \},$$

where $\chi(I) \in \{0, 1\}^E$ is the characteristic vector of I with $\chi(I)_e = 1$ if and only if $e \in I$. Independence polytopes of matroids are central objects in the field of combinatorial optimization. It is well-known that all non-trivial facet-defining inequalities for $P(\mathcal{M})$ are of the form $\sum_{i \in S} x_i \leq r(S)$ with $S \subseteq E$, where r denotes the rank function of \mathcal{M} . Furthermore, any linear function can be maximized over $P(\mathcal{M})$ by a simple greedy algorithm involving only a linear number of independence oracle calls. For more results on independence polytopes of matroids, see, e.g., Schrijver [14].

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In this sense, independence polytopes of matroids are well-understood. One might wonder whether all such polytopes admit polynomial size extended formulations. Unfortunately, this question was answered negatively by Rothvoss [11] who showed that there exists a family of independence polytopes of matroids having extension complexity growing exponentially in the dimension. On the positive side, there are only a few interesting classes of matroids for which we know that the corresponding independence polytopes admit polynomial size extensions. As we will see, using Martin’s [9] extended formulation for the spanning-forest polytope, it is easy to derive quadratic size extended formulations for independence polytopes of graphic and cographic matroids. Recently, this has been generalized by Iwata et al. [6] to the class of “sparsity matroids” and later in [1] to the even more general class of “count matroids”. In this work, we derive quadratic-size extended formulations for independence polytopes of all regular matroids – another, well-known superclass of (co-)graphic matroids. While the notion of extension complexity only captures existence, the extended formulations presented in this paper indeed can be constructed by a polynomial-time algorithm.

Outline. In Section 2, we recall known results implying quadratic-size extended formulations for independence polytopes of graphic and cographic matroids. Section 3 contains basic definitions and facts about linear and regular matroids including 1-, 2- and 3-sums of regular matroids and Seymour’s decomposition theorem. Then, in Section 4, we give an alternative characterization of independent sets in a 1-, 2- or 3-sum. This allows us to derive quadratic-size extended formulations for independence polytopes of regular matroids, as our main result. Finally, we give remarks on applications to related classes of matroids and pose some open questions in Section 5.

2. Graphic and Cographic Matroids. Classic examples of matroids are *graphic matroids*. Given an undirected graph $G = (V, E)$, the graphic matroid of G has ground set E , where a set of edges is independent if and only if it does not contain a cycle. In other words, a set of edges is independent if it is contained in a spanning forest (i.e., a cycle-free subgraph that has the same connected components as G). Thus, if $\mathcal{M}(G)$ is the graphic matroid of some graph G , we have

$$P(\mathcal{M}(G)) = \text{conv} \{x \in \{0, 1\}^E \mid x \leq \chi(F) \text{ for some spanning forest } F\}. \quad (2.1)$$

In order to derive an extended formulation for $P(\mathcal{M}(G))$, we make use of the following simple result concerning the monotonization of 0/1-polytopes. Though it can probably be considered folklore, we include a brief proof, since we are not aware of any appropriate reference.

LEMMA 2.1. *For $Y \subseteq \{0, 1\}^n$, $P := \text{conv}(Y)$, and*

$$P^\downarrow := \text{conv} \{x \in \{0, 1\}^n \mid x \leq y \text{ for some } y \in Y\},$$

we have

$$P^\downarrow = \{x \in \mathbb{R}_+^n \mid \exists y \in P : x \leq y\},$$

thus $\text{xc}(P^\downarrow) \leq \text{xc}(P) + 2n$.

Proof. Let us define $Q := \{x \in \mathbb{R}_+^n \mid \exists y \in P : y \leq x\}$. For every $c \in \mathbb{R}^n$, setting $\bar{c}_i := \max\{c_i, 0\}$ for all $i \in [n]$ yields

$$\begin{aligned} \max \{\langle c, x \rangle \mid x \in P^\downarrow\} &= \max \{\langle c, x \rangle \mid x \in \{0, 1\}^n, x \leq y \text{ for some } y \in Y\} \\ &= \max \{\langle \bar{c}, y \rangle \mid y \in Y\} \\ &= \max \{\langle \bar{c}, y \rangle \mid y \in P\} \\ &= \max \{\langle c, x \rangle \mid x \in Q\}, \end{aligned}$$

and thus $P^\downarrow = Q$. \square

Together with Equation (2.1), Lemma 2.1 implies

$$\text{xc}(\mathcal{M}(G)) \leq \text{xc}(P_{\text{sp.forests}}(G)) + 2|E|, \quad (2.2)$$

where $P_{\text{sp.forests}}(G)$ is the *spanning-forest polytope* of G , i.e., the convex hull of characteristic vectors of spanning forests of G .

Closely related to graphic matroids are *cographic matroids*, which are the duals of graphic matroids. Given an undirected graph $G = (V, E)$, the cographic matroid of G also has ground set E , where now a set of edges is independent if and only if it is contained in the complement of a spanning forest. Thus, if $\mathcal{M}^*(G)$ is the cographic matroid of some graph G , we obtain

$$P(\mathcal{M}^*(G)) = \text{conv} \{x \in \{0, 1\}^E \mid x \leq \mathbb{1} - \chi(F) \text{ for some spanning forest } F\},$$

where $\mathbb{1}$ denotes the all-ones vector. Hence, again by Lemma 2.1, this implies

$$\begin{aligned} \text{xc}(P(\mathcal{M}^*(G))) &\leq \text{xc}(\mathbb{1} - P_{\text{sp.forests}}(G)) + 2|E| \\ &= \text{xc}(P_{\text{sp.forests}}(G)) + 2|E|. \end{aligned} \quad (2.3)$$

As Martin [9] showed, the spanning-forest polytope of a graph $G = (V, E)$ admits an extended formulation of size $\mathcal{O}(|V| \cdot |E|)$. Because the spanning-forest polytope of a graph is the Cartesian product of the spanning-forest polytopes of its connected components, and because one has $|V| \leq |E| - 1$ for every connected graph $G = (V, E)$, we obtain the estimate $\text{xc}(P_{\text{sp.forests}}(G)) \leq \mathcal{O}(|E|^2)$. Using Inequality (2.2) and Inequality (2.3), we conclude:

PROPOSITION 2.2. *For any graphic or cographic matroid \mathcal{M} on ground set E we have*

$$\text{xc}(P(\mathcal{M})) \leq \mathcal{O}(|E|^2).$$

Moreover, a result by Williams [17] even provides linear-size extended formulations for spanning-forest polytopes in the case of planar graphs. Thus, if G is a planar graph, we can improve the bound in Proposition 2.2 to $\text{xc}(P(\mathcal{M})) \leq \mathcal{O}(|E|)$.

3. Regular Matroids. A much more general family of matroids comprising graphic and cographic matroids is the class of *linear matroids*. Given a matrix $A \in \mathbb{F}^{p \times q}$ with entries in some field \mathbb{F} , we denote by

$$\mathcal{M}_{\mathbb{F}}(A) := \{I \subseteq [p + q] \mid (\mathbb{I}, A)_{*, I} \text{ has full column-rank over } \mathbb{F}\}$$

the (set of independent sets of the) matroid defined by A , where \mathbb{I} is the $p \times p$ -identity matrix. Therefore, the cardinality of the ground set of $\mathcal{M}_{\mathbb{F}}(A)$ is $p + q$, i.e., the number columns of the *identity-extension* (\mathbb{I}, A) of the matrix A . Note that this use of notation differs, e.g., from Oxley [10], but is in accordance to Schrijver's book [13, Chap. 19]. Here, we allow A to have $q = 0$ columns (in which case $\mathcal{M}_{\mathbb{F}}(A)$ is a *free matroid* on p elements with all subsets being independent) or $p = 0$ rows (in which case $\mathcal{M}_{\mathbb{F}}(A)$ is a matroid on q elements such that the empty set is the only independent set), but we will always have $p + q > 0$. If a matroid \mathcal{M} is isomorphic to $\mathcal{M}_{\mathbb{F}}(A)$ for some matrix A and some field \mathbb{F} , we say that \mathcal{M} can be *represented* (by A) over \mathbb{F} . The class of linear matroids consists of all matroids that can be represented over some field.

In this paper, we focus on the well-known class of matroids that can be represented over every field, namely *regular matroids*. It can be shown that a matroid is regular if and only if it can be represented by a totally-unimodular matrix over \mathbb{R} (see, e.g., [13, Chap. 19]). Note that for a totally-unimodular matrix A , the matroid $\mathcal{M}_{\mathbb{F}}(A)$ does not depend on the specific choice of the field \mathbb{F} . Thus, we will mainly work over the most simple field \mathbb{F}_2 , with two elements.

Key examples of regular matroids are graphic matroids. Let $G = (V, E)$ be an undirected connected graph. Choosing some $T \subseteq E$ that forms a spanning tree of G and assigning some orientation to all edges in E , let us construct a matrix $A \in \{0, 1, -1\}^{T \times E}$ as follows: For every pair of (directed) edges $t \in T$ and $e = (v, w) \in E$, set the entry $A_{t,e}$ to 1 or -1 if the path from v to w in T passes through t in forward or backward direction, respectively, and to 0 if it does not pass through t at all. It can be shown that $\mathcal{M}(G)$ is (isomorphic to) $\mathcal{M}_{\mathbb{R}}(A)$ and that A is totally unimodular. In particular, this implies that $\mathcal{M}(G)$ is regular.

Not every regular matroid is graphic or cographic. However, it turns out that all remaining regular matroids can be constructed from only graphic matroids, cographic matroids and matroids of size at most ten.

3.1. Seymour's Decomposition Theorem. From its basic definition, there seems to be no hint concerning crucial properties of regular matroids to exploit in order to obtain polynomial-size extended formulations for the corresponding independence polytopes. Fortunately, there is a very strong structural characterization of regular matroids due to Seymour that provides suitable access for our purpose. In order to state the result, we need to define a few operations on regular matroids. In what follows, all matrices and operations are considered over \mathbb{F}_2 . For convenience, we will therefore write $\mathcal{M}(A) := \mathcal{M}_{\mathbb{F}_2}(A)$.

Let \mathcal{M} , \mathcal{M}_1 and \mathcal{M}_2 be binary matroids, i.e., matroids represented over \mathbb{F}_2 . We say that \mathcal{M} is a *1-sum* of \mathcal{M}_1 and \mathcal{M}_2 if there exist matrices A, B such that

$$\mathcal{M}_1 = \mathcal{M}(A), \quad \mathcal{M}_2 = \mathcal{M}(B), \quad \mathcal{M} = \mathcal{M} \begin{pmatrix} A & \mathbb{O} \\ \mathbb{O} & B \end{pmatrix}$$

holds. We say that \mathcal{M} is a *2-sum* of \mathcal{M}_1 and \mathcal{M}_2 if there exist matrices A, B and column vectors a, b such that

$$\mathcal{M}_1 = \mathcal{M} \begin{pmatrix} a & A \end{pmatrix}, \quad \mathcal{M}_2 = \mathcal{M} \begin{pmatrix} b^\top \\ B \end{pmatrix}, \quad \mathcal{M} = \mathcal{M} \begin{pmatrix} A & ab^\top \\ \mathbb{O} & B \end{pmatrix}$$

holds. Finally, we say that \mathcal{M} is a *3-sum* of \mathcal{M}_1 and \mathcal{M}_2 if there exist matrices A, B and column vectors a, b, c, d such that

$$\mathcal{M}_1 = \mathcal{M} \begin{pmatrix} a & a & A \\ 0 & 1 & c^\top \end{pmatrix}, \quad \mathcal{M}_2 = \mathcal{M} \begin{pmatrix} 0 & 1 & b^\top \\ d & d & B \end{pmatrix}, \quad \mathcal{M} = \mathcal{M} \begin{pmatrix} A & ab^\top \\ dc^\top & B \end{pmatrix}$$

holds. In each of the above definitions, we allow A to consist of no columns and B to consist of no rows. Seymour's characterization of regular matroids yields the following:

THEOREM 3.1 (Regular-Matroid Decomposition Theorem [15], see [13, Thm. 19.6]). *For every regular matroid \mathcal{M} there exists a rooted binary tree T whose nodes are binary matroids such that*

- the root of T is \mathcal{M} ,
- each non-leaf node of T is a k -sum of its two children for some $k \in \{1, 2, 3\}$,

- each leaf of T is a graphic matroid, a cographic matroid or has size at most ten.
- each leaf of T has a ground set of cardinality at least three, and
- whenever a non-leaf node of T is a 3-sum of its children, both children have ground sets of cardinality at least seven.

Moreover, for input matrix $A \in \mathbb{F}_2^{m \times n}$ such a decomposition of $\mathcal{M} = \mathcal{M}(A)$ can be computed in time polynomially bounded in m and n .

In Schrijver's book [13], the above statement is formulated in terms of a decomposition theorem for totally-unimodular matrices, for which he allows certain additional operations on matrices such as adding an all-zero row or column to a matrix A , adding a unit-vector as a row or column to A , or repeating a row or column of A . However, it can be easily seen that all these operations can be interpreted as 2-sums of $\mathcal{M}(A)$ with certain matroids on ground sets of cardinality three. Furthermore, in our definitions of k -sums, we only require the *existence* of matrices defining the components of the sum. Hence, we do not have to allow row/column permutations, pivots or scaling (by -1) of rows or columns. Finally, we also do not need the transpose a matrix as a particular operation: Given some decomposition tree T as in Theorem 3.1, suppose we want to transpose a matrix defining a matroid \mathcal{M} at some non-leaf node of T . By our definition of k -sums, this can be simulated by simply transposing the matrices defining \mathcal{M}_1 and \mathcal{M}_2 (and swapping the summands). Thus, the transposition can be propagated down to the leaf nodes of T resulting in a decomposition tree \tilde{T} , where we replaced some matrices defining leaves by their transposes. Note that the sums of the cardinalities of the ground sets of leaf nodes of T and \tilde{T} coincide. Furthermore, since the transpose of a matrix corresponds to taking the dual of the induced matroid, \tilde{T} is still a decomposition tree as in Theorem 3.1.

4. Main Result. As described in Section 2, we already have small extended formulations for the independence polytopes of the leaf nodes in decompositions as in Theorem 3.1. (Note that the leaf nodes that are not graphic or cographic have size bounded by some constant.) Given a decomposition tree, we are going to construct an extended formulation for the independence polytope of the root node whose size is small in terms of the sum of the sizes of the extended formulations of the independence polytopes of the leaf nodes. In order to deduce our main result from such a construction later, we first bound the sizes of components in decomposition trees.

LEMMA 4.1. *Let \mathcal{M} be a regular matroid on ground set E , and let T be a decomposition tree of \mathcal{M} as in Theorem 3.1. Then the sum of the cardinalities of the ground sets of leaf nodes of T can be bounded linearly in $|E|$.*

Proof. Let $f(n)$ denote the largest sum of cardinalities of the ground sets of leaf nodes in any decomposition tree as in Theorem 3.1 for a regular matroid whose ground set has cardinality n . Defining

$$g(n) := \max \left(\{n\} \cup \{g(t) + g(n-t) \mid 3 \leq t \leq n-3\} \right. \\ \left. \cup \{g(t+1) + g(n-t+1) \mid 2 \leq t \leq n-2\} \right. \\ \left. \cup \{g(t+3) + g(n-t+3) \mid 4 \leq t \leq n-4\} \right)$$

for all $n \geq 2$, and setting $g(1) := 1$, we read off from Theorem 3.1 that we have $f(n) \leq g(n)$ for all $n \geq 1$ (note that whenever a node with ground set of size n is the 1-, 2-, or 3-sum of its two children with ground sets of sizes n_1 and n_2 , then $n_1 + n_2$ equals n minus 0, 2, or 6, respectively). Inspecting the function g more closely, we

find that we have $g(7) = 15$, $g(8) = 30$ and

$$g(n) = \max \left(\{g(t+3) + g(n-t+3) \mid 4 \leq t \leq n-4\} \right),$$

for all $n \geq 9$. From this one deduces $g(n) = 15(n-6)$ for all $n \geq 7$ by induction. \square

Next, for our construction it is necessary to characterize the independent sets of k -sums. We use the symbol \uplus in order to emphasize when a union is taken of two sets with empty intersection.

LEMMA 4.2. *Let $\mathcal{M} = (E, \mathcal{I})$, $\mathcal{M}_1 = (E_1, \mathcal{I}_1)$, and $\mathcal{M}_2 = (E_2, \mathcal{I}_2)$ be binary matroids with $E_1 \cap E_2 = \emptyset$ such that \mathcal{M} is a k -sum of \mathcal{M}_1 and \mathcal{M}_2 . Then the independent sets of \mathcal{M} can be characterized (up to isomorphism) as follows:*

- $k = 1$: $\mathcal{I} = \{I_1 \uplus I_2 \mid I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}$
- $k = 2$: \mathcal{M} is a 1-sum of a minor of \mathcal{M}_1 and a minor of \mathcal{M}_2 ; or there exist elements $r_1 \in E_1, r_2 \in E_2$ satisfying

$$\begin{aligned} \mathcal{I} = \{ & (I_1 \setminus \{r_1\}) \uplus (I_2 \setminus \{r_2\}) : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2, \\ & |I_1 \cap \{r_1\}| + |I_2 \cap \{r_2\}| = 1 \}. \end{aligned}$$

- $k = 3$: \mathcal{M} is a 2-sum of a minor of \mathcal{M}_1 and a minor of \mathcal{M}_2 ; or there exist pairwise distinct elements $r_1, p_1, q_1 \in E_1, r_2, p_2, q_2 \in E_2$ satisfying

$$\begin{aligned} \mathcal{I} = \{ & (I_1 \setminus \{r_1, p_1, q_1\}) \uplus (I_2 \setminus \{r_2, p_2, q_2\}) : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2, \\ & |I_1 \cap \{r_1\}| + |I_2 \cap \{r_2\}| = 1, \\ & |I_1 \cap \{p_1\}| + |I_2 \cap \{p_2\}| = 1, \\ & |I_1 \cap \{q_1\}| + |I_2 \cap \{q_2\}| = 1 \}. \end{aligned}$$

Proof. Note that the statement for the case $k = 1$ follows trivially from the definition of a 1-sum. Let us consider the case $k = 2$ and suppose that we have $\mathcal{M}_1 = \mathcal{M}(a \ A)$, $\mathcal{M}_2 = \mathcal{M}(b^T \ B)$ and $\mathcal{M} = \mathcal{M}\left(\begin{smallmatrix} A & ab^T \\ \mathbb{0} & B \end{smallmatrix}\right)$. The identity-extension of $\left(\begin{smallmatrix} A & ab^T \\ \mathbb{0} & B \end{smallmatrix}\right)$ (after permuting columns) is

$$\begin{pmatrix} \mathbb{I} & A & \mathbb{0} & ab^T \\ \mathbb{0} & \mathbb{0} & \mathbb{I} & B \end{pmatrix}.$$

Denote the elements corresponding to the first column of $(a \ A)$ and the first column of the identity-extension of $\begin{pmatrix} b^T \\ B \end{pmatrix}$ (being the first unit vector) by r_1 and r_2 , respectively. With this notation, we may assume that we have $E = (E_1 \setminus \{r_1\}) \uplus (E_2 \setminus \{r_2\})$. In addition, note that if $a = \mathbb{0}$ holds, then \mathcal{M} is a 1-sum of $\mathcal{M}(A)$ and $\mathcal{M}(B)$, which are minors of \mathcal{M}_1 and \mathcal{M}_2 , respectively. Thus, we may further assume that $a \neq \mathbb{0}$ holds and obtain that a subset of E is independent (in \mathcal{M}) if and only if it is of the form $J_1 \uplus J_2$ with $J_1 = I_1 \setminus \{r_1\}$ and $J_2 = I_2 \setminus \{r_2\}$ where $I_1 \in \mathcal{I}_1$ and (due to $a \neq \mathbb{0}$) $I_2 \in \mathcal{I}_2$ such that

$$\text{span}_{J_1} \begin{pmatrix} \mathbb{I} & A \\ \mathbb{0} & \mathbb{0} \end{pmatrix} \cap \text{span}_{J_2} \begin{pmatrix} \mathbb{0} & ab^T \\ \mathbb{I} & B \end{pmatrix} = \{\mathbb{0}\}$$

holds, where $\text{span}_J(\cdot)$ denotes the \mathbb{F}_2 -subspace spanned by the columns corresponding to J . Because we have (with $\text{span}(\cdot)$ denoting the \mathbb{F}_2 -subspace spanned by all columns)

$$\text{span} \begin{pmatrix} \mathbb{I} & A \\ \mathbb{0} & \mathbb{0} \end{pmatrix} \cap \text{span} \begin{pmatrix} \mathbb{0} & ab^T \\ \mathbb{I} & B \end{pmatrix} \setminus \{\mathbb{0}\} \subseteq \left\{ \begin{pmatrix} a \\ \mathbb{0} \end{pmatrix} \right\},$$

the latter condition is equivalent to

$$\begin{pmatrix} a \\ \mathbb{O} \end{pmatrix} \notin \text{span}_{J_1} \begin{pmatrix} \mathbb{I} & A \\ \mathbb{O} & \mathbb{O} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a \\ \mathbb{O} \end{pmatrix} \notin \text{span}_{J_2} \begin{pmatrix} \mathbb{O} & ab^\top \\ \mathbb{I} & B \end{pmatrix},$$

which is equivalent to (recall $a \neq \mathbb{O}$)

$$a \notin \text{span}_{J_1} (\mathbb{I} \ A) \quad \text{or} \quad \begin{pmatrix} 1 \\ \mathbb{O} \end{pmatrix} \notin \text{span}_{J_2} \begin{pmatrix} \mathbb{O} & b^\top \\ \mathbb{I} & B \end{pmatrix},$$

and thus to

$$J_1 \cup \{r_1\} \in \mathcal{I}_1 \quad \text{or} \quad J_2 \cup \{r_2\} \in \mathcal{I}_2.$$

Hence, we obtain

$$\begin{aligned} \mathcal{I} &= \{(I_1 \setminus \{r_1\}) \uplus (I_2 \setminus \{r_2\}) : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2, \\ &\quad |I_1 \cap \{r_1\}| + |I_2 \cap \{r_2\}| \geq 1\} \\ &= \{(I_1 \setminus \{r_1\}) \uplus (I_2 \setminus \{r_2\}) : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2, \\ &\quad |I_1 \cap \{r_1\}| + |I_2 \cap \{r_2\}| = 1\}, \end{aligned}$$

where the last equality follows from the fact that

$$(E_1 \uplus E_2, \{I_1 \uplus I_2 \mid I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\})$$

is an independence system (in fact, a matroid that is the direct sum of matroids).

For the remaining case $k = 3$, let $\mathcal{M}_1 = \mathcal{M} \begin{pmatrix} a & a & A \\ \mathbb{O} & 1 & c^\top \end{pmatrix}$, $\mathcal{M}_2 = \mathcal{M} \begin{pmatrix} \mathbb{O} & 1 & b^\top \\ d & d & B \end{pmatrix}$ and $\mathcal{M} = \mathcal{M} \begin{pmatrix} A & ab^\top \\ dc^\top & B \end{pmatrix}$. The identity extension of $\begin{pmatrix} A & ab^\top \\ dc^\top & B \end{pmatrix}$ (after permuting columns) is

$$\begin{pmatrix} \mathbb{I} & A & \mathbb{O} & ab^\top \\ \mathbb{O} & dc^\top & \mathbb{I} & B \end{pmatrix}.$$

Let us denote certain elements corresponding to the columns of the identity extensions of $\begin{pmatrix} a & a & A \\ \mathbb{O} & 1 & c^\top \end{pmatrix}$ and $\begin{pmatrix} \mathbb{O} & 1 & b^\top \\ d & d & B \end{pmatrix}$, respectively, as follows:

$$\begin{array}{ccc} \begin{pmatrix} 1 & \mathbb{O} & a & a & A \\ \mathbb{O} & \mathbb{I} & 0 & 1 & c^\top \end{pmatrix} & & \begin{pmatrix} 1 & \mathbb{O} & 0 & 1 & b^\top \\ \mathbb{O} & \mathbb{I} & d & d & B \end{pmatrix} \\ \uparrow & \uparrow & \uparrow \\ p_1 & r_1 & q_1 \end{array} \quad \begin{array}{ccc} \begin{pmatrix} 1 & \mathbb{O} & 0 & 1 & b^\top \\ \mathbb{O} & \mathbb{I} & d & d & B \end{pmatrix} \\ \uparrow & \uparrow & \uparrow \\ r_2 & p_2 & q_2 \end{array}$$

With this notation, we may assume that we have $E = (E_1 \setminus \{r_1, p_1, q_1\}) \uplus (E_2 \setminus \{r_2, p_2, q_2\})$. In addition, note that if $d = \mathbb{O}$ holds, then \mathcal{M} is a 2-sum of $\mathcal{M} \begin{pmatrix} a & a \\ \mathbb{O} & 1 \end{pmatrix}$ and $\mathcal{M} \begin{pmatrix} b^\top \\ B \end{pmatrix}$, which are minors of \mathcal{M}_1 and \mathcal{M}_2 , respectively. A similar argument holds for the case $a = \mathbb{O}$. Thus, we may further assume that $d \neq \mathbb{O}$ and $a \neq \mathbb{O}$ holds. In this case, a subset of E is independent (in \mathcal{M}) if and only if it is of the form $J_1 \uplus J_2$ with $J_1 = I_1 \setminus \{r_1, p_1, q_1\}$ and $J_2 = I_2 \setminus \{r_2, p_2, q_2\}$ where (due to $d \neq \mathbb{O}$) $I_1 \in \mathcal{I}_1$ and (due to $a \neq \mathbb{O}$) $I_2 \in \mathcal{I}_2$ such that

$$\text{span}_{J_1} \begin{pmatrix} \mathbb{I} & A \\ \mathbb{O} & dc^\top \end{pmatrix} \cap \text{span}_{J_2} \begin{pmatrix} \mathbb{O} & ab^\top \\ \mathbb{I} & B \end{pmatrix} = \{\mathbb{O}\}$$

holds. Because we have

$$\text{span} \begin{pmatrix} \mathbb{I} & A \\ \mathbb{O} & dc^\top \end{pmatrix} \cap \text{span} \begin{pmatrix} \mathbb{O} & ab^\top \\ \mathbb{I} & B \end{pmatrix} \setminus \{\mathbb{O}\} \subseteq \left\{ \begin{pmatrix} a \\ \mathbb{O} \end{pmatrix}, \begin{pmatrix} \mathbb{O} \\ d \end{pmatrix}, \begin{pmatrix} a \\ d \end{pmatrix} \right\},$$

the latter condition is equivalent to

$$\begin{aligned} & \left[\begin{pmatrix} a \\ \mathbb{O} \end{pmatrix} \notin \text{span}_{J_1} \begin{pmatrix} \mathbb{I} & A \\ \mathbb{O} & dc^\top \end{pmatrix} \text{ or } \begin{pmatrix} a \\ \mathbb{O} \end{pmatrix} \notin \text{span}_{J_2} \begin{pmatrix} \mathbb{O} & ab^\top \\ \mathbb{I} & B \end{pmatrix} \right] \text{ and} \\ & \left[\begin{pmatrix} \mathbb{O} \\ d \end{pmatrix} \notin \text{span}_{J_1} \begin{pmatrix} \mathbb{I} & A \\ \mathbb{O} & dc^\top \end{pmatrix} \text{ or } \begin{pmatrix} \mathbb{O} \\ d \end{pmatrix} \notin \text{span}_{J_2} \begin{pmatrix} \mathbb{O} & ab^\top \\ \mathbb{I} & B \end{pmatrix} \right] \text{ and} \\ & \left[\begin{pmatrix} a \\ d \end{pmatrix} \notin \text{span}_{J_1} \begin{pmatrix} \mathbb{I} & A \\ \mathbb{O} & dc^\top \end{pmatrix} \text{ or } \begin{pmatrix} a \\ d \end{pmatrix} \notin \text{span}_{J_2} \begin{pmatrix} \mathbb{O} & ab^\top \\ \mathbb{I} & B \end{pmatrix} \right], \end{aligned}$$

which due to $a \neq \mathbb{O}$ and $d \neq \mathbb{O}$ is equivalent to

$$\begin{aligned} & \left[\begin{pmatrix} a \\ 0 \end{pmatrix} \notin \text{span}_{J_1} \begin{pmatrix} \mathbb{I} & A \\ 0 & c^\top \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ \mathbb{O} \end{pmatrix} \notin \text{span}_{J_2} \begin{pmatrix} 0 & b^\top \\ \mathbb{I} & B \end{pmatrix} \right] \text{ and} \\ & \left[\begin{pmatrix} \mathbb{O} \\ 1 \end{pmatrix} \notin \text{span}_{J_1} \begin{pmatrix} \mathbb{I} & A \\ 0 & c^\top \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ d \end{pmatrix} \notin \text{span}_{J_2} \begin{pmatrix} 0 & b^\top \\ \mathbb{I} & B \end{pmatrix} \right] \text{ and} \\ & \left[\begin{pmatrix} a \\ 1 \end{pmatrix} \notin \text{span}_{J_1} \begin{pmatrix} \mathbb{I} & A \\ 0 & c^\top \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ d \end{pmatrix} \notin \text{span}_{J_2} \begin{pmatrix} 0 & b^\top \\ \mathbb{I} & B \end{pmatrix} \right], \end{aligned}$$

and thus to

$$\begin{aligned} & [J_1 \cup \{r_1\} \in \mathcal{I}_1 \text{ or } J_2 \cup \{r_2\} \in \mathcal{I}_2] \text{ and} \\ & [J_1 \cup \{p_1\} \in \mathcal{I}_1 \text{ or } J_2 \cup \{p_2\} \in \mathcal{I}_2] \text{ and} \\ & [J_1 \cup \{q_1\} \in \mathcal{I}_1 \text{ or } J_2 \cup \{q_2\} \in \mathcal{I}_2]. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \mathcal{I} &= \{(I_1 \setminus \{r_1, p_1, q_1\}) \uplus (I_2 \setminus \{r_2, p_2, q_2\}) : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2, \\ & \quad |I_1 \cap \{r_1\}| + |I_2 \cap \{r_2\}| \geq 1, \\ & \quad |I_1 \cap \{p_1\}| + |I_2 \cap \{p_2\}| \geq 1, \\ & \quad |I_1 \cap \{q_1\}| + |I_2 \cap \{q_2\}| \geq 1\} \\ &= \{(I_1 \setminus \{r_1, p_1, q_1\}) \uplus (I_2 \setminus \{r_2, p_2, q_2\}) : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2, \\ & \quad |I_1 \cap \{r_1\}| + |I_2 \cap \{r_2\}| = 1, \\ & \quad |I_1 \cap \{p_1\}| + |I_2 \cap \{p_2\}| = 1, \\ & \quad |I_1 \cap \{q_1\}| + |I_2 \cap \{q_2\}| = 1\}, \end{aligned}$$

where the last equality again follows from the fact that

$$(E_1 \uplus E_2, \{I_1 \uplus I_2 \mid I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\})$$

is an independence system. \square

Finally, we bound the extension complexities of independence polytopes of k -sums in terms of their summands.

LEMMA 4.3. *Let \mathcal{M} be a k -sum of \mathcal{M}_1 and \mathcal{M}_2 for some $k \in \{1, 2, 3\}$. Then we have*

$$\text{xc}(P(\mathcal{M})) \leq \text{xc}(P(\mathcal{M}_1)) + \text{xc}(P(\mathcal{M}_2)).$$

Proof. Let $\mathcal{M} = (E, \mathcal{I})$ be a k -sum of $\mathcal{M}_1 = (E_1, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E_2, \mathcal{I}_2)$ (with $E_1 \cap E_2 = \emptyset$). First, observe that if some matroid \mathcal{M}'' is a minor of \mathcal{M}' , then $P(\mathcal{M}'')$ can be obtained by intersecting $P(\mathcal{M}')$ with a face of the 0/1-cube. Hence, $P(\mathcal{M}'')$ is a coordinate projection of a face of $P(\mathcal{M}')$ and therefore $\text{xc}(P(\mathcal{M}'')) \leq \text{xc}(P(\mathcal{M}'))$. Thus, by Lemma 4.2, it remains to consider the case in which there exist pairwise distinct elements $e_1, \dots, e_t \in E_1$ and pairwise distinct elements $f_1, \dots, f_t \in E_2$ such that

$$E = (E_1 \setminus \{e_1, \dots, e_t\}) \uplus (E_2 \setminus \{f_1, \dots, f_t\})$$

and

$$\begin{aligned} \mathcal{I} = \{ & (I_1 \setminus \{e_1, \dots, e_t\}) \uplus (I_2 \setminus \{f_1, \dots, f_t\}) : \\ & I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2, \\ & |I_1 \cap \{e_i\}| + |I_2 \cap \{f_i\}| \geq 1 \\ & \text{for all } i \in \{1, \dots, t\} \} \end{aligned}$$

holds. Thus, setting

$$Q := \{(x, y) \in [0, 1]^{E_1} \times [0, 1]^{E_2} \mid x_{e_i} + y_{f_i} = 1 \ \forall i = 1, \dots, t\},$$

we obtain that $P(\mathcal{M})$ is a coordinate projection of

$$\begin{aligned} & \text{conv} \left((P(\mathcal{M}_1) \times P(\mathcal{M}_2)) \cap Q \cap (\{0, 1\}^{E_1} \times \{0, 1\}^{E_2}) \right) \\ & = (P(\mathcal{M}_1) \times P(\mathcal{M}_2)) \cap Q, \end{aligned}$$

where the equality follows from Edmonds' intersection theorem for matroid polytopes [2] and the fact that $P(\mathcal{M}_1) \times P(\mathcal{M}_2)$ and Q are (faces of) matroid polytopes. In particular, we established

$$\begin{aligned} \text{xc}(P(\mathcal{M})) & \leq \text{xc} \left((P(\mathcal{M}_1) \times P(\mathcal{M}_2)) \cap Q \right) \\ & = \text{xc} \left(\{(x, y) \in P(\mathcal{M}_1) \times P(\mathcal{M}_2) \mid x_{e_i} + y_{f_i} = 1 \ \forall i = 1, \dots, t\} \right) \\ & \leq \text{xc} \left(P(\mathcal{M}_1) \times P(\mathcal{M}_2) \right) \\ & \leq \text{xc}(P(\mathcal{M}_1)) + \text{xc}(P(\mathcal{M}_2)). \end{aligned}$$

□

We remark that [5, Lemma 3.4] gives a similar result on the structure of independence polytopes of matroids arising from 2-sums. We are now ready to prove our main result.

THEOREM 4.4. *For any regular matroid \mathcal{M} on ground set E , we have*

$$\text{xc}(P(\mathcal{M})) \leq \mathcal{O}(|E|^2).$$

Proof. Let $\mathcal{M}_1 = (E_1, \mathcal{I}_1), \dots, \mathcal{M}_k = (E_k, \mathcal{I}_k)$ be the leaf nodes in some decomposition tree as in Theorem 3.1. By Lemma 4.3, we have that

$$\text{xc}(P(\mathcal{M})) \leq \sum_{i=1}^k \text{xc}(P(\mathcal{M}_i))$$

holds. Because there is a constant $\gamma > 0$ with $\text{xc}(P(\mathcal{M}_i)) \leq \gamma \cdot |E_i|^2$ for each $i = 1, \dots, k$ (recall that each leaf is graphic, cographic or has size bounded by 10), and $\sum_{i=1}^k |E_i|$ can be bounded linearly in $|E|$ due to Lemma 4.1, we can further estimate

$$\sum_{i=1}^k \text{xc}(P(\mathcal{M}_i)) \leq \gamma \cdot \sum_{i=1}^k |E_i|^2 \leq \gamma \cdot \left(\sum_{i=1}^k |E_i| \right)^2 = \mathcal{O}(|E|^2),$$

which gives the claim. \square

Suppose we are given some matrix $A \in \mathbb{F}_2^{m \times n}$ defining a regular matroid $\mathcal{M} = \mathcal{M}(A)$. It is possible to compute a decomposition tree for \mathcal{M} as in Theorem 3.1 – including \mathbb{F}_2 -matrices representing all nodes – in time polynomial in m and n , see, e.g., [13, Chap. 19–20]. In the next step, for each matrix defining a (co-)graphic leaf node, we compute a graph inducing the same (co-)graphic matroid. This can be also done in polynomial time, see, e.g., [16]. Since the presented quadratic-size extended formulations for independence polytopes of graphic and cographic matroids can be easily constructed given the underlying graphs, and since the leaf nodes not being graphic or cographic have bounded size, we can thus compute quadratic-size extended formulations for the independence polytopes of all leaf nodes in polynomial time. Together with Lemma 4.2 and Lemma 4.3, it is now easy to propagate these extended formulations through the tree until we obtain an extended formulation for $P(\mathcal{M})$. The proof of Theorem 4.4 shows that this extended formulation has size quadratic in the size of the ground set of \mathcal{M} . Thus, a quadratic-size extended formulation for $P(\mathcal{M})$ can be computed in time polynomial in m and n .

5. Concluding Remarks. A natural question that arises in our context is for which other classes of matroids there are (similar) constructions of polynomial-size extended formulations for the associated independence polytopes. In Section 4, we have seen that whenever a matroid \mathcal{M} can be decomposed by means of 1-, 2- and 3-sums, the extension complexity of $P(\mathcal{M})$ can be bounded by the sum of the extension complexities of the leaf nodes' independence polytopes. However, not many classes of matroids are known that admit decompositions using only 1-, 2- and 3-sums and starting from simple building blocks – as in case of regular matroids. As an obvious generalization of regular matroids, linear matroids over \mathbb{F}_2 do not seem to have such decompositions.

For any fixed field \mathbb{F} , does $\text{xc}(P(\mathcal{M}))$ grow polynomially (in the dimension) for every \mathbb{F} -linear matroid \mathcal{M} ?

As mentioned in the introduction, we know from [11] that there *exists* a family of independence polytopes of matroids whose extension complexities grow exponentially in their dimension. Moreover, following the argumentation in [11], a random sequence of independence polytopes of matroids has this property. However, no such family is known explicitly. We would like to highlight a class of matroids that might contain interesting candidates: Given an undirected graph G , let $\mathcal{M}_{\text{match}}(G)$ be the *matching matroid* [8] of G whose independent sets are the subsets of nodes of G that can be covered by some matching. It is easy to see that the extension complexity of $\mathcal{M}_{\text{match}}(G)$ can be bounded in terms of the extension complexity of the matching polytope of G . Clearly, this neither implies a polynomial-size upper bound (for general graphs) nor any lower bound on $\text{xc}(\mathcal{M}_{\text{match}}(G))$.

Does there exist a family of graphs G for which $\text{xc}(\mathcal{M}_{\text{match}}(G))$ grows exponentially (in the dimension)?

Given a candidate matroid \mathcal{M} , the question arises, how to prove a non-trivial lower bound on $\text{xc}(P(\mathcal{M}))$. A technique that has been established to provide simple proofs for super-polynomial lower bounds on the extension complexities of many combinatorial polytopes (starting from the correlation polytope, see, e.g., [7]), is to prove lower bounds on the *rectangle covering number*. In terms of the independence polytopes of matroids $\mathcal{M} = (E, \mathcal{I})$ with rank function r , a *rectangle* can be defined as a set $\mathcal{F} \times \mathcal{V}$, where \mathcal{F} is a set of non-empty subsets of E , and \mathcal{V} is a set of independent sets such that

$$|I \cap S| \leq r(S) - 1 \tag{5.1}$$

holds for all $(S, I) \in \mathcal{F} \times \mathcal{V}$. Up to an additive term of $\mathcal{O}(|E|)$, the rectangle covering number of $P(\mathcal{M})$ is defined as the smallest number of rectangles needed to cover all pairs (S, I) with $S \subseteq E$, $I \in \mathcal{I}$ that satisfy (5.1) and is known to provide a lower bound on the extension complexity of $P(\mathcal{M})$, see [3]. Unfortunately, it turns out that the rectangle-covering number of independence polytopes of matroids cannot provide super-polynomial bounds:

PROPOSITION 5.1. *Given a matroid $\mathcal{M} = (E, \mathcal{I})$, the rectangle-covering number of $P(\mathcal{M})$ is at most $\mathcal{O}(|E|^2)$.*

Proof. Observe that a pair (S, I) with $S \subseteq E$, $I \in \mathcal{I}$ satisfies (5.1) if and only if

- there exists some $e \in S \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$, or
- there exist some $e \in S \setminus I$ and $f \in I \setminus S$ such that $(I \setminus \{f\}) \cup \{e\} \in \mathcal{I}$.

Thus, each such pair is contained in a rectangle of type

$$\{S \subseteq E \mid e \in S\} \times \{I \in \mathcal{I} \mid e \notin I, I \cup \{e\} \in \mathcal{I}\}$$

for some $e \in E$, or in a rectangle of type

$$\{S \subseteq E \mid e \in S, f \notin S\} \times \{I \in \mathcal{I} \mid e \notin I, f \in I, (I \setminus \{f\}) \cup \{e\} \in \mathcal{I}\}$$

for some $e, f \in E$. \square

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