ON "ON AN EFFICIENT IMPLEMENTATION OF THE FACE ALGORITHM FOR LINEAR PROGRAMMING"

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This paper is dedicated to Professor Yu-Da Hu for his 80th birthday.

Abstract. Zhang et al ([2]) recently propose another approach to the face algorithm [1]. This note gives a modification of the result.

1. Introduction

The recently proposed face algorithm [1] moves from face to face in the underlying polyhedron until reaching an optimal face. Unlike the simplex algorithm, therefore, it yields iterates that are not necessarily vertices but boundary points on the associated faces. It is attractive that only a single triangular system is solved in each iteration, compared with four such systems in the simplex algorithm. It appears very promising in computational tests.

The face algorithm is benefited from the so-called "reduced problem". This special form can be obtained by regarding f as a variable and bringing $f = c^{\mathrm{T}}x$ into the constraint part, i.e.,

$$\begin{array}{ll} \min & f, \\ \text{s.t.} & \begin{pmatrix} A & 0 \\ c^{\mathrm{T}} & -1 \end{pmatrix} \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \quad x \geq 0. \end{array} \tag{1.0.1}$$

It might be well to introduce

$$A := \begin{pmatrix} A \\ c^{\mathrm{T}} \end{pmatrix}, \quad b := \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

Thereafter f is indexed by $n+1$, and deemed the same as x_{n+1} So, $a_{n+1} = -e_{m+1}$, and the problems can be written as a so-called "reduced problem":

$$\begin{array}{ll} \min & f, \\ \text{s.t.} & \left(A \,\vdots\, a_{n+1}\right) \begin{pmatrix} x \\ f \end{pmatrix} = b, \quad x \geq 0. \end{array} \tag{1.0.2}$$

were $(A\ a_{n+1}) \in \mathcal{R}^{(m+1)\times(n+1)}$, $b \in \mathcal{R}^{m+1}$, rank $(A\ a_{m+1}) = m+1,\ m < n$. Note that the reduced problem is of a standard from, but its objective function involves only a single (free) objective variable f.

Therefore, we now consider the following problem:

$$\begin{array}{ll} \min & x_{n+1}, \\ \text{s.t.} & Ax = b, \quad x_j \geq 0,\ j = 1, \cdots, n. \end{array} \tag{1.0.3}$$

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where $A \in \mathcal{R}^{(m+1)\times(n+1)}$, $b \in \mathcal{R}^{m+1}$, rank $A = m+1$, $m < n$, $a_{n+1} = Ae_{n+1} = -e_{m+1}$. The assumption on the rank of A is not essential, and can be dropped.

Assume that $B \in \mathcal{R}^{(m+1)\times k}$, $N \in \mathcal{R}^{(m+1)\times(n-k+1)}$ is a partition of A, where

$$\text{rank } B = m+1, \quad m+1 \le k \le n+1.$$

Respectively, B and N are called *face matrix* and *nonface matrix*; the associated variables are called *face variables* and *nonface variables.* Without confusion, denote the associated index sets also by B and N. It is always assumed that the kth (last) column of the face matrix equals the $(n+1)$th (last) column of A, that is,

$$Be_k = -e_{m+1}. \tag{1.0.4}$$

Before the presentation of the variant, we briefly outline the face algorithm in the next section (for more details, the reader is referred to [1]).

2. Face Algorithm

The face, associated with the face matrix B, is defined by

$$P_B = \{x \in A \mid Bx_B = b;\ x_s \ge 0,\ s = 1, \cdots, k-1,\ x_N = 0\}. \tag{2.0.5}$$

A point on the face is called *face point.* Thus, an 0-dimensional face has an unique face point, vertex. A face is level face if the objective value is constant over it. A level face is an *optimal face* if the related objective value equals the optimal value.

Assume that $\bar{x}$ is the current face point on P_B, i.e., $\bar{x} \in P_B$. $\bar{x}_B \in P_B$ is often used in place of $\bar{x} \in P_B$, for simplicity.

For the minimization over the face, e_k is the objective gradient. Thus the orthogonal projection of e_k onto the null space of B is

$$\Delta_B = Pe_k = e_k - B^{\mathrm{T}}y, \qquad y = -(BB^{\mathrm{T}})^{-1}e_{m+1}, \tag{2.0.6}$$

where $P = I - B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}B$ is the projection matrix. If $-\Delta_B$ is nonzero, therefore, it is a good choice for being the search direction in x_B-space.

The computation of Δ_B is central to the face algorithm. One way is by solving the $(m+1)\times(m+1)$ system

$$BB^{\mathrm{T}}y = -e_{m+1}. \tag{2.0.7}$$

Since B is of full row rank, there exists the Cholesky factorization (see, e.g., [5])

$$BB^{\mathrm{T}} = LL^{\mathrm{T}}, \tag{2.0.8}$$

where L is nonsingular lower triangular. Thereby, (2.0.7) becomes

$$LL^{\mathrm{T}}y = -e_{m+1}, \tag{2.0.9}$$

which can be handled via two triangular systems, i.e.,

$$Lv = -e_{m+1}, \qquad L^{\mathrm{T}}y = v.$$

The solution to the first system is readily available, that is, $v = -(1/\nu)e_{m+1}$, where ν is the $(m+1)$th diagonal of L. Consequently, there is only one triangular system to be solved, i.e.,

$$L^{\mathrm{T}}y = -(1/\nu)e_{m+1} \tag{2.0.10}$$

Assume that $\Delta_B \ne 0$. It can be shown that the problem is lower unbounded if index set

$$J = \{j \in B \mid \Delta_j > 0, j \ne n+1\}$$

is empty. Otherwise, a new iterate can be obtained by the following line search scheme

$$\hat{x}_B = \bar{x}_B - \alpha \Delta_B, \tag{2.0.11}$$

where stepsize α and index j are such that

$$\alpha = \bar{x}_j / \Delta_j = \min_{t \in J} \bar{x}_t / \Delta_t \geq 0, \tag{2.0.12}$$

which is the largest possible stepsize for the new iterate remaining within the feasible region. Note that $\hat{x}_j = 0$.

Then, B and N are updated accordingly by moving j from B to N. It can be shown that the new face matrix $\breve{B} \in \mathcal{R}^{(m+1)\times(k-1)}$ is again of rank $m+1$. Therefore, there still exists the Cholesky factorization

$$\breve{B}\breve{B}^{\mathrm{T}} = \breve{L}\breve{L}^{\mathrm{T}}.$$

Saunders [6] offers a procedure to obtain the new factor from its predecessor.

In case of $\Delta_B = 0$, it can be shown that P_B is a level face; in particular, it is the case whenever $k = m+1$. This is a point to check for optimality. In fact, if, further, it holds that

$$\bar{z}_N = -N^{\mathrm{T}}\bar{y} \geq 0,$$

then $(\bar{x}_B, \bar{x}_N)$ and $(\bar{z}_B, \bar{z}_N, \bar{y})$ $(\bar{x}_N = 0,\ \bar{z}_B = 0)$ are a pair of primal and dual optimal solutions, and P_B is an optimal face to (1.0.3).

If $\Delta_B = 0$ but $\bar{z}_N = -N^{\mathrm{T}}\bar{y} \not\geq 0$, optimality of the level face P_B can not be asserted, but a column index j can be determined such that

$$j \in \arg\min_{t \in N}\ \bar{z}_t.$$

The face and nonface index sets are then updated by move index j from N to B. Thus, the q-indexed column is added to the face matrix. Denote the resulting new face matrix by The according update of the Cholesky factor is quite simple (see [4] or [1]).

We do not go into details of updating Cholesky factors, but only point out the fact that both the procedures involve orthogonal transformations.

The overall steps is put into the following algorithm.

Algorithm 1.(Face algorithm) Initial $(B,\ N)$, $m+1 < k \leq n+1$, feasible solution x and Cholesky factor L of BB^{T}. This algorithm solves the reduced problem (1.0.3).

1. Solves upper triangular system $L^{\mathrm{T}}y = -(1/\nu)e_{m+1}$, where ν is the $(m+1)$th diagonal of L.
2. Compute $\Delta_B = e_k - B^{\mathrm{T}}y$.
3. Go to step 11 if $\Delta_B = 0$, .
4. Stop if $J = \{j \in B \mid \Delta_j > 0,\ j \neq n+1\} = \emptyset$ (unbounded problem).
5. Determine stepsize α and index p such that $\alpha = x_p/\Delta_p = \min_{j \in J} x_j/\Delta_j$.
6. Update (B, N) by moving p from B to N.
7. If $\alpha \neq 0$, update $x_B = x_B - \alpha\Delta_B$.
8. Update L.
9. Set $k = k - 1$.
10. Go to step 1 if $k > m+1$.
11. Compute $z_N = -N^{\mathrm{T}}\bar{y}$.
12. Stop if $z_N \geq 0$ (optimality achieved).
13. Determine index $q \in \arg\min_{j \in N}\ z_j$.

14. Update L.
15. Moving q from N to B.
16. Set $k = k + 1$.
17. Go to step 1.

3. Variant of the Face Algorithm

With favorable computational results, Zhang et al ([2]) recently propose another approach to the face algorithm. The only difference is in the way to compute y (see the second expression of 2.0.6): their approach updates $(BB^T)^{-1}$ rather than the Cholesky factor. Consequently, it involves no Cholesky factorization, but a vector sequence and a scalar sequence instead. However, the update of the search direction Δ_B is somewhat curmbersome, bringing about extra column permutations. In this note, we will compute Δ_B directly rather than recursively. Some other changes will also be made relevantly.

Assume that $\breve{B}$ resulted from dropping a column $a_j, j \in B$ from the predecessor B, in some contracting iteration. Then it holds that

$$\breve{B}^T = \sum_{j \neq r \in B} a_r a_r^t = \sum_{r \in B} a_r a_r^t - a_j a_j^T = BB^T - a_j a_j^T.$$

As $\hat{B}\hat{B}^T$ is nonsingular, Sherman-Morrison formula (**??**) can be used to obtain

$$(\breve{B}\breve{B}^T)^{-1} = (BB^T)^{-1} + \frac{(BB^T)^{-1} a_j a_j^T (BB^T)^{-1}}{1 - a_j^T (BB^T)^{-1} a_j}. \tag{3.0.13}$$

If $\hat{B}$ is yielded from entering a column $a_j, j \in N$ to B, in some expending iteration, we obtain similarly,

$$(\hat{B}\hat{B}^T)^{-1} = (BB^T)^{-1} - \frac{(BB^T)^{-1} a_j a_j^T (BB^T)^{-1}}{1 + a_j^T (BB^T)^{-1} a_j}. \tag{3.0.14}$$

The preceding two formulas may serve as a basis for the variant.

Let $l \geq 1$ be an integer and let $B^{(l)}$ be the face matrix at the lth iteration. Assume that a column $a^{(l)}$ was selected to leave or enter $B^{(l)}$, yielding the next face matrix $B^{(l+1)}$. According to (3.0.13) and (3.0.14), the update of the inverse can be written uniformly as follows:

$$(B^{(l+1)}(B^{(l+1)})^T)^{-1} = (B^{(l)}(B^{(l)})^T)^{-1} + \eta^{(l)} h^{(l)} (h^{(l)})^T, \tag{3.0.15}$$

where

(3.0.16)

$$h^{(l)} = (B^{(l)}(B^{(l)})^T)^{-1} a^{(l)}, \qquad \eta^{(l)} = \begin{cases} 1/(1 - (a^{(1)})^T h^{(l)}) & \text{if } a^{(l)} \text{ leaves} \\ -1/(1 + (a^{(l)})^T h^{(l)}) & \text{if } a^{(l)} \text{ enters} \end{cases}$$

Let $B^{(1)}$ be the initial face matrix. Assume that Cholesky factorization is available:

$$B^{(1)}(B^{(1)})^T = LL^T. \tag{3.0.17}$$

Thus, the key vector

$$y^{(1)} = -(B^{(1)}(B^{(1)})^T)^{-1} e_{m+1} = -(LL^T)^{-1} e_{m+1} \tag{3.0.18}$$

can be computed by only solving an upper triangular system, i.e.,

$$L^T y = -(1/\nu)e_{m+1},$$

where ν is the $(m+1)$th diagonal of L.

Thereby, the first iteration can be carried out just as in the standard face algorithm; but some extra steps are required to prepare for the next iteration. To do so, $\bar{y}^{(2)}$ is yielded from updating $\bar{y}^{(1)}$. Setting $l = 1$ in (3.0.15) and noting (3.0.17) gives

$$(B^{(2}(B^{(2)})^T)^{-1} = (LL^T)^{-1} + \eta^{(1)}h^{(1)}(h^{(1)})^T.$$

where

$$h^{(1)} = (LL^T)^{-1}a^{(1)}, \qquad \eta^{(1)} = \begin{cases} 1/(1-(a^{(1)})^T h^{(1)}) & \text{if } a^{(1)} \text{ leaves} \\ -1/(1+(a^{(1)})^T h^{(l)}) & \text{if } a^{(1)} \text{ enters} \end{cases}$$

From the preceding and (3.0.18), the update follows, i.e.,

$$y^{(2)} = -(B^{(2}(B^{(2)})^T)^{-1}e_{m+1} = y^{(1)} - (\eta^{(1)}h^{(1)}_{m+1})h^{(1)}.$$

For iteration $l \geq 2$, the update is easily obtained from (3.0.15), i.e.,

$$y^{(l+1)} = -(B^{(l+1}(B^{(l+1)})^T)^{-1}e_{m+1} = y^{(l)} - (\eta^{(l)}h^{(l)}_{m+1})h^{(l)}. \tag{3.0.19}$$

But $)h^{(l)}$ and $(\eta^{(l)}$ will not directly computed from (3.0.16), which involves the current inverse $(B^{(l)}(B^{(l)})^T)^{-1}$; instead, they are obtained from $(h^{(t)}, \eta^{(t)}), t = 1, \cdots, l-1$, resulting from the previous $(l-1)$ iterations. In fact, it holds that

$$\begin{aligned} (B^{(l)}(B^{(l)})^T)^{-1} &= (B^{(l-2)}(B^{(l-2)})^T)^{-1} + \eta^{(l-2)}h^{(l-2)}(h^{(l-2)})^T + \eta^{(l-1)}h^{(l-1)}(h^{(l-1)})^T \\ &\vdots \\ &= (LL^T)^{-1} + \sum_{t=1}^{l-1} \eta^{(t)}h^{(t)}(h^{(t)})^T, \end{aligned}$$

from which and (3.0.16), it follows that

$$\begin{aligned} h^{(l)} &= (B^{(l)}(B^{(l)})^T)^{-1}a^{(l)} \\ &= (LL^T)^{-1}a^{(l)} + \sum_{t=1}^{l-1}(\eta^{(t)}(a^{(l)})^T h^{(t)})h^{(t)} \\ \eta^{(l)} &= \begin{cases} 1/(1-(a^{(l)})^T h^{(l)}) & \text{if } a^{(l)} \text{ leaves} \\ -1/(1+(a^{(l)})^T h^{(l)}) & \text{if } a^{(l)} \text{ enters} \end{cases} \end{aligned}$$

It is clear that (3.0.19) together with (3.0.20) is valid for all $l \geq 1$

The overall step are summarized into the following algorithm.

Algorithm 2. (Variant of face algorithm) $l = 1, B = A,\ N = \emptyset,\ k = n+1$, feasible solution x and Cholesky factor L of AA^{T}. This algorithm solves the reduced problem (1.0.3).

1. Compute $L^T y = -(1/\nu)e_{m+1}$, where ν is the $(m+1)$th diagonal of L.
2, compute $\Delta_B = e_k - B^{\mathrm{T}}y$.
3. Go to step 9 if $\Delta_B = 0$, .
4. Stop if $J = \{j \in B \mid \Delta_j > 0,\ j \neq n+1\} = \emptyset$
5. Determine stepsize α and index j such that $\alpha = x_j/\Delta_j = \min_{t \in J} x_t/\Delta_t$.
6. Update (B, N) by braining j from B to N, and set $k = k-1$.
7. If $\alpha \neq 0$, update $x_B = x_B - \alpha\Delta_B$.

8. Set $s = 0$, and go to step 13.
9. Compute $z_N = -N^{\mathrm{T}}y$.
10. Determine index $z_j = \min_{t\in N}\ z_t$.
11. Stop if $z_j \geq 0$.
12. Bring j from N to B, and set $k = k+1$ and $s = 1$.
13. Solve $Lv = a_j$, and then $L^T w = v$.
14. Compute $h^{(l)} = w + \sum_{t=1}^{l-1}(\eta^{(t)} a_j^T h^{(t)})h^{(t)}$
15. Compute $\gamma = a_j^T h^{(l)}$.
16. If $s = 0$, set $\eta^{(1)} = 1/(1-\gamma)$; else set $\eta^{(1)} = -1/(1+\gamma)$.
17. Compute $y = y - (\eta^{(l)} h_{m+1}^{(l)})h^{(l)}$.
18. Set $l = l+1$.
19. If $k > m+1$, go to step 2; else to step 9.

Note 1. In step 12, x is updated by simply inserting zero in accordance with the entering a_j.

In theory, the preceding Algorithm and the face algorithm create the same iterates. So we cite the result given by [1] without proof.

Theorem 3.0.1. *Under the nondegeneracy assumption, Algorithm* **??** *terminates either at*

(i) step 4, detecting lower unboundedness of (1.0.3); or at

(ii) step 11, giving an optimal face together with a pair of primal and dual optimal solutions.

Example 3.0.2. Solve the following problem by Algorithm 2:

$$\begin{array}{lrrrrrrrrcr} \min & f = -2x_1 & 0x_2 & -4x_3 & 3x_4 & -2x_5 & 0x_6 & -1x_7 & & \\ \text{s.t.} & 0x_1 & 6x_2 & -5x_3 & 4x_4 & 0x_5 & -1x_6 & -2x_7 & = & 2 \\ & -2x_1 & 2x_2 & 0x_3 & -7x_4 & -6x_5 & 0x_6 & 6x_7 & = & -7 \\ & 0x_1 & 3x_2 & -6x_3 & 0x_4 & -3x_5 & -2x_6 & -5x_7 & = & -13 \\ & & & & & & & x_j & \geq & 0 \quad j = 1,\dots,7. \end{array}$$

Answer: Initial: $k = 8,\ x = (1,1,1,1,1,1,1,-6)^T$

$$A = \begin{pmatrix} & 6 & -5 & 4 & & -1 & -2 & \\ -2 & 2 & & -7 & -6 & & 6 & \\ & 3 & -6 & & -3 & -2 & -5 & \\ -2 & & -4 & 3 & -2 & & -1 & -1 \end{pmatrix}$$

$AA^T = LL^T$, where

$$L = \begin{pmatrix} -2943/325 & 0 & 0 & 0 \\ 1914/619 & -3224/295 & 0 & 0 \\ -6500/981 & -1323/998 & 3037/497 & 0 \\ -1607/428 & -417/7474 & 1350/821 & -7580/1777 \end{pmatrix}$$

Iteration l = 1:

1. $-1/\nu = 1777/7580, \qquad y = (74/6463, -146/96477, 157/10616, -133/2420)^T$.
2. $\Delta_B = (-1027/9093, -422/3835, -320/4333, 523/4821, -328/4395, 99/2413, 219/4297, 791/837)^T$.
5. $\alpha = \min\{1/(523/4821), 1/(99/2413), 1/(219/4297)\}$
$= 1/(523/4821) = 4821/523, j = 4$.

6.

$$B = \begin{pmatrix} & 6 & -5 & & -1 & -2 & \\ -2 & 2 & & -6 & & 6 & \\ & 3 & -6 & -3 & -2 & -5 & \\ -2 & & -4 & -2 & & -1 & -1 \end{pmatrix} \qquad N = \begin{pmatrix} 4 \\ -7 \\ 0 \\ 3 \end{pmatrix}, \qquad k = 7,\ s = 0.$$

7. $x = x - \delta\alpha = (1539/754, 562/279, 1232/733, 3359/1990, 998/1605, 1387/2616, -6473/440)^T$.
13. $a_j = (4, -7, 0, 3)^T$, $\quad v = (-1300/2943, 946/1835, -659/1795, -764/1651)^T$
$w = (1498/26503, -39/1057, -243/2722, 523/4821)^T$.
14. $h1 = w$.
15. $\gamma = a_1^T h1 = 4075/5032$.
16. $\eta 1 = 1/(1-\gamma) = 5032/957$.
17. $y = y - (eta1 * h1(4)) * h1 = (-165/7936, 257/13157, 289/4398, -315/2696)^T$.
18. $l = 2$.
19. $k == 7 > 4$.
Iteration l = 2:
2. $\Delta_B = (-289/1485, -253/2270, -924/5219, 374/4637, 1360/12293, 183/3457, 2381/2696)^T$.
5. $\alpha = \min\{(3359/1990)/(374/4637), (998/1605)/(1360/12293), (1387/2616)/(183/3457)\}$
$= (998/1605)/(1360/12293) = 2962/527, j = 6$.
6.

$$B = \begin{pmatrix} & 6 & -5 & & -2 & \\ -2 & 2 & & -6 & 6 & \\ & 3 & -6 & -3 & -5 & \\ -2 & & -4 & -2 & -1 & -1 \end{pmatrix} \qquad N = \begin{pmatrix} 4 & -1 \\ -7 & \\ 0 & -2 \\ 3 & \end{pmatrix}, \qquad k = 6,\ s = 0.$$

7. $x = x - \delta\alpha = (1696/541, 1801/682, 1263/472, 321/260, 762/3275, -3089/157)^T$.
13. $a_j = (-1, 0, -2, 0)^T$, $\quad v = (325/2943, 173/5537, -207/1031, -493/2817)^T$
$w = (83/22533, 39/17285, -155/3531, 99/2413)^T$.
14. $h2 = w + (eta1 * (a' * h1)) * h1 = (714/17873, -107/4996, -353/3489, 77/696)^T$.
15. $\gamma = a^T h2 = 971/5979$.
16. $\eta 2 = 1016/851$.
17. $y = y - (eta2 * h2(4)) * h2 = (-83/3184, 103/4606, 342/4325, -601/4572)^T$.
18. $l = 3$.
19. $k == 6 > 4$.
Iteration l = 3:
2. $\Delta_B = (-2729/12508, -347/2764, -137/754, 502/4627, 233/3002, 1262/1453)^T$.
5. $\alpha = \min\{(321/260)/(502/4627), (762/3275)/(233/3002)\} = (762/3275)/(233/3002)$
$= 1346/449, j = 7$.
6.

$$B = \begin{pmatrix} & 6 & -5 & & \\ -2 & 2 & & -6 & \\ & 3 & -6 & -3 & \\ -2 & & -4 & -2 & -1 \end{pmatrix} \qquad N = \begin{pmatrix} 4 & -1 & -2 \\ -7 & & 6 \\ 0 & -2 & -5 \\ 3 & & -1 \end{pmatrix}, \qquad k = 5,\ s = 0.$$

7. $x = x - \delta\alpha = (413/109, 1587/526, 3607/1120, 2318/2549, -2317/104)^T$.
13. $a_j = (-2, 6, -5, -1)^T$, $\quad v = (343/1553, -812/1669, -789/1153, -1042/4793)^T$,
$w = (383/5736, 1363/22906, -337/2681, 314/6161)^T$.
14. $h3 = w + (eta1 * (a' * h1)) * h1 + (eta2 * (a' * h2)) * h2 = (469/6114, 627/11599, -545/3636, 233/3002)^T$.
15. $\gamma = a^T h3 = 1774/2105$.
16. $\eta 3 = 2105/331$.

17. $y = y-(eta3*h3(4))*h3 = (-148/2315, -183/42364, 1078/7043, -393/2315)^T$.
18. $l = 4$.
19. $k == 5 > 4$.
Iteration l = 4:
2. $\Delta_B = (-313/899, -450/6721, -381/4742, 211/2251, 1922/2315)^T$.
5. $\alpha = \min\{(2318/2549)/(211/2251)\} = 3347/345, j = 5$.
6.

$$B = \begin{pmatrix} & 6 & -5 & & \\ -2 & 2 & & & \\ & 3 & -6 & & \\ -2 & & -4 & -1 & \end{pmatrix} \qquad N = \begin{pmatrix} 4 & -1 & -2 & \\ -7 & & 6 & -6 \\ 0 & -2 & -5 & -3 \\ 3 & & -1 & -2 \end{pmatrix}, \qquad k = 4,\ s = 0.$$

7. $x = x - \delta\alpha = (52582/7337, 11/3, 592133/148033, -91/3)^T$.
13. $a_j = (0, -6, -3, -2)^T$, $v = (0, 885/1612, -692/1861, 446/1401)^T$, $w = (148/3257, -86/1915, -123/3017, -323/4328)^T$.
14. $h4 = w + (eta1*(a'*h1))*h1 + (eta2*(a'*h2))*h2 + (eta3*(a'*h3))*h3 = (282/2315, -105/926, -431/2604, 214/2283)^T$.
15. $\gamma = a^T h4 = 3272/3307$.
16. $\eta 4 = 3307/35$.
17. $y = y-(eta4*h4(4))*h4 = (-3297/2885, 21512/21513, 1143/706, -25539/25540)^T$.
18. $l = 5$.
19. $k = m + 1 = 4$.
Iteration l = 5:
9. $\bar{z}_N = -N^{\mathrm{T}} y = (9107/625, 2619/1250, -456/383, 8857/1000)^T$.
10. $\min = -456/383 < 0, j = 7$.
11.

$$B = \begin{pmatrix} & 6 & -5 & -2 & \\ -2 & 2 & & 6 & \\ & 3 & -6 & -5 & \\ -2 & & -4 & -1 & -1 \end{pmatrix} \qquad N = \begin{pmatrix} 4 & -1 & \\ -7 & & -6 \\ 0 & -2 & -3 \\ 3 & & -2 \end{pmatrix}, \qquad k = 5,\ s = 1.$$

13. $a_j = (-2, 6, -5, -1)^T$, $v = (343/1553, -812/1669, -789/1153, -1042/4793)^T$, $w = (383/5736, 1363/22906, -337/2681, 314/6161)^T$.
14. $h5 = w + \sum_{t=1}^{4}(etat*(a'*ht))*ht = (-1721/1012, 1419/596, 1693/838, -794/667)^T$.
15. $\gamma = a^T h5 = 2852/325$.
16. $\eta 5 = -1/(1 + gamma) = -325/3177$.
17. $y = y - (eta5 * h5(4)) * h5 = (-1208/1291, 529/745, 1443/1051, -348/407)^T$.
18. $l = 6$.
19. $k == 5 > 4$.
Iteration l = 6:
2. $\Delta_B = (-1882/6491, 356/4735, 340/2443, -458/3755, 600/4139)^T$.
5. $\alpha = \min\{(11/3)/(356/4735), (592133/148033)/(340/2443)\}$
$= (592133/148033)/(340/2443) = 6553/228, j = 3$.
6.

$$B = \begin{pmatrix} & 6 & -2 & & \\ -2 & 2 & 6 & & \\ & 3 & -5 & & \\ -2 & & -1 & -1 & \end{pmatrix} \qquad N = \begin{pmatrix} 4 & -1 & & -5 \\ -7 & & -6 & \\ 0 & -2 & -3 & -6 \\ 3 & & -2 & -4 \end{pmatrix}, \qquad k = 4,\ s = 0.$$

7. $x = x - \delta\alpha = (95154/6139, 1045/694, 1255/358, -65032/1885)^T$.
13. $a_j = (-5, 0, -6, -4)^T$, $\quad v = (1625/2943, 172/1101, -489/1400, 671/2130)^T$,
$w = (-83/13219, -763/81217, -116/3111, -320/4333)^T$.
14. $h6 = w + \sum_{t=1}^{5}(etat*(a'*ht))*ht = (383/1615, -216/1193, -1273/3351, 83/1413)^T$.
15. $\gamma = a^T h6 = 674/785$.
16. $\eta 6 = -785/1459$.
17. $y = y - (eta6 * h6(4)) * h6 = (-556/599, 2399/3406, 1625/1194, -523/613)^T$.
18. $l = 7$.
19. $k = m + 1 = 4$.

Iteration l = 7:

9. $\bar{z}_N = -N^{\mathrm{T}}y = (3977/355, 1087/606, 1512/229, 336/2999)^T \geq 0$.
11. Optimal solution $x = (15.4999, 1.5058, 0, 0, 0, 0, 3.5056)^T, f = -34.4997$.

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