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On measures of size for convex cones

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Abstract. By using an axiomatic approach we formalize the concept of size index for closed convex cones in the Euclidean space $\mathbb{R}^n$. We review a dozen of size indices disseminated through the literature, commenting on the advantages and disadvantages of each choice.

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1 Introduction

Let $\mathcal{C}_n$ denote the set of closed convex cones in $\mathbb{R}^n$. For avoiding trivialities one assumes that the dimension n is at least equal to three. The vector space $\mathbb{R}^n$ is equipped with the usual inner product $\langle y, x\rangle := y^T x$ and the associated norm $\|x\| := \langle x, x\rangle^{1/2}$. Throughout this work, the calligraphic letter $\mathcal{K}$ refers to a nonempty subset of $\mathcal{C}_n$ that is stable under orthogonal transformations, i.e.,

$$K \in \mathcal{K} \text{ and } U \in \mathbb{O}(n) \text{ imply } U(K) \in \mathcal{K},$$

where $\mathbb{O}(n)$ is the set of orthogonal matrices of order n and $U(K) := \{Ux : x \in K\}$ is the image of K under U. In applications one usually takes $\mathcal{K}$ as the whole set $\mathcal{C}_n$ or any of the following subsets:

$$\begin{aligned} \mathcal{N}_n &:= \{K \in \mathcal{C}_n : K \text{ is nontrivial}\}, \\ \mathcal{P}_n &:= \{K \in \mathcal{C}_n : K \text{ is proper}\}, \\ \mathcal{E}_n &:= \{K \in \mathcal{C}_n : K \text{ is ellipsoidal}\}. \end{aligned}$$

That a closed convex cone is nontrivial means that it is different from the zero cone and different from the whole space $\mathbb{R}^n$. A proper cone is a closed convex cone that it is pointed and has nonempty interior. Ellipsoidal cones are defined as in Stern and Wolkowicz [35].

The question addressed in this work is, roughly speaking, as follows: how to define in a meaningful and operationally convenient way a function $\Sigma : \mathcal{K} \to \mathbb{R}$ whose aim is to measure the size of each member of $\mathcal{K}$? There is no universal agreement on how to understand the concept of size of a closed convex cone, but probably everyone agrees that:

$$\begin{cases} \text{a small perturbation in a cone should not produce a big change in its size,} \\ \text{the size of a cone should be a nonnegative real,} \\ \text{the size of a cone should be nondecreasing with respect to set inclusion,} \\ \text{the size of a cone should be invariant under orthogonal transformations.} \end{cases} \tag{1}$$

Since the first requirement in (1) is about continuity, one needs to equip $\mathcal{C}_n$ with a topology. In what follows, all topological issues on $\mathcal{C}_n$ refer to the truncated Pompeiu-Hausdorff metric

$$\delta(P,Q) := \mathrm{haus}(P \cap \mathbb{B}_n, Q \cap \mathbb{B}_n).$$

Here, $\mathbb{B}_n$ is the closed unit ball of $\mathbb{R}^n$ and

$$\mathrm{haus}(C,D) := \max\left\{\max_{x\in C} \mathrm{dist}(x,D), \max_{x\in D} \mathrm{dist}(x,C)\right\}$$

is the classical Pompeiu-Hausdorff distance between a pair of nonempty compact subsets of $\mathbb{R}^n$. Convergence with respect to the truncated Pompeiu-Hausdorff metric is equivalent to convergence in the Painlevé-Kuratowski sense. As a consequence of Blaschke's selection theorem, the metric space $(\mathcal{C}_n, \delta)$ is compact.

Definition 1.1. *A* size index *on $\mathcal{K}$ is a function $\Sigma : \mathcal{K} \to \mathbb{R}$ satisfying the following axioms:*

A_1. Σ *is continuous,*

A_2. $\Sigma(K) \geq 0$ *for all* $K \in \mathcal{K}$,

A_3. Σ *is nondecreasing with respect to set inclusion,*

A_4. $\Sigma(U(K)) = \Sigma(K)$ *for all* $K \in \mathcal{K}$ *and* $U \in \mathbb{O}(n)$.

The above set of axioms is a bare minimum. In order to enrich the quality of a size index, one may consider some additional properties like strictness, Lipschitz continuity, modularity, and polar-reversibility. These are useful properties for theoretical and practical purposes, but only a few size indices meet all these extra requirements. Strictness, Lipschitz continuity, and modularity, are defined as follows.

Definition 1.2. *A size index Σ on $\mathcal{K}$ is called:*

i) Strict *if, in axiom* A_3*, nondecreasingness is changed by increasingness.*

ii) Lipschitz continuous *if the following supremum is finite:*

$$\mathrm{lip}_\delta(\Sigma, \mathcal{K}) := \sup_{\substack{P,Q\in\mathcal{K}\\ P\neq Q}} \frac{|\Sigma(P) - \Sigma(Q)|}{\delta(P,Q)}.$$

iii) Modular *if, for all $P, Q \in \mathcal{K}$ such that $P \cup Q \in \mathcal{K}$ and $P \cap Q \in \mathcal{K}$, one has*

$$\Sigma(P) + \Sigma(Q) \;=\; \Sigma(P \cup Q) + \Sigma(P \cap Q). \tag{2}$$

Modularity can be broken down into submodularity and supermodularity. For instance, Σ is submodular if the first sum in (2) is greater than or equal to the second sum. The definition of modularity is inspired from the inclusion-exclusion rule

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

of probability theory. Modularity helps, for instance, for computing the size of the union of two members of $\mathcal{K}$, provided the union and the intersection of these members remain in $\mathcal{K}$.

Polar-reversibility means, roughly speaking, that one can establish a certain link between the size of a convex cone and the size of its polar cone. The formal definition of polar-reversibility is given in Section 4. The organization of this survey paper on size indices is as follows.

- Section 2 presents a battery of examples of size indices disseminated in the literature. For each example of size index, we briefly comment on strictness, modularity and Lipschitz continuity.
- Valuable information on the behavior of a size index Σ can be obtained by evaluating Σ on revolution cones. This theme is elaborated in Section 3.2. In Section 3.1 we discuss different ways of measuring the size of an ellipsoidal cone.
- Section 4 discusses the concept of polar-reversibility, as well as a related notion called polar-compatibility.

2 Examples of size indices

A linear subspace is a particular instance of a closed convex cone. As shown in the next proposition, if one wishes to measure the size of a linear subspace, then there are not too many choices: one must use a function that depends only on the dimension of the linear subspace.

Proposition 2.1. *Let $\mathcal{L}_n$ be the set of linear subspaces of $\mathbb{R}^n$. Then Σ is a size index on $\mathcal{L}_n$ if and only if there exists a (necessarily unique) nonnegative nondecreasing function $\psi : \{0, 1, \ldots, n\} \to \mathbb{R}$ such that*

$$\Sigma(K) = \psi(\mathrm{dim}K) \quad \text{for all } K \in \mathcal{L}_n. \tag{3}$$

Proof. Necessity. Let Σ be a size index on $\mathcal{L}_n$. Let P and Q be two linear subspaces of $\mathbb{R}^n$ with the same dimension, say $d \in \{0, 1, \ldots, n\}$. Then there exists $U \in \mathbb{O}(n)$ such that $Q = U(P)$. Hence, $\Sigma(P) = \Sigma(Q)$ by axiom A_4. In other words, the function $\Sigma : \mathcal{L}_n \to \mathbb{R}$ is constant on each subset of the form

$$\mathcal{L}_{n,d} := \{K \in \mathcal{L}_n : \mathrm{dim}K = d\}.$$

This fact leads to the representation (3), with ψ nonnegative, by axiom A_2, and nondecreasing, by axiom A_3. *Sufficiency.* The function $\dim : \mathcal{L}_n \to \mathbb{R}$ satisfies all the axioms of a size index. The continuity of this function may seem surprising at first sight, but we recall that $\mathcal{L}_n = \cup_{d=0}^{n} \mathcal{L}_{n,d}$ is a union of sets that are open in $\mathcal{C}_n$, cf. [16, Proposition 6.3]. Since ψ is nonnegative and nondecreasing, the composite function $\psi \circ \dim : \mathcal{L}_n \to \mathbb{R}$ is also a size index. One may see ψ as a continuous function on the set $\{0, 1, \ldots, n\}$ equipped with the discrete metric. ☐

The next corollary, which is a reformulation of Proposition 2.1, asserts that a size index on $\mathcal{L}_n$ can be identified with a vector in the nonnegative orthant of $\mathbb{R}^{n+1}$.

Corollary 2.2. *Σ is a size index on $\mathcal{L}_n$ if and only if there exists a (necessarily unique) entrywise nonnegative vector $u \in \mathbb{R}^{n+1}$ such that*

$$\Sigma(K) = \sum_{k=0}^{\mathrm{dim}K} u_k \quad \text{for all } K \in \mathcal{L}_n.$$

The dimension of a closed convex cone K is defined as the dimension of the linear subspace spanned by K. The function $\dim : \mathcal{C}_n \to \mathbb{R}$ satisfies all the axioms of a size index, except for the continuity requirement. Corollary 6.6 in [16] shows that $\dim : \mathcal{C}_n \to \mathbb{R}$ is lower-semicontinuous, but not upper-semicontinuous. If one wishes to measure the size of a closed convex cone, then there are plenty of options for consideration. Perhaps the first idea that comes to mind is to use the maximal angle of the cone or, more generally, a suitable function of the maximal angle.

Definition 2.3. *Let $K \in \mathcal{N}_n$. The* maximal angle *of K is defined as the nonnegative number*

$$\theta_{\max}(K) := \max_{u,v \in K \cap \mathbb{S}_n} \arccos\langle u, v\rangle, \tag{4}$$

where $\mathbb{S}_n$ is the unit sphere of $\mathbb{R}^n$.

The need of evaluating the maximal angle of a convex cone arises frequently in applications, see for instance Peña and Renegar [26], Iusem and Seeger [17, 18], and Clarke et al. [5, 6]. Computing the expression (4) is a matter of solving a nonconvex optimization problem. This may be difficult or not depending on the specific structure of the convex cone under consideration. For instance, if K is a polyhedral cone with a moderate number of extreme rays, then (4) can be evaluated at a reasonable computational cost by solving finitely many generalized eigenvalue problems, cf. [20, Theorem 3]. Another “easy” case is presented in the example below.

Example 2.4. A topheavy cone in $\mathbb{R}^n$ is a proper cone that can be represented as the epigraph

$$\mathrm{epi}(\Phi) := \{(z, t) \in \mathbb{R}^n : \Phi(z) \leq t\} \tag{5}$$

of some norm Φ on $\mathbb{R}^{n-1}$. As shown in Seeger [28, Theorem 5.2], the maximal angle of (5) admits the explicit formula

$$\theta_{\max}(\mathrm{epi}(\Phi)) = \arccos\left[\frac{\alpha_\Phi^2 - 1}{\alpha_\Phi^2 + 1}\right],$$

where $\alpha_\Phi := \min_{\|z\|=1} \Phi(z)$.

Additional rules for computing maximal angles of convex cones can be found in [21]. Without further ado, we state:

Theorem 2.5. *The function $\theta_{\max} : \mathcal{N}_n \to \mathbb{R}$ is a Lipschitz continuous size index. It is neither strict nor modular.*

Proof. Clearly, $\theta_{\max}$ is nonnegative, nondecreasing with respect to set inclusion, and invariant under orthogonal transformations. Its continuity was shown in [17]. Lipschitz continuity was proven later in [30, Theorem 2]. Let $P = \mathbb{R}_+ \times \mathbb{R}_+^2 \times \{0\}^{n-3}$ and $Q = \mathbb{R}_- \times \mathbb{R}_+^2 \times \{0\}^{n-3}$, the component $\{0\}^{n-3}$ being dropped when $n = 3$. Note that

$$P \cup Q = \mathbb{R} \times \mathbb{R}_+^2 \times \{0\}^{n-3}, \quad P \cap Q = \{0\} \times \mathbb{R}_+^2 \times \{0\}^{n-3}$$

are both in $\mathcal{N}_n$. A direct computation yields

$$\theta_{\max}(P) = \theta_{\max}(Q) = \pi/2, \quad \theta_{\max}(P \cup Q) = \pi, \quad \theta_{\max}(P \cap Q) = \pi/2.$$

This example shows that $\theta_{\max}$ is neither strict nor modular. □

Mathematical statements concerning maximal angles can be rephrased in terms of half-diameters, and viceversa. The half-diameter of $K \in \mathcal{N}_n$ is defined as the number

$$\mathrm{hd}(K) := (1/2)\,\mathrm{diam}(K \cap \mathbb{S}_n),$$

where $\mathrm{diam}(C)$ stands for the diameter of a nonempty compact set C.

Corollary 2.6. *The function* hd : $\mathcal{N}_n \to \mathbb{R}$ *is a Lipschitz continuous size index. It is neither strict nor modular.*

Proof. It suffices to observe that

$$\mathrm{hd}(K) = \sin\left[\frac{\theta_{\max}(K)}{2}\right],$$

that is to say, $\mathrm{hd}(K)$ is a Lipschitz continuous increasing function of $\theta_{\max}(K)$. □

Solidity indices in the sense of Iusem and Seeger [17, Definition 6.1] provide additional examples of size indices. The role of a solidity index is to measure the degree of solidity of a convex cone. One says that $K \in \mathcal{N}_n$ is solid if its interior is nonempty, otherwise one says that K is flat. Solidity indices satisfy not just the axioms A_1, A_2, A_3, and A_4, but other more specialized axioms as well. The details can be consulted in [17]. The next definition recalls two popular solidity indices used in the literature.

Definition 2.7. *Let $K \in \mathcal{N}_n$. The* metric solidity index *and the* Frobenius solidity index *of K are defined respectively by*

$$\pi_{\mathrm{met}}(K) := \min_{Q \in \mathcal{N}_n^{\mathrm{flat}}} \delta(K, Q), \tag{6}$$

$$\pi_{\mathrm{frob}}(K) := \max_{x \in K \cap \mathbb{S}_n} \mathrm{dist}(x, \mathrm{bd}(K)), \tag{7}$$

where $\mathrm{bd}(K)$ *stands for the boundary of K and*

$$\mathcal{N}_n^{\mathrm{flat}} := \{Q \in \mathcal{N}_n : Q \text{ is not solid}\}.$$

The interpretation of the metric solidity index is clear: $\pi_{\rm met}(K)$ corresponds to the distance from K to the set of nonsolid nontrivial closed convex cones in $\mathbb{R}^n$. Hence, $\pi_{\rm met} : \mathcal{N}_n \to \mathbb{R}$ is a distance function and, as such, it is nonexpansive:

$$|\pi_{\rm met}(P) - \pi_{\rm met}(Q)| \leq \delta(P,Q) \quad \text{for all } P,Q \in \mathcal{N}_n.$$

The minimum in (6) is attained because the problem under consideration concerns the minimization of a continuous function on a compact set. In order to evaluate (6) one can rely on the formula

$$\pi_{\rm met}(K) = \cos\left[\frac{\theta_{\max}(K^\circ)}{2}\right] \tag{8}$$

established in [22, Corollary 2]. Here, K° stands for the polar cone of K, i.e.,

$$K^\circ := \{y \in \mathbb{R}^n : \langle y, x\rangle \leq 0 \text{ for all } x \in K\}.$$

Concerning the geometric interpretation of the Frobenius solidity index, note that (7) corresponds to the radius of a largest ball contained in K and centered at a unit vector. The Frobenius solidity index is a function arising in different mathematical contexts and, therefore, it is known under different names, cf. [2, 7, 9, 10, 14]. Calculus rules for computing (7) have been developed in a number of papers, see for instance [15]. As mentioned in [17, Proposition 6.3], a minimax argument leads to the characterization

$$\pi_{\rm frob}(K) = \min_{y \in {\rm co}(K^\circ \cap \mathbb{S}_n)} \|y\|\,, \tag{9}$$

where "co" stands for the convex hull operation. Geometrically speaking, the minimum on the right-hand side of (9) is a coefficient that measures the degree of pointedness of K°.

Example 2.8. Suppose that one wishes to evaluate the size of the upward ellipsoidal cone

$$E_M := \left\{(z,t) \in \mathbb{R}^n : \sqrt{\langle z, Mz\rangle} \leq t\right\}$$

associated to a symmetric positive definite matrix M of order $n-1$. The combination of [15, Proposition 2.6] and [19, Theorem 8.6] yields

$$\pi_{\rm met}(E_M) \;=\; \pi_{\rm frob}(E_M) \;=\; [1 + \lambda_{\max}(M)]^{-1/2}\,,$$

where $\lambda_{\max}(M)$ denotes the largest eigenvalue of M. More generally, if Φ is a norm on $\mathbb{R}^{n-1}$, then

$$\pi_{\rm met}({\rm epi}(\Phi)) \;=\; \pi_{\rm frob}({\rm epi}(\Phi)) \;=\; \left[1 + \beta_\Phi^2\right]^{-1/2},$$

where $\beta_\Phi := \max_{\|z\|=1} \Phi(z)$. Beware, however, that $\pi_{\rm met}$ and $\pi_{\rm frob}$ may differ substantially for convex cones that are not topheavy.

Theorem 2.9. *The functions $\pi_{\rm met} : \mathcal{N}_n \to \mathbb{R}$ and $\pi_{\rm frob} : \mathcal{N}_n \to \mathbb{R}$ are Lipschitz continuous size indices. They are neither strict nor modular.*

Proof. The functions $\pi_{\rm met}$ and $\pi_{\rm frob}$ are known to be solidity indices, cf. [17]. They satisfy in particular all the axioms of a size index. As mentioned before, $\pi_{\rm met}$ is a nonexpansive function with respect to the truncated Pompeiu-Hausdorff metric. By combining (9) and Proposition 13 in [23], one gets

$$|\pi_{\rm frob}(P) - \pi_{\rm frob}(Q)| \leq \tau(P^\circ, Q^\circ)\,,$$

where τ stands for the spherical metric on $\mathcal{N}_n$, i.e.,

$$\tau(P,Q) := {\rm haus}\,(P \cap \mathbb{S}_n, Q \cap \mathbb{S}_n).$$

But the spherical metric is Lipschitz equivalent to the truncated Pompeiu-Hausdorff metric. In fact,

$$\delta(P,Q) \;\leq\; \tau(P,Q) \;\leq\; 2\,\delta(P,Q) \quad \text{for all } P,Q \in \mathcal{N}_n.$$

So, by using the inequality $\tau \leq 2\,\delta$ and the Walkup-Wets isometry formula (cf.[37, Theorem 1])

$$\delta(P^\circ, Q^\circ) = \delta(P, Q),$$

one derives the Lipschitz condition

$$|\pi_{\rm frob}(P) - \pi_{\rm frob}(Q)| \leq 2\,\delta(P, Q) \quad \text{for all } P, Q \in \mathcal{N}_n.$$

For proving the second part of the theorem, we consider the convex cones $P = \mathbb{R}_+ \times \mathbb{R}^2_+ \times \mathbb{R}^{n-3}$ and $Q = \mathbb{R}_- \times \mathbb{R}^2_+ \times \mathbb{R}^{n-3}$, the component $\mathbb{R}^{n-3}$ being dropped when $n = 3$. Note that

$$P \cup Q = \mathbb{R} \times \mathbb{R}^2_+ \times \mathbb{R}^{n-3}, \quad P \cap Q = \{0\} \times \mathbb{R}^2_+ \times \mathbb{R}^{n-3}$$

are both in $\mathcal{N}_n$. A matter of computation yields

$$\pi_{\rm met}(P) = \pi_{\rm met}(Q) = 1/\sqrt{2}, \quad \pi_{\rm met}(P \cup Q) = 1/\sqrt{2}, \quad \pi_{\rm met}(P \cap Q) = 0,$$
$$\pi_{\rm frob}(P) = \pi_{\rm frob}(Q) = 1/\sqrt{3}, \quad \pi_{\rm frob}(P \cup Q) = 1/\sqrt{2}, \quad \pi_{\rm frob}(P \cap Q) = 0,$$

This proves that $\pi_{\rm met}$ and $\pi_{\rm frob}$ are not modular. Finally, $\pi_{\rm met}$ and $\pi_{\rm frob}$ are not strict, because both functions are identically zero on $\mathcal{N}_n^{\rm flat}$. □

An alternative way of measuring the size of a convex cone is by evaluating its volume. Since a convex cone is an unbounded set, the concept of volume must be handled with care. The simplest way of proceeding is to intersect the convex cone with a ball centered at the origin.

Definition 2.10. *Let $K \in \mathcal{N}_n$. The* ball-truncated volume *of K is defined by*

$$\text{btv}(K) := \text{vol}_n(K \cap \mathbb{B}_n), \tag{10}$$

where vol_n stands for the n-dimensional Lebesgue measure.

The computation of ball-truncated volumes of convex cones is a theme discussed in Gourion and Seeger [11] and Ribando [27]. Up to a positive multiplicative factor, the ball-truncated volume of K is equal to the solid angle of K. Another way of characterizing the expression (10) is by using the n-dimensional Gaussian measure. As pointed out in [11, Proposition 5.1], one can write

$$\frac{\text{btv}(K)}{\text{vol}_n(\mathbb{B}_n)} = \mathbb{P}[\,\mathbf{g} \in K] = \frac{1}{(2\pi)^{n/2}} \int_K e^{-\frac{1}{2}\|x\|^2} dx, \tag{11}$$

where $\mathbf{g} \sim \textsc{Normal}\,(0, I_n)$ is an n-dimensional random vector following a normal distribution.

Example 2.11. A cantilever in $\mathbb{R}^n$ is a polyhedral cone of the form

$$K = \{x \in \mathbb{R}^n : \langle a, x\rangle \geq 0,\ \langle b, x\rangle \geq 0\},$$

where $a, b \in \mathbb{R}^n$ are non-collinear unit vectors. One has

$$\text{btv}(K) = \left(1 - \frac{\arccos\langle a, b\rangle}{\pi}\right) \frac{\text{vol}_n(\mathbb{B}_n)}{2}.$$

The above formula is easily obtained by using the probabilistic characterization (11) and the fact that a normal distribution is spherically symmetric.

Theorem 2.12. *The function* $\text{btv} : \mathcal{N}_n \to \mathbb{R}$ *is a size index. Furthermore,*

(a) btv *is Lipschitz continuous and modular.*

(b) btv *is not strict on $\mathcal{N}_n$, but it is strict as size index on the smaller set $\mathcal{P}_n$.*

Proof. The function $\mathrm{btv} : \mathcal{N}_n \to \mathbb{R}$ clearly satisfies the axioms A_2, A_3, and A_4. Lipschitz continuity is proven in [12, Theorem 2.2]. Modularity follows from the representation formula (11) and the inclusion-exclusion rule of probability theory. Note that btv is identically zero on $\mathcal{N}_n^{\mathrm{flat}}$, so it is not strict on the whole set $\mathcal{N}_n$. Consider two distinct cones $P, Q \in \mathcal{P}_n$ such that $P \subseteq Q$. Since the set difference $Q \backslash P$ has nonempty interior, one sees that

$$\mathrm{btv}(Q) - \mathrm{btv}(P) = \frac{\mathrm{vol}_n(\mathbb{B}_n)}{(2\pi)^{n/2}} \int_{Q \backslash P} e^{-\frac{1}{2}\|x\|^2} dx$$

is positive. □

One of the main drawback of the function btv is that the evaluation of a ball-truncated volume is computationally expensive in general. The following concept of volume is more tractable from a numerical point of view and, in addition, it enjoys a number of interesting theoretical properties.

Definition 2.13. *Let $K \in \mathcal{P}_n$. The* canonical volume *of K is the positive real*

$$\mathrm{Vol}(K) := \min_{y \in \mathbb{S}_n} v_K(y), \tag{12}$$

where v_K is the extended-real-valued function on $\mathbb{R}^n$ given by

$$v_K(y) := \mathrm{vol}_n(\{x \in K : \langle y, x \rangle \leq 1\}).$$

Practitioners of interior point methods in conic programming know that v_K behaves as a barrier function for the dual cone

$$K^* := \{y \in \mathbb{R}^n : \langle y, x \rangle \geq 0 \text{ for all } x \in K\}.$$

As shown by Güler [13, Theorem 4.1], one can write

$$v_K(y) = \frac{1}{n!} \int_K e^{-\langle y, x \rangle} dx$$

for all $y \in \mathrm{int}(K^*)$. Such an integral representation formula has many useful consequences. For instance, it helps to prove that the minimization problem (12) admits exactly one solution. Such a solution is denoted by $c(K)$ and it is called the volumetric center of K^*. If $c(K)$ is known, then one can evaluate the canonical volume of K by using the formula

$$\mathrm{Vol}(K) = \frac{1}{n} \mathrm{vol}_{n-1}(\{x \in K : \langle c(K), x \rangle = 1\}). \tag{13}$$

Canonical volumes and volumetric centers have been the object of a long work by Torki and Seeger [31, 32].

Theorem 2.14. *The function* $\mathrm{Vol} : \mathcal{P}_n \to \mathbb{R}$ *is a strict size index. However, it is neither Lipschitz continuous nor modular.*

Proof. The function Vol is clearly nonnegative and nondecreasing with respect to set inclusion. Continuity and invariance under orthogonal transformations are proven in [31, Theorem 5.7] and [31, Proposition 3.2], respectively. Example 5.9 in [31] shows that Vol is not Lipschitz continuous. For proving strictness, consider two distinct cones $P, Q \in \mathcal{P}_n$ such that $P \subseteq Q$. Note that

$$c(Q) \in \mathrm{int}(Q^*) \subseteq \mathrm{int}(P^*).$$

One gets

$$n!\,\mathrm{Vol}(Q) \;=\; \int_Q e^{-\langle c(Q),\, x \rangle} dx \;=\; a_1 + a_2,$$

with

$$\begin{aligned} a_1 \;&:=\; \int_P e^{-\langle c(Q),\, x \rangle} dx \;\geq\; n!\,\mathrm{Vol}(P), \\ a_2 \;&:=\; \int_{Q \backslash P} e^{-\langle c(Q),\, x \rangle} dx \;>\; 0, \end{aligned}$$

the last inequality being due to the fact that $Q\backslash P$ has nonempty interior. This shows that $\mathrm{Vol}(P)$ is smaller than $\mathrm{Vol}(Q)$. To see that Vol is not modular on $\mathcal{P}_n$, we construct a counter-example in dimension $n = 3$. Similar counter-examples can be constructed in higher dimensions. Consider the proper polyhedral cones P_ε and Q_ε that are generated by the columns of the matrices

$$A_\varepsilon = \begin{bmatrix} 1 & -1 & 1 & -\varepsilon \\ -1 & 1 & 1 & -\varepsilon \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B_\varepsilon = \begin{bmatrix} 1 & -1 & -1 & \varepsilon \\ -1 & 1 & -1 & \varepsilon \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

respectively. Here, the entry $\varepsilon \in [0,1]$ is viewed as a parameter. Regardless of the choice of ε, the polyhedral cone

$$P_\varepsilon \cup Q_\varepsilon = \{x \in \mathbb{R}^3 : |x_1| \leq x_3,\ |x_2| \leq x_3\}$$

is proper and symmetric with respect to the line generated by $e_3 = (0,0,1)^T$. Hence, $c(P_\varepsilon \cup Q_\varepsilon) = e_3$. By applying formula (13), one readily gets $\mathrm{Vol}(P_\varepsilon \cup Q_\varepsilon) = 4/3$. If ε is different from 0, then $P_\varepsilon \cap Q_\varepsilon$ is equal to the proper polyhedral cone generated by the columns of the matrix

$$\begin{bmatrix} 1 & -1 & \varepsilon & -\varepsilon \\ -1 & 1 & \varepsilon & -\varepsilon \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Note that $P_\varepsilon \cap Q_\varepsilon$ is symmetric with respect to the line generated by e_3. Thus, $c(P_\varepsilon \cap Q_\varepsilon) = e_3$ and formula (13) yields $\mathrm{Vol}(P_\varepsilon \cap Q_\varepsilon) = (4/3)\varepsilon$. Computing $\mathrm{Vol}(P_\varepsilon)$ and $\mathrm{Vol}(Q_\varepsilon)$ is a bit more complicated. Note however that

$$\lim_{\varepsilon \to 0} \delta(P_\varepsilon, P_0) = 0 \ \text{ and } \ \lim_{\varepsilon \to 0} \delta(Q_\varepsilon, Q_0) = 0,$$

where P_0 is the simplicial cone generated by the first three columns of A_ε and Q_0 is the simplicial cone generated by the first three columns of B_ε. The volumetric center and the canonical volume of a simplicial cone can be computed as explained in [32, Section 2]. One gets

$$\begin{aligned} c(P_0) &= \frac{\sqrt{2}}{6}(1,1,4)^T, & \mathrm{Vol}(P_0) &= \frac{3}{8}\sqrt{2}, \\ c(Q_0) &= \frac{\sqrt{2}}{6}(-1,-1,4)^T, & \mathrm{Vol}(Q_0) &= \frac{3}{8}\sqrt{2}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{Vol}(P_\varepsilon) + \mathrm{Vol}(Q_\varepsilon) &\approx \mathrm{Vol}(P_0) + \mathrm{Vol}(Q_0) = \frac{3}{4}\sqrt{2} \\ \mathrm{Vol}(P_\varepsilon \cup Q_\varepsilon) + \mathrm{Vol}(P_\varepsilon \cap Q_\varepsilon) &= \frac{4}{3}(1+\varepsilon) \approx \frac{4}{3} \end{aligned}$$

whenever ε is near 0. This shows that Vol is not modular on $\mathcal{P}_n$. □

The canonical volume function is a rare example of size index that is not Lipschitz continuous. The lack of Lipschitz continuity in a size index is not truly a big handicap. Ordinary continuity is often times enough for all practical purposes. Lack of modularity is not too bothersome either. The function Vol remains, after all, a good choice as tool for measuring the size of a proper cone. A great advantage of the canonical volume function is the fact of being strict and not too expensive to evaluate in general.

Example 2.15. Suppose that one wishes to evaluate the size of the upward ellipsoidal cone E_M associated to a symmetric positive definite matrix M of order $n-1$. The formula

$$\mathrm{Vol}(E_M) = \frac{\mathrm{vol}_{n-1}(\mathbb{B}_{n-1})}{n} \frac{1}{\sqrt{\det M}} \tag{14}$$

provides a cheap way of computing the canonical volume of E_M. By contrast, there is no explicit formula that helps with the computation of $\mathrm{btv}(E_M)$. For evaluating the ball-truncated volume of E_M one must rely on Monte Carlo methods or on expensive numerical integration techniques.

Example 2.16. More generally, if Φ is a norm on $\mathbb{R}^{n-1}$, then

$$\mathrm{Vol}(\mathrm{epi}(\Phi)) \;=\; \frac{1}{n}\,\mathrm{vol}_{n-1}(\{z\in\mathbb{R}^{n-1}:\Phi(z)\le 1\}).$$

So, one needs to compute the usual volume of the closed unit ball associated to Φ.

We add next an example of size index whose definition is probabilistic in nature. In what follows, $\mathbb{E}X$ stands for the mathematical expectation of a random variable X.

Definition 2.17. *Let $K\in\mathcal{N}_n$. The* Gaussian width *of K is given by*

$$\Omega(K) := \mathbb{E}\left[\max_{x\in K\cap\mathbb{S}_n}\langle \mathbf{g},x\rangle\right], \tag{15}$$

with $\mathbf{g}\sim$ Normal $(0,I_n)$.

The concept of Gaussian width for convex cones arises in a number of applications, for instance in the analysis of certain linear inverse problems; cf. Tropp [36] and Chandrasekaran et al. [4]. Note that

$$\Omega(K) \;=\; \omega(K\cap\mathbb{S}_n) \;=\; \omega(\mathrm{co}(K\cap\mathbb{S}_n)),$$

where $\omega(C) := \mathbb{E}\left[\max_{x\in C}\langle\mathbf{g},x\rangle\right]$ is the Gaussian width of a nonempty compact set C. Gaussian widths of compact sets have been abundantly studied in the literature. Evaluating the mathematical expectation (15) is a very cumbersome task in general. Since $\mathbf{g}$ follows a normal distribution, one can write

$$\Omega(K) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\Psi^*_{K\cap\mathbb{S}_n}(y)\,e^{-\frac{1}{2}\|y\|^2}dy, \tag{16}$$

with Ψ^*_C standing for the support function of a nonempty compact set C.

Example 2.18. Let $K\in\mathcal{N}_n$ be a linear subspace of dimension $d\in\{1,\ldots,n-1\}$. By evaluating (16) one gets $\Omega(K)=\ell_d$, where

$$\ell_d := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\|\xi\|\,e^{-\frac{1}{2}\|\xi\|^2}d\xi \;=\; \sqrt{2}\,\frac{\Gamma((d+1)/2)}{\Gamma(d/2)}$$

corresponds to the mathematical expectation of a chi-distribution with d degrees of freedom. The symbol Γ refers to the usual Euler Gamma function. Beware that $\Omega(K)$ is not equal to $\sqrt{d}$, as wrongly asserted in [4, Section 3.2].

Example 2.19. Let $K\in\mathcal{N}_n$ be a half-space. By applying an orthogonal transformation if necessary, one may assume that K is equal to the upward oriented half-space $\{x\in\mathbb{R}^n: x_n\ge 0\}$. In such a case,

$$\Psi^*_{K\cap\mathbb{S}_n}(y) = \begin{cases} \|y\| & \text{if } y_n\ge 0,\\ [y_1^2+\ldots+y_{n-1}^2]^{1/2} & \text{if } y_n\le 0,\end{cases}$$

and a direct evaluation of (16) yields

$$\Omega(K) = (\ell_{n-1}+\ell_n)/2.$$

The next lemma shows that $\Omega:\mathcal{N}_n\to\mathbb{R}$ is Lipschitz continuous.

Lemma 2.20. *For all $P,Q\in\mathcal{N}_n$, one has*

$$|\Omega(P)-\Omega(Q)|\le 2\,\ell_n\delta(P,Q),$$

where ℓ_n is the mathematical expectation of a chi-distribution with n degrees of freedom.

Proof. The length $\|\mathbf{g}\|$ and the orientation $\mathbf{g}/\|\mathbf{g}\|$ of a random vector $\mathbf{g} \sim \textsc{Normal}\,(0, I_n)$ are stochastically independent. Hence, for an arbitrary nonempty compact set C in $\mathbb{R}^n$, one has

$$\frac{1}{\ell_n}\,\omega(C) \;=\; \widehat{\omega}\,(C) \;:=\; \mathbb{E}\left[\max_{x\in C}\langle \mathbf{u}, x\rangle\right], \tag{17}$$

where $\mathbf{u} \sim \textsc{Uniform}\,(\mathbb{S}_n)$ and $\ell_n = \mathbb{E}\|\mathbf{g}\|$. For any pair C, D of nonempty compact sets, one has

$$\mathrm{haus}\,(C, D) \;\geq\; \mathrm{haus}\,(\mathrm{co}(C), \mathrm{co}(D)) \;=\; \max_{u\in\mathbb{S}_n} |\Psi^*_C(u) - \Psi^*_D(u)|.$$

Hence,

$$\begin{aligned}
\Psi^*_{P\cap\mathbb{S}_n}(\mathbf{u}) - \Psi^*_{Q\cap\mathbb{S}_n}(\mathbf{u}) \;&\leq\; \max_{u\in\mathbb{S}_n} |\Psi^*_{P\cap\mathbb{S}_n}(u) - \Psi^*_{Q\cap\mathbb{S}_n}(u)| \\
&=\; \mathrm{haus}\,(\mathrm{co}(P\cap\mathbb{S}_n), \mathrm{co}(Q\cap\mathbb{S}_n)) \\
&\leq\; \tau(P, Q) \\
&\leq\; 2\delta(P, Q).
\end{aligned}$$

By passing to mathematical expectations and multiplying by the constant ℓ_n, one gets

$$\Omega(P) - \Omega(Q) \leq 2\,\ell_n\,\delta(P, Q).$$

It remains now to exchange the roles of P and Q. □

The mathematical expectation on the right-hand side of (17) has an interesting geometric interpretation. As observed in [4, Section 3.2], one can write

$$2\,\widehat{\omega}\,(C) = \int_{\mathbb{S}_n} \left[\max_{x\in C}\langle u, x\rangle - \min_{x\in C}\langle u, x\rangle\right] d\sigma(u), \tag{18}$$

where the integral is with respect to the Haar measure on $\mathbb{S}_n$. The above integral is known as the mean width of the set C. The combination of (17) and (18) leads to the formula

$$\Omega(K) = \frac{\ell_n}{2}\int_{\mathbb{S}_n} \left[\max_{x\in K\cap\mathbb{S}_n}\langle u, x\rangle - \min_{x\in K\cap\mathbb{S}_n}\langle u, x\rangle\right] d\sigma(u). \tag{19}$$

The next lemma shows that $\Omega(K)$ does not depend on the dimension of the space $\mathbb{R}^n$ in which the convex cone K is embedded.

Lemma 2.21. *Let $d \in \{1, \ldots, n-1\}$. If R is a nonzero closed convex cone in $\mathbb{R}^d$, then*

$$\Omega(R \times \{0\}^{n-d}) = \mathbb{E}\left[\max_{w\in R\cap\mathbb{S}_d}\langle \mathbf{v}, w\rangle\right],$$

with $\mathbf{v} \sim \textsc{Normal}\,(0, I_d)$.

Proof. Since $K = R \times \{0\}^{n-d}$ belongs to $\mathcal{N}_n$ and

$$\Psi^*_{K\cap\mathbb{S}_n}(\xi, \eta) = \Psi^*_{R\cap\mathbb{S}_d}(\xi)$$

for all $(\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^{n-d}$, the integral (16) becomes

$$\Omega(K) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^d\times\mathbb{R}^{n-d}} \Psi^*_{R\cap\mathbb{S}_d}(\xi)\, e^{-\frac{1}{2}(\|\xi\|^2+\|\eta\|^2)} d\xi d\eta.$$

By applying Fubini's theorem one ends up with

$$\Omega(K) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} \Psi^*_{R\cap\mathbb{S}_d}(\xi)\, e^{-\frac{1}{2}\|\xi\|^2} d\xi.$$

The term on the right-hand side corresponds of course to the Gaussian width of R as convex cone in $\mathbb{R}^d$. □

Example 2.22. Let $K = \mathrm{cone}\{a, b\}$ be generated by a pair of non-collinear unit vectors of $\mathbb{R}^n$. Let β be the angle between a and b. Note that K is a polyhedral cone of dimension two in a space of dimension n. By using an orthogonal transformation if necessary, one can convert K into a polyhedral cone of the form $R \times \{0\}^{n-2}$, with

$$R = \mathrm{cone}\{(1,0), (\cos\beta, \sin\beta)\}.$$

By relying on Lemma 2.21 one obtains

$$\Omega(K) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \Psi^*_{R\cap\mathbb{S}_2}(\xi)\, e^{-\frac{1}{2}\|\xi\|^2} d\xi.$$

The above integral can be computed by using polar coordinates. One gets

$$\Omega(K) = \frac{1}{\sqrt{2\pi}}\left[\frac{\beta}{2} + \sin\left(\frac{\beta}{2}\right)\right]. \tag{20}$$

We now are ready to state:

Theorem 2.23. *The function $\Omega : \mathcal{N}_n \to \mathbb{R}$ is a size index. Furthermore, it is Lipschitz continuous and strict, but not modular.*

Proof. Clearly, Ω is nondecreasing with respect to set inclusion and invariant under orthogonal transformations. Formula (19) shows that Ω is nonnegative. Lipschitz continuity is taken care by Lemma 2.20. For proving strictness, consider two distinct cones $P, Q \in \mathcal{N}_n$ such that $P \subseteq Q$. It is not difficult to check that $C := \mathrm{co}(P \cap \mathbb{S}_n)$ is strictly contained in $D := \mathrm{co}(Q \cap \mathbb{S}_n)$. Hence, the mean width of C is smaller than the mean width of D. In such a case, one gets

$$\Omega(P) \;=\; \ell_n\, \widehat{\omega}\,(C) \;<\; \ell_n\, \widehat{\omega}\,(D) \;=\; \Omega(Q).$$

The lack of modularity of Ω can be shown with the help of a counter-example. Consider for instance

$$P = \mathbb{R}_+ \times \mathbb{R}_+ \times \{0\}^{n-2} = \mathrm{cone}\{e_1, e_2\},$$
$$Q = \mathbb{R}_- \times \mathbb{R}_+ \times \{0\}^{n-2} = \mathrm{cone}\{-e_1, e_2\}.$$

By using formula (20) one gets

$$\Omega(P) \;=\; \Omega(Q) \;=\; \frac{1}{\sqrt{2\pi}}\left[\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right] \approx 0.5954.$$

On the other hand, $\Omega(P \cap Q) = 0$ and

$$\Omega(P \cup Q) \;=\; \frac{1}{2}(\ell_1 + \ell_2) \;=\; \frac{1}{2}\left[\frac{\sqrt{2}}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{\sqrt{2}}\right] \approx 1.0256.$$

This completes the proof of the theorem. □

Remark 2.24. The classical mean width function defined by the integral (18) behaves in a modular manner on the hyperset of nonempty convex compact subsets of $\mathbb{R}^n$. Indeed, if C and D are nonempty convex compact sets such that $C \cup D$ is convex, then

$$\widehat{\omega}\,(C) + \widehat{\omega}\,(D) \;=\; \widehat{\omega}\,(C \cup D) + \widehat{\omega}\,(C \cup D). \tag{21}$$

Unfortunately, the conic version Ω does not inherit the modularity of the classical mean width function.

We next introduce a collection $\{\mu_m\}_{m\geq 1}$ of size indices whose construction is based on probabilistic considerations. Recall that, in the parlance of probability theory, the term $\mathbb{E}(X^m)$ is referred to as the m-th order raw moment of a random variable X.

Definition 2.25. *Let m be a positive integer and $K \in \mathcal{C}_n$. The m-th* order Gaussian projection length *of K is given by*

$$\mu_m(K) := \mathbb{E}\left\{\|\Pi_K(\mathbf{g})\|^m\right\}, \tag{22}$$

where $\mathbf{g} \sim$ NORMAL $(0, I_n)$ and Π_K is the metric projector onto K.

Since the vector $\mathbf{g}$ in Definition 2.25 follows a normal distribution, the term (22) is equal to the multiple integral

$$\mu_m(K) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \|\Pi_K(x)\|^m \, e^{-\frac{1}{2}\|x\|^2} dx. \tag{23}$$

Of special interest are the first-order and the second-order Gaussian projection lengths. The coefficient $\mu_1(K)$ is introduced without name in Chandrasekaran et al. [4], where it is used as upper bound for the Gaussian width of K. The second-order raw moment

$$\mu_2(K) = \mathbb{E}\left\{\|\Pi_K(\mathbf{g})\|^2\right\}$$

is known as the statistical dimension of K. The theory of statistical dimensions for convex cones is nowadays an active field of research, see [1, 24].

Computing a Gaussian projection length amounts in practice to evaluate the integral (23), and this is often times fairly difficult. Three easy cases are displayed below. In what follows, the notation $\ell_{d,m}$ refers to the m-th order raw moment of a chi-distribution with d degrees of freedom, i.e.,

$$\ell_{d,m} = 2^{m/2}\, \frac{\Gamma((d+m)/2)}{\Gamma(d/2)}\,.$$

The computation of $\ell_{d,m}$ offers no difficulty, see Table 1.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$
$m=1$	$\sqrt{\frac{2}{\pi}}$	$\sqrt{\frac{\pi}{2}}$	$2\sqrt{\frac{2}{\pi}}$	$\frac{3}{2}\sqrt{\frac{\pi}{2}}$	$\frac{8}{3}\sqrt{\frac{2}{\pi}}$	$\frac{15}{8}\sqrt{\frac{\pi}{2}}$
$m=2$	1	2	3	4	5	6
$m=3$	$2\sqrt{\frac{2}{\pi}}$	$3\sqrt{\frac{\pi}{2}}$	$8\sqrt{\frac{2}{\pi}}$	$\frac{15}{2}\sqrt{\frac{\pi}{2}}$	$16\sqrt{\frac{2}{\pi}}$	$\frac{105}{8}\sqrt{\frac{\pi}{2}}$
$m=4$	3	8	15	24	35	48
$m=5$	$8\sqrt{\frac{2}{\pi}}$	$15\sqrt{\frac{\pi}{2}}$	$48\sqrt{\frac{2}{\pi}}$	$\frac{105}{2}\sqrt{\frac{\pi}{2}}$	$128\sqrt{\frac{2}{\pi}}$	$\frac{945}{8}\sqrt{\frac{\pi}{2}}$
$m=6$	15	48	105	192	315	480

Table 1: Selected values of $\ell_{d,m}$.

Example 2.26. Let K be a d-dimensional linear subspace of $\mathbb{R}^n$. Then the projected vector $\Pi_K(\mathbf{g})$ follows a normal distribution on K and its length $\|\Pi_K(\mathbf{g})\|$ follows a chi-distribution with d degrees of freedom. Hence,

$$\mu_m(K) = \ell_{d,m}, \tag{24}$$

where one adopts the convention $\ell_{0,m} = 0$. In particular, $\mu_1(K) = \ell_d$ and $\mu_2(K) = d$.

Example 2.27. Let K be a ray in $\mathbb{R}^n$. By applying an orthogonal transformation if necessary, one may suppose that K is the ray generated by e_n, the n-th canonical vector of $\mathbb{R}^n$. In such a case,

$$\|\Pi_K(x)\| = \max\{0, x_n\}.$$

By substituting this information into (23) and simplifying, one gets

$$\mu_m(K) \;=\; \frac{1}{\sqrt{2\pi}} \int_0^\infty t^m e^{-\frac{1}{2}t^2} dt \;=\; \frac{1}{2}\,\ell_{1,m}.$$

In particular, $\mu_1(K) = 1/\sqrt{2\pi}$ and $\mu_2(K) = 1/2$.

Example 2.28. The m-th order Gaussian projection length of the nonnegative orthant of $\mathbb{R}^n$ is given by

$$\mu_m(\mathbb{R}^n_+) = \frac{1}{2^n}\sum_{d=0}^{n} C^n_d\, \ell_{d,m} \tag{25}$$

with $C^n_d := n!/(d!(n-d)!)$ denoting the usual binomial coefficient. In particular,

$$\begin{aligned}\mu_1(\mathbb{R}^n_+) &= \frac{1}{2^n}\sum_{d=0}^{n} C^n_d\, \ell_d\,,\\ \mu_2(\mathbb{R}^n_+) &= \frac{1}{2^n}\sum_{d=0}^{n} C^n_d\, d \;=\; \frac{n}{2}\,.\end{aligned}$$

Formula (25) is obtained by substituting

$$\|\Pi_{\mathbb{R}^n_+}(x)\| = \left[\sum_{i=1}^{n} (\max\{0, x_i\})^2\right]^{1/2}$$

into (23) and carrying out the integration over each one of the 2^n orthants of $\mathbb{R}^n$. The details are omitted.

The term (22) can be written in several equivalent ways. For instance, Moreau's orthogonal decomposition theorem (cf. [25]) yields

$$\mu_m(K) = \mathbb{E}\left\{[\mathrm{dist}(\mathbf{g}, K^\circ)]^m\right\}.$$

Since $\mathrm{dist}(\,\cdot\,, K^\circ)$ is equal to the support function of the set $K \cap \mathbb{B}_n$, one can also write

$$\mu_m(K) = \mathbb{E}\left\{\left[\max_{x\in K\cap\mathbb{B}_n} \langle \mathbf{g}, x\rangle\right]^m\right\}. \tag{26}$$

By using the characterization (26) one can easily prove that $\mu_m : \mathcal{C}_n \to \mathbb{R}$ is Lipschitz continuous.

Lemma 2.29. *Let m be a positive integer. Then, for all $P, Q \in \mathcal{C}_n$, one has*

$$|\mu_m(P) - \mu_m(Q)| \le m\, \ell_{n,m}\, \delta(P, Q).$$

In particular,

$$\begin{aligned}|\mu_1(P) - \mu_1(Q)| &\le \ell_n\, \delta(P, Q),\\ |\mu_2(P) - \mu_2(Q)| &\le 2n\, \delta(P, Q).\end{aligned}$$

Proof. The case $m = 1$ is easy. It suffices to pass to mathematical expectations in

$$\Psi^*_{P\cap\mathbb{B}_n}(\mathbf{g}) - \Psi^*_{Q\cap\mathbb{B}_n}(\mathbf{g}) \;\le\; \|\mathbf{g}\| \sup_{u\in\mathbb{S}_n} \left|\Psi^*_{P\cap\mathbb{B}_n}(u) - \Psi^*_{Q\cap\mathbb{B}_n}(u)\right| \;=\; \|\mathbf{g}\|\, \delta(P, Q)\,.$$

For proving the case $m \ge 2$, one writes the identity

$$a^m - b^m = (a-b)\sum_{k=0}^{m-1} a^k b^{m-1-k}$$

with $a := \Psi^*_{P\cap\mathbb{B}_n}(\mathbf{g})$ and $b := \Psi^*_{Q\cap\mathbb{B}_n}(\mathbf{g})$. Since a and b are smaller than or equal to $\|\mathbf{g}\|$, one gets

$$\left[\Psi^*_{P\cap\mathbb{B}_n}(\mathbf{g})\right]^m - \left[\Psi^*_{Q\cap\mathbb{B}_n}(\mathbf{g})\right]^m \;\le\; m\|\mathbf{g}\|^m \delta(P, Q).$$

Again, one passes to mathematical expectations. □

Theorem 2.30. *For any positive integer m, the function $\mu_m : \mathcal{C}_n \to \mathbb{R}$ is a size index. Furthermore, it is Lipschitz continuous and strict.*

Proof. Let $K \in \mathcal{C}_n$ and $U \in \mathbb{O}(n)$. If $\mathbf{g} \sim \textsc{Normal}\,(0, I_n)$, then $U^T\mathbf{g} \sim \textsc{Normal}\,(0, I_n)$ and

$$\left\|\Pi_{U(K)}(\mathbf{g})\right\|^m = \left\|\Pi_K(U^T\mathbf{g})\right\|^m .$$

A passage to mathematical expectations in the above line shows that μ_m in invariant under orthogonal transformations. For proving strictness, consider two distinct cones $P, Q \in \mathcal{C}_n$ such that $P \subseteq Q$. Clearly, $\mathrm{dist}(x, P^\circ) \leq \mathrm{dist}(x, Q^\circ)$ for all $x \in \mathbb{R}^n$, but $\mathrm{dist}(\,\cdot\,, P^\circ)$ and $\mathrm{dist}(\,\cdot\,, Q^\circ)$ are not the same function. Hence,

$$\mu_m(Q) - \mu_m(P) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left([\mathrm{dist}(x, Q^\circ)]^m - [\mathrm{dist}(x, P^\circ)]^m\right) e^{-\frac{1}{2}\|x\|^2} dx$$

is positive. Lipschitz continuity is taken care by Lemma 2.29. □

As a complement to Theorem 2.30, we state below a partial result on modularity.

Proposition 2.31. *The size indices $\mu_1 : \mathcal{C}_n \to \mathbb{R}$ and $\mu_2 : \mathcal{C}_n \to \mathbb{R}$ are modular.*

Proof. That μ_2 is modular is a statement that can be found in [1]. For proving the modularity of μ_1, consider a pair $P, Q \in \mathcal{C}_n$ such that $P \cup Q$ is convex. Then $C := P \cap \mathbb{B}_n$ and $D := Q \cap \mathbb{B}_n$ are nonempty convex compact sets such that $C \cup D$ is convex. Note that $\mu_1(P) = \omega(C)$, $\mu_1(Q) = \omega(D)$, and

$$\begin{aligned} \mu_1(P \cup Q) &= \omega((P \cup Q) \cap \mathbb{B}_n) = \omega(C \cup D), \\ \mu_1(P \cap Q) &= \omega((P \cap Q) \cap \mathbb{B}_n) = \omega(C \cap D). \end{aligned}$$

So, multiplying on each side of (21) by ℓ_n, one gets

$$\mu_1(P) + \mu_1(Q) = \mu_1(P \cup Q) + \mu_1(P \cap Q),$$

which is the desired conclusion. □

We do not know whether $\mu_m : \mathcal{C}_n \to \mathbb{R}$ is modular when $m \geq 3$. This particular issue is left as an open question. The next proposition gives a characterization of $\mu_m(K)$ in terms of the conic intrinsic volumes $\nu_0(K), \nu_1(K), \ldots, \nu_n(K)$. For simplicity in the presentation we restrict the attention to the particular case in which K is polyhedral. We suppose that the reader is familiar with the theory of faces of polyhedral cones. By definition, the face of K associated to a point $x \in K$ is the unique face of K that contains x in its relative interior.

Definition 2.32. *Let $K \in \mathcal{C}_n$ be a polyhedral cone. Consider the discrete random variable*

$$\mathbf{d}_K := \text{dimension of the face of } K \text{ associated to } \Pi_K(\mathbf{g}),$$

where $\mathbf{g} \sim \textsc{Normal}\,(0, I_n)$. For each $d \in \{0, 1, \ldots, n\}$, one defines the d-th conic intrinsic volume of K as the number

$$\nu_d(K) := \mathbb{P}[\mathbf{d}_K = d].$$

In other words, $\nu_d(K)$ is equal to the probability that $\Pi_K(\mathbf{g})$ be in the relative interior of a d-dimensional face of K. Clearly, the $\nu_d(K)$'s form a probability distribution on $\{0, 1, \ldots, n\}$, i.e.,

$$\nu_d(K) \geq 0 \quad \text{and} \quad \sum_{d=0}^{n} \nu_d(K) = 1.$$

Conic intrinsic volumes capture fundamental information on the geometric structure of polyhedral cones. General material concerning the theory of conic intrinsic volumes can be found in [1, 24].

Proposition 2.33. *Let m be a positive integer and $K \in \mathcal{C}_n$ be a polyhedral cone. Then*

$$\mu_m(K) = \mathbb{E}[\ell_{\mathbf{d}_K, m}] = \sum_{d=0}^{n} \ell_{d,m}\, \nu_d(K). \tag{27}$$

Proof. This result follows by combining (24) and a formula by McCoy and Trapp [24, Lemma 8.1] that goes under the name of master Steiner formula for polyhedral cones. □

In view of the convention $\ell_{0,m} = 0$, the sum in (27) starts in fact with $d = 1$. Proposition 2.33 is stated in [24, Proposition 4.3] for the special case $m = 2$. Observe that (25) is a particular instance of (27). Indeed, it is not difficult to check that

$$\nu_d(\mathbb{R}^n_+) = \frac{1}{2^n}\, C^n_d \quad \text{for } d = 0, 1, \ldots, n.$$

Another example in the same vein is as follows.

Example 2.34. Let $K = \mathrm{cone}\{a, b\} \subseteq \mathbb{R}^n$ be a two-dimensional cone as in Example 2.22. Then

$$\nu_d(K) = \begin{cases} (1/2) - (\beta/2\pi) & \text{if } d = 0, \\ 1/2 & \text{if } d = 1, \\ \beta/2\pi & \text{if } d = 2, \\ 0 & \text{otherwise,} \end{cases}$$

where β is the angle between a and b. Hence,

$$\mu_m(K) = \frac{1}{2}\,\ell_{1,m} + \frac{\beta}{2\pi}\,\ell_{2,m}.$$

Table 2 collects all the information on size indices that has been presented up to now. Size indices for ellipsoidal cones will be introduced in the next section.

Σ	$\mathcal{K}$	strict	Lipschitz	modular
$\theta_{\max}$	$\mathcal{N}_n$	no	yes	no
hd	$\mathcal{N}_n$	no	yes	no
π_{met}	$\mathcal{N}_n$	no	yes	no
π_{frob}	$\mathcal{N}_n$	no	yes	no
btv	$\mathcal{N}_n$	no	yes	yes
	$\mathcal{P}_n$	yes	yes	yes
Vol	$\mathcal{P}_n$	yes	no	no
Ω	$\mathcal{N}_n$	yes	yes	no
μ_1	$\mathcal{C}_n$	yes	yes	yes
μ_2	$\mathcal{C}_n$	yes	yes	yes

Table 2: Examples of size indices and some of their properties.

3 Size indices for revolution cones and ellipsoidal cones

3.1 Size and semiaxes lengths in ellipsoidal cones

Ellipsoids are convex bodies with a lot of symmetry in them. An ellipsoid is fully determined by a collection of mutually orthogonal vectors, called the semiaxes of the ellipsoid, and a collection of positive reals, called the semiaxes lengths of the ellipsoid. The semiaxes take care of the orientation of the ellipsoid, but the "size" of the ellipsoid depends only on the semiaxes lengths. This elementary observation can be extended from ellipsoids to ellipsoidal cones.

In this work, ellipsoidal cones are understood in the sense of Stern and Wolkowicz [35]. Equivalently, an ellipsoidal cone in $\mathbb{R}^n$ is a set of the form

$$U(E_M) = \{Ux : x \in E_M\}, \tag{28}$$

where $U \in \mathbb{O}(n)$ and M is a symmetric positive definite matrix of order $n-1$. Recall that E_M refers to the upward ellipsoidal cone associated to M, see Example 2.8. To each ellipsoidal cone K in $\mathbb{R}^n$ one can associate a vector s_K, called the axial symmetry center of K. By definition, s_K is the unit vector in K that generates the symmetry axis of K. For instance, the axial symmetry center of E_M is equal to e_n. More generally, the axial symmetry center of (28) is equal to the last column of the matrix U. If one intersects an ellipsoidal cone K with the hyperplane

$$\mathcal{H}_K := \{x \in \mathbb{R}^n : \langle s_K, x\rangle = 1\},$$

then one gets an $(n-1)$ - dimensional ellipsoid

$$\mathrm{e}(K) := \{x \in K : \langle s_K, x\rangle = 1\}.$$

We insist upon the fact that $\mathrm{e}(K)$ is full dimensional as convex set in the hyperplane $\mathcal{H}_K$, but not as convex set in the underlying space $\mathbb{R}^n$. In what follows, the notation $\gamma(K)$ refers to the vector in $\mathrm{int}(\mathbb{R}^{n-1}_+)$ whose components are the semiaxes lengths of the ellipsoid $\mathrm{e}(K)$. To avoid ambiguities, we suppose that the semiaxes lengths are arranged in nondecreasing order:

$$\gamma_1(K) \leq \ldots \leq \gamma_{n-1}(K).$$

The next lemma, inspired from [33, Section 7], asserts that a size index on $\mathcal{E}_n$ has necessarily the form $\Sigma = f \circ \gamma$ for a suitable vector function f.

Lemma 3.1. *The following statements are equivalent:*

(a) Σ *is a size index on* $\mathcal{E}_n$*.*

(b) There exists a (necessarily unique) function $f : \mathrm{int}(\mathbb{R}^{n-1}_+) \to \mathbb{R}$ *that is continuous, permutation invariant, nonnegative, nondecreasing in each argument, and such that*

$$\Sigma(K) = f(\gamma(K)) \quad \text{for all } K \in \mathcal{E}_n. \tag{29}$$

Proof. Let Σ be a size index on $\mathcal{E}_n$. As mentioned before, any $K \in \mathcal{E}_n$ can be represented as in (28). The pair (U, M) representing K is not unique, but the eigenvalues of M are unique. Consider the spectral decomposition $M = V\Lambda V^T$, where Λ is a diagonal matrix containing the eigenvalues of M arranged in nonincreasing order

$$\lambda_1(M) \geq \ldots \geq \lambda_{n-1}(M),$$

and the columns of $V \in \mathbb{O}(n-1)$ are formed with corresponding eigenvectors. One can check that

$$K = \left(U \begin{bmatrix} V & 0 \\ 0 & 1 \end{bmatrix}\right)(E_\Lambda).$$

Note that K is the image of E_Λ under an orthogonal matrix. Since the functions γ_i's are invariant under orthogonal transformations, one has

$$\gamma_i(K) = \gamma_i(E_\Lambda) = \frac{1}{\sqrt{\lambda_i(M)}} \tag{30}$$

for all $i \in \{1, \ldots, n-1\}$. On the other hand, in view of the axiom A_4, the term $\Sigma(K) = \Sigma(E_\Lambda)$ is a function of the eigenvalues of M. But the eigenvalues of M can be expressed in terms of the semiaxes lengths of $e(K)$. This shows that Σ can be represented in the composite form (29) for some permutation invariant function f defined on $\mathrm{int}(\mathbb{R}^{n-1}_+)$. Consider the ellipsoidal cone

$$E(w) := \left\{(z,t) \in \mathbb{R}^n : \left[(z_1/w_1)^2 + \ldots + (z_{n-1}/w_{n-1})^2\right]^{1/2} \leq t\right\}$$

described by a vector $w \in \mathrm{int}(\mathbb{R}^{n-1}_+)$. Observe that

$$\gamma(E(w)) = \gamma(E(w^{\uparrow})) = w^{\uparrow},$$

where $w^{\uparrow}$ is a nondecreasing rearrangement of w. If one evaluates both sides of (29) at the cone $E(w)$, then one gets

$$f(w) = f_\Sigma(w) := \Sigma\left(E(w)\right).$$

Thanks to the axioms imposed on Σ, the function f_Σ is continuous, nonnegative, and nondecreasing in each argument. This completes the proof of the implication (a) $\Rightarrow$ (b). The proof of the reverse implication is easy and omitted. $\square$

If Σ is a size index on a set $\mathcal{K}$ that contains the ellipsoidal cones, then f_Σ is referred to as the elliptic core of Σ. For instance, by combining (14) and (30) one sees that the elliptic core of the size index $\mathrm{Vol} : \mathcal{P}_n \to \mathbb{R}$ is given by

$$f_{\mathrm{Vol}}(w) := \mathrm{Vol}(E(w)) = \frac{\mathrm{vol}_{n-1}(\mathbb{B}_{n-1})}{n}\, \Pi_{i=1}^{n-1} w_i\,.$$

The representation formula (29) provides a simple mechanism for constructing size indices for ellipsoidal cones.

Theorem 3.2. *The following functions ϱ_0, ϱ_1, and ϱ_2, are size indices on $\mathcal{E}_n$:*

$$\varrho_0(K) := \Pi_{i=1}^{n-1}\gamma_i(K), \tag{31}$$

$$\varrho_1(K) := \sum_{i=1}^{n-1}\gamma_i(K), \tag{32}$$

$$\varrho_2(K) := \sum_{i=1}^{n-1}[\gamma_i(K)]^2. \tag{33}$$

These three size indices are strict, but none of them is Lipschitz continuous.

Proof. For the first part of the theorem, one can apply Lemma 3.1. Note that $\varrho_j = f_j \circ \gamma$ with

$$f_0(w) := \Pi_{i=1}^{n-1} w_i, \quad f_1(w) := \sum_{i=1}^{n-1} w_i, \quad f_2(w) := \sum_{i=1}^{n-1} w_i^2.$$

The size index (31) is not entirely new. Up to multiplication by a positive factor, ϱ_0 is equal to the canonical volume function. Indeed,

$$\varrho_0(K) = \frac{n}{\mathrm{vol}_{n-1}(\mathbb{B}_{n-1})}\,\mathrm{Vol}(K) \quad \text{for all } K \in \mathcal{E}_n.$$

So, for the second part of the theorem one can concentrate on (32) and (33). Note that f_1 and f_2 are increasing in each argument. This proves that ϱ_1 and ϱ_2 are strict. Counter-examples for Lipschitz continuity are easy to construct. Consider for instance the ellipsoidal cones

$$P_k := \{(z,t) \in \mathbb{R}^n : (1/a_k)\|z\| \le t\},$$
$$Q_k := \{(z,t) \in \mathbb{R}^n : (1/b_k)\|z\| \le t\},$$

with

$$a_k := \tan\left(\frac{\pi}{2} - \frac{1}{k}\right), \quad b_k := \tan\left(\frac{\pi}{2} - \frac{2}{k}\right).$$

One can check that both quotients

$$\frac{\varrho_1(P_k) - \varrho_1(Q_k)}{\delta(P_k, Q_k)} = \frac{(n-1)(a_k - b_k)}{\sin(1/k)}$$

$$\frac{\varrho_2(P_k) - \varrho_2(Q_k)}{\delta(P_k, Q_k)} = \frac{(n-1)(a_k^2 - b_k^2)}{\sin(1/k)}$$

go to infinity as k goes to infinity. $\square$

Fisher [8, Section 3] and Calafiore [3, Section V.A] suggest the sum of squared semiaxes lengths as measure of size of an ellipsoid. This choice leads to consider the associated criterion (33) as measure of size of an ellipsoidal cone. The motivation behind the size index (32) is in the same vein. In Weber and Schröecker [38] one finds other ways of measuring the size of an ellipsoid in terms of its semiaxes lengths.

3.2 The circular core of a size index

For better understanding the nature of a size index Σ, it is helpful to compute $\Sigma(K)$ for an arbitrary proper revolution cone K. By proceeding in such a way one gets a preliminary insight on the behavior of Σ. Recall that a proper revolution cone in $\mathbb{R}^n$ is a set of the form

$$\operatorname{rev}(\theta, y) := \{x \in \mathbb{R}^n : (\cos\theta)\|x\| \leq \langle y, x\rangle\},$$

where $y \in \mathbb{R}^n$ is a unit vector that generates the revolution axis and $\theta \in]0, \pi/2[$ is a parameter called the half-aperture angle of the cone. In this section we assume that Σ is well defined for any proper revolution cone. Since a size index is invariant under orthogonal transformations, one can concentrate on proper revolutions cones

$$R(\theta) := \operatorname{rev}(\theta, e_n) = \left\{(z, t) \in \mathbb{R}^n : (\tan\theta)^{-1}\|z\| \leq t\right\}$$

with revolution axis generated by e_n.

Definition 3.3. *Let Σ be a size index on $\mathcal{K}$. The* circular core *of Σ is the function $h_\Sigma :]0, \pi/2[\to \mathbb{R}$ given by $h_\Sigma(\theta) := \Sigma(R(\theta))$.*

Evaluating h_Σ is often times easy, but not always. Some examples are displayed below.

Example 3.4. Among the easiest cases we mention

$$\theta_{\max}(R(\theta)) = 2\theta, \qquad \operatorname{hd}(R(\theta)) = \sin(\theta), \tag{34}$$

$$\pi_{\rm met}(R(\theta)) = \sin\theta, \quad \pi_{\rm frob}(R(\theta)) = \sin\theta. \tag{35}$$

The formulas in (34) are straightforward. The formulas in (35) can be found in [19, Proposition 5.1] and [15, Proposition 2.4], respectively. Note that two different size indices may have the same circular core.

Example 3.5. Other easy cases include

$$\operatorname{Vol}(R(\theta)) = \operatorname{Vol}(\mathbb{L}_n)(\tan\theta)^{n-1}, \tag{36}$$

$$\varrho_1(R(\theta)) = (n-1)\,\tan\theta, \tag{37}$$

$$\varrho_2(R(\theta)) = (n-1)\,(\tan\theta)^2, \tag{38}$$

where $\operatorname{Vol}(\mathbb{L}_n) = (1/n)\operatorname{vol}_{n-1}(\mathbb{B}_{n-1})$ is the canonical volume of the n-dimensional Lorentz cone

$$\mathbb{L}_n := \{(z, t) \in \mathbb{R}^n : \|z\| \leq t\}.$$

Formulas (36), (37), and (38), are due to the fact that $\gamma_i(R(\theta)) = \tan\theta$ for all $i \in \{1, \ldots, n-1\}$.

Example 3.6. As shown by Shannon [34], the ball-truncated volume of $R(\theta)$ is given by

$$\operatorname{btv}(R(\theta)) = \frac{\operatorname{vol}_n(\mathbb{B}_n)}{2\kappa_n}\, F_n(\theta),$$

where

$$F_n(\theta) := \int_0^\theta (\sin t)^{n-2} dt\,,$$

$$\kappa_n := \int_0^{\frac{\pi}{2}} (\sin t)^{n-2} dt = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma\left((n-1)/2\right)}{\Gamma\left(n/2\right)}.$$

The next proposition characterizes the circular core of μ_m. The particular case $m = 2$ is considered already in [1, 24].

Proposition 3.7. *For any positive integer m, one has*

$$\mu_m(R(\theta)) = \frac{\ell_{n,m}}{2\kappa_n} \int_0^\pi (\sin t)^{n-2} [\zeta(\theta, t)]^m dt, \tag{39}$$

where

$$\zeta(\theta, t) := \begin{cases} 1 & \text{if } 0 \leq t \leq \theta, \\ \cos(t - \theta) & \text{if } \theta \leq t \leq \theta + (\pi/2), \\ 0 & \text{if } \theta + (\pi/2) \leq t \leq \pi. \end{cases}$$

Proof. The expression (22) can be rewritten as

$$\mu_m(K) = \ell_{n,m} \, \mathbb{E}\left\{\|\Pi_K(\mathbf{u})\|^m\right\}, \tag{40}$$

with $\mathbf{u} \sim \textsc{Uniform}(\mathbb{S}_n)$. A direct computation shows that

$$\left\|\Pi_{R(\theta)}(\mathbf{u})\right\| = \zeta(\theta, \mathbf{s}),$$

where $\mathbf{s} = \arccos\langle \mathbf{u}, e_n \rangle$ is a random variable with density function $f_\mathbf{s} : \mathbb{R} \to \mathbb{R}$ given by

$$f_\mathbf{s}(t) = \begin{cases} \frac{1}{2\kappa_n} (\sin t)^{n-2} & \text{if } t \in [0, \pi], \\ 0 & \text{if } t \notin [0, \pi]. \end{cases}$$

Hence,

$$\mu_m(R(\theta)) \;=\; \ell_{n,m} \, \mathbb{E}\left\{[\zeta(\theta, \mathbf{s})]^m\right\} \;=\; \ell_{n,m} \int_{-\infty}^{\infty} [\zeta(\theta, t)]^m \, f_\mathbf{s}(t) \, dt.$$

This proves the announced formula (39). ☐

4 Polar-reversibility and polar-compatibility

In addition to stability under orthogonal transformations, in this section one assumes that $\mathcal{K}$ is also stable under polarity, i.e., $K \in \mathcal{K}$ implies $K^\circ \in \mathcal{K}$. Observe that $\mathcal{C}_n, \mathcal{N}_n, \mathcal{P}_n$, and $\mathcal{E}_n$, are all stable under polarity. The map $K \mapsto K^\circ$ is clearly decreasing with respect to set inclusion. Intuitively speaking, if the size of a closed convex cone K is large, then the size of the polar cone K° should be small, and viceversa. This idea is captured by the notion of polar-reversibility introduced in the definition below.

Definition 4.1. *A size index Σ on $\mathcal{K}$ is* polar-reversible *if its range*

$$\Sigma(\mathcal{K}) := \{\Sigma(K) : K \in \mathcal{K}\}$$

is not a singleton and there exists a decreasing function $g : \Sigma(\mathcal{K}) \to \Sigma(\mathcal{K})$ such that

$$\Sigma(K^\circ) = g(\Sigma(K)) \quad \textit{for all } K \in \mathcal{K}. \tag{41}$$

Polar-reversibility is a desirable property for a size index. The relation (41) implies that the size of K° is fully determined by the size of K, and viceversa. The proof of the next proposition is easy and therefore omitted.

Proposition 4.2. *Let Σ be a polar-reversible size index on $\mathcal{K}$. Then the function g in Definition 4.1 is unique. Furthermore, g is an involution on $\Sigma(\mathcal{K})$.*

There is an easy way of identifying the function g that serves as candidate for testing the polar-reversibility. Suppose that any proper revolution cone belongs to $\mathcal{K}$. By evaluating both sides of (41) at the proper revolution cone $R(\theta)$ one gets

$$h_\Sigma((\pi/2) - \theta) = g(h_\Sigma(\theta)).$$

If h_Σ is increasing, then $h_\Sigma :]0, \pi/2[\to h_\Sigma(\,]0, \pi/2[\,)$ is a bijection and the change of variables $r = h_\Sigma(\theta)$ leads to the representation formula

$$g(r) = h_\Sigma\left((\pi/2) - h_\Sigma^{-1}(r)\right)$$

for all $r \in h_\Sigma(\,]0, \pi/2[\,)$. In other words, g can be expressed in terms of the circular core of Σ. The proposition below displays some examples of polar-reversible size indices.

Proposition 4.3. *The following statements are true:*

(a) Every strict size index on $\mathcal{L}_n$ is polar-reversible.

(b) μ_2 is a polar-reversible size index on $\mathcal{C}_n$. In fact, for all $K \in \mathcal{C}_n$, one has

$$\mu_2(K^\circ) + \mu_2(K) = n. \tag{42}$$

(c) Vol *is polar-reversible as size index on $\mathcal{E}_n$. In fact, for all $K \in \mathcal{E}_n$, one has*

$$\mathrm{Vol}(K^\circ)\mathrm{Vol}(K) = [\mathrm{Vol}(\mathbb{L}_n)]^2. \tag{43}$$

Proof. Let Σ be a strict size index on $\mathcal{L}_n$. From Proposition 2.1 one sees that $\Sigma = \psi \circ \dim$ for some nonnegative increasing function

$$\psi : \{0, 1, \ldots, n\} \to \{\psi(0), \psi(1), \ldots, \psi(n)\}.$$

For all $K \in \mathcal{L}_n$, one has $\dim(K^\circ) = n - \dim K$, and therefore

$$\Sigma(K^\circ) \;=\; \psi(n - \psi^{-1}(\Sigma(K))).$$

This takes care of (a). Formulas (42) and (43) are stated in [1, Proposition 4.1] and [31, Corollary 4.7], respectively. □

Beware that Vol is not polar-reversible as size index on $\mathcal{P}_n$. As shown in [32, Theorem 5.3], the equality (43) does not hold beyond the class of ellipsoidal cones. Polar-reversibility is a very strong requirement in general. The great majority of size indices introduced in Section 2 are not polar-reversible. Instead of satisfying binding equalities like (42) or (43), the terms $\Sigma(K^\circ)$ and $\Sigma(K)$ are usually connected only by means of inequalities. The next proposition illustrates this point.

Proposition 4.4. *The following statements are true:*

(a) For all $K \in \mathcal{P}_n$, one has

$$\mathrm{Vol}(K^\circ)\mathrm{Vol}(K) \leq [\mathrm{Vol}(\mathbb{L}_n)]^2. \tag{44}$$

(b) For all $K \in \mathcal{N}_n$, one has

$$\theta_{\max}(K^\circ) + \theta_{\max}(K) \;\geq\; \pi, \tag{45}$$

$$[\mathrm{hd}(K^\circ)]^2 + [\mathrm{hd}(K)]^2 \;\geq\; 1, \tag{46}$$

$$[\pi_{\mathrm{met}}(K^\circ)]^2 \,+\, [\pi_{\mathrm{met}}(K)]^2 \;\leq\; 1, \tag{47}$$

$$[\pi_{\mathrm{frob}}(K^\circ)]^2 \,+\, [\pi_{\mathrm{frob}}(K)]^2 \;\leq\; 1. \tag{48}$$

$$[\Omega(K^\circ)]^2 \,+\, [\Omega(K)]^2 \;\leq\; \ell_n^2. \tag{49}$$

(c) For all $K \in \mathcal{C}_n$, one has

$$[\mu_1(K^\circ)]^2 + [\mu_1(K)]^2 \;\leq\; \ell_n^2. \tag{50}$$

(d) For all integer $m \geq 3$ and $K \in \mathcal{C}_n$, one has

$$\left[\frac{\mu_m(K^\circ)}{\ell_{n,m}}\right]^{\frac{2}{m}} + \left[\frac{\mu_m(K)}{\ell_{n,m}}\right]^{\frac{2}{m}} \;\geq\; 1. \tag{51}$$

Proof. Part (a) is proven in [32, Theorem 5.3]. The relation (44) can be viewed as a conic version of the celebrated Blaschke-Santaló inequality for convex bodies. We now prove (d). From (40) one sees that

$$\frac{\mu_m(K^\circ)}{\ell_{n,m}} \;=\; \mathbb{E}\left\{\|\Pi_{K^\circ}(\mathbf{u})\|^m\right\} \;=\; \mathbb{E}\Upsilon_m\left(\|\Pi_K(\mathbf{u})\|^m\right),$$

where $\mathbf{u} \sim \textsc{Uniform}\,(\mathbb{S}_n)$ and $\Upsilon_m(t) := \left(1 - t^{2/m}\right)^{m/2}$ for $t \in [0,1]$. If $m \geq 3$, then Υ_m is convex and Jensen's inequality yields

$$\mathbb{E}\Upsilon_m\left(\|\Pi_K(\mathbf{u})\|^m\right) \;\geq\; \Upsilon_m\left(\mathbb{E}\left\{\|\Pi_K(\mathbf{u})\|^m\right\}\right).$$

Hence,

$$\frac{\mu_m(K^\circ)}{\ell_{n,m}} \;\geq\; \Upsilon_m\left(\frac{\mu_m(K)}{\ell_{n,m}}\right) \;=\; \left[1 - \left(\frac{\mu_m(K)}{\ell_{n,m}}\right)^{2/m}\right]^{m/2}.$$

For proving (c) we apply the previous reasoning with $m = 1$. Note that $\Upsilon_1(t) = [1 - t^2]^{1/2}$ is concave and, therefore, Jensen's inequality must be written in the reverse order. We now take care of (b). The inequalities (45) and (46) are borrowed from [17, Theorem 7.2] and [17, Lemma 7.1], respectively. The inequality (47) is obtained by combining (8) and (45). Finally, (48) and (49) are derived from (47) and (50), respectively. To see this, note that $\pi_{\rm frob}(K) \leq \pi_{\rm met}(K)$ and $\Omega(K) \leq \mu_1(K)$ for all $K \in \mathcal{N}_n$. □

We mention in passing that (49) is sharper than the inequality of Chandradekaran et al. [4, Lemma 3.7]. The next proposition opens a parenthesis and states two curious characterizations of the expression $\mu_m(K^\circ)$.

Proposition 4.5. *The following statements are true:*

(a) If $K \in \mathcal{C}_n$ is a polyhedral cone and m is a positive integer, then

$$\mu_m(K^\circ) \;=\; \sum_{d=0}^{n} \ell_{n-d,m}\, \nu_d(K). \tag{52}$$

(b) If $K \in \mathcal{C}_n$ and $m = 2p$ for some positive integer p, then

$$\frac{\mu_m(K^\circ)}{\ell_{n,m}} = 1 + \sum_{k=1}^{p} (-1)^k\, C_k^p\, \frac{\mu_{2k}(K)}{\ell_{n,2k}}. \tag{53}$$

Proof. Let $K \in \mathcal{C}_n$ be a polyhedral cone. As mentioned in [1], the conic intrinsic volumes reverse under polarity, i.e., $\nu_d(K^\circ) = \nu_{n-d}(K)$ for $d \in \{0, 1, \ldots, n\}$. Hence, Proposition 2.33 yields

$$\mu_m(K^\circ) \;=\; \sum_{d=0}^{n} \ell_{d,m}\, \nu_d(K^\circ) \;=\; \sum_{d=0}^{n} \ell_{d,m}\, \nu_{n-d}(K).$$

A change of variables in the last sum leads to (52). Consider now an arbitrary $K \in \mathcal{C}_n$ and $m = 2p$. The equality (40) asserts that

$$\mu_m(K) = \ell_{n,m} \int_{\mathbb{S}_n} \|\Pi_K(u)\|^m \, d\sigma(u),$$

where the integral is with respect to the Haar measure on $\mathbb{S}_n$. If one changes K by K°, then one gets

$$\begin{aligned}
\frac{\mu_m(K^\circ)}{\ell_{n,m}} \;&=\; \int_{\mathbb{S}_n} \|\Pi_{K^\circ}(u)\|^{2p}\, d\sigma(u) \;=\; \int_{\mathbb{S}_n} \left(1 - \|\Pi_K(u)\|^2\right)^p d\sigma(u) \\
&=\; \int_{\mathbb{S}_n} \sum_{k=0}^{p} (-1)^k\, C_k^p\, \|\Pi_K(u)\|^{2k}\, d\sigma(u) \\
&=\; \sum_{k=0}^{p} (-1)^k\, C_k^p \int_{\mathbb{S}_n} \|\Pi_K(u)\|^{2k}\, d\sigma(u).
\end{aligned}$$

This leads to the advertised formula (53). □

We have not found in the literature any relationship between $\mathrm{btv}(K^\circ)$ and $\mathrm{btv}(K)$. To the best of our knowledge, the next result is new.

Proposition 4.6. *Let the dimension n be even. Then, for all $K \in \mathcal{N}_n$, one has*

$$\mathrm{btv}(K^\circ) + \mathrm{btv}(K) \;\leq\; \frac{\mathrm{vol}_n(\mathbb{B}_n)}{2}. \tag{54}$$

Proof. The proof of (54) is based on elegant arguments taken from the theory of conic intrinsic volumes. Consider first the case in which $K \in \mathcal{N}_n$ is a polyhedral cone. One may suppose that K is not a linear subspace, otherwise (54) holds trivially. For any dimension n, be even or not, one has

$$\begin{aligned} \nu_0(K) &= \mathbb{P}[\mathbf{g} \in K^\circ] = \frac{\mathrm{btv}(K^\circ)}{\mathrm{vol}_n(\mathbb{B}_n)}, \\ \nu_n(K) &= \mathbb{P}[\mathbf{g} \in K] \;\;= \frac{\mathrm{btv}(K)}{\mathrm{vol}_n(\mathbb{B}_n)}, \end{aligned}$$

where $\mathbf{g} \sim \textsc{Normal}\,(0, I_n)$. A remarkable feature of the conic intrinsic volumes of a polyhedral cone K, different from a linear subspace, is satisfying the Gauss-Bonnet identities

$$\sum_{\substack{d=0 \\ d \text{ even}}}^{n} \nu_d(K) = \frac{1}{2} \quad \text{and} \quad \sum_{\substack{d=1 \\ d \text{ odd}}}^{n} \nu_d(K) = \frac{1}{2}\,.$$

Now, if $n \geq 4$ is even, then the first Gauss-Bonnet identity becomes

$$\frac{\mathrm{btv}(K^\circ)}{\mathrm{vol}_n(\mathbb{B}_n)} + \frac{\mathrm{btv}(K)}{\mathrm{vol}_n(\mathbb{B}_n)} = \frac{1}{2} - q(K),$$

where

$$q(K) := \sum_{\substack{d=2 \\ d \text{ even}}}^{n-2} \nu_d(K)$$

is a nonnegative real. This proves (54) under the assumption of polyhedrality. The non-polyhedral case follows then by a density argument. Indeed, as a consequence of [29, Theorem 4.4], any $K \in \mathcal{N}_n$ can be written as limit of a sequence $\{K_k\}_{k\in\mathbb{N}} \subseteq \mathcal{N}_n$ of polyhedral cones. In such a case, $\{K_k^\circ\}_{k\in\mathbb{N}}$ converges to K° and a passage to the limit in

$$\mathrm{btv}(K_k^\circ) + \mathrm{btv}(K_k) \;\leq\; \frac{\mathrm{vol}_n(\mathbb{B}_n)}{2}$$

leads to (54). □

Since polar-reversibility is a very strong hypothesis for a size index, we suggest to have a look at polar-compatibility, which is a less demanding requirement. Roughly speaking, a size index $\Sigma : \mathcal{K} \to \mathbb{R}$ is polar-compatible if, for each each $K \in \mathcal{K}$, the following two conditions are in force:

$$\begin{cases} \Sigma(K) \text{ and } \Sigma(K^\circ) \text{ are not both near } \sup_{\mathcal{K}} \Sigma, \\ \Sigma(K) \text{ and } \Sigma(K^\circ) \text{ are not both near } \inf_{\mathcal{K}} \Sigma. \end{cases} \tag{55}$$

Several size indices mentioned in Section 2 satisfy either the first or the second condition in (55), but not both conditions at the same time. This observation leads us to split polar-compatibility into two sub-concepts: upper polar-compatibility and lower polar-compatibility.

Definition 4.7. *Let Σ be a size index on $\mathcal{K}$. One says that Σ is* upper polar-compatible (UPC) *if there is no sequence $\{P_k\}_{k\in\mathbb{N}}$ in $\mathcal{K}$ such that*

$$\lim_{k\to\infty} \Sigma(P_k) = \lim_{k\to\infty} \Sigma(P_k^\circ) = \sup_{\mathcal{K}}\Sigma, \tag{56}$$

and lower polar-compatible (LPC) *if there is no sequence $\{Q_k\}_{k\in\mathbb{N}}$ in $\mathcal{K}$ such that*

$$\lim_{k\to\infty} \Sigma(Q_k) = \lim_{k\to\infty} \Sigma(Q_k^\circ) = \inf_{\mathcal{K}}\Sigma.$$

Beware that $\sup_{\mathcal{K}}\Sigma$ and $\inf_{\mathcal{K}}\Sigma$ are not necessarily attained. Polar-compatibility is perhaps better understood when $\mathcal{K}$ is closed, because in such a case $\Sigma : \mathcal{K} \to \mathbb{R}$ attains its extremal values and there is no need of using optimizing sequences.

Proposition 4.8. *Let $\mathcal{K}$ be closed in the metric space $(\mathcal{C}_n, \delta)$. One has:*

(a) Σ *is* UPC *if and only if there is no $P \in \mathcal{K}$ such that*

$$\Sigma(P) = \Sigma(P^\circ) = \max_{\mathcal{K}}\Sigma. \tag{57}$$

(b) Σ *is* LPC *if and only if there is no $Q \in \mathcal{K}$ such that*

$$\Sigma(Q) = \Sigma(Q^\circ) = \min_{\mathcal{K}}\Sigma.$$

(c) If Σ is polar-reversible, then Σ is both UPC *and* LPC*.*

Proof. $\mathcal{K}$ is compact because it is a closed set in a compact metric space. Suppose that (56) holds for some sequence $\{P_k\}_{k\in\mathbb{N}}$ in $\mathcal{K}$. Taking a subsequence if necessary, one may assume that $\{P_k\}_{k\in\mathbb{N}}$ converges to some $P \in \mathcal{K}$. In such a case, $\{P_k^\circ\}_{k\in\mathbb{N}}$ converges P° and (56) leads to (57). This proves the "if part" of (a). The "only if " part is obvious. The proof of (b) is similar to that of (a). We now take care of (c). Since Σ is nonconstant, $\min_{\mathcal{K}}\Sigma$ is smaller than $\max_{\mathcal{K}}\Sigma$. Let g be as in Definition 4.1. Then

$$\max_{K\in\mathcal{K}} \Sigma(K) = \max_{K\in\mathcal{K}} g(\Sigma(K^\circ)) = \max_{r\in\Sigma(\mathcal{K})} g(r) = g\left(\min_{\mathcal{K}}\Sigma\right).$$

It follows that $\max_{\mathcal{K}}\Sigma > g(\max_{\mathcal{K}}\Sigma)$. This inequality implies that Σ is UPC. Indeed, if (57) holds for some $P \in \mathcal{K}$, then

$$\Sigma(P) = \max_{\mathcal{K}}\Sigma \ > \ g(\max_{\mathcal{K}}\Sigma) = g(\Sigma(P^\circ)) = \Sigma(P),$$

a clear contradiction. In a similar way one can prove the inequality $\min_{\mathcal{K}}\Sigma < g(\min_{\mathcal{K}}\Sigma)$ and the lower upper-compatibility of Σ. □

Proposition 4.9. *Let Σ be a size index on $\mathcal{N}_n$.*

(a) Σ *is* UPC *if no cantilever in $\mathbb{R}^n$ achieve $\max_{\mathcal{N}_n}\Sigma$.*

(b) Σ *is* LPC *if no two-dimensional pointed cone in $\mathbb{R}^n$ achieve $\min_{\mathcal{N}_n}\Sigma$.*

Proof. In order to prove (a), suppose that $\Sigma(P) = \Sigma(P^\circ) = \max_{\mathcal{N}_n}\Sigma$ for some $P \in \mathcal{N}_n$. Changing the roles of P and P° if necessary, one may assume that P is not a half-space. In such a case, there exists a cantilever K in $\mathbb{R}^n$ such that $P \subseteq K$. Since Σ is nondecreasing with respect to set inclusion, one has $\Sigma(K) \geq \Sigma(P) = \max_{\mathcal{N}_n}\Sigma$. Hence, the cantilever K is a maximizer of the function $\Sigma : \mathcal{N}_n \to \mathbb{R}$. The proof of (b) is similar. □

Concerning the polar-compatibility of the size indices discussed in Section 2, the situation known to us is as summarized in the next theorem, see also Table 3.

Theorem 4.10. *The following statements are true:*

(a) $\theta_{\max} : \mathcal{N}_n \to \mathbb{R}$ *and* $\mathrm{hd} : \mathcal{N}_n \to \mathbb{R}$ *are* LPC*, but not* UPC.

(b) $\pi_{\mathrm{met}} : \mathcal{N}_n \to \mathbb{R}$ *and* $\pi_{\mathrm{frob}} : \mathcal{N}_n \to \mathbb{R}$ *are* UPC*, but not* LPC.

(c) $\mathrm{btv} : \mathcal{N}_n \to \mathbb{R}$ *is* UPC*, but not* LPC.

(d) $\Omega : \mathcal{N}_n \to \mathbb{R}$ *is both* UPC *and* LPC*.*

(e) For all positive integer m, $\mu_m : \mathcal{C}_n \to \mathbb{R}$ is both UPC *and* LPC*.*

(f) $\mathrm{Vol} : \mathcal{P}_n \to \mathbb{R}$ *is* UPC*.*

Proof. The relation (45) shows that $\theta_{\max}(K)$ and $\theta_{\max}(K^\circ)$ cannot both be equal to $\min_{\mathcal{N}_n} \theta_{\max} = 0$. However, it is possible to have both $\theta_{\max}(K)$ and $\theta_{\max}(K^\circ)$ equal to $\max_{\mathcal{N}_n} \theta_{\max} = \pi$. To see this, just take $K \in \mathcal{N}_n$ as a linear subspace or, more generally, as a closed convex cone that is neither pointed nor solid. The case of the size index hd is treated in a similar way, but now one uses the relation (46). For proving the upper polar-compatibility of $\pi_{\rm met}$ and $\pi_{\rm frob}$, one uses the relations (47) and (48), respectively. The upper polar-compatibility of btv is obtained by combining Proposition 4.9(a), Example 2.11, and

$$\max_{\mathcal{N}_n} \mathrm{btv} = \mathrm{vol}_n(\mathbb{B}_n)/2.$$

Alternatively, one can use the relation (54) in case the dimension n is even. The lower polar-compatibility of Ω is because a two-dimensional pointed cone in $\mathbb{R}^n$ does not achieve $\min_{\mathcal{N}_n} \Omega = 0$, see Example 2.22. The upper polar-compatibility of Ω is because a cantilever in $\mathbb{R}^n$ does not achieve

$$\max_{\mathcal{N}_n} \Omega \;=\; \Omega(\text{half-space in } \mathbb{R}^n) \;=\; (\ell_{n-1} + \ell_n)/2.$$

Recall that Ω is a strict size index and a cantilever is strictly contained in a half-space. The lower polar-compatibility of μ_m is because this size index admits the zero cone $\{0\}$ as unique minimizer, cf. Example 2.27. The upper polar-compatibility of μ_m is because this size index admits the whole space $\mathbb{R}^n$ as unique maximizer. Finally, the relation (44) shows that Vol is UPC. □

Remark 4.11. We suspect that Vol : $\mathcal{P}_n \to \mathbb{R}$ is also LPC, but we do not have a formal proof of this fact. At present time, we are only able to prove that there is a positive constant r_n such that

$$\mathrm{Vol}(K^\circ)\mathrm{Vol}(K) \geq r_n \quad \text{for all } K \in \mathcal{A}_n,$$

where $\mathcal{A}_n$ denotes the set of proper cones in $\mathbb{R}^n$ that are axially symmetric. Thus, we know that Vol is LPC as size index on $\mathcal{A}_n$.

Σ	$\mathcal{K}$	extremal values		polar-compatibility	
		$\inf_{\mathcal{K}} \Sigma$	$\sup_{\mathcal{K}} \Sigma$	lower	upper
$\theta_{\max}$	$\mathcal{N}_n$	0	π	yes	no
hd	$\mathcal{N}_n$	0	1	yes	no
$\pi_{\rm met}$	$\mathcal{N}_n$	0	1	no	yes
$\pi_{\rm frob}$	$\mathcal{N}_n$	0	1	no	yes
btv	$\mathcal{N}_n$	0	$(1/2)\,\mathrm{vol}_n(\mathbb{B}_n)$	no	yes
Vol	$\mathcal{P}_n$	0	∞	?	yes
	$\mathcal{A}_n$	0	∞	yes	yes
Ω	$\mathcal{N}_n$	0	$(\ell_{n-1} + \ell_n)/2$	yes	yes
μ_m	$\mathcal{C}_n$	0	$\ell_{n,m}$	yes	yes

Table 3: Polar-compatibility properties.

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