ON THE CONVERGENCE OF THE SAKAWA-SHINDO ALGORITHM IN STOCHASTIC CONTROL

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Abstract. We analyze an algorithm for solving stochastic control problems, based on Pontryagin's maximum principle, due to Sakawa and Shindo in the deterministic case and extended to the stochastic setting by Mazliak. We assume that either the volatility is an affine function of the state, or the dynamics are linear. We obtain a monotone decrease of the cost functions as well as, in the convex case, the fact that the sequence of controls is minimizing, and converges to an optimal solution if it is bounded.

1. Introduction

In this work we consider an extension of an algorithm for solving deterministic optimal control problems introduced by Sakawa and Shindo in [19], and analyzed by Bonnans [7]. This algorithm has been adapted to a class of stochastic optimal control problems in Mazliak [15]. We extend here the analysis of the latter to more general situations appearing naturally in applications, and obtain stronger results regarding the convergence of iterates.

We introduce now the problem and the algorithm. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space on which an m-dimensional standard Brownian motion $W(\cdot)$ is defined. We suppose that $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ $(T > 0)$ is the natural filtration, augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$, associated to $W(\cdot)$ and we recall that $\mathbb{F}$ is right-continuous. Let us consider the following controlled Stochastic Differential Equation (SDE):

$$\begin{cases} \mathrm{d}y(t) = f(y(t), u(t), t, \omega)dt + \sigma(y(t), u(t), t, \omega)dW(t) \quad t \in (0, T), \\ y(0) = y_0 \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^r \times [0, T] \times \Omega \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times \mathbb{R}^r \times [0, T] \times \Omega \to \mathbb{R}^{n\times m}$ are given maps. In the notation above $y \in \mathbb{R}^n$ denotes the state function and $u \in \mathbb{R}^r$ the control. We define the cost functional

$$J(u) = \mathbb{E}\left\{\int_0^T \ell(y(t), u(t), t)dt + g(y(T))\right\}. \tag{1.2}$$

where $\ell : \mathbb{R}^n \times \mathbb{R}^r \times [0, T] \times \Omega \to \mathbb{R}$ and $g : \mathbb{R}^n \times \Omega \to \mathbb{R}$ are given. Precise definition of the state and control spaces, and assumptions over the data will be provided in the next sections.

Let U_{ad} be a non-empty closed, convex subset of $\mathbb{R}^r$ and

$$\mathcal{U} := \left\{u \in (\mathbb{H}^2)^r;\ u(t, \omega) \in U_{ad},\ \text{for almost all } (t, \omega) \in (0, T) \times \Omega\right\}, \tag{1.3}$$

where

$$\mathbb{H}^2 := \left\{v \in L^2([0, T] \times \Omega); \quad \text{the process } (t, \omega) \in [0, T] \times \Omega \mapsto v(t, \omega) \text{ is } \mathbb{F}\text{-adapted}\right\}.$$

The control problem that we will consider is

$$\text{Min } J(u) \ \text{ subject to } u \in \mathcal{U}. \tag{1.4}$$

The *Hamiltonian* of the problem is defined by

$$\begin{array}{rcl} H : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times [0, T] \times \Omega & \to & \mathbb{R}, \\ (y, u, p, q, t, \omega) & \mapsto & \ell(y, u, t, \omega) + p \cdot f(y, u, t, \omega) + q \cdot \sigma(y, u, t, \omega), \end{array} \tag{1.5}$$

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where $p \cdot f(y,u,t,\omega)$ is the scalar product in $\mathbb{R}^n$ and $q \cdot \sigma(y,u,t,\omega) := \sum_{j=1}^m q^j \cdot \sigma^j(y,u,t,\omega)$. Finally, given (y,u) satisfying (1.1) let us define the adjoint state $(p(\cdot),q(\cdot)) \in (\mathbb{H}^2)^n \times \left(\mathbb{H}^2\right)^{n\times m}$ as the unique solution of the following Backward Stochastic Differential Equation (BSDE)

$$\left\{ \begin{array}{l} \mathrm{d}p(t) = -\nabla_y H(y(t),u(t),p(t),q(t),t,\omega)\mathrm{d}t + q(t)\mathrm{d}W(t) \quad t \in (0,T), \\ p(T) = \nabla_y g(y(T)). \end{array} \right. \tag{1.6}$$

Given $\epsilon > 0$, we define the *augmented Hamiltonian* as

$$\begin{array}{rcl} K_\epsilon : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times [0,T] \times \Omega & \to & \mathbb{R}, \\ (y,u,v,p,q,t,\omega) & \mapsto & H(y,u,p,q,t,\omega) + \frac{1}{2\epsilon}\left|u-v\right|^2. \end{array} \tag{1.7}$$

We consider the following algorithm to solve (1.4).

Algorithm:

(1) Let some admissible control $u^0(\cdot)$ and a sequence $\{\epsilon_k\}$ of positive numbers be given. Set $k=0$. Using (1.1), compute the state $y^0(\cdot)$ associated to u^0.
(2) Compute p^k and q^k, the adjoint state, solution of (1.6), associated to u^k and y^k.
(3) Set $k = k+1$. Compute u^k and y^k such that y^k is the state corresponding to u^k and

$$u^k(t,\omega) = \mathrm{argmin}\left\{ K_{\epsilon_k}(y^k(t,\omega),u,u^{k-1}(t,\omega),p^{k-1}(t,\omega),q^{k-1}(t,\omega),t,\omega)\ ;\ u \in U_{ad} \right\}, \tag{1.8}$$

for almost all $(t,\omega) \in [0,T]\times\Omega$. We will see in Section 3 that u^k is well defined if ϵ_k is small enough.
(4) Stop if some convergence test is satisfied. Otherwise, go to (2).

The main idea of the algorithm is to compute at each step a new control that minimizes the augmented Hamiltonian K_ϵ which depends on H and on a quadratic term that penalizes the distance to the current control. We can prove that this is a descent method and that the distance to a gradient and projection step tends to zero. Consequently, in the convex framework the algorithm is shown to be globally convergent in the weak topology of $(\mathbb{H}^2)^r$. Step 3 in the algorithm reveals its connection with the extension of the Pontryagin maximum principle [18] to the stochastic setting. We refer the reader to Kushner and Schweppe [14], Kushner [12, 13], Bismut [4, 6], Haussmann [11] and Bensoussan [2, 3] for the the initial works in this area. Afterwards, general extensions were established by Peng [17] and by Cadenillas and Karatzas [10]. The stochastic maximum principle usually involves two pairs of adjoint processes (see [17]). Nevertheless, the gradient of the cost function depends only on one pair of adjoint processes and since we suppose that $\mathcal{U}$ is convex, the first order necessary condition at a local optimum u^* depends only on $\nabla J(u^*)$ (see [20, p. 119-120] for a more detailed discussion).

In this paper we work with two types of assumptions. The first one supposes that σ in (1.1) does not depend on u and that the cost functions ℓ and g are Lipschitz. In the second assumption we suppose that the functions f and σ involved in (1.1) are affine with respect to (y,u). Thus, our results are a significant extension of those of [15]. Let us explain now our main improvements, referring to Remark 2.1(i) for other technical differences. In [15] the author studies a restricted form of our first assumption, he shows that if in addition σ is independent of the state y and the problem is convex, then, except for some subsequence, the iterates u^k converges weakly to a solution of (1.4). If σ depends on the state y, it is proven in [15, Theorem 5] that given $\varepsilon > 0$, the algorithm can be suitably modified in such a way that every weak limit point $\hat{u}$ of u^k is an ε-approximated optimal solution, i.e. $J(\hat{u}) \le J(u)+\varepsilon$ for all $u \in \mathcal{U}$. We show in Theorem 4.6 that such a modification is unnecessary as we prove that the sequence of iterates u^k, generated by the algorithm described above, satisfies that each weak limit point of u^k *solves* (1.4). Moreover, as we said before, in our second assumption we suppose that σ can depend in an affine manner on the control u and the state y and the Lipschtiz assumption on the cost terms ℓ and g are removed. This implies that the sequences of adjoint states p^k are not bounded almost surely, which is the basic ingredient in the proof of the main results in [15]. Finally, let us underline that in Corollary 4.4 we prove that some weak forms of optimality conditions are satisfied for both assumptions. In the convex case, this allows us to prove that the iterates u^k form a *minimizing sequence*, a result that is absent in [15] and also in the deterministic case studied in [7]. Of course, this implies that if in addition J is strongly convex, we have strong convergence of the sequence u^k.

The article is organized as follows: in Section 2 we state the main assumptions that we make in the entire paper. In Section 3, we prove that the algorithm is well-defined. In the last section we analyse the convergence of the method. We show that the sequence of costs generated by the algorithm is decreasing and convergent. Finally, under some convexity assumptions, we can prove that every weak limit point of the sequence of iterates solves (1.4).

2. Main assumptions

Let us first fix some standard notations. Endowed with the natural scalar product in $L^2([0,T]\times\Omega)$ denoted by $(\cdot,\cdot)$, $\mathbb{H}^2$ is a Hilbert space. We denote by $\|\cdot\|$ the $L^2([0,T]\times\Omega)$ norm on $(\mathbb{H}^2)^l$, for any $l\in\mathbb{N}$. As usual and if the context is clear, we omit the dependence on ω of the stochastic processes. We set $\mathbb{S}^2$ for the subspace of $\mathbb{H}^2$ of continuous processes x satisfying that $\mathbb{E}\left(\sup_{t\in[0,T]}|x(t)|^2\right)<\infty$. Finally, given an Euclidean space $\mathbb{R}^l$, we denote by $|\cdot|$ the Euclidean norm and by $\mathcal{B}(\mathbb{R}^l)$ the Borel sigma-algebra.

Let us now fix the standing assumptions that will be imposed from now on.

(H1) Assumptions for the dynamics:

(a) The maps $\varphi=f,\sigma$ are $\mathcal{B}(\mathbb{R}^n\times\mathbb{R}^r\times[0,T])\otimes\mathcal{F}_T$-measurable.

(b) For all $(y,u)\in\mathbb{R}^n\times\mathbb{R}^r$ the process $[0,T]\times\Omega\ni(t,\omega)\mapsto\varphi(y,u,t,\omega)$ is $\mathbb{F}$-adapted.

(c) For almost all $(t,\omega)\in[0,T]\times\Omega$ the map $(y,u)\mapsto\varphi(y,u,t,\omega)$ is C^2 and there exist a constant $L>0$ and a process $\rho_\varphi\in\mathbb{H}^2$ such that for almost all $(t,\omega)\in[0,T]\times\Omega$ and for all $y,\bar{y}\in\mathbb{R}^n$ and $u,\bar{u}\in U_{ad}$ we have

$$
\left\{\begin{array}{l}
|\varphi(y,u,t,\omega)|\leq L\left[|y|+|u|+\rho_\varphi(t,\omega)\right],\\
|\varphi_y(y,u,t,\omega)|+|\varphi_u(y,u,t,\omega)|\leq L,\\
|\varphi_{yy}(y,u,t,\omega)-\varphi_{yy}(\bar{y},\bar{u},t,\omega)|\leq L\left(|y-\bar{y}|+|u-\bar{u}|\right),\\
|\varphi_{yy}(y,u,t,\omega)|+|\varphi_{yu}(y,u,t,\omega)|+|\varphi_{uu}(y,u,t,\omega)|\leq L.
\end{array}\right. \tag{2.1}
$$

(H2) Assumptions for the cost:

(a) The maps ℓ and g are respectively $\mathcal{B}(\mathbb{R}^n\times\mathbb{R}^r\times[0,T])\otimes\mathcal{F}_T$ and $\mathcal{B}(\mathbb{R}^n)\otimes\mathcal{F}_T$ measurable.

(b) For all $(y,u)\in\mathbb{R}^n\times\mathbb{R}^r$ the process $[0,T]\times\Omega\ni(t,\omega)\mapsto\ell(y,u,t,\omega)$ is $\mathbb{F}$-adapted.

(c) For almost all $(t,\omega)\in[0,T]\times\Omega$ the map $(y,u)\mapsto\ell(y,u,t,\omega)$ is C^2, and there exists $L>0$ and a process $\rho_\ell(\cdot)\in\mathbb{H}^2$ such that for all $y,\bar{y}\in\mathbb{R}^n$ and $u\in U_{ad}$

$$
\left\{\begin{array}{l}
|\ell(y,u,t,\omega)|\leq L\left[|y|+|u|+\rho_\ell(t,\omega)\right]^2,\\
|\ell_y(y,u,t,\omega)|+|\ell_u(y,u,t,\omega)|\leq L\left[|y|+|u|+\rho_\ell(t,\omega)\right],\\
|\ell_{yy}(y,u,t,\omega)|+|\ell_{yu}(y,u,t,\omega)|+|\ell_{uu}(y,u,t,\omega)|\leq L,\\
|\ell_{yy}(y,u,t,\omega)-\ell_{yy}(\bar{y},u,t,\omega)|\leq L\,|y-\bar{y}|.
\end{array}\right. \tag{2.2}
$$

(d) For almost all $\omega\in\Omega$ the map $y\mapsto g(y,\omega)$ is C^2 and there exists $L>0$ such that for all $y,\bar{y}\in\mathbb{R}^n$ and almost all $\omega\in\Omega$,

$$
\left\{\begin{array}{l}
|g(y,\omega)|\leq L\left[|y|+1\right]^2,\\
|g_y(y,\omega)|\leq L\left[|y|+1\right],\\
|g_{yy}(y,\omega)|\leq L,\\
|g_{yy}(y,\omega)-g_{yy}(\bar{y},\omega)|\leq L\,|y-\bar{y}|.
\end{array}\right. \tag{2.3}
$$

(H3) At least one of the following assumptions holds true:

(a) For all $(y,u)\in\mathbb{R}^n\times\mathbb{R}^r$ and almost all $(t,\omega)\in[0,T]\times\Omega$ we have

$$
\sigma_u(y,u,t,\omega)\equiv 0 \quad\text{and}\quad \sigma_{yy}(y,t,\omega)\equiv 0. \tag{2.4}
$$

Moreover, the following Lipschitz condition holds true: there exists $L\geq 0$ such that for almost all $(t,\omega)\in[0,T]\times\Omega$, and for all $y,\bar{y}\in\mathbb{R}^n$ and $u,\bar{u}\in U_{ad}$,

$$
\left\{\begin{array}{l}
|\ell(y,u,t,\omega)-\ell(\bar{y},\bar{u},t,\omega)|\leq L(|y-\bar{y}|+|u-\bar{u}|),\\
|g(y,\omega)-g(\bar{y},\omega)|\leq L\,|y-\bar{y}|.
\end{array}\right. \tag{2.5}
$$

(b) For $\varphi=f,\sigma$ and for almost all $(t,\omega)\in[0,T]\times\Omega$ the map $(y,u)\mapsto\varphi(y,u,t,\omega)$ is affine.

Remark 2.1. (i) *Our assumptions* **(H1)**-**(H2)**-**(H3)**(a) *are weaker than those in* [15], *where it is supposed that ℓ and g are bounded and, in the statement of the main results, $\sigma_y \equiv 0$. In addition, the data ℓ, g, f, σ do not depend explicitly on ω and the set U is assumed to be bounded.*
(ii) *Under assumption* **(H1)**, *for any $u \in \mathcal{U}$ the state equation (1.1) admits a unique strong solution in $(\mathbb{S}^2)^n$, see* [16, Proposition 2.1]. *Also, by the estimates in* [16, Proposition 2.1] *and assumption* **(H2)** *the function J is well defined. Moreover, equation (1.6) can be written as*

$$\begin{cases} dp(t) = -\left[\nabla_y \ell(y(t),u(t),t) + f_y(y(t),u(t),t)^\top p(t) + \sum_{j=1}^m \sigma_y^j(y(t),t)^\top q_j(t)\right] \mathrm{d}t + q(t)\mathrm{d}W(t) \\ p(T) = \nabla_y g(y(T)) \end{cases} \tag{2.6}$$

and under **(H1)**-**(H2)**, *it has a unique solution $(p,q) \in (\mathbb{S}^2)^n \times (\mathbb{H}^2)^{n\times m}$ (see* [5] *and* [20, Theorem 2.2, p. 349]).

3. Well-posedness of the algorithm

The aim of this section is to prove that the iterates of the Sakawa-Shindo algorithm are well defined. We need first the following lemma.

Lemma 3.1. *Under assumptions* **(H1)-(H2)** *and* **(H3)**-(a), *there exists $C>0$ such that the solution (p,q) of (2.6) satisfies*

$$|p(t)| \leq C, \qquad \forall t \in [0,T], \ \mathbb{P} - a.s. \tag{3.1}$$

Proof. By Îto's formula and (2.6), we have that

$$\begin{aligned} |p(t)|^2 &= 2\int_t^T p(s)\cdot[\nabla_y \ell(y(s),u(s),s) + f_y(y(s),u(s),s)^\top p(s) + \sum_{j=1}^m \sigma_y^j(y(s),s)^\top q_j(s)]ds \\ &\quad - \int_t^T \sum_{j=1}^m |q_j(s)|^2 \mathrm{d}s - 2\int_t^T p(s)\cdot q(s)\mathrm{d}W(s) + |p(T)|^2, \qquad \forall t\in[0,T], \ \mathbb{P}-\text{a.s.} \end{aligned} \tag{3.2}$$

Using that $p(T) = \nabla_y g(y(T))$, our assumptions and the Cauchy-Schwarz and Young inequalities imply that for any $\varepsilon > 0$ we have

$$\begin{aligned} |p(t)|^2 &\leq 2\int_t^T [L|p(s)| + L|p(s)|^2 + \sum_{j=1}^m L|p(s)||q_j(s)|]\mathrm{d}s \\ &\quad - \int_t^T \sum_{j=1}^m |q_j(s)|^2 \mathrm{d}s - 2\int_t^T p(s)\cdot q(s)\mathrm{d}W(s) + L^2 \\ &\leq L^2 T + \int_t^T (1+2L)|p(s)|^2 + \sum_{j=1}^m [\tfrac{L^2}{\varepsilon}|p(s)|^2 + \varepsilon|q_j(s)|^2]\mathrm{d}s \\ &\quad - \int_t^T \sum_{j=1}^m |q_j(s)|^2 \mathrm{d}s - 2\int_t^T p(s)\cdot q(s)\mathrm{d}W(s) + L^2 \\ &= L^2(T+1) + \left(1+2L+\tfrac{mL^2}{\varepsilon}\right)\int_t^T |p(s)|^2\mathrm{d}s + (\varepsilon-1)\sum_{j=1}^m \int_t^T |q_j(s)|^2 \mathrm{d}s \\ &\quad -2\int_t^T p(s)\cdot q(s)\mathrm{d}W(s) \end{aligned} \tag{3.3}$$

Choosing $\varepsilon < 1$, we get

$$|p(t)|^2 \leq C_1 + C_2\int_t^T |p(s)|^2 ds - 2\int_t^T p(s)\cdot q(s)\mathrm{d}W(s). \tag{3.4}$$

for some constants C_1, $C_2 > 0$. Now fix $\bar{t} \in [0,T]$ and define $r(t) := \mathbb{E}\left(|p(t)|^2 | \mathcal{F}_{\bar{t}}\right) \geq 0$ for all $t \geq \bar{t}$. Combining (3.4) and [1, Lemma 3.1] we have

$$r(t) \leq C_1 + C_2\int_t^T r(s)ds. \tag{3.5}$$

Thus, by Gronwall's Lemma there exists $C>0$ independent of (t,ω) and $\bar{t}$ such that $r(t)\leq C$ for all $\bar{t} \leq t \leq T$ and so in particular $|p(\bar{t})|^2 = r(\bar{t}) \leq C$ for a.a. ω. Since $\bar{t}$ is arbitrary and p admits a continuous version, the result follows. ☐

Lemma 3.2. *Consider the mapping*

$$u_\epsilon : \mathbb{R}^n \times P \times \mathbb{R}^{n\times m} \times U_{ad} \times [0,T] \times \Omega \quad \to \mathbb{R}^r \tag{3.6}$$

where $P := B(0,C)$ *(ball in* $\mathbb{R}^n$*) in case (a), and* $P = \mathbb{R}^n$ *in case (b), defined by*

$$u_\epsilon(y,p,q,v,t,\omega) := \operatorname{argmin}\{K_\epsilon(y,u,v,p,q,t,\omega)\ ;\ u \in U_{ad}\}. \tag{3.7}$$

Under assumptions **(H1)-(H2)** *and* **(H3)***, there exist* $\epsilon_0 > 0$*,* $\alpha > 0$ *and* $\beta > 0$ *independent of* (t,ω)*, with* $\beta = 0$ *if* **(H3)***(a) is verified, such that, if* $\epsilon < \epsilon_0$*,* u_ε *is well defined and for a.a.* $(t,\omega) \in [0,T] \times \Omega$ *and all* $(y_i,p_i,q_i,v_i) \in \mathbb{R}^n \times P \times \mathbb{R}^{n\times m} \times U_{ad}$*,* $i = 1,2$*:*

$$|u_\epsilon(y_2,p_2,q_2,v_2,t) - u_\epsilon(y_1,p_1,q_1,v_1,t)| \le 2\,|v_2 - v_1| + \alpha\,(|y_2 - y_1| + |p_2 - p_1|) + \beta\,|q_2 - q_1|\,. \tag{3.8}$$

Proof. We follow [7, 15]. Setting $w := (y,p,q,v)$, element of $E := \mathbb{R}^n \times P \times \mathbb{R}^{n\times m} \times U_{ad}$, we can rewrite K_ϵ as $K_\epsilon(u,w)$. We claim that for ϵ small enough:

$$D^2_{uu}K_\epsilon(u,w)(u',u') \ge (1/\epsilon - 2C_1)|u'|^2 \ge \frac{1}{2\epsilon}|u'|^2 \quad \text{for all } w \in E \text{ and } u' \text{ in } \mathbb{R}^m. \tag{3.9}$$

This holds if **(H3)**(a) holds since p, f_{uu} and ℓ_{uu} are bounded, and also if **(H3)**(b) since f and σ are affine and ℓ_{uu} is bounded. By (3.9), K_ϵ is a strongly convex function of u with modulus $1/(2\epsilon)$, and hence, for all u_1, $u_2 \in U_{ad}$:

$$\left(D_uK_\epsilon(u_2,w) - D_uK_\epsilon(u_1,w)\right)(u_2 - u_1) \ge \frac{1}{2\epsilon}\,|u_2 - u_1|^2\,. \tag{3.10}$$

On the other hand, for $i = 1,2$, take $w_i = (y_i,p_i,q_i,v_i) \in E$ and denote $u_i := u_\epsilon(y_i,p_i,q_i,v_i)$. Then

$$D_uK_\epsilon(u_i,w_i)\,(u_{3-i} - u_i) \ge 0. \tag{3.11}$$

Summing these inequalities for $i = 1,2$ with (3.10) in which we set $w := w_1$, we obtain that

$$\begin{aligned} \frac{1}{2\epsilon}\,|u_2 - u_1|^2 &\le \left(D_uK_\epsilon(u_2,w_1) - D_uK_\epsilon(u_2,w_2)\right)(u_2 - u_1)\,, \\ &\le |D_uK_\epsilon(u_2,w_1) - D_uK_\epsilon(u_2,w_2)|\,|u_2 - u_1|\,. \end{aligned} \tag{3.12}$$

Since $D_uK_\epsilon(u,w) = (1/\varepsilon)(u - v) + H_u(y,u,p,q)$,

$$|u_2 - u_1| \le 2\,|v_2 - v_1| + 2\epsilon\,|\nabla_u H(y_1,u_2,p_1,q_1) - \nabla_u H(y_2,u_2,p_2,q_2)|\,. \tag{3.13}$$

Inequality (3.2) easily follows from (3.13) and our assumptions. □

Theorem 3.3. *Under assumptions* **(H1)-(H3)***, there exists* $\epsilon_0 > 0$ *such that, if* $\epsilon_k < \epsilon_0$ *for all* k*, then the algorithm defines a uniquely defined sequence* $\{u^k\}$ *of admissible controls.*

Proof. Given u^0, let us define $\bar f : \mathbb{R}^n \times [0,T] \times \Omega \to \mathbb{R}^n$ and $\bar\sigma : \mathbb{R}^n \times [0,T] \times \Omega \to \mathbb{R}^{n\times m}$ as

$$\begin{aligned} \bar f(y) &:= f(y, u_\epsilon(y,p^0,q^0,u^0)), \\ \bar\sigma(y) &:= \sigma(y, u_\epsilon(y,p^0,q^0,u^0)). \end{aligned} \tag{3.14}$$

Assumption **(H1)** and Lemma 3.2 imply that the SDE

$$\left\{ \begin{aligned} \mathrm{d}y(t) &= \bar f(y)\mathrm{d}t + \bar\sigma(y)\mathrm{d}W(t) \quad t \in [0,T], \\ y(0) &= y_0, \end{aligned} \right. \tag{3.15}$$

has a unique strong solution (e.g. [20, Theorem 6.16, p. 49]). Therefore $u^1 := u_\epsilon(y,p^0,q^0,u^0)$ is uniquely defined; so is u^k by induction. □

4. Convergence

In this section we prove our main results. If $\sup_k \epsilon_k \leq \epsilon_0$ where ϵ_0 is small enough, then the cost function decreases with the iterates (see Theorem 4.2 and Theorem 4.3). Moreover, if the problem is convex, then any weak limit point of the sequence u^k solves the problem (see Theorem 4.6). We will need the following elementary Lemma.

Lemma 4.1. *Under assumption* **(H1)**, *there exists* $C > 0$ *such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|y^k(t) - y^{k-1}(t)\right|^2\right] \leq C\left\|u^k - u^{k-1}\right\|^2. \tag{4.1}$$

Proof. Define

$$\delta y^k := y^k - y^{k-1} \quad \text{and} \quad \delta u^k := u^k - u^{k-1}.$$

By [16, Proposition 2.1] there exists $C > 0$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\delta y^k(t)\right|^2\right] \leq C\left[\mathbb{E}\left(\int_0^T \left|f(y^{k-1}(t), u^k(t), t) - f(y^{k-1}(t), u^{k-1}(t), t)\right| \mathrm{d}t\right)^2\right. \tag{4.2}$$

$$\left. + \mathbb{E}\int_0^T \left|\sigma(y^{k-1}(t), u^k(t), t) - \sigma(y^{k-1}(t), u^{k-1}(t), t)\right|^2 \mathrm{d}t\right]. \tag{4.3}$$

Assumption **(H1)**-(c) and the Cauchy-Schwarz inequality imply directly (4.1).

□

Theorem 4.2. *Under assumptions* **(H1)-(H3)**, *there exists* $\alpha > 0$ *such that any sequence generated by the algorithm satisfies*

$$J(u^k) - J(u^{k-1}) \leq -\left(\frac{1}{\epsilon_k} - \alpha\right)\left\|u^k - u^{k-1}\right\|^2. \tag{4.4}$$

Proof. We drop the variable t when there is no ambiguity. We have,

$$\begin{aligned} J(u^k) - J(u^{k-1}) \; = \; & \mathbb{E}\Big[\textstyle\int_0^T \big[H(y^k, u^k, p^{k-1}, q^{k-1}) - H(y^{k-1}, u^{k-1}, p^{k-1}, q^{k-1}) \\ & -p^{k-1}\cdot(f(y^k, u^k) - f(y^{k-1}, u^{k-1})) \\ & -q^{k-1}\cdot(\sigma(y^k) - \sigma(y^{k-1}))\big]\mathrm{d}t + g(y^k(T)) - g(y^{k-1}(T))\Big]. \end{aligned} \tag{4.5}$$

Define $\delta y^k := y^k - y^{k-1}$ and $\delta u^k := u^k - u^{k-1}$. By Îto's formula, almost surely we have

$$\begin{aligned} p^{k-1}(T)\cdot\delta y^k(T) \; = \; & p^{k-1}(0)\cdot\delta y^k(0) + \textstyle\int_0^T \big[p^{k-1}\cdot\big(f(y^k, u^k) - f(y^{k-1}, u^{k-1})\big) \\ & -H_y(y^{k-1}, u^{k-1}, p^{k-1}, q^{k-1})\delta y^k \\ & +q^{k-1}\cdot\big(\sigma(y^k) - \sigma(y^{k-1})\big)\big]\mathrm{d}t + \\ & \textstyle\int_0^T \big[p^{k-1}\cdot\big(\sigma(y^k) - \sigma(y^{k-1})\big) + q^{k-1}\cdot\delta y^k\big]\mathrm{d}W(t). \end{aligned} \tag{4.6}$$

Then, replacing in (4.5) and using that $\delta y^k(0) = 0$ we get

$$\begin{aligned} J(u^k) - J(u^{k-1}) \; = \; & \mathbb{E}\Big[\textstyle\int_0^T \big[H(y^k, u^k, p^{k-1}, q^{k-1}) - H(y^{k-1}, u^{k-1}, p^{k-1}, q^{k-1}) \\ & -H_y(y^{k-1}, u^{k-1}, p^{k-1}, q^{k-1}, t)\delta y^k(t)\big]\mathrm{d}t \\ & -p^{k-1}(T)\cdot\delta y^k(T) + g(y^k(T)) - g(y^{k-1}(T))\Big]. \end{aligned} \tag{4.7}$$

Moreover, we have

$$\begin{aligned} \Delta \; := \; & H(y^k, u^k, p^{k-1}, q^{k-1}) - H(y^{k-1}, u^{k-1}, p^{k-1}, q^{k-1}) \\ = \; & H(y^k, u^k, p^{k-1}, q^{k-1}) - H(y^k, u^k - \delta u^k, p^{k-1}, q^{k-1}) \\ & +H(y^{k-1} + \delta y^k, u^{k-1}, p^{k-1}, q^{k-1}) - H(y^{k-1}, u^{k-1}, p^{k-1}, q^{k-1}) \\ = \; & \Delta_y - \Delta_u + H_y(y^{k-1}, u^{k-1}, p^{k-1}, q^{k-1})\delta y^k + H_u(y^k, u^k, p^{k-1}, q^{k-1})\delta u^k, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned}\Delta_u &:= H(y^k, u^k - \delta u^k, p^{k-1}, q^{k-1}) - H(y^k, u^k, p^{k-1}, q^{k-1}) \\ &\quad + H_u(y^k, u^k, p^{k-1}, q^{k-1})\delta u^k\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\Delta_y &:= H(y^{k-1} + \delta y^k, u^{k-1}, p^{k-1}, q^{k-1}) - H(y^{k-1}, u^{k-1}, p^{k-1}, q^{k-1}) \\ &\quad - H_y(y^{k-1}, u^{k-1}, p^{k-1}, q^{k-1})\delta y^k.\end{aligned} \tag{4.10}$$

Replacing in (4.7) we obtain

$$\begin{aligned}J(u^k) - J(u^{k-1}) &= \mathbb{E}\Big[\textstyle\int_0^T \left[H_u(y^k, u^k, p^{k-1}, q^{k-1})\delta u^k - \Delta_u + \Delta_y\right] \mathrm{d}t \\ &\quad - p^{k-1}(T) \cdot \delta y^k(T) + g(y^k(T)) - g(y^{k-1}(T))\Big].\end{aligned} \tag{4.11}$$

Since in both cases in **(H3)** we have $\sigma_{uu} \equiv 0$, we get

$$\begin{aligned}\Delta_u &= \textstyle\int_0^1 (1-s) H_{uu}(y^k, u^k + s\delta u^k, p^{k-1}, q^{k-1})(\delta u^k, \delta u^k)\mathrm{d}s \\ &= \textstyle\int_0^1 (1-s)\left[\ell_{uu}(y^k, u^k + s\delta u^k)(\delta u^k, \delta u^k) + p^{k-1} \cdot f_{uu}(y^k, u^k + s\delta u^k)(\delta u^k, \delta u^k)\right] \mathrm{d}s.\end{aligned} \tag{4.12}$$

If **(H3)**-(a) holds true, Lemma 3.1 and assumptions **(H1)-(H2)** imply

$$\Delta_u \geq -\frac{L}{2}\left|\delta u^k\right|^2 - \frac{CL}{2}\left|\delta u^k\right|^2. \tag{4.13}$$

On the other hand, if **(H3)**-(b) holds true, we have $f_{uu} \equiv 0$, and so **(H2)** implies

$$\Delta_u \geq -\frac{L}{2}\left|\delta u^k\right|^2. \tag{4.14}$$

Now, for both cases in **(H3)** we have $\sigma_{yy} \equiv 0$. Thus,

$$\begin{aligned}\Delta_y &= \textstyle\int_0^1 (1-s) H_{yy}(y^{k-1} + s\delta y^k, u^{k-1}, p^{k-1}, q^{k-1})(\delta y^k, \delta y^k)\mathrm{d}s \\ &= \textstyle\int_0^1 (1-s)[\ell_{yy}(y^{k-1} + s\delta y^k, u^{k-1})(\delta y^k, \delta y^k) \\ &\qquad + p^{k-1} \cdot f_{yy}(y^{k-1} + s\delta y^k, u^{k-1})(\delta y^k, \delta y^k)]\mathrm{d}s.\end{aligned} \tag{4.15}$$

If **(H3)**-(a) holds, Lemma 3.1 and **(H1)-(H2)** imply

$$\Delta_y \leq \frac{L}{2}\left|\delta y^k\right|^2 + \frac{CL}{2}\left|\delta y^k\right|^2. \tag{4.16}$$

If **(H3)**-(b) holds, then $f_{yy} \equiv 0$ and so **(H2)** implies $\Delta_y \leq \frac{L}{2}\left|\delta y^k\right|^2$. In conclusion, there exists $C_4 > 0$ such that

$$\Delta_u \geq -C_4\left|\delta u^k\right|^2 \quad \text{and} \quad \Delta_y \leq C_4\left|\delta y^k\right|^2. \tag{4.17}$$

Then, combining (4.11), and (4.17), we deduce

$$\begin{aligned}J(u^k) - J(u^{k-1}) &\leq \mathbb{E}\Big(\textstyle\int_0^T \left[H_u(y^k, u^k, p^{k-1}, q^{k-1})\delta u^k + C_4\left|\delta u^k\right|^2 + C_4\left|\delta y^k\right|^2\right] \mathrm{d}t \\ &\quad - p^{k-1}(T) \cdot \delta y^k(T) + g(y^k(T)) - g(y^{k-1}(T))\Big).\end{aligned} \tag{4.18}$$

Since u^k minimizes K_{ϵ_k} we have,

$$D_u K_{\epsilon_k}(y^k, u^k, u^{k-1}, p^{k-1}, q^{k-1})\delta u^k \leq 0, \quad \text{a.e. } t \in [0,T], \ \mathbb{P}\text{-a.s.} \tag{4.19}$$

then,

$$\begin{aligned}H_u(y^k, u^k, p^{k-1}, q^{k-1})\delta u^k &= D_u K_{\epsilon_k}(y^k, u^k, u^{k-1}, p^{k-1}, q^{k-1})\delta u^k - \tfrac{1}{\epsilon_k}\left|\delta u^k\right|^2 \\ &\leq -\tfrac{1}{\epsilon_k}\left|\delta u^k\right|^2, \quad \text{a.e. } t \in [0,T], \ \mathbb{P}\text{-a.s.}\end{aligned} \tag{4.20}$$

By assumption **(H2)**-(d) and the definition of $p^{k-1}(T)$, there exists $C_5 > 0$ such that

$$-p^{k-1}(T) \cdot \delta y^k(T) + g(y^k(T)) - g(y^{k-1}(T)) \leq C_5\left|\delta y^k(T)\right|^2. \tag{4.21}$$

Then, by (4.20) and (4.21) we obtain

$$J(u^k) - J(u^{k-1}) \leq \mathbb{E}\left[\int_0^T \left[\left(C_4 - \frac{1}{\epsilon_k}\right)\left|\delta u^k(t)\right|^2 + C_4\left|\delta y^k(t)\right|^2\right] \mathrm{d}t + C_5\left|\delta y^k(T)\right|^2\right]. \tag{4.22}$$

Then, the conclusion follows from Lemma 4.1. □

Now we consider the projection map $P_{\mathcal{U}} : (\mathbb{H}^2)^r \to \mathcal{U} \subset (\mathbb{H}^2)^r$, i.e. for any $u \in (\mathbb{H}^2)^r$,

$$P_{\mathcal{U}}(u) := \operatorname{argmin}\{\|u - v\| \ ; \ v \in \mathcal{U}\}.$$

By [9, Lemma 6.2],

$$P_{\mathcal{U}}(u)(t,\omega) = P_{U_{ad}}(u(t,\omega)), \quad \text{a.e. } t \in [0,T], \ \mathbb{P}-\text{a.s.}, \tag{4.23}$$

where $P_{U_{ad}} : \mathbb{R}^r \to U_{ad} \subset \mathbb{R}^r$ is the projection map in $\mathbb{R}^r$. We have the following result:

Theorem 4.3. *Assume that J is bounded from below and that assumptions* **(H1)-(H3)** *hold true. Then there exists $\epsilon_0 > 0$ such that, if $\epsilon_k < \epsilon_0$, any sequence generated by the algorithm satisfies:*

(1) $J(u^k)$ is a nonincreasing convergent sequence,
(2) $\|u^k - u^{k-1}\| \to 0$,
(3) $\|u^k - P_{\mathcal{U}}(u^k - \epsilon_k \nabla J(u^k))\| \to 0$.

Proof. The first two items are a consequence of Theorem 4.2 and the fact that J is bounded from below. Since u^k minimizes K_{ϵ_k} we have

$$D_u K_{\epsilon_k}(y^k, u^k, u^{k-1}, p^{k-1}, q^{k-1})(v - u^k) \geq 0, \quad \forall v \in U_{ad}, \ \text{a.e. } t \in [0,T], \ \mathbb{P}-\text{a.s.} \tag{4.24}$$

then,

$$\left(u^k - u^{k-1} + \epsilon_k \nabla_u H(y^k, u^k, p^{k-1}, q^{k-1}), v - u^k\right) \geq 0, \quad \forall v \in U_{ad}, \ \text{a.e. } t \in [0,T], \ \mathbb{P}-\text{a.s.} \tag{4.25}$$

and so

$$u^k = P_{\mathcal{U}}\left(u^{k-1} - \epsilon_k \nabla_u H(y^k, u^k, p^{k-1}, q^{k-1})\right). \tag{4.26}$$

By [9, Proposition 8] we know that

$$\nabla J(u^{k-1}) = \nabla_u H(y^{k-1}, u^{k-1}, p^{k-1}, q^{k-1}) \quad \text{in } \mathbb{H}^2, \tag{4.27}$$

then, by (4.26),

$$\begin{aligned} u^{k-1} - P_{\mathcal{U}}(u^{k-1} - \epsilon_k \nabla J(u^{k-1})) \ &= \ u^{k-1} - u^k + P_{\mathcal{U}}\left(u^{k-1} - \epsilon_k \nabla_u H(y^k, u^k, p^{k-1}, q^{k-1})\right) \\ &\quad -P_{\mathcal{U}}\left(u^{k-1} - \epsilon_k \nabla_u H(y^{k-1}, u^{k-1}, p^{k-1}, q^{k-1})\right). \end{aligned} \tag{4.28}$$

As $P_{\mathcal{U}}$ is non-expansive in $\mathbb{H}^2$, we obtain

$$\begin{aligned} \left\|u^{k-1} - P_{\mathcal{U}}\left(u^{k-1} - \epsilon_k \nabla J(u^{k-1})\right)\right\| &\leq \left\|u^{k-1} - u^k\right\| \\ &\quad + \epsilon_k \left\|\nabla_u H(y^k, u^k, p^{k-1}, q^{k-1}) - \nabla_u H(y^{k-1}, u^{k-1}, p^{k-1}, q^{k-1})\right\|. \end{aligned} \tag{4.29}$$

Now, let us estimate the last term in the previous inequality. By **(H3)**, considering any of the two cases, we have $\sigma_{uy} \equiv \sigma_{uu} \equiv 0$. Therefore, for a.e. $t \in [0,T]$ there exist $(\hat{y}, \hat{u}) \in \mathbb{R}^n \times U_{ad}$ such that

$$\begin{aligned} &\left|\nabla_u H(y^k, u^k, p^{k-1}, q^{k-1}) - \nabla_u H(y^{k-1}, u^{k-1}, p^{k-1}, q^{k-1})\right| \\ &\quad = \left|H_{uy}(\hat{y}, \hat{u}, p^{k-1}, q^{k-1})(y^k - y^{k-1}) + H_{uu}(\hat{y}, \hat{u}, p^{k-1}, q^{k-1})(u^k - u^{k-1})\right| \\ &\quad = \left|\left(\ell_{uy}(\hat{y}, \hat{u}) + {p^{k-1}}^\top f_{uy}(\hat{y}, \hat{u})\right)(y^k - y^{k-1}) + \left(\ell_{uu}(\hat{y}, \hat{u}) + {p^{k-1}}^\top f_{uu}(\hat{y}, \hat{u})\right)(u^k - u^{k-1})\right| \\ &\quad \leq C\left(\left|y^k - y^{k-1}\right| + \left|u^k - u^{k-1}\right|\right), \end{aligned} \tag{4.30}$$

where in the last inequality, if **(H3)**-(a) holds we use Lemma 3.1 and assumptions **(H1)-(H2)**, and if **(H3)**-(b) holds we use the fact that $f_{uy} \equiv f_{uu} \equiv 0$ and **(H2)**. We conclude that

$$\begin{aligned} &\left\|\nabla_u H(y^k, u^k, p^{k-1}, q^{k-1}) - \nabla_u H(y^{k-1}, u^{k-1}, p^{k-1}, q^{k-1})\right\|^2 \\ &\qquad \leq 2C^2\left(\left\|u^k - u^{k-1}\right\|^2 + \left\|y^k - y^{k-1}\right\|^2\right). \end{aligned} \tag{4.31}$$

Using (4.29)-(4.31), Lemma 4.1 yields the existence of $C > 0$ such that

$$\left\|u^{k-1} - P_{\mathcal{U}}\left(u^{k-1} - \epsilon_k \nabla J(u^{k-1})\right)\right\| \leq C\left\|u^{k-1} - u^k\right\|, \tag{4.32}$$

which proves the last assertion. □

Corolary 4.4. *Assume that the sequence generated by the algorithm u^k is bounded, $\epsilon_k < \epsilon_0$ and $\liminf \epsilon_k \geq \epsilon > 0$. Then, for every bounded sequence $v^k \in \mathcal{U}$ we have that*

$$\liminf_{k\to\infty} \left(\nabla J(u^k), v^k - u^k\right) \geq 0. \tag{4.33}$$

In particular

$$\liminf_{k\to\infty} \left(\nabla J(u^k), v - u^k\right) \geq 0 \quad \forall\, v \in \mathcal{U} \tag{4.34}$$

and in the unconstrained case $\mathcal{U} = (\mathbb{H}^2)^r$

$$\lim_{k\to\infty} \left\|\nabla J(u^k)\right\| = 0. \tag{4.35}$$

Proof. We define for each k,

$$w_k := P_{\mathcal{U}}(u^k - \epsilon_k \nabla J(u^k)), \tag{4.36}$$

then we have

$$0 \leq \left(w^k - u^k + \epsilon_k \nabla J(u^k), v^k - w^k\right), \quad \forall v^k \in \mathcal{U}. \tag{4.37}$$

By Theorem 4.3(3), we know $\|u^k - w^k\| \to 0$. Using the fact that $\nabla J(u^k) = \nabla_u H(y^k, u^k, p^k, q^k)$, and the boundedness of u^k, which in particular yields the boundedness of (p^k, q^k) in $(\mathbb{S}^2)^n \times (\mathbb{H}^2)^{n\times m}$ (see e.g. [16, Proposition 3.1] or [20, Chapter 7]), we can deduce that $\nabla J(u^k)$ is a bounded sequence in $(\mathbb{H}^2)^r$. Finally, since the sequence v^k is bounded by assumption, we can conclude

$$0 \leq \liminf_{k\to\infty} \left(w^k - u^k + \epsilon_k \nabla J(u^k), v^k - w^k\right) = \epsilon \liminf_{k\to\infty} \left(\nabla J(u^k), v^k - u^k\right), \tag{4.38}$$

which proves (4.33). Inequality (4.34) follows from (4.33) by taking $v^k \equiv v$, for any fixed $v \in \mathcal{U}$, and identity (4.35) follows by letting $v^k = u^k - \nabla J(u^k)$, which is bounded in $(\mathbb{H}^2)^r$. □

Remark 4.5. *From* (4.36) *and the fact that $\|u^k - w^k\| \to 0$, it follows that in the unconstrained case,* (4.35) *holds true even if u^k is not bounded.*

Under convexity assumptions we obtain a convergence result.

Theorem 4.6. *Assume that $J(u)$, defined as above, is convex and bounded from below. Moreover, suppose that $\epsilon_k < \epsilon_0$, where ϵ_0 is given by Theorem 4.3, and $\liminf \epsilon_k > 0$. Then any weak limit point $\bar{u}$ of $\{u^k\}$ is an optimal control.*

As a consequence, if $\{u^k\}_{k\in\mathbb{N}}$ has bounded subsequence, then $J(u^k) \to \min_{u\in\mathcal{U}} J(u)$.

Proof. Consider a subsequence $\{u^{k_1}\}$ that converges weakly to $\bar{u}$, then $\{u^{k_1}\}$ is bounded. By the convexity of J, and the previous Corollary, for all $v \in \mathcal{U}$ we obtain

$$J(v) \geq \liminf_{k_1\to\infty} \left\{J(u^{k_1}) + \left(\nabla J(u^{k_1}), v - u^{k_1}\right)\right\} \geq \liminf_{k_1\to\infty} J(u^{k_1}) \geq J(\bar{u}), \tag{4.39}$$

by the weak lower semi-continuity of J, which is implied by the convexity and continuity of J. This proves the first assertion. In order to prove the second one, take $\hat{u} \in \mathcal{U}$ and a subsequence $\{u^{k_2}\}$ such that u^{k_2} converges weakly to $\hat{u}$ as $k_2 \to \infty$. Then, by the first assertion

$$\min_{u\in\mathcal{U}} J(u) = J(\hat{u}) \geq \liminf_{k_2\to\infty} \left\{J(u^{k_2}) + \left(\nabla J(u^{k_2}), \hat{u} - u^{k_2}\right)\right\} \geq \liminf_{k_2\to\infty} J(u^{k_2}) \geq J(\hat{u}).$$

Theorem 4.3(1) implies that $\lim_{k\to\infty} J(u^k) = \liminf_{k_2\to\infty} J(u^{k_2}) = \min_{u\in\mathcal{U}} J(u)$. The result follows. □

Remark 4.7. *The fact that any weak limit point is an optimal control can also be obtained with Theorem 4.3(3) and the arguments in* [7].

If J is strongly convex in $(\mathbb{H}^2)^r$ we obtain strong convergence of the iterates.

Corolary 4.8. *If in addition $J(u)$ is strongly convex, then the whole sequence converges strongly to the unique optimal control.*

Proof. Since J is strongly convex, classical arguments imply the existence of a unique u^* such that $J(u^*) = \min_{u \in \mathcal{U}} J(u)$. Moreover, since Theorem 4.3(1) implies that $J(u^k) \leq J(u^0)$, for all $k \in \mathbb{N}$, the strong convexity of J implies that the whole sequence u^k is bounded and is a minimizing sequence. The result follows from the classical argument that a minimizing sequence of a strongly convex problem converge strongly (see e.g. [8, Proof of Lemma 2.33(ii)]). □

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