$\varepsilon-$Strictly Subdifferential of Set-valued Map and Its Application *†

Li Yu

Institute of Mathematics and Computer of Science, Yichun University, Yichun 336000, China

Abstract In this paper, firstly, the concept of $\varepsilon-$strictly efficient subdifferential for set-valued map is introduced in Hausdorff locally convex topological vector spaces. Secondly, a characterization of this subdifferential by scalarization and the generalized $\varepsilon-$ Moreau-Rockafellar type theorem for set-valued maps are established. Finally, the necessary optimality condition of the constraint set-valued optimization problem for $\varepsilon-$ strictly efficient solutions is obtained in terms of Lagrange multiplier by using the concept of $\varepsilon-$ strictly efficient subdifferential for set-valued map.

Key words set-valued map; $\varepsilon-$strictly efficiency; subdifferential; $\varepsilon-$ Moreau-Rockafellar type theorem Lagrange multiplier;

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1 Introduction

In recent years, some authors have been interested in studying subdifferential to characterize the optimality conditions for set-valued optimization problem. Tanino[1] introduced the notion of weak subdifferentials for set-valued maps. Based on Tanino's work, some authors[2-5] introduced some new concepts of generalized subdifferential such as cone subdifferential and cone weak subdifferential of the point sets mapping, weak Benson proper efficient subgradient, cone-Henig efficient subdifferential and globally proper efficient subdifferential. Under the concepts of the above generalized subdifferential, some optimality conditions were established.

It is well known that mathematical models are an approximation of the practical situations modelled, the use of an algorithm in a computer to solve an vector optimization problem usually leads to an approximation of the solution, therefore, approximate solutions

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†E-mail address: yulilyy@163.com

play an important role in vector optimization theory. Recently, different types of approximate solutions have been introduced and many interesting results have been obtained. Valyi[6] introduced different types of approximate efficient solutions and derived Hurwitz-type saddle theorems. Rong and Wu[7-8] introduced $\varepsilon-$ weakly efficient solutions of vector optimization problems with set-valued maps and established scalarization theorem, $\varepsilon-$ saddle point theorems and $\varepsilon-$ duality theorem for $\varepsilon-$ weakly efficient solutions of vector optimizations with set-valued maps. Li and Xu[9] introduced the concept of $\varepsilon-$ strictly minimal efficient solution of vector optimization problems with set-valued maps. Wang[10] defined $\varepsilon-$ strongly efficient point of set and obtained the optimality conditions of $\varepsilon-$ strongly efficient solutions of vector optimization with generalized equality constraints and inequality constraints. Xu and Han[11] studied some properties of $\varepsilon-$ strictly minimal efficient points. Since subdifferential is an important role to characterize optimality conditions for vector optimality problem, it is necessary to generalized the concepts of subdifferential and apply them to study the $\varepsilon-$ solutions for set-valued optimality problem. Taa[12] studied $\varepsilon-$ subdifferential for set-valued maps and $\varepsilon-$ weak Pareto optimality for multiobjective optimization.

In this paper, our purpose is to introduce $\varepsilon-$ strictly efficient subdifferential for set-valued maps and derive an $\varepsilon-$ optimality condition for vector optimization problems with set-valued maps by using the concept of $\varepsilon-$ strictly efficient subdifferential. This paper is divided into three sections. In section 2, we give some definitions and results. In section 3, we define $\varepsilon-$ strictly efficient subdifferential and present a characterization of this subdifferential by scalarization and the generalized $\varepsilon-$ Moreau-Rockafellar type theorem for set-valued maps. In section 4, we establish the necessary optimality condition of the constraint set-valued optimization problem for $\varepsilon-$ strictly efficient solutions in terms of Lagrange multiplier by using the concept of $\varepsilon-$ strictly efficient subdifferential for set-valued map.

2 Preliminaries

Throughout this paper, let X be a topological linear space, Y and Z be Hausdorff locally convex topological vector spaces, Y^* and Z^* be the topological dual spaces of Y and Z, respectively. For a set $M \subset Y$, we write

$$\text{cone } M = \{\lambda m : \lambda \geq 0, m \in M\}.$$

The closure and interior of set M are denoted by cl M and int M, respectively. Let C be a pointed convex cone of Y, the positive dual cone C^* of C is denoted by $C^* = \{y^* \in Y^* : \langle y^*, y \rangle \geq 0, \forall y \in C\}$. A convex subset B of a cone C is a base of C if $0 \notin \text{cl}B$ and $C = \text{cone}B$. Write $B^{st} = \{f \in Y^* : \exists t > 0 \text{ such that } f(b) \geq t, \forall b \in B\}$.

Definition 2.1[12] Let M be a nonempty subset of Y, $y \in M$ and $\varepsilon \in C$. y is said to be an $\varepsilon-$ weak Pareto minimal point of M with respect to C, written as $y \in \varepsilon - W.\text{Min}(M, C)$, if

$$(M - y + \varepsilon) \cap (-\text{int } C) = \emptyset.$$

Definition 2.2[9] Let M be a nonempty subset of Y, B be a base of C, $\varepsilon \in C$. $y \in M$ is called an $\varepsilon-$ strictly minimal efficient point of M with respect to B ,written as $y \in \varepsilon - F\min(M, B)$, if there is a neighborhood U of 0 such that

$$\text{cl } [\text{cone}(M + \varepsilon - y)] \cap (U - B) = \emptyset. \tag{1}$$

Remark 2.1[9] The equality (1) is equivalent to

$$\text{cone}(M + \varepsilon - y) \cap (U - B) = \emptyset.$$

Let $F : X \to 2^Y$ be a set-valued map. The set

$$\text{dom } F := \{x \in X : F(x) \neq \emptyset\}$$

is called the domain of F. The set

$$\text{graph } F := \{(x, y) \in X \times Y : x \in \text{dom}F, \ y \in F(x)\}$$

is called the graph of F. The set

$$\text{epi } F := \{(x, y) \in X \times Y : x \in \text{dom}F, \ y \in F(x) + C\}$$

is called the epigraph of F.

Definition 2.3[13] Let $F : X \longrightarrow 2^Y$ be a set-valued map. F is said to be $C-$ convex on X, if for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$,

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C.$$

Lemma 2.1[9] Let B be a base of C, $\varepsilon \in C$, $\emptyset \neq M \subset Y$. Then

$$\varepsilon - F\min(M, B) = \varepsilon - F\min(M + C, B).$$

Lemma 2.2[4] Let $F : X \to 2^Y$ be a set-valued map and $x_0 \in \text{dom } F$, then the following three conditions are satisfied as long as one, we have $\text{int}(\text{epi } F) \neq \emptyset$.

(I) there exists a $\hat{y} \in F(x_0)$ such that F is lower semicontinuous at $(x_0, \hat{y})$.

(II) there exists a $a \in Y$ such that $F(X) \subset a - C$.

(III) there exists a map $f : X \to Y$ such that $f(x) \in F(x)(\forall x \in X)$, and f is continuous at a neighborhood $U(x_0)$ of x_0.

3 $\varepsilon-$ Subdifferentials of set-valued maps

In this section, we introduce the concept of the $\varepsilon-$strictly efficient subdifferential for set-valued map. In the next, by using the Lemma 2.2, we propose the existence theorem for $\varepsilon-$strictly efficient subdifferential. Finally, we establish a characterization of this subdifferential by scalarization, and we obtain the generalized $\varepsilon-$ Moreau-Rockafellar type theorem of $\varepsilon-$strictly efficiency for set-valued maps.

Definition 3.1[**12**] Let $F: X \to 2^Y$ be a set-valued map, $\varepsilon \in C,\ \bar{x} \in X$ and $\bar{y} \in F(\bar{x})$. A linear operator $T \in L(X, Y)$ is said to be an $\varepsilon-$ weak subgradient for $\bar{y}$ of F at $\bar{x}$ if

$$\bar{y} - T(\bar{x}) \in \varepsilon - W.\text{Min}[\cup_{x \in X}(F(x) - T(x)), C].$$

The set of all $\varepsilon-$ weak subgradients for $\bar{y}$ of F at $\bar{x}$ is called the $\varepsilon-$ weak subdifferential for $\bar{y}$ of F at $\bar{x}$ and is denoted by $\partial^C_{\varepsilon-W}F(\bar{x}, \bar{y})$. Moreover, F is said to be $\varepsilon-$ weakly subdifferential at $\bar{x}$ if for all $\bar{y} \in F(\bar{x}),\ \partial^C_{\varepsilon-W}F(\bar{x}, \bar{y}) \neq \emptyset$.

Definition 3.2 Let $F: X \to 2^Y$ be a set-valued map, $\varepsilon \in C,\ \bar{x} \in X$ and $\bar{y} \in F(\bar{x})$. A linear operator $T \in L(X, Y)$ is said to be an $\varepsilon-$strictly efficient subgradient for $\bar{y}$ of F at $\bar{x}$ if

$$\bar{y} - T(\bar{x}) \in \varepsilon - F\text{min}[\cup_{x \in X}(F(x) - T(x)), B].$$

The set of all $\varepsilon-$ strictly efficient subgradients for $\bar{y}$ of F at $\bar{x}$ is called the $\varepsilon-$ strictly efficient subdifferential for $\bar{y}$ of F at $\bar{x}$ and is denoted by $\partial^B_{\varepsilon-F}F(\bar{x}, \bar{y})$. Moreover, F is said to be $\varepsilon-$strictly efficient subdifferential at $\bar{x}$ if for all $\bar{y} \in F(\bar{x}),\ \partial^B_{\varepsilon-F}F(\bar{x}, \bar{y}) \neq \emptyset$.

Theorem 3.1 Let $F: X \to 2^Y$ be $C-$ convex on X, $(x_0, y_0) \in$ graph F, $x_0 \in$ int X, $y_0 \in \varepsilon - F\text{min}(F(x_0), B)$, then so long as lemma 2.2 condition satisfies one of them, we have $\partial^B_{\varepsilon-F}F(x_0, y_0) \neq \emptyset$.

Proof Since $y_0 \in \varepsilon - F\text{min}(F(x_0), B)$, there exists an open convex neighborhood U of 0 such that

$$\text{cone}(F(x_0) + \varepsilon - y_0) \cap (U - B) = \emptyset. \tag{2}$$

Define

$$A = \{(x, y) \in X \times Y : y \in F(x) + \text{cone}(B - U)\}.$$

Then, since F is $C-$ convex and $C \subset$ cone $(B - U)$, A is a convex set. It follows from Lemma 2.2 and epi $F \subset A$ that int $A \neq \emptyset$. In what follows, we prove that $(x_0, y_0 - \varepsilon) \notin \text{int}A$. Indeed, if there exists $U_1 \in N(0_Y)$ such that $(x_0, y_0 - \varepsilon + U_1) \subset A$, since cone$(B - U)$ is a cone, there exists a $-d \in \text{cone}(B - U)\backslash\{0\}$ such that $d \in U_1$, then

$$y_0 - \varepsilon + d \in F(x_0) + \text{cone}(B - U).$$

therefore, there exist $y_1 \in F(x_0), d_1 \in \text{cone}(B - U)$ such that

$$y_0 - \varepsilon + d = y_1 + d_1.$$

This implies $y_1 - y_0 + \varepsilon = d - d_1 \in -\text{cone}(B-U)\backslash\{0\} \subset \text{cone}(U-B)\backslash\{0\}$, which is a contradiction to (2). Thus, $(x_0, y_0 - \varepsilon) \notin \text{int}A$. By the separation theorem for convex sets, there exists $(f,g) \in X^* \times Y^* \backslash (0_{X^*}, 0_{Y^*})$ such that

$$f(x) + g(y) \geq f(x_0) + g(y_0 - \varepsilon), \quad \forall x \in X, y \in F(x) + \text{cone}(B-U). \tag{3}$$

which is equivalent to

$$f(x) + g(y+\varepsilon) \geq f(x_0) + g(y_0), \quad \forall x \in X, y \in F(x) + \text{cone}(B-U). \tag{4}$$

From $y+\varepsilon \in F(x)+\text{cone}(B-U)+\varepsilon \subset F(x)+\text{cone}(B-U)+\text{cone}(B-U) \subset F(x)+\text{cone}(B-U)$, we can let $\hat{y} = y + \varepsilon$, (4) yields

$$f(x) + g(\hat{y}) \geq f(x_0) + g(y_0), \quad \forall x \in X, \hat{y} \in F(x) + \text{cone}(B-U). \tag{5}$$

We may claim that $g \neq 0_{Y^*}$. Otherwise, it follows that

$$f(x - x_0) \geq 0, \quad \forall x \in X. \tag{6}$$

For a positive real number $\lambda_0 > 0$, let $v \in X$ be an arbitrary vector. Taking $x = \pm\lambda_0 v + x_0$ in (6), we get $f(\pm\lambda_0 v) \geq 0$. This shows that $f = 0$, a contradiction to the fact $(f,g) \neq (0_{X^*}, 0_{Y^*})$. In addition, taking $x = x_0, \hat{y} = y_0 + q$ in (5), we obtain that

$$g(q) \geq 0, \quad \forall q \in \text{cone } (B-U).$$

Since $g \neq 0_{Y^*}$ and U is an open convex neighborhood of 0, there exists a $u_0 \in U$ such that

$$g(u_0) = t > 0.$$

consequently

$$g(b) \geq g(u_0) = t, \quad \forall b \in B.$$

we get

$$g \in B^{st}.$$

Since $g \in B^{st}$, we can choose a $b_0 \in B$ such that $y_0 = b_0/g(b_0)$, then

$$g(y_0) = 1. \tag{7}$$

Define the linear operator $T : X \to Y$ by

$$T(x) = -f(x)y_0. \tag{8}$$

Set

$$\tilde{U} = \{y \in Y | g(y) < \frac{t}{2}\}.$$

In what follows, we prove

$$\text{cone}(\cup_{x\in X}(F(x)-T(x)+\varepsilon-y_0+T(x_0)+C))\cap(\tilde{U}-B)=\emptyset.$$

Indeed, if there exists

$$\tilde{y}\in \text{cone } (\cup_{x\in X}(F(x)-T(x)+\varepsilon-y_0+T(x_0)+C))\cap(\tilde{U}-B),$$

then, $\tilde{y}\neq 0$, and

$$\tilde{y}=\lambda(y_1-T(x_1)+\varepsilon-y_0+T(x_0)+c_1)=u_1-b_1, \tag{9}$$

where $\lambda>0,\ x_1\in X,\ y_1\in F(x_1),\ u_1\in\tilde{U},\ b_1\in B\ ,c_1\in C$. Since $\tilde{U}$ is a neighborhood of 0 and

$$g(\tilde{u}-b)<\frac{t}{2}-t<0,\quad \forall \tilde{u}\in\tilde{U},\ b\in B. \tag{10}$$

Combing with (9) and (10), we obtain

$$\lambda g(y_1-T(x_1)+\varepsilon-y_0+T(x_0)+c_1)<0. \tag{11}$$

It follows from $g\in B^{st}$ and $\varepsilon\in C$ that $g(\lambda\varepsilon)\geq 0$. By (7), (8) and (11), we have

$$\lambda(f(x_1)+g(y_1+c_1)-f(x_0)-g(y_0))<0.$$

which together with $\lambda>0$ leads to

$$f(x_1)+g(y_1+c_1)<f(x_0)+g(y_0),\quad (x_1,y_1+c_1)\in \text{epi}F\subset A.$$

a contradiction to (5). Thus, it follows from Definition 3.2 and Lemma 2.1 that $T\in\partial^B_{\varepsilon-F}F(x_0,y_0)$. The proof is completed.

Now, we start with our first main result in this section, a characterization of $\varepsilon-$ strictly efficient subdifferential by scalarization.

Theorem 3.2 Let $\varepsilon\in C,\ F:X\to 2^Y$ be $C-$ convex on X, $x_0\in X,\ y_0\in F(x_0)$. Then $T\in\partial^B_{\varepsilon-F}F(x_0,y_0)$ if and only if there exists $y^*\in B^{st}$ such that

$$\langle y^*,y-y_0+\varepsilon\rangle\geq\langle y^*,T(x-x_0)\rangle,\quad \forall x\in X, y\in F(x). \tag{12}$$

Proof Necessity. Since $T\in\partial^B_{\varepsilon-F}F(x_0,y_0)$, by the Definition 3.2, there exists an open convex neighborhood U of 0 such that

$$\text{cone}(\cup_{x\in X}(F(x)-T(x)+\varepsilon-y_0+T(x_0)))\cap(U-B)=\emptyset.$$

Since $U-B$ is an open convex set, we get

$$\text{cl cone}(\cup_{x\in X}(F(x)-T(x)+\varepsilon-y_0+T(x_0)))\cap(U-B)=\emptyset.$$

Since $\text{cone}(\cup_{x\in X}(F(x)-T(x)+\varepsilon-y_0+T(x_0)))$ is a convex set, $\text{cl cone}(\cup_{x\in X}(F(x)-T(x)+\varepsilon-y_0+T(x_0)))$ is a convex set. By the separation theorem for convex sets, there exists $y^*\in Y^*\backslash\{0_{Y^*}\}$ such that

$$\langle y^*,y\rangle \geq \langle y^*,y^{'}\rangle,\quad \forall y\in \text{cl cone}(\cup_{x\in X}(F(x)-T(x)+\varepsilon-y_0+T(x_0))),\ y^{'}\in U-B. \quad (13)$$

Since $\text{cl cone}(\cup_{x\in X}(F(x)-T(x)+\varepsilon-y_0+T(x_0)))$ is a cone and on which y^* has low bound, we have

$$\langle y^*,y\rangle \geq 0,\quad \forall y\in \text{cl cone}(\cup_{x\in X}(F(x)-T(x)+\varepsilon-y_0+T(x_0))). \quad (14)$$

and

$$\langle y^*,B\rangle \geq \langle y^*,U\rangle. \quad (15)$$

Since $y^*\neq 0$, there exists a $\bar{u}\in U$ such that

$$\langle y^*,\bar{u}\rangle = t>0.$$

Consequently

$$\langle y^*,b\rangle \geq \langle y^*,\bar{u}\rangle = t,\quad \forall b\in B$$

Thus, $y^*\in B^{st}$, and by (14), (12) holds.

Sufficiency. Suppose there exists $y^*\in B^{st}$ such that (12) holds. Therefore, there exists $t>0$ such that

$$\langle y^*,b\rangle \geq t,\quad \forall b\in B.$$

Set

$$U=\{y\in Y|\langle y^*,y\rangle < \frac{t}{2}\}.$$

In what follows, we prove

$$\text{cone}(\cup_{x\in X}(F(x)-T(x)+\varepsilon-y_0+T(x_0)))\cap(U-B)=\emptyset.$$

Indeed, if there exists

$$\hat{y}\in \text{cone}(\cup_{x\in X}(F(x)-T(x)+\varepsilon-y_0+T(x_0)))\cap(U-B)$$

then $\hat{y}\neq 0$, and

$$\hat{y}=\tilde{\lambda}(\tilde{y}-T(\tilde{x})+\varepsilon-y_0+T(x_0))=\tilde{u}-\tilde{b}, \quad (16)$$

where $\tilde{\lambda}>0,\ \tilde{x}\in X,\ \tilde{y}\in F(\tilde{x})$ $,\tilde{u}\in U,\ \tilde{b}\in B$. Since U is an open convex neighborhood of 0 and

$$\langle y^*,u-b\rangle < 0,\quad \forall u\in U,\ b\in B. \quad (17)$$

Combing with (16) and (17), we have

$$\langle y^*,\tilde{\lambda}(\tilde{y}-T(\tilde{x})-y_0+\varepsilon+T(x_0))\rangle < 0.$$

which together with $\tilde{\lambda} > 0$ that

$$\langle y^*, \tilde{y} - T(\tilde{x}) - y_0 + \varepsilon + T(x_0)\rangle < 0.$$

a contradiction to (12). Therefore, $T \in \partial^B_{\varepsilon-F} F(x_0, y_0)$. The proof is completed.

Now, we can prove the following generalized $\varepsilon-$ Moreau-Rockafellar type theorem for $\varepsilon-$ strictly efficiency of Theorem 3.4 by using the generalized $\varepsilon-$ Moreau-Rockafellar type theorem for $\varepsilon-$ weak subgradients of the set-valued maps in [12].

Lemma 3.1[12] Let F_1 and F_2 be two set-valued maps from X into Y with F_1 and F_2 are $C-$ convex on X. If $\text{int}(\text{epi}F_1) \cap \text{epi}\ (F_2) \neq \emptyset$, then for $\varepsilon \in C$, $\bar{x} \in \text{dom}\ F_1 \cap \text{dom}\ F_2$, $\bar{y}_1 \in F_1(\bar{x})$ and $\bar{y}_2 \in F_2(\bar{x})$ we have

$$\partial^C_{\varepsilon-W}(F_1+F_2)(\bar{x}, \bar{y}_1+\bar{y}_2) \subset \bigcup\{\partial^C_{\varepsilon_1-W} F_1(\bar{x}, \bar{y}_1) + \partial^C_{\varepsilon_2-W} F_2(\bar{x}, \bar{y}_2) :$$

$$\varepsilon_1, \varepsilon_2 \in C,\ \varepsilon_1+\varepsilon_2 \in \varepsilon + Y\backslash \text{int}\ C\}.$$

Lemma 3.2[12] Let F_1 and F_2 be two set-valued maps from the set $E := \{x \in X : F_1(x) \neq \emptyset \text{ and } F_2(x) \neq \emptyset\}$ into Y, and F_1 and F_2 be $C-$ convex on E. If F_1 is connected at some $x_0 \in \text{int}\ E$. then for $\bar{x} \in E$, $\bar{y}_1 \in F_1(\bar{x})$ and $\bar{y}_2 \in F_2(\bar{x})$, we have

$$\partial^C_{\varepsilon-W}(F_1+F_2)(\bar{x}, \bar{y}_1+\bar{y}_2) \subset \bigcup\{\partial^C_{\varepsilon_1-W} F_1(\bar{x}, \bar{y}_1) + \partial^C_{\varepsilon_2-W} F_2(\bar{x}, \bar{y}_2) :$$

$$\varepsilon_1, \varepsilon_2 \in C,\ \varepsilon_1+\varepsilon_2 \in \varepsilon + Y\backslash \text{int}\ C\}.$$

According to Lemma 3.1 and Lemma 3.2, it is easy to prove the following Theorem 3.3.

Theorem 3.3 Let F_1 and F_2 be two set-valued maps from the set $E := \{x \in X : F_1(x) \neq \emptyset \text{ and } F_2(x) \neq \emptyset\}$ into Y, and F_1 and F_2 be $C-$ convex on E. If F_1 satisfies one of three conditions of Lemma 2.2, then for $\bar{x} \in E$, $\bar{y}_1 \in F_1(\bar{x})$ and $\bar{y}_2 \in F_2(\bar{x})$, we have

$$\partial^C_{\varepsilon-W}(F_1+F_2)(\bar{x}, \bar{y}_1+\bar{y}_2) \subset \bigcup\{\partial^C_{\varepsilon_1-W} F_1(\bar{x}, \bar{y}_1) + \partial^C_{\varepsilon_2-W} F_2(\bar{x}, \bar{y}_2) :$$

$$\varepsilon_1, \varepsilon_2 \in C,\ \varepsilon_1+\varepsilon_2 \in \varepsilon + Y\backslash \text{int}\ C\}.$$

Theorem 3.4 Let F_1 and F_2 be two set-valued maps from the set $E := \{x \in X : F_1(x) \neq \emptyset \text{ and } F_2(x) \neq \emptyset\}$ into Y, and F_1 and F_2 be $C-$ convex on E. If F_1 satisfies one of three conditions of Lemma 2.2, then for $\bar{x} \in E$, $\bar{y}_1 \in F_1(\bar{x})$ and $\bar{y}_2 \in F_2(\bar{x})$, we have

$$\partial^B_{\varepsilon-F}(F_1+F_2)(\bar{x}, \bar{y}_1+\bar{y}_2) \subset \bigcup\{\partial^B_{\varepsilon_1-F} F_1(\bar{x}, \bar{y}_1) + \partial^B_{\varepsilon_2-F} F_2(\bar{x}, \bar{y}_2) :$$

$$\varepsilon_1, \varepsilon_2 \in C,\ \varepsilon_1+\varepsilon_2 \in \varepsilon + Y\backslash \text{int}\ C\}. \tag{18}$$

Proof Let $T \in L(X, Y)$ and

$$T \in \partial^B_{\varepsilon-F}(F_1+F_2)(\bar{x}, \bar{y}_1+\bar{y}_2). \tag{19}$$

In order to prove (18), we can only prove: there exist $\varepsilon_1,\ \varepsilon_2 \in C,\ \varepsilon_1+\varepsilon_2 \in \varepsilon + Y\backslash\text{int } C,\ T_i \in \partial^B_{\varepsilon_i - F} F_i(\bar{x}, \bar{y}_i),\ i = 1,2$ such that $T = T_1 + T_2$. It follows from (19) that, there exists a neighborhood U of 0 such that

$$\text{cone } (F_1(E) + F_2(E) - T(E) + \varepsilon - \bar{y}_1 - \bar{y}_2 + T(\bar{x})) \cap (U - B) = \emptyset.$$

It is clear that

$$(F_1(E) + F_2(E) - T(E) + \varepsilon - \bar{y}_1 - \bar{y}_2 + T(\bar{x})) \cap (-\text{int cone } (B - U)) = \emptyset. \tag{20}$$

Thus

$$T \in \partial^{\text{cone } (B-U)}_{\varepsilon - W}(F_1 + F_2)(\bar{x}, \bar{y}_1 + \bar{y}_2). \tag{21}$$

Since F_1 and F_2 are $C-$ convex on E and $C \subset$ cone $(B-U)$, $F_1,\ F_2$ are cone $(B-U)-$ convex on E. Therefore, the conditions of Theorem 3.3 are all satisfied. Thus

$$\partial^{\text{cone } (B-U)}_{\varepsilon - W}(F_1 + F_2)(\bar{x}, \bar{y}_1 + \bar{y}_2) \subset \bigcup\{\partial^{\text{cone } (B-U)}_{\varepsilon_1 - W} F_1(\bar{x}, \bar{y}_1) + \partial^{\text{cone } (B-U)}_{\varepsilon_2 - W} F_2(\bar{x}, \bar{y}_2) :$$

$$\varepsilon_1, \varepsilon_2 \in C,\ \varepsilon_1 + \varepsilon_2 \in \varepsilon + Y\backslash\text{int } C\}. \tag{22}$$

It follows from (21) and (22) that there exist $\varepsilon_1,\ \varepsilon_2 \in C,\ \varepsilon_1+\varepsilon_2 \in \varepsilon + Y\backslash\text{int } C$ and continuous linear operator $T_i \in \partial^{\text{cone } (B-U)}_{\varepsilon_i - W} F_i(\bar{x}, \bar{y}_i)$ such that $T = T_1 + T_2$. In what follows, we prove

$$T_i \in \partial^B_{\varepsilon_i - F} F_i(\bar{x}, \bar{y}_i), \quad i = 1, 2.$$

Since $T_i \in \partial^{\text{cone } (B-U)}_{\varepsilon_i - W} F_i(\bar{x}, \bar{y}_i)$, we have

$$(F_i(E) - T_i(E) - \bar{y}_i + T_i(\bar{x}) + \varepsilon_i) \cap (-\text{int cone } (B - U)) = \emptyset. \tag{23}$$

It is clear that

$$\text{cl cone } (F_i(E) - T_i(E) - \bar{y}_i + T_i(\bar{x}) + \varepsilon_i) \cap (-\text{int cone } (B - U)) = \emptyset. \tag{24}$$

By the separation theorem of convex sets, there exists $0 \neq f \in Y^*$ such that

$$\langle f, \text{cl cone } (F_i(E) - T_i(E) - \bar{y}_i + T_i(\bar{x}) + \varepsilon_i)\rangle \geq \langle f, -\text{int cone } (B - U)\rangle. \tag{25}$$

Since cl cone $(F_i(E) - T_i(E) - \bar{y}_i + T_i(\bar{x}) + \varepsilon_i)$ is a convex cone and on which f has low bound, we have $\langle f, \text{cl cone } (F_i(E) - T_i(E) - \bar{y}_i + T_i(\bar{x}) + \varepsilon_i)\rangle \geq 0$, which implies that

$$\langle f, F_i(E) - T_i(E) - \bar{y}_i + T_i(\bar{x}) + \varepsilon_i\rangle \geq 0. \tag{26}$$

That is

$$\langle f, F_1(E) - T_1(E) - \bar{y}_1 + T_1(\bar{x}) + \varepsilon_1\rangle \geq 0. \tag{27}$$

and

$$\langle f, F_2(E) - T_2(E) - \bar{y}_2 + T_2(\bar{x}) + \varepsilon_2 \rangle \geq 0. \tag{28}$$

By (27) and (28), we have

$$\langle f, F_1(E) + F_2(E) - T_1(E) - T_2(E) - \bar{y}_1 - \bar{y}_2 + T_1(\bar{x}) + T_2(\bar{x}) + \varepsilon_1 + \varepsilon_2 \rangle \geq 0. \tag{29}$$

On the other hand, by (25), we know that $\langle f, -\text{int cone } (B - U)\rangle \leq 0$, which implies that $f \in (\text{int cone } (B - U))^* \subset \text{int } C^*$. By Proposition 2.1 in [14], $f \in B^{st}$. Therefore, there exists $t > 0$ such that

$$\langle f, b \rangle \geq t, \quad \forall b \in B.$$

Let

$$U = \{y \in Y | \langle f, y \rangle < \frac{t}{2}\}.$$

In what follows, we verify

$$\text{cone } (F_i(E) - T_i(E) + \varepsilon_i - \bar{y}_i + T_i(\bar{x})) \cap (U - B) = \emptyset, \quad i = 1, 2.$$

Indeed, if there exists

$$\tilde{y} \in \text{cone } (F_i(E) - T_i(E) + \varepsilon_i - \bar{y}_i + T_i(\bar{x})) \cap (U - B)$$

then $\tilde{y} \neq 0$, and

$$\tilde{y} = \lambda_i(F_i(\tilde{x}_i) - T_i(\tilde{x}_i) + \varepsilon_i - \bar{y}_i + T_i(\bar{x})) = u_i - b_i, \tag{30}$$

where $\tilde{x}_i \in E,\ u_i \in U\ , b_i \in B$. Since U is an open convex neighborhood of 0 and

$$\langle f, u - b \rangle < 0, \quad \forall u \in U,\ b \in B. \tag{31}$$

Combing with (30) and (31), we have

$$\langle f, \lambda_i(F_i(\tilde{x}_i) - T_i(\tilde{x}_i) + \varepsilon_i - \bar{y}_i + T_i(\bar{x})) \rangle < 0.$$

which together with $\lambda_i > 0$ that

$$\langle f, F_i(\tilde{x}_i) - T_i(\tilde{x}_i) + \varepsilon_i - \bar{y}_i + T_i(\bar{x})) \rangle < 0, \quad \tilde{x}_i \in E, \quad i = 1, 2. \tag{32}$$

(32) yields that

$$\langle f, F_1(\tilde{x}_1) - T_1(\tilde{x}_1) + \varepsilon_1 - \bar{y}_1 + T_1(\bar{x}) \rangle < 0,\ \tilde{x}_1 \in E. \tag{33}$$

and

$$\langle f, F_2(\tilde{x}_2) - T_2(\tilde{x}_2) + \varepsilon_2 - \bar{y}_2 + T_2(\bar{x}) \rangle < 0,\ \tilde{x}_2 \in E. \tag{34}$$

By (33) and (34), we have

$$\langle f, F_1(\tilde{x}_1) + F_2(\tilde{x}_2) - T_1(\tilde{x}_1) - T_2(\tilde{x}_2) + \varepsilon_1 + \varepsilon_2 - \bar{y}_1 - \bar{y}_2 + T_1(\bar{x}) + T_2(\bar{x}) \rangle < 0,\ \tilde{x}_1,\ \tilde{x}_2 \in E.$$

a contradiction to (29), so $T_i \in \partial^B_{\varepsilon_i - F} F_i(\bar{x}, \bar{y}_i),\ i = 1, 2$. The proof is completed.

4 Optimality Condition

Let C and D be pointed convex cones of Y and Z, respectively and $\text{int } D \neq \emptyset$. Suppose $F : X \to 2^Y$ is $C-$ convex on X and $G : X \to 2^Z$ is $D-$ convex on X. Consider the following constrained set-valued optimization problem (P):

$$C - \min F(x)$$

$$\text{s.t. } x \in A =: \{x \in X : G(x) \cap (-D) \neq \emptyset\}.$$

Definition 4.1 Let $\bar{x} \in A,\ \bar{y} \in F(\bar{x}),\ \bar{x}$ is called an $\varepsilon-$ strictly efficient solution of (P) with respect to B, if there is a neighborhood U of 0 such that

$$\text{cl cone}(F(X) + \varepsilon - \bar{y}) \cap (U - B) = \emptyset.$$

Definition 4.2 We say that condition (CQ2) holds for problem (P), if for any $z^* \in D^* \backslash \{0\}$, there exists $x \in X$ such that

$$\langle z^*, z \rangle < 0, \quad \forall z \in G(x).$$

Theorem 4.1 Let $\bar{x} \in X,\ \bar{y} \in F(\bar{x}),\ \bar{h} \in G(\bar{x}) \cap (-D)$ and condition (CQ2) holds. If $\bar{x}$ is an $\varepsilon-$ strictly efficient solution of (P) with respect to B, then there exist $y^* \in B^{st}, z^* \in D^*$ such that $(y^*, z^*) \neq (0_{Y^*}, 0_{Z^*}), \langle z^*, \bar{h} \rangle = 0$ and

$$0 \in \partial^B_{\varepsilon - F}[y^* \circ F + z^* \circ G](\bar{x}, \langle y^*, \bar{y} + \langle z^*, \bar{h} \rangle).$$

Proof Since $\bar{x}$ is an $\varepsilon-$ strictly efficient solution of (P) with respect to B, there exists an open convex neighborhood U of 0 such that

$$\text{cone}(F(X) + \varepsilon - \bar{y}) \cap (U - B) = \emptyset.$$

It is clear that

$$(F(X) + \varepsilon - \bar{y}) \cap -\text{int cone}(B - U) = \emptyset.$$

Therefore

$$F(x) \subset \bar{y} - \varepsilon + Y \backslash \{-\text{int cone}(B - U)\},\ \forall x \in X. \tag{35}$$

Define

$$\Omega = \{(p, q) \in Y \times Z : \exists x \in X,\ \text{s.t. } (F(x) - p) \cap (\bar{y} - \varepsilon - \text{int cone}(B - U)) \neq \emptyset,$$

$$(G(x) - q) \cap (-\text{int } D) \neq \emptyset\}.$$

The proof of this theorem consists of several steps. First, we prove four important properties of Ω and then we apply a separation theorem in order to obtain the desired result.

Step 1 $\Omega \neq \emptyset$. In fact, for any $r \in \text{int cone}(B-U)$ and $s \in \text{int } D$, we have $(r+\varepsilon, s+\bar{h}) \in \Omega$.

Step 2 $(0_Y, 0_Z) \notin \Omega$. In fact, otherwise there exists $\hat{x} \in X$ such that

$$F(\hat{x}) \cap (\bar{y} - \varepsilon - \text{int cone}(B-U)) \neq \emptyset; \; G(\hat{x}) \cap (-\text{int } D) \neq \emptyset.$$

Thus, there exist $\hat{y} \in F(\hat{x})$ and $\hat{h} \in G(\hat{x})$ such that

$$\hat{y} \in \bar{y} - \varepsilon - \text{int cone}(B-U); \quad \hat{h} \in -\text{int } D \subset -D.$$

a contradiction to (35).

Step 3 Ω is a convex set. Let $\lambda \in [0,1]$, $(p_1, q_1) \in \Omega$ and $(p_2, q_2) \in \Omega$. It follows from the definition of Ω that, there exist $x_1, x_2 \in X$ such that

$$(F(x_1) - p_1) \cap (\bar{y} - \varepsilon - \text{int cone}(B-U)) \neq \emptyset; \; (G(x_1) - q_1) \cap (-\text{int } D) \neq \emptyset,$$

and

$$(F(x_2) - p_2) \cap (\bar{y} - \varepsilon - \text{int cone}(B-U)) \neq \emptyset; \; (G(x_2) - q_2) \cap (-\text{int } D) \neq \emptyset.$$

Therefore, there exist $z_1, z_2 \in \bar{y} - \varepsilon - \text{int cone}(B-U)$ and $t_1, t_2 \in -\text{int } D$ such that

$$z_1 \in F(x_1) - p_1; \quad t_1 \in G(x_1) - q_1,$$

and

$$z_2 \in F(x_2) - p_2; \quad t_2 \in G(x_2) - q_2.$$

Define $x =: \lambda x_1 + (1-\lambda)x_2$, $z =: \lambda z_1 + (1-\lambda)z_2$, $t =: \lambda t_1 + (1-\lambda)t_2$, $p =: \lambda p_1 + (1-\lambda)p_2$ and $q =: \lambda q_1 + (1-\lambda)q_2$. Since F is $C-$ convex on X and $C \subset$ cone $(B-U)$, F is cone $(B-U)-$ convex on X. Since G is $D-$ convex on X, we have

$$z \in F(x) - p + \text{cone}(B-U); \quad t \in G(x) - q + D.$$

Thus, there exist $u_0 \in \text{cone}(B-U)$ and $v_0 \in D$ such that

$$z - u_0 \in F(x) - p; \quad t - v_0 \in G(x) - q.$$

On the other hand, since $z = \lambda z_1 + (1-\lambda)z_2$, z_1, $z_2 \in \bar{y} - \varepsilon - \text{int cone}(B-U)$ and $t = \lambda t_1 + (1-\lambda)t_2, t_1, t_2 \in -\text{int } D$, we get

$$z - u_0 \in \bar{y} - \varepsilon - \text{int cone}(B-U) - \text{cone}(B-U) \subset \bar{y} - \varepsilon - \text{int cone}(B-U),$$

and

$$t - v_0 \in -\text{int } D - D \subset -\text{int } D.$$

Hence

$$(F(x) - p) \cap (\bar{y} - \varepsilon - \text{int cone}(B - U)) \neq \emptyset, \quad (G(x) - q) \cap (-\text{int } D) \neq \emptyset.$$

It yields that $(p, q) \in \Omega$. Thus, Ω is a convex set.

Step 4 Ω is an open set. In fact, let $(\tilde{p}, \tilde{q}) \in \Omega$. By the definition of Ω that, there exists $\tilde{x} \in X$ such that

$$(F(\tilde{x}) - \tilde{p}) \cap (\bar{y} - \varepsilon - \text{int cone}(B - U)) \neq \emptyset; \quad (G(\tilde{x}) - \tilde{q}) \cap (-\text{int } D) \neq \emptyset.$$

Thus, there exist $\tilde{z} \in \bar{y} - \varepsilon - \text{int cone}(B - U)$ and $\tilde{t} \in -\text{int } D$ such that

$$\tilde{z} \in F(\tilde{x}) - \tilde{p}; \quad \tilde{t} \in G(\tilde{x}) - \tilde{q}.$$

Since int cone$(B - U)$ and int D are open sets, there exists an open ball $B_{Y\times Z}(0_{Y\times Z}, R)$ such that for any $(a, b) \in B_{Y\times Z}(0_{Y\times Z}, R)$, we have

$$\tilde{z} - a \in \bar{y} - \varepsilon - \text{int cone}(B - U); \quad \tilde{z} - a \in F(\tilde{x}) - (\tilde{p} + a),$$

and

$$\tilde{t} - b \in -\text{int } D; \quad \tilde{t} - b \in G(\tilde{x}) - (\tilde{q} + b).$$

Thus, for any $(a, b) \in B_{Y\times Z}(0_{Y\times Z}, R)$, we have

$$(F(\tilde{x}) - (\tilde{p} + a)) \cap (\bar{y} - \varepsilon - \text{int cone}(B - U)) \neq \emptyset,$$

and

$$(G(\tilde{x}) - (\tilde{q} + b)) \cap (-\text{int } D) \neq \emptyset.$$

which implies that $(\tilde{p}, \tilde{q}) + B_{Y\times Z}(0_{Y\times Z}, R) \subset \Omega$, so Ω is an open set.

Step 5 In this step, using separation theorem of convex sets to finish the proof of Theorem 4.1. Since Ω is an open convex set with $\Omega \neq \emptyset$ and $(0_Y, 0_Z) \notin \Omega$, there exists $(y^*, z^*) \in Y^* \times Z^* \backslash (0_{Y^*}, 0_{Z^*})$ such that

$$\langle (y^*, z^*), (p, q) \rangle \geq 0, \quad \forall (p, q) \in \Omega.$$

For arbitrary $x \in X$, $y \in F(x)$, $h \in G(x)$, $r \in \text{int cone}(B - U)$, $s \in \text{int } D$ and $\rho, \sigma > 0$. Setting

$$p = y + \varepsilon - \bar{y} + \rho r; \quad q = h + \sigma s.$$

Obviously, $(p, q) \in \Omega$ and

$$\langle y^*, y + \varepsilon - \bar{y} + \rho r \rangle + \langle z^*, h + \sigma s \rangle \geq 0. \tag{36}$$

Taking $x = \bar{x}, y = \bar{y}$ and $h = \bar{h}$ in (36), we have

$$\rho\langle y^*, \frac{1}{\rho}\varepsilon + r\rangle + \langle z^*, \bar{h}\rangle + \sigma\langle z^*, s\rangle \geq 0. \tag{37}$$

Since $\rho > 0$ and $\sigma > 0$ are arbitrary, (37) yields

$$\langle z^*, \bar{h}\rangle \geq 0. \tag{38}$$

and

$$\sigma\langle z^*, s\rangle + \langle z^*, \bar{h}\rangle \geq 0. \tag{39}$$

Combining with (38) and (39), we get that $\langle z^*, s\rangle \geq 0$. it yields that $z^* \in D^*$. On the other hand, from $\bar{h} \in -D$, we have

$$\langle z^*, \bar{h}\rangle \leq 0. \tag{40}$$

Thus, by (38) and (40), we have

$$\langle z^*, \bar{h}\rangle = 0.$$

Since $\frac{1}{\rho}\varepsilon + r \in C + \text{int cone}(B - U) \subset \text{cone}(B - U)$. Setting $\bar{r} = \frac{1}{\rho}\varepsilon + r$ in (37), it yields

$$\rho\langle y^*, \bar{r}\rangle + \langle z^*, \bar{h}\rangle + \sigma\langle z^*, s\rangle \geq 0, \quad \forall \bar{r} \in \text{int cone}(B - U). \tag{41}$$

By (41), we obtain that $\langle y^*, \bar{r}\rangle \geq 0$, it yields that $y^* \in (\text{cone}(B - U))^*$. Furthermore, from (36), we can see that

$$\langle y^*, y + \varepsilon - \bar{y}\rangle + \langle z^*, h\rangle \geq 0, \quad \forall x \in X, \forall y \in F(x), \forall h \in G(x). \tag{42}$$

In what follows, we prove that $y^* \neq 0_{Y^*}$. If not, then $z^* \in D^* \backslash \{0\}$. It follows from the definition of condition (CQ2) that there exist a $x^{'} \in X$ such that

$$\langle z^*, z^{'}\rangle < 0, \quad \forall z^{'} \in G(x^{'}). \tag{43}$$

On the other hand, by (42), we have

$$\langle z^*, h\rangle \geq 0, \quad \forall x \in X, \; h \in G(x).$$

a contradiction to (43). Thus, $y^* \in (\text{cone}(B - U))^* \backslash \{0\}$. Therefore,

$$\langle y^*, b\rangle \geq \langle y^*, u\rangle, \quad \forall b \in B, u \in U.$$

Since $y^* \neq 0_{Y^*}$ and U is an open convex neighborhood of 0, there exists $u_0 \in U$, such that

$$\langle y^*, u_0\rangle = t > 0.$$

Consequently

$$\langle y^*, b\rangle \geq t, \quad \forall b \in B.$$

That is

$$y^* \in B^{st}.$$

By (42), we have

$$\langle y^*, y\rangle + \langle z^*, h\rangle \geq \langle y^*, \bar{y}\rangle + \langle z^*, \bar{h}\rangle - \langle y^*, \varepsilon\rangle, \quad \forall x \in X, \forall y \in F(x), \forall h \in G(x).$$

Since $\langle y^*, \bar{y}\rangle + \langle z^*, \bar{h}\rangle \in \cup_{x\in X}[\langle y^*, F(x)\rangle + \langle z^*, G(x)\rangle]$,

$$\langle y^*, \bar{y}\rangle + \langle z^*, \bar{h}\rangle \in \min\ \cup_{x\in X}\ [\langle y^*, F(x)\rangle + \langle z^*, G(x)\rangle].$$

which is equivalent to

$$0 \in \partial^B_{\varepsilon-F}(y^* \circ F + z^* \circ G)(\bar{x}, \langle y^*, \bar{y}\rangle + \langle z^*, \bar{h}\rangle).$$

Thus the proof of Theorem 4.1 is completed.

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