Bridging the Gap Between Multigrid, Hierarchical, and Receding-Horizon Control

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Abstract: We analyze the structure of the Euler-Lagrange conditions of a lifted long-horizon optimal control problem. The analysis reveals that the conditions can be solved by using block Gauss-Seidel schemes and we prove that such schemes can be implemented by solving sequences of short-horizon problems. The analysis also reveals that a receding-horizon control scheme is equivalent to performing a single Gauss-Seidel sweep. We also derive a strategy that uses adjoint information from a coarse long-horizon problem to correct the receding-horizon scheme and we observe that this strategy can be interpreted as a hierarchical control architecture in which a high-level controller transfers long-horizon information to a low-level, short-horizon controller. Our results bridge the gap between multigrid, hierarchical, and receding-horizon control.

Keywords: Euler-Lagrange, Gauss-Seidel, multigrid, receding-horizon, hierarchical control

1. BASIC NOTATION AND SETTING

We consider the following *long-horizon* optimal control problem:

$$\min_{u(\cdot)} \int_0^T \varphi(z(\tau), u(\tau))d\tau \tag{1a}$$

s.t.

$$\dot{z}(\tau) = f(z(\tau), u(\tau)),\ \tau \in [0, T] \tag{1b}$$

$$z(0) = \bar{z}. \tag{1c}$$

Here, $z(\cdot)$ are the states, $u(\cdot)$ are the controls, and the mappings $\varphi(\cdot)$ and $f(\cdot)$ are assumed to be smooth. We *lift* the long-horizon problem by partitioning the horizon T into n stages. This lifting approach was proposed by Bock and Plitt (1984) in the context of multiple-shooting. We define the sets $\mathcal{N} := \{0..n-1\}$ and $\mathcal{N}^- := \mathcal{N} \setminus \{n-1\}$; and we assume the stages to be of equal length $h := T/n$. The partitioning gives rise to the *lifted* problem,

$$\min_{u_k(\cdot)} \sum_{k\in\mathcal{N}} \int_0^h \varphi(z_k(\tau), u_k(\tau))d\tau \tag{2a}$$

s.t.

$$\dot{z}_k(\tau) = f(z_k(\tau), u_k(\tau)),\ k \in \mathcal{N},\ \tau \in [0, h] \tag{2b}$$

$$z_{k+1}(0) = z_k(h),\ k \in \mathcal{N}^- \tag{2c}$$

$$z_0(0) = \bar{z}. \tag{2d}$$

To simplify our analysis, we transcribe the lifted problem into a finite-dimensional nonlinear programming problem by applying an implicit Euler scheme with m inner stages of equal length $\delta := h/m$. We define the sets of inner points $\mathcal{M} := \{0..m-1\}$. We thus obtain the discretized long-horizon problem,

$$\min_{u_{k,j}} \sum_{k\in\mathcal{N}} \sum_{j\in\mathcal{M}} \varphi(z_{k,j+1}, u_{k,j+1}) \tag{3a}$$

s.t.

$$(\nu_{k,j+1})\ z_{k,j+1} = z_{k,j} + \delta f(z_{k,j+1}, u_{k,j+1}),\ k \in \mathcal{N}, j \in \mathcal{M} \tag{3b}$$

$$(\lambda_k)\ z_{k,0} = z_{k-1,m},\ k \in \mathcal{N}. \tag{3c}$$

Here, $\nu_{k,j}$ are the dual variables of the inner dynamic equations (3b), and λ_k are the dual variables of the transition equations between stages (3c). The dual variables are scaled by the constant $1/\delta$. We use the dummy variable $z_{-1,m} := \bar{z}$ to simplify notation. We denote the discretized long-horizon problem (3) as $\mathcal{P}$.

Despite advances in computational methods for optimal control, the long-horizon problem (3) can be difficult or impossible to solve in real time (reviews on the topic are presented by Diehl et al. (2009) and Zavala and Biegler (2009)). Traditionally, long-horizon complexity is addressed by using a receding-horizon scheme. In particular, one can solve the following short-horizon problems sequentially for $k = 0, ..., N-1$:

$$\min_{u_{k,j}} \sum_{j\in\mathcal{M}} \varphi(z_{k,j+1}, u_{k,j+1}) \tag{4a}$$

s.t.

$$(\nu_{k,j+1})\ z_{k,j+1} = z_{k,j} + \delta f(z_{k,j+1}, u_{k,j+1}),\ j \in \mathcal{M} \tag{4b}$$

$$(\lambda_k)\ z_{k,0} = z^{\ell}_{k-1,m}. \tag{4c}$$

Here, $z^{\ell}_{k-1,m}$ is fixed and is obtained from the solution of the problem at $k-1$. We will show that this receding-horizon scheme is a block Gauss-Seidel iteration applied to the solution of the Euler-Lagrange conditions of (3). This observation will help us derive strategies to modify the short-horizon problem (4) so that the receding-horizon scheme better approximates the solution of (3).

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2. STRUCTURE OF EULER-LAGRANGE CONDITIONS

We group variables by stages by defining the vectors $\mathbf{z}_k := (z_{k,0}, ..., z_{k,m})$, $\mathbf{u}_k := (u_{k,1}, ..., u_{k,m})$, and $\nu_k := (\nu_{k,1}, ..., \nu_{k,m})$. We thus obtain the block form of $\mathcal{P}$,

$$\min_{\mathbf{u}_k} \quad \sum_{k\in\mathcal{N}} \phi(\mathbf{z}_k, \mathbf{u}_k) \tag{5a}$$

s.t.

$$(\nu_k) \quad 0 = \chi(\mathbf{z}_k, \mathbf{u}_k), \; k \in \mathcal{N} \tag{5b}$$

$$(\lambda_k) \quad \overline{\Pi}_k \mathbf{z}_k = \underline{\Pi}_k \, \mathbf{z}_{k-1}, \; k \in \mathcal{N}, \tag{5c}$$

where the structure of the mappings $\phi(\cdot)$ and $\chi(\cdot)$ are given by:

$$\phi(\mathbf{z}_k, \mathbf{u}_k) := \sum_{j\in\mathcal{M}} \varphi(z_{k,j+1}, u_{k,j+1}) \tag{6a}$$

$$\chi(\mathbf{z}_k, \mathbf{u}_k) := \begin{bmatrix} z_{k,1} - z_{k,0} - \delta f(z_{k,1}, u_{k,1}) \\ \vdots \\ z_{k,m} - z_{k,m-1} - \delta f(z_{k,m}, u_{k,m}) \end{bmatrix}. \tag{6b}$$

The coefficient matrices $\overline{\Pi}_k$ and $\underline{\Pi}_k$ satisfy $\overline{\Pi}_k \mathbf{z}_k = z_{k,0}$ and $\underline{\Pi}_k \mathbf{z}_{k-1} = z_{k-1,m}$. To enable compact notation, we also define the fixed dummy vector $\mathbf{z}_{-1}$ satisfying $\underline{\Pi}_0 \, \mathbf{z}_{-1} = z_{-1,m} = \bar{z}$.

The Lagrange function of $\mathcal{P}$ is given by

$$\mathcal{L}(\mathbf{z}_k, \mathbf{u}_k, \nu_k, \lambda_k) := \sum_{k\in\mathcal{N}} \phi(\mathbf{z}_k, \mathbf{u}_k) - \nu_k^T \chi(\mathbf{z}_k, \mathbf{u}_k) - \lambda_k^T (\overline{\Pi}_k \mathbf{z}_k - \underline{\Pi}_k \mathbf{z}_{k-1}), \tag{7}$$

and its first-order optimality conditions are

$$0 = \nabla_z \phi_k - \nabla_z \chi_k^T \nu_k - \overline{\Pi}_k^T \lambda_k + \underline{\Pi}_{k+1}^T \lambda_{k+1}, \; k \in \mathcal{N}^- \tag{8a}$$

$$0 = \nabla_z \phi_{n-1} - \nabla_z \chi_{n-1}^T \nu_{n-1} - \overline{\Pi}_{n-1}^T \lambda_{n-1} \tag{8b}$$

$$0 = \nabla_u \phi_k - \nabla_u \chi_k^T \nu_k, \; k \in \mathcal{N} \tag{8c}$$

$$0 = \chi(\mathbf{z}_k, \mathbf{u}_k), \; k \in \mathcal{N} \tag{8d}$$

$$0 = \overline{\Pi}_k \mathbf{z}_k - \underline{\Pi}_k \mathbf{z}_{k-1}, \; k \in \mathcal{N}. \tag{8e}$$

Here, $\nabla_z \phi_k := \nabla_{\mathbf{z}_k} \phi(\cdot)$, $\nabla_u \phi_k := \nabla_{\mathbf{u}_k} \phi(\cdot)$, $\nabla_z \chi_k := \nabla_{\mathbf{z}_k} \chi(\cdot)$, and $\nabla_u \chi_k := \nabla_{\mathbf{u}_k} \chi(\cdot)$. System (8) is the discrete-time version of the Euler-Lagrange conditions of the lifted problem (2). Moreover, the dual variables $\nu_{k,j}, \lambda_k$ can be tied together to form discrete-time profiles of the adjoint variables of the lifted problem. These properties are discussed in the book of Biegler (2010).

We consider the following compact form of the Euler-Lagrange conditions (8),

$$F_k(z_{k-1,m}, z_{k,m}, \lambda_k, \lambda_{k+1}) = 0, \; k \in \mathcal{N}^- \tag{9a}$$

$$F_{n-1}(z_{n-2,m}, \lambda_{n-1}) = 0. \tag{9b}$$

Here, we have dropped the dependence on the non-coupling variables between stages from the notation. We can see that coupling between stages $k-1$, k, and $k+1$ is introduced only through $z_{k-1,m}$ and λ_{k+1}. For clarity, we write $F_k(z_{k-1,m}, z_{k,m}, \lambda_k, \lambda_{k+1}) = 0$ explicitly:

$$0 = \nabla_z \phi_k - \nabla_z \chi_k^T \nu_k - \overline{\Pi}_k^T \lambda_k + \underline{\Pi}_{k+1}^T \lambda_{k+1} = 0 \tag{10a}$$

$$0 = \nabla_u \phi_k - \nabla_u \chi_k^T \nu_k \tag{10b}$$

$$0 = \chi(\mathbf{z}_k, \mathbf{u}_k) \tag{10c}$$

$$0 = \overline{\Pi}_k \mathbf{z}_k - \underline{\Pi}_k \mathbf{z}_{k-1} \tag{10d}$$

for $k \in \mathcal{N}^-$. For $F_{n-1}(z_{n-2,m}, \lambda_{n-1}) = 0$ we have

$$0 = \nabla_z \phi_{n-1} - \nabla_z \chi_{n-1}^T \nu_{n-1} - \overline{\Pi}_{n-1}^T \lambda_{n-1} = 0 \tag{11a}$$

$$0 = \nabla_u \phi_{n-1} - \nabla_u \chi_{n-1}^T \nu_{n-1} \tag{11b}$$

$$0 = \chi(\mathbf{z}_{n-1}, \mathbf{u}_{n-1}) \tag{11c}$$

$$0 = \overline{\Pi}_{n-1} \mathbf{z}_{n-1} - \underline{\Pi}_{n-1} \mathbf{z}_{n-2}. \tag{11d}$$

The block structure of the Euler-Lagrange conditions is not affected by the presence of inequality constraints and algebraic states and equations. The concepts presented next apply to this more general setting as well.

3. BLOCK GAUSS-SEIDEL SCHEMES

Our key observation is that we can solve the Euler-Lagrange conditions (9) (or equivalently (8)) of the long-horizon problem using *block*, nonlinear Gauss-Seidel schemes.

Assume that the adjoints λ_k are fixed to $\lambda_k^\ell = 0$ for $k \in \mathcal{N}$. At $k = 0$ and with fixed $z_{-1}^\ell = \bar{z}$ we solve the following optimal control problem:

$$\min_{z_{k,j}, u_{k,j}} \quad \sum_{j\in\mathcal{M}} \varphi(z_{k,j+1}, u_{k,j+1}) + \delta(\lambda_{k+1}^\ell)^T z_{k,m} \tag{12a}$$

s.t.

$$(\nu_{k,j}) \; z_{k,j+1} = z_{k,j} + \delta f(z_{k,j+1}, u_{k,j+1}), \; j \in \mathcal{M} \tag{12b}$$

$$(\lambda_k) \qquad z_{k,0} = z_{k-1,m}^\ell. \tag{12c}$$

We refer to this problem as $\mathcal{P}_k$ and introduce the notation $(z_{k,m}^{\ell+1}, \lambda_k^{\ell+1}) \leftarrow \mathcal{P}_k(z_{k-1,m}^\ell, \lambda_{k+1}^\ell)$ to indicate the inputs and outputs of problem $\mathcal{P}_k$. The primal-dual solution of $\mathcal{P}_k$ solves block k of the Euler-Lagrange conditions (10) for fixed initial state $z_{k-1,m}^\ell$ and adjoint $\lambda_{k+1} = \lambda_{k+1}^\ell$.

From the solution of $\mathcal{P}_k$ we obtain the terminal state $z_{k,m}^{\ell+1}$ and we use this as initial state for $\mathcal{P}_{k+1}$ to compute $(z_{k+1,n}^\ell, \lambda_{k+1}^\ell) \leftarrow \mathcal{P}_k(z_{k,m}^{\ell+1}, \lambda_{k+2}^\ell)$. We continue the recursion until reaching the last stage, $k = n-1$. At this stage we solve problem $\mathcal{P}_{n-1}$:

$$\min_{z_{n-1,j}, u_{n-1,j}} \quad \sum_{j\in\mathcal{M}} \varphi(z_{n-1,j+1}, u_{n-1,j+1}) \tag{13a}$$

s.t.

$$(\nu_{n-1,j}) \; z_{n-1,j+1} = z_{n-1,j} + \delta f(z_{n-1,j+1}, u_{n-1,j+1}), \; j \in \mathcal{M} \tag{13b}$$

$$(\lambda_{n-1}) \qquad z_{n-1,0} = z_{n-2,m}^\ell. \tag{13c}$$

The primal-dual solution of $\mathcal{P}_{n-1}$ solves the optimality system (11) for fixed initial state $z_{n-2,m}^\ell$ obtained from the solution of $\mathcal{P}_{n-2}$. With this step we have updated all the state (primal) $z_k^{\ell+1}$ and adjoint $\lambda_k^{\ell+1}$ variables. We return to the first stage $k = 0$ and repeat the recursion to obtain $z_k^{\ell+2}, \lambda_k^{\ell+2}$. We repeat this procedure n_{GS} times. We summarize the Gauss-Seidel scheme below.

Gauss-Seidel Scheme A

I) GIVEN $\bar{z}$, set counter $\ell \leftarrow 0$, set $z_{-1,m}^\ell \leftarrow \bar{z}$, and set $\lambda_k^\ell \leftarrow 0$ for $k = 0, ..., n-1$. FOR $\ell = 0, ..., n_{GS}$ DO:

II) FOR $k = 0, ..., n-2$ SOLVE $(z_{k,m}^{\ell+1}, \lambda_k^{\ell+1}) \leftarrow \mathcal{P}_k(z_{k-1,m}^{\ell+1}, \lambda_{k+1}^\ell)$.

III) FOR $k = n-1$ SOLVE $(z_{n-1,m}^{\ell+1}, \lambda_{n-1}^{\ell+1}) \leftarrow \mathcal{P}_{n-1}(z_{n-2,m}^\ell, 0)$.

IV) SET $\ell \leftarrow \ell + 1$ and RETURN TO Step II).

A key observation that we make is that the *Gauss-Seidel scheme can be implemented by using off-the-shelf optimization tools*. This is because all that is needed is the ability to solve the nonlinear programming problems $\mathcal{P}_k$. In particular, no internal linear algebra manipulations are needed.

The structure of the block Gauss-Seidel scheme also reveals that a *receding-horizon control scheme is equivalent to performing a single Gauss-Seidel sweep* (iteration) with adjoints $\lambda_k^\ell = 0,\ k \in \mathcal{N}$. The adjoints λ_{k+1}^ℓ encode important global information of the future horizon beyond h for problem $\mathcal{P}_k$. In particular, the adjoints can be interpreted as *terminal costs* and the term $(\lambda_{k+1}^\ell)^T z_{k,m}$ can be seen as a *cost-to-go*. This information is neglected by the receding-horizon scheme and this can lead to a poor approximation of the long-horizon solution. A key insight that we gain from our analysis is that we can correct the receding-horizon scheme to better approximate the long-horizon solution if we are capable of obtaining estimates of the adjoints λ_k^ℓ. Moreover, in the ideal case where our adjoint estimates are optimal, the corrected receding-horizon scheme (a Gauss-Seidel sweep) will deliver the optimal long-horizon state profiles.

We can obtain refined estimates of the adjoints λ_k^ℓ by performing multiple Gauss-Seidel iterations of the scheme previously discussed. This can be seen as a *self-correcting* receding-horizon scheme. Gauss-Seidel updates can be performed in different ways so we have some degree of flexibility. For instance, Borzı (2003) proposed the following forward-backward scheme:

Gauss-Seidel Scheme B

I) GIVEN $\bar{z}$, set counter $\ell \leftarrow 0$, set $z_{-1,m}^\ell \leftarrow \bar{z}$, and set $\lambda_k^\ell \leftarrow 0$ for $k = 0, ..., n-1$. FOR $\ell = 0, ..., n_{GS}$ DO:
Forward Sweep
II) FOR $k = 0, ..., n-2$ SOLVE
$(z_{k,m}^{\ell+1}, \cdot) \leftarrow \mathcal{P}_k(z_{k-1,m}^{\ell+1}, \lambda_{k+1}^\ell)$.
III) FOR $k = n-1$ SOLVE
$(z_{n-1,m}^{\ell+1}, \lambda_{n-1}^{\ell+1}) \leftarrow \mathcal{P}_{n-1}(z_{n-2,m}^\ell, 0)$.
Backward Sweep
IV) FOR $k = n-2, n-3, ..., 0$ SOLVE
$(\cdot, \lambda_k^{\ell+1}) \leftarrow \mathcal{P}_k(z_{k-1,m}^{\ell+1}, \lambda_{k+1}^\ell)$.
V) SET $\ell \leftarrow \ell + 1$ and RETURN TO Step II).

In Gauss-Seidel Scheme B, the forward sweep updates the states while the backward sweep updates the adjoints. In Gauss-Seidel Scheme A the states and adjoints are updated simultaneously. Forward-backward schemes are also typically used in the solution of continuous-time Euler-Lagrange conditions. For a description of these approaches the reader is referred to the book of Bryson and Ho (1975).

4. COARSENING-BASED CORRECTION

Gauss-Seidel (receding-horizon) schemes provide the computational advantage that they need to solve only short-horizon problems in order to approximate the solution of the long-horizon problem. However, Gauss-Seidel schemes are well known for exhibiting slow convergence or no convergence at all. We propose to aid convergence by using adjoint estimates λ_k^ℓ obtained from the solution of a *coarsened* long-horizon problem to correct the short-horizon problems. Coarsening is a standard concept used in multigrid optimal control of partial differential equations (PDEs). Such concepts are discussed extensively by Borzì and Schulz (2009). To perform coarsening, we consider a coarse grid with m_c elements and $m_c \ll m$ (thus reducing computational complexity). We define the coarse set as $\mathcal{M}_c := \{0..m_c - 1\}$ and the corresponding coarsened long-horizon problem $\mathcal{P}_c$. We use the notation $(\lambda_0^\ell, ..., \lambda_{n-1}^\ell) \leftarrow \mathcal{P}^c(\bar{z})$ to indicate the inputs and outputs of $\mathcal{P}_c$. We consider the following scheme:

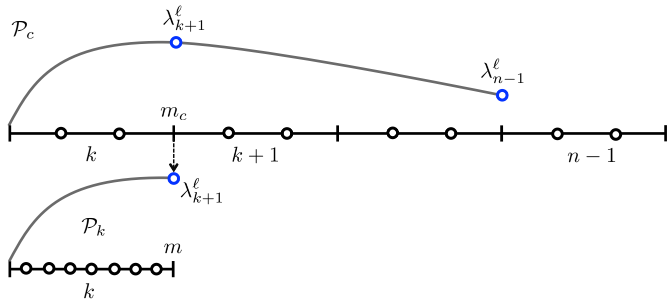

Fig. 1. Transfer of adjoint information from coarse long-horizon problem to short-horizon problems.

Corrected Gauss-Seidel Scheme A

I) GIVEN $\bar{z}$, set counter $\ell \leftarrow 0$, set $z_{-1,m}^\ell \leftarrow \bar{z}$. FOR $\ell = 0, ..., n_{GS}$ DO:
II) SOLVE $(\lambda_0^\ell, ..., \lambda_{n-1}^\ell) \leftarrow \mathcal{P}_c(\bar{z})$.
III) FOR $k = 0, ..., n-2$ SOLVE
$(z_{k,m}^{\ell+1}, \lambda_k^{\ell+1}) \leftarrow \mathcal{P}_k(z_{k-1,m}^{\ell+1}, \lambda_{k+1}^\ell)$.
IV) FOR $k = n-1$ SOLVE
$(z_{n-1,m}^{\ell+1}, \lambda_{n-1}^{\ell+1}) \leftarrow \mathcal{P}_{n-1}(z_{n-2,m}^\ell, 0)$.
V) SET $\ell \leftarrow \ell + 1$ and RETURN TO Step III).

In this scheme, $\mathcal{P}_c$ transfers global (long-horizon) information of problem $\mathcal{P}$ to the local (short-horizon) problems $\mathcal{P}_k$. This is depicted in Figure 1. This approach can be seen as a *hierarchical control scheme* where a coarse-grained high-level controller supervises a fine-grained low-level controller. In other words, the coarse long-horizon problem provides terminal costs to the receding-horizon controller so that this better approximates the solution of the long-horizon problem. In this hierarchical controller dual information is transferred; as opposed to traditional hierarchical control schemes that transfer state information. For a review on hierarchical control see the paper by Scattolini (2009). The lifting approach proposed in this work *avoids the need to perform interpolation* of state and adjoint profiles in order to move from a coarse grid to a fine grid (as is typically done in multigrid control for PDEs). In our approach, all that is needed from the coarse problem are the adjoints λ_k^ℓ at the stage transition points.

While our analysis borrows concepts of multigrid control of PDEs such as Gauss-Seidel recursions and coarsening, *our contributions* are the following:

- We demonstrate how to use multigrid concepts in a more general setting that might involve constraints and sets of differential and algebraic equations.

- We establish a connection between receding-horizon control and Gauss-Seidel schemes.
- We demonstrate that coarsening schemes can be used to construct hierarchical control schemes.
- We demonstrate how to implement multigrid concepts by using off-the-shelf optimization tools.

5. NUMERICAL STUDY

We illustrate the concepts using the well-studied nonlinear CSTR reactor problem:

$$\min \int_0^T \alpha_c(c(\tau)-\hat{c})^2 + \alpha_t(t(\tau)-\hat{t})^2 + \alpha_u(u(\tau)-\hat{u})^2 d\tau \tag{14a}$$

s.t.

$$\dot{c} = \frac{1-c(\tau)}{\theta} - p_k \cdot \exp\left(-\frac{p_E}{t(\tau)}\right) \cdot c(\tau) \tag{14b}$$

$$\dot{t} = \frac{t_f - t(\tau)}{\theta} + p_k \cdot \exp\left(-\frac{p_E}{t(\tau)}\right) \cdot c(\tau) - p_\alpha \cdot u(\tau) \cdot (t(\tau) - t_c) \tag{14c}$$

$$0 \le c(\tau) \le 1,\ 0 \le t(\tau) \le 1,\ 200 \le u(\tau) \le 500 \tag{14d}$$

$$c(0) = \bar{c}, \quad t(0) = \bar{t}. \tag{14e}$$

The system states are the concentration of reactant $c(\cdot)$ and the temperature of reacting mixture $t(\cdot)$. The control is the cooling water flow $u(\cdot)$. After lifting, we denote the adjoint associated with (14b) as λ_c and the adjoint associated with (14c) as λ_t. The model parameters are given by $\alpha_c, \alpha_t, \alpha_u, p_\alpha, p_E, t_f, t_c$, and p_k and can be found in the work of Zavala and Anitescu (2010). The objective function is of Bolza type with the desired targets $\hat{c}, \hat{t}$ and $\hat{u}$. We highlight the nonlinearity of the system and the presence of bounds. All optimization problems were implemented on AMPL and solved with IPOPT.

We partition the time horizon in $n = 10$ stages and discretize each stage using an implicit Euler scheme with $m = 10$ grid points. In Figures 2 and 3 we present the control and temperature profiles for Gauss-Seidel Scheme A. As can be seen, the profiles obtained with a standard receding-horizon scheme (first iteration of Gauss-Seidel scheme) severely deviates from the optimal ones. The Gauss-Seidel scheme approximates the long-horizon solution well after three iterations.

In Figure 4 we plot the error of the Gauss-Seidel Schemes A and B. The error is defined as the Euclidean norm of the error profiles for the states and controls. As can be seen, both schemes have similar performance and converge to error levels of 10^{-4}. This result is surprising since one would expect that performing an additional backward sweep in Scheme B would be beneficial. Despite the nonlinearity and presence of bounds, both Gauss-Seidel Schemes converge to the optimal profiles.

In Figure 5 we present the optimal and approximate adjoint profiles obtained from coarsening. The coarse profiles are obtained by discretizing the stage using $m_c = 2$ points (compared to the $m = 10$ points used for the optimal profile). As can be seen, the adjoint profiles have exhibit some errors but the overall structure is preserved. We use the coarse adjoint profiles to correct the Gauss-Seidel scheme

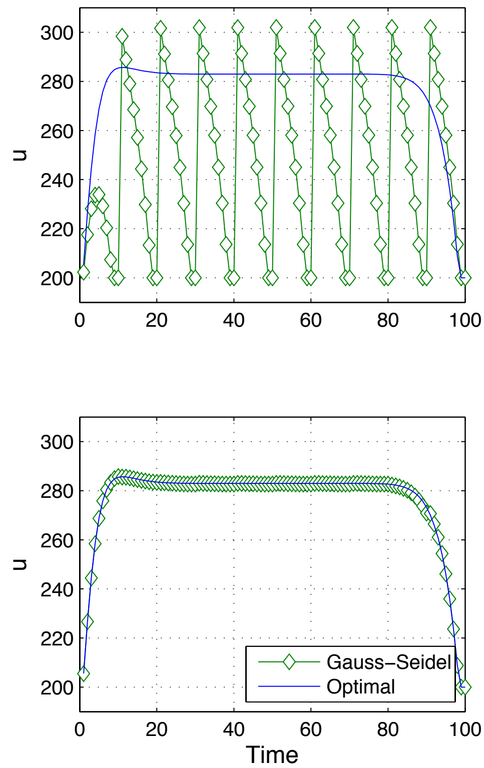

Fig. 2. Gauss-Seidel A convergence for control. First iteration (top) and third iteration (bottom).

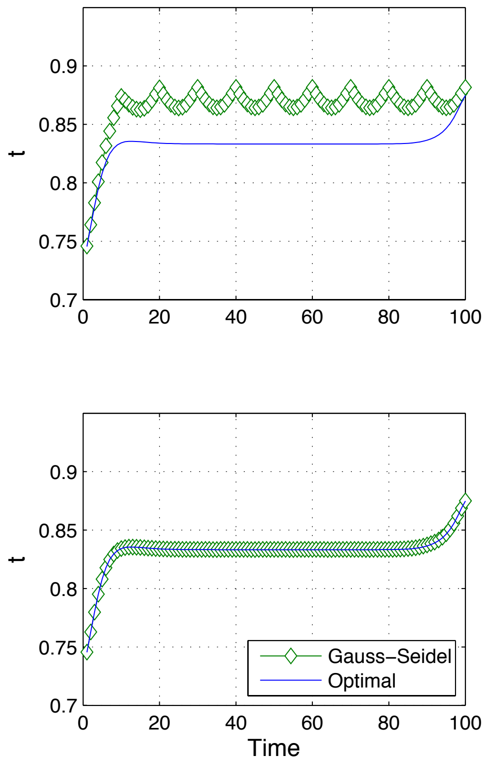

Fig. 3. Gauss-Seidel A convergence for temperature. First iteration (top) and third iteration (bottom).

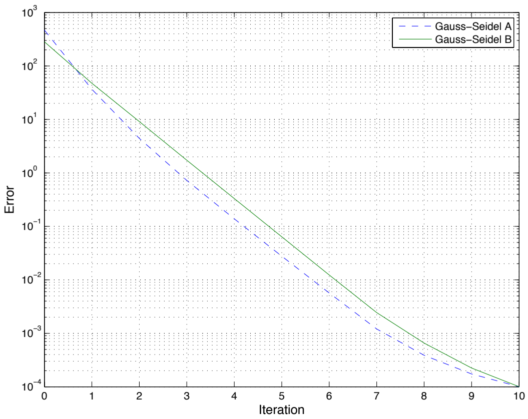

Fig. 4. Convergence of Gauss-Seidel schemes A and B.

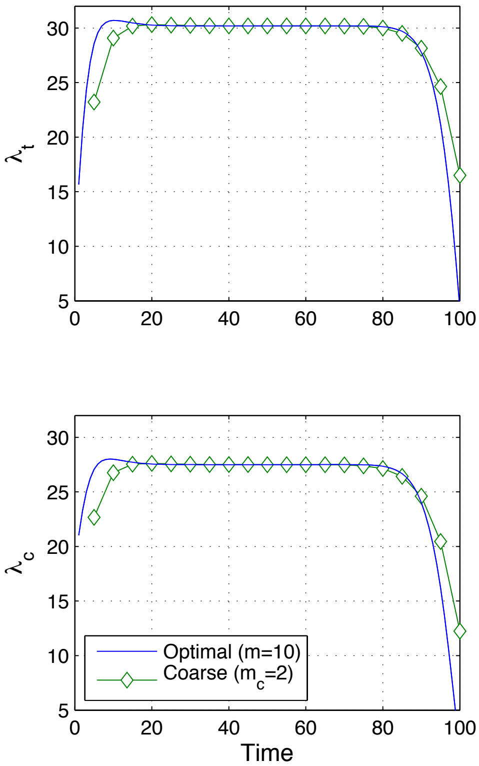

Fig. 5. Optimal and coarsened adjoint profiles.

A. In Figure 6 we present the convergence of Gauss-Seidel with and without correction. The top graph presents the profiles after one iteration and the bottom graph after two iterations. As can be seen, correction dramatically aids convergence. In Figure 7 we present the convergence of the corrected and uncorrected Gauss-Seidel Scheme A. For the corrected case we consider an additional case in which we refine the coarse grid by using $m_c = 4$ points. As can be seen, the initial error is reduced by two orders of magnitude in both cases.

6. CONCLUSIONS AND FUTURE WORK

We presented an analysis of the Euler-Lagrange conditions for a lifted optimal control problem. This enabled us to derive block Gauss-Seidel schemes that enable the solution of long-horizon problems by solving sequences of short-horizon problems. Our analysis revealed that a receding-horizon scheme is equivalent to performing a Gauss-Seidel sweep of the Euler-Lagrange conditions. We have also used our analysis to derive strategies to correct adjoint profiles by using coarsening. This approach enabled us to accelerate Gauss-Seidel convergence and can be interpreted as a hierarchical control structure in which a coarse high-level controller transfers long-horizon information to a low-level, short-horizon controller. Our results thus bridge the gap between multigrid, hierarchical, and receding-horizon control. As part of future work, we would like to gain additional insight on conditions guaranteeing convergence of Gauss-Seidel schemes and we will use multigrid control concepts to design alternative control architectures.

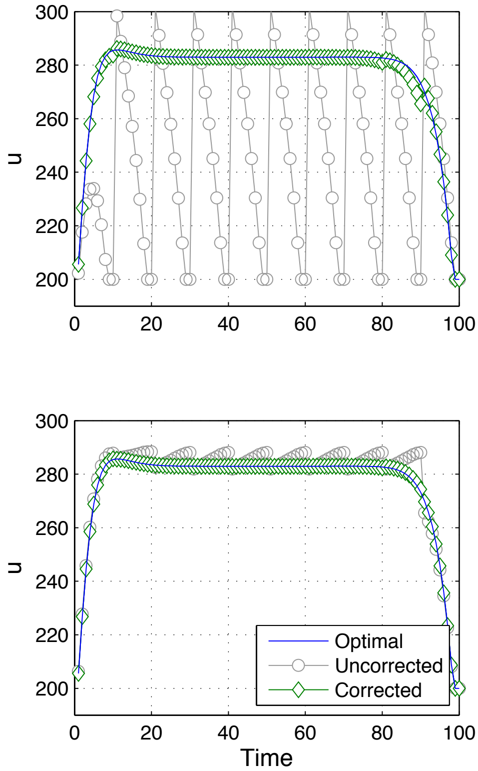

Fig. 6. Convergence of Gauss-Seidel Scheme A with and without correction. First iteration (top) and second iteration (bottom).

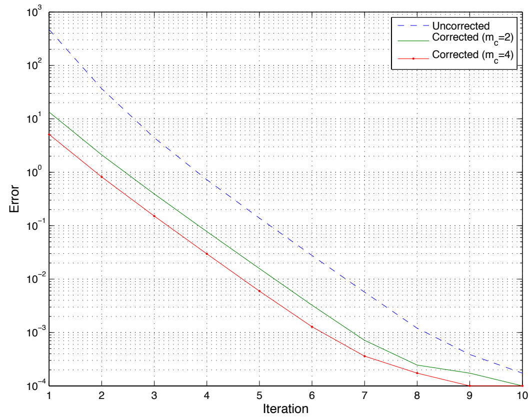

Fig. 7. Convergence of Gauss-Seidel Scheme A with and without correction.

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