A Polyhedral Study of the Integrated Minimum-Up/-Down Time and Ramping Polytope

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Abstract

In this paper, we consider the polyhedral structure of the integrated minimum-up/-down time and ramping polytope for the unit commitment problem. Our studied generalized polytope includes minimum-up/-down time constraints, generation ramp-up/-down rate constraints, logical constraints, and generation upper/lower bound constraints. We derive strong valid inequalities by utilizing the structures of the unit commitment problem, and these inequalities, plus trivial inequalities described in the original formulation, are sufficient to provide the convex hull descriptions for variant two-period and three-period problems corresponding to different minimum-up/-down time limits and parameter assumptions. In addition, more generalized strong valid inequalities (including one, two, and three continuous variable cases respectively) are introduced to strengthen the multi-period formulations, and we further prove these inequalities are facet-defining under certain mild conditions. Finally, extensive computational experiments are conducted to verify the effectiveness of our proposed strong valid inequalities on solving both the network-constrained unit commitment problem and the self-scheduling unit commitment problem, for which our derived approach outperforms the default CPLEX significantly.

Key words: strong valid inequalities; polyhedral study; unit commitment; convex hull

1 Introduction

As a fundamental optimization problem in power system operations, the unit commitment (UC) problem decides the unit status (online/offline) and schedules the particular power generation amount for each unit over a finite discrete horizon to satisfy the load (energy demand) with a minimum total cost. Each unit should satisfy associated physical restrictions, such as generation upper/lower limits, ramp rate limits, and minimum-up/-down time limits. In general, the UC problem can be formulated as a large-scale mixed-integer linear program (MILP) and has been attracting interests from both academia and industry. Several traditional approaches, such as dynamic programming [14, 21], Lagrangian relaxation [25, 4], branch-and-bound [8, 5], genetic algorithms [12, 22], and simulated annealing [26, 15], have been developed to solve the UC problem. Detailed reviews of these approaches to solve the UC problem can be found in [18, 20].

Recently, optimization algorithm developments on power system operations are switched from Lagrangian relaxation to MILP approaches due to MILP's ease of development and maintenance, ability to specify accurate solutions, and exact modeling of complex functionality [16]. Therefore, MILP has been widely adopted by the Independent System Operators (ISO) recently in US [11, 2] and creates more than 500 million annual savings [16]. In particular, MILP arises as a promising approach to formulate and solve the unit commitment problem [1]. The earliest MILP UC formulation was proposed in the 1960s [10], and further improvements has been developed until recently. For instance, in [7], an exact and computationally efficient MILP formulation is provided to address the single-generator self-scheduling unit commitment problem in order to maximize the total profit. In [9], security-constrained UC problems are modeled and solved through the MILP approach for large-scale power systems with multiple generators.

As indicated in [11, 24], a strong MILP formulation plays a significant role in improving the solution quality, as strong (tightening) formulations reduce the feasible region of the linear programming (LP) relaxation of the original problem and improve the LP relaxation bounds. In addition, strong valid inequalities (e.g., facet-defining inequalities) will help speed up the branch-and-cut algorithm to obtain an optimal mixed-integer solution. There has been research progress on developing strong formulations for the unit commitment problem by exploring its special structure. For instance, in [13], alternating up/down inequalities are proposed to strengthen the minimum-up/-down time polytope of the unit commitment problem. In [19], the convex hull of the minimum-up/-down polytope considering start-up costs is provided, in which additional start-up and shut-down variables are introduced to provide the integral formulation. Recently, several new families of strong valid inequalities are proposed in [17, 6] to tighten the ramping polytope of the unit commitment problem.

Following this direction, in this paper, we focus on deriving strong cutting planes to help solve the unit commitment problem by exploring the polyhedral structure of its feasible scheduling region. More specifically, we consider the polyhedral structure of the feasible region of a generator including both the minimum-up/-down time and ramping polytopes. This integrated polytope minimum-up/-down constraints, logical constraints, power generation upper/lower bound constraints, and generation ramp-up/-down rate constraints. To describe the polytope for each generator, we let T be the number of time periods for the whole operational horizon, L (ℓ) be minimum-up (-down) time limits of the generator, $\overline{C}$ ($\underline{C}$) be its generation upper (lower) bound when it is online, $\overline{V}$ be its start-up/shut-down ramp rate, and V be its ramp-up/-down rate in stable generation region. In addition, we let (x, y, u) be the decision variables to represent the generator's status, in which continuous variable x represents the generation amount, binary variable y represents the generator's online/offline status (i.e., $y_t = 1$ means the generator is online at t and $y_t = 0$ otherwise), and binary variable u represents whether the generator starts up or not (i.e., $u_t = 1$ means the generator starts up at t and $u_t = 0$ otherwise). Thus we focus on the following integrated minimum-up/-down time and ramping polytope:

$$
\begin{aligned}
P := \Big\{(x, y, u) \in \mathbb{R}_+^T \times \mathbb{B}^T \times \mathbb{B}^{T-1} : \quad & \sum_{i=t-L+1}^{t} u_i \leq y_t, \ \forall t \in [L+1, T]_{\mathbb{Z}}, && \text{(1a)} \\
& \sum_{i=t-\ell+1}^{t} u_i \leq 1 - y_{t-\ell}, \ \forall t \in [\ell+1, T]_{\mathbb{Z}}, && \text{(1b)} \\
& y_t - y_{t-1} - u_t \leq 0, \ \forall t \in [2, T]_{\mathbb{Z}}, && \text{(1c)} \\
& -x_t + \underline{C} y_t \leq 0, \ \forall t \in [1, T]_{\mathbb{Z}}, && \text{(1d)} \\
& x_t - \overline{C} y_t \leq 0, \ \forall i \in [1, T]_{\mathbb{Z}}, && \text{(1e)} \\
& x_t - x_{t-1} \leq V y_{t-1} + \overline{V}(1 - y_{t-1}), \ \forall t \in [2, T]_{\mathbb{Z}}, && \text{(1f)} \\
& x_{t-1} - x_t \leq V y_t + \overline{V}(1 - y_t), \ \forall i \in [2, T]_{\mathbb{Z}} \Big\}, && \text{(1g)}
\end{aligned}
$$

where constraints (1a) and (1b) describe the minimum-up and minimum-down time limits [13, 19], respectively (i.e., if the generator starts up at time $t - L + 1$, it should keep online in the following L consecutive time periods until time t; if the generator shuts down at time $t - \ell + 1$, it should keep offline in the following ℓ consecutive time periods until time t), constraints (1c) describe the logical relationship between y and u, constraints (1d) and (1e) describe the generation lower and upper bound, respectively, and constraints (1f) and (1g) describe the generation ramp-up and ramp-down rate limits, respectively. Note here that, in our polytope description, there is no start-up decision corresponding to the first-time period. In this way, the derived inequalities can be applied to each time period and can be used recursively. Meanwhile, considering the physical characteristics of a thermal generator, without loss of generality, we can assume $\underline{C} < \overline{V} < \underline{C} + V$ and $\overline{C} - \underline{C} - V \geq 0$. In addition, we assume $\overline{C} - \overline{V} - V \geq 0$ so that the generator can ramp up at least once after its start-up, which is also reasonable for most thermal generators. For notation convenience, we define ϵ as an arbitrarily small positive real number and $[a, b]_{\mathbb{Z}}$ as the set of integer numbers between integers a and b, i.e., $\{a, a+1, \cdots, b\}$ with $[a, b]_{\mathbb{Z}} = \emptyset$ if $a > b$. Finally, we let $\text{conv}(P)$ represent the convex hull description of P.

Before describing the details of our derived strong formulations, we report the convex hull description of the two-period problem as follows:

Theorem 1 *For $T = 2$ and $L = \ell = 1$, conv(P) can be described as follows:*

$$
\begin{aligned}
Q_2 := \Big\{ (x, y, u) \in \mathbb{R}^5 : \ & u_2 \geq 0,\ u_2 \geq y_2 - y_1, && \text{(2a)} \\
& u_2 \leq y_2,\ y_1 + u_2 \leq 1, && \text{(2b)} \\
& x_1 \geq \underline{C} y_1,\ x_2 \geq \underline{C} y_2, && \text{(2c)} \\
& x_1 \leq \overline{V} y_1 + (\overline{C} - \overline{V})(y_2 - u_2), && \text{(2d)} \\
& x_2 \leq \overline{C} y_2 - (\overline{C} - \overline{V}) u_2, && \text{(2e)} \\
& x_2 - x_1 \leq (\underline{C} + V) y_2 - \underline{C} y_1 - (\underline{C} + V - \overline{V}) u_2, && \text{(2f)} \\
& x_1 - x_2 \leq \overline{V} y_1 - (\overline{V} - V) y_2 - (\underline{C} + V - \overline{V}) u_2 \Big\}. && \text{(2g)}
\end{aligned}
$$

Proof: An alternative formulation of this convex hull is provided in [6], where the convex hulls considering ramp-up and ramp-down polytopes separately are provided with corresponding proofs, and thus the proofs are omitted here. ■

Remark 1 *Since the start-up decision is not considered in the first-time period in Q_2, the strong valid inequalities in Q_2 (e.g., (2d) - (2g)) can be applied to any two consecutive time periods.*

In the remaining part of this paper, we derive strong valid inequalities and the further convex hull descriptions for the three-period problems by considering different minimum-up/-down time limits in Section 2. In Section 3, we extend our study to derive strong valid inequalities so as to strengthen the general multi-period formulations. Following these, in Section 4 we perform computational studies to verify the effectiveness of our proposed strong valid inequalities. Finally, we conclude our study in Section 5.

2 Strengthening Three-period Formulations

In this section, we perform the polyhedral study for the three-period formulation, i.e., $T = 3$ in P, and propose convex hull descriptions for variant cases with different minimum-up/-down time limits. We first study the case in which $L = \ell = 2$ in the original polytope, which is the most complicated one among the cases in which $L = \ell = 1$, $L = 1$ and $\ell = 2$, $L = 2$ and $\ell = 1$, and $L = \ell = 2$. Since the formulations of the strong valid inequalities are different for $\overline{C} - \underline{C} - 2V \geq 0$ and $\overline{C} - \underline{C} - 2V < 0$ cases, we firstly study the case $\overline{C} - \underline{C} - 2V \geq 0$. Under this setting, the corresponding formulation can be described as follows:

$$
\begin{aligned}
P_3^2 := \Big\{ \quad & (x, y, u) \in \mathbb{R}_+^3 \times \mathbb{B}^3 \times \mathbb{B}^2 : \\
& u_2 + u_3 \leq y_3, && \text{(3a)} \\
& y_1 + u_2 + u_3 \leq 1, && \text{(3b)} \\
& u_2 \geq y_2 - y_1, \ u_3 \geq y_3 - y_2, && \text{(3c)} \\
& x_1 \geq \underline{C} y_1, \ x_2 \geq \underline{C} y_2, \ x_3 \geq \underline{C} y_3, && \text{(3d)} \\
& x_1 \leq \overline{C} y_1, \ x_2 \leq \overline{C} y_2, \ x_3 \leq \overline{C} y_3, && \text{(3e)} \\
& x_2 - x_1 \leq V y_1 + \overline{V}(1 - y_1), \ x_3 - x_2 \leq V y_2 + \overline{V}(1 - y_2), && \text{(3f)} \\
& x_1 - x_2 \leq V y_2 + \overline{V}(1 - y_2), \ x_2 - x_3 \leq V y_3 + \overline{V}(1 - y_3) \Big\}. && \text{(3g)}
\end{aligned}
$$

For P_3^2, we first provide the strong valid inequalities in the following proposition. Then we provide a linear programming description Q_3^2 and further prove that Q_3^2 provides the convex hull description for P_3^2.

Proposition 1 *For P_3^2, the following inequalities*

$$x_1 \leq \overline{V} y_1 + V(y_2 - u_2) + (\overline{C} - \overline{V} - V)(y_3 - u_3 - u_2), \tag{4}$$

$$x_2 \leq \overline{V} y_2 + (\overline{C} - \overline{V})(y_3 - u_3 - u_2), \tag{5}$$

$$x_3 \leq \overline{C}y_3 - (\overline{C} - \overline{V})u_3 - (\overline{C} - \overline{V} - V)u_2, \tag{6}$$

$$x_2 - x_1 \leq \overline{V}y_2 - \underline{C}y_1 + (\underline{C} + V - \overline{V})(y_3 - u_3 - u_2), \tag{7}$$

$$x_3 - x_2 \leq (\underline{C} + V)y_3 - \underline{C}y_2 - (\underline{C} + V - \overline{V})u_3, \tag{8}$$

$$x_1 - x_2 \leq \overline{V}y_1 - (\overline{V} - V)y_2 - (\underline{C} + V - \overline{V})u_2, \tag{9}$$

$$x_2 - x_3 \leq \overline{V}y_2 - \underline{C}y_3 + (\underline{C} + V - \overline{V})(y_3 - u_3 - u_2), \tag{10}$$

$$x_3 - x_1 \leq (\underline{C} + 2V)y_3 - \underline{C}y_1 - (\underline{C} + 2V - \overline{V})u_3 - (\underline{C} + V - \overline{V})u_2, \tag{11}$$

$$x_1 - x_3 \leq \overline{V}y_1 - \underline{C}y_3 + V(y_2 - u_2) + (\underline{C} + V - \overline{V})(y_3 - u_3 - u_2), \tag{12}$$

$$x_1 - x_2 + x_3 \leq \overline{V}y_1 - (\overline{V} - V)y_2 + \overline{V}y_3 + (\overline{C} - \overline{V})(y_3 - u_3 - u_2), \tag{13}$$

are valid for conv(P_3^2).

Proof: To prove the validity of (4), we discuss the following two possible cases in terms of the value of y_1:

1) If $y_1 = 0$, then $x_1 = 0$ due to (3e). It follows that (4) is valid since $y_2 - u_2 \geq 0$ due to (3c) and (3a) and $y_3 - u_3 - u_2 \geq 0$ due to (3a).

2) If $y_1 = 1$, then $u_2 = u_3 = 0$ due to (3b). We consider the following three possible cases based on when the generator shuts down:

   (1) If the generator shuts down at the second time period, i.e., $y_2 = 0$, then we have $y_3 = 0$ since minimum-down time limit $\ell = 2$. Inequality (4) converts to $x_1 \leq \overline{V}$, which is valid due to ramp-down constraints (3g).

   (2) If the generator shuts down at the third time period, i.e., $y_3 = 0$ and $y_2 = 1$, then inequality (4) converts to $x_1 \leq \overline{V} + V$, which is valid due to ramp-down constraints (3g).

   (3) If the generator does not shut down, i.e., $y_2 = y_3 = 1$, then inequality (4) converts to $x_1 \leq \overline{C}$, which is valid due to (3e).

We can use the similar argument as above for (4) to prove that inequalities (5) and (6) are valid.

To prove the validity of (7), we discuss the following two possible cases in terms of the value of y_2:

1) If $y_2 = 0$, then $x_2 = 0$ due to (3e). It follows that inequality (7) is valid since $x_1 \geq \underline{C} y_1$ due to (3d), $y_3 - u_3 - u_2 \geq 0$ due to (3a), and $\underline{C} + V > \overline{V}$.

2) If $y_2 = 1$, then $u_3 = 0$ due to constraints (3b) and (3c) (i.e., $y_2 \leq y_1 + u_2 \leq 1 - u_3$). We further discuss the following two possible cases in terms of the value of u_2:

   (1) If $u_2 = 1$, then $y_1 = 0$ due to (3b) and $y_3 - u_3 - u_2 = 0$ due to (3a). It follows that inequality (7) converts to $x_2 \leq \overline{V}$, which is valid due to ramp-up constraints (3f).

   (2) If $u_2 = 0$, then $y_1 = 1$ due to (3c) (i.e., $y_1 \geq y_2 - u_2$). If $y_3 = 1$, then (7) converts to $x_2 - x_1 \leq V$, which is valid due to ramp-up constraints (3f); if $y_3 = 0$, then (7) converts to $x_2 - x_1 \leq \overline{V} - \underline{C}$, which is valid since $x_2 \leq \overline{V}$ due to (3f) and $x_1 \geq \underline{C}$ due to (3d).

We can use the similar argument for (7) to prove that inequality (8) is valid.

To prove the validity of (9), we discuss the following four possible cases in terms of the values of y_1 and y_2:

1) If $y_1 = y_2 = 1$, then $u_2 = 0$ due to (3b). Inequality (9) converts to $x_1 - x_2 \leq V$, which is valid following ramp-down constraints (3g).

2) If $y_1 = 1$ and $y_2 = 0$, then $u_2 = 0$ due to (3b). Inequality (9) converts to $x_1 \leq \overline{V}$, which is valid following ramp-down constraints (3g).

3) If $y_1 = 0$ and $y_2 = 1$, then $u_2 = 1$ due to (3c). Inequality (9) converts to $x_2 \geq \underline{C}$, which is valid following (3d).

4) If $y_1 = y_2 = 0$, (9) is clearly valid.

We can use the similar argument for (9) to prove that inequality (10) is valid.

To prove the validity of (11), we discuss the following four possible cases in terms of the values of y_1 and y_3:

1) If $y_1 = y_3 = 1$, then $u_2 = u_3 = 0$ due to (3b). Inequality (11) converts to $x_3 - x_1 \leq 2V$, which is valid following ramp-up constraints (3f).

2) If $y_1 = 1$ and $y_3 = 0$, then $u_2 = u_3 = 0$ due to (3b). Inequality (11) converts to $x_1 \leq \underline{C}$, which is valid following (3d).

3) If $y_1 = 0$ and $y_3 = 1$, then $u_2 + u_3 = 1$ due to (3a) - (3c). If $u_2 = 1$, i.e., $u_3 = 0$, then (11) converts to $x_3 \leq \overline{V} + V$, which is valid following ramp-up constraints (3f); if $u_3 = 1$, i.e., $u_2 = 0$, then (11) converts to $x_3 \leq \overline{V}$, which is valid following ramp-up constraints (3f).

4) If $y_1 = y_3 = 0$, (11) is clearly valid.

We can use the similar argument for (11) to prove that inequality (12) is valid.

To prove the validity of (13), we discuss the following two possible cases in terms of the value of y_3:

1) If $y_3 = 0$, then $u_2 = u_3 = 0$ due to (3a). It follows that inequality (13) converts to $x_1 - x_2 \leq \overline{V} y_1 - (\overline{V} - V) y_2$, which can be proved to be valid following inequality (9).

2) If $y_3 = 1$, then $u_2 + u_3 \leq 1$ due to (3a). We further discuss the following three possible cases based on when the generator starts up:

   (1) If $u_2 = 0$ and $u_3 = 1$, then $y_1 = y_2 = 0$ due to (3b) and (3c). It follows that (13) converts to $x_3 \leq \overline{V}$, which is valid due to ramp-up constraints (3f).

   (2) If $u_2 = 1$ and $u_3 = 0$, then $y_1 = 0$ due to (3b). It follows that (13) converts to $x_3 - x_2 \leq V$, which is valid due to ramp-up constraints (3f).

   (3) If $u_2 = u_3 = 0$, then $y_1 = y_2 = 1$ due to (3c). It follows that (13) converts to $x_1 - x_2 + x_3 \leq V + \overline{C}$, which is valid since $x_1 - x_2 \leq V$ due to ramp-down constraints (3g) and $x_3 \leq \overline{C}$ due to (3e).

In sum, this completes the proof. ■

Now, through utilizing inequalities (4) - (13), we introduce the linear programming description of $\mathrm{conv}(P_3^2)$ by adding trivial inequalities as follows:

$$Q_3^2 := \Big\{ (x, y, u) \in \mathbb{R}^8 : (3a) - (3d), (4) - (13), \\ u_2 \geq 0,\ u_3 \geq 0 \Big\}. \tag{14}$$

Note here that the nonnegativity of x in Q_3^2 is guaranteed by (3a), (3c) - (3d), and (14). In the following, we show that Q_3^2 describes the convex hull of P_3^2, i.e., $Q_3^2 = \mathrm{conv}(P_3^2)$. We first provide the following preliminary results.

Proposition 2 *Q_3^2 is full-dimensional.*

Proof: We prove that $\dim(Q_3^2) = 8$, because there are eight decision variables in Q_3^2. We generate nine affinely independent points in Q_3^2. Since $0 \in Q_3^2$, we generate other eight linearly independent points in Q_3^2 as shown in Table 1. ■

Table 1: Eight linearly independent points in Q_3^2

x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3
$\underline{C}$	0	0	1	0	0	0	0
$\underline{C}+\epsilon$	0	0	1	0	0	0	0
$\underline{C}$	$\underline{C}$	0	1	1	0	0	0
$\underline{C}+\epsilon$	$\underline{C}+\epsilon$	0	1	1	0	0	0
$\underline{C}$	$\underline{C}$	$\underline{C}$	1	1	1	0	0
$\underline{C}+\epsilon$	$\underline{C}+\epsilon$	$\underline{C}+\epsilon$	1	1	1	0	0
0	$\underline{C}$	$\underline{C}$	0	1	1	1	0
0	0	$\underline{C}$	0	0	1	0	1

Proposition 3 *Every inequality in Q_3^2 is facet-defining for conv(P_3^2).*

Proof: The facet-defining proofs for inequalities (3a) - (3d) and (14) are trivial and thus omitted here. For inequalities (4) - (13), we provide eight affinely independent points in $\text{conv}(P_3^2)$ that satisfy each inequality at equality. Since $0 \in \text{conv}(P_3^2)$, we generate other seven linearly independent points in P_3^2, as shown in Tables 2 - 6. In particular, for inequalities (11) and (12), we consider $\overline{C} - \underline{C} - 2V > 0$ to avoid the redundancy.

Table 2: Linearly independent points for inequalities (4) and (5)

(4)								(5)							
x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3	x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3
$\overline{V}$	0	0	1	0	0	0	0	$\underline{C}$	0	0	1	0	0	0	0
$\overline{V}+V$	$\overline{V}$	0	1	1	0	0	0	$\underline{C}+\epsilon$	0	0	1	0	0	0	0
$\overline{C}$	$\overline{C}$	$\overline{C}$	1	1	1	0	0	$\overline{V}$	$\overline{V}$	0	1	1	0	0	0
0	$\underline{C}$	$\underline{C}$	0	1	1	1	0	$\overline{C}$	$\overline{C}$	$\overline{C}$	1	1	1	0	0
0	$\underline{C}+\epsilon$	$\underline{C}+\epsilon$	0	1	1	1	0	0	$\overline{V}$	$\overline{V}$	0	1	1	1	0
0	0	$\underline{C}$	0	0	1	0	1	0	0	$\underline{C}$	0	0	1	0	1
0	0	$\underline{C}+\epsilon$	0	0	1	0	1	0	0	$\underline{C}+\epsilon$	0	0	1	0	1

■

Proposition 4 *Every extreme point in Q_3^2 is integral at y and u.*

Table 3: Linearly independent points for inequalities (6) and (7)

(6)								(7)							
x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3	x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3
$\underline{C}$	0	0	1	0	0	0	0	$\underline{C}$	0	0	1	0	0	0	0
$\underline{C}+\epsilon$	0	0	1	0	0	0	0	$\underline{C}$	$\overline{V}$	0	1	1	0	0	0
$\underline{C}$	$\underline{C}$	0	1	1	0	0	0	$\underline{C}$	$\underline{C}+V$	$\underline{C}$	1	1	1	0	0
$\underline{C}+\epsilon$	$\underline{C}+\epsilon$	0	1	1	0	0	0	$\underline{C}+\epsilon$	$\underline{C}+V+\epsilon$	$\underline{C}+\epsilon$	1	1	1	0	0
$\overline{C}$	$\overline{C}$	$\overline{C}$	1	1	1	0	0	0	$\overline{V}$	$\overline{V}$	0	1	1	1	0
0	$\overline{V}$	$\overline{V}+V$	0	1	1	1	0	0	0	$\underline{C}$	0	0	1	0	1
0	0	$\overline{V}$	0	0	1	0	1	0	0	$\underline{C}+\epsilon$	0	0	1	0	1

Table 4: Linearly independent points for inequalities (8) and (9)

(8)								(9)							
x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3	x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3
$\underline{C}$	0	0	1	0	0	0	0	$\overline{V}$	0	0	1	0	0	0	0
$\underline{C}+\epsilon$	0	0	1	0	0	0	0	$\underline{C}+V$	$\underline{C}$	0	1	1	0	0	0
$\underline{C}$	$\underline{C}$	0	1	1	0	0	0	$\underline{C}+V+\epsilon$	$\underline{C}+\epsilon$	0	1	1	0	0	0
$\underline{C}$	$\underline{C}$	$\underline{C}+V$	1	1	1	0	0	$\underline{C}+V$	$\underline{C}$	$\underline{C}$	1	1	1	0	0
$\underline{C}+\epsilon$	$\underline{C}+\epsilon$	$\underline{C}+V+\epsilon$	1	1	1	0	0	0	$\underline{C}$	$\underline{C}$	0	1	1	1	0
0	$\underline{C}$	$\underline{C}+V$	0	1	1	1	0	0	0	$\underline{C}$	0	0	1	0	1
0	0	$\overline{V}$	0	0	1	0	1	0	0	$\underline{C}+\epsilon$	0	0	1	0	1

Table 5: Linearly independent points for inequalities (10) and (11)

(10)								(11)							
x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3	x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3
$\underline{C}$	0	0	1	0	0	0	0	$\underline{C}$	0	0	1	0	0	0	0
$\underline{C}+\epsilon$	0	0	1	0	0	0	0	$\underline{C}$	$\underline{C}$	0	1	1	0	0	0
$\overline{V}$	$\overline{V}$	0	1	1	0	0	0	$\underline{C}$	$\underline{C}+\epsilon$	0	1	1	0	0	0
$\underline{C}+V$	$\underline{C}+V$	$\underline{C}$	1	1	1	0	0	$\underline{C}$	$\underline{C}+V$	$\underline{C}+2V$	1	1	1	0	0
$\underline{C}+V+\epsilon$	$\underline{C}+V+\epsilon$	$\underline{C}+\epsilon$	1	1	1	0	0	$\underline{C}+\epsilon$	$\underline{C}+V+\epsilon$	$\underline{C}+2V+\epsilon$	1	1	1	0	0
0	$\overline{V}$	$\underline{C}$	0	1	1	1	0	0	$\overline{V}$	$\overline{V}+V$	0	1	1	1	0
0	0	$\underline{C}$	0	0	1	0	1	0	0	$\overline{V}$	0	0	1	0	1

Proof: It is sufficient to prove that every point $z \in Q_3^2$ can be written as $z = \sum_{s \in S} \lambda_s z^s$ for some $\lambda_s \geq 0$ and $\sum_{s \in S} \lambda_s = 1$, where $z^s \in Q_3^2, s \in S$ with y and u binary and S is the index set for the candidate points.

For a given point $z = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{u}_2, \bar{u}_3) \in Q_3^2$, we pick $z^1, z^2, \cdots, z^6 \in Q_3^2$ such that $z^1 = (\hat{x}_1, 0, 0, 1, 0, 0, 0, 0)$, $z^2 = (\hat{x}_2, \hat{x}_3, 0, 1, 1, 0, 0, 0)$, $z^3 = (\hat{x}_4, \hat{x}_5, \hat{x}_6, 1, 1, 1, 0, 0)$, $z^4 = (0, \hat{x}_7, \hat{x}_8, 0, 1, 1, 1, 0)$, $z^5 = (0, 0, \hat{x}_9, 0, 0, 1, 0, 1)$, and $z^6 = (0, 0, 0, 0, 0, 0, 0, 0)$. In addition, we let $\lambda_1 = \bar{y}_1 - \bar{y}_2 + \bar{u}_2$, $\lambda_2 = \bar{y}_2 - \bar{y}_3 + \bar{u}_3$, $\lambda_3 = \bar{y}_3 - \bar{u}_2 - \bar{u}_3$, $\lambda_4 = \bar{u}_2$, $\lambda_5 = \bar{u}_3$, and $\lambda_6 = 1 - \bar{y}_1 - \bar{u}_2 - \bar{u}_3$.

Table 6: Linearly independent points for inequalities (12) and (13)

(12)								(13)							
x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3	x_1	x_2	x_3	y_1	y_2	y_3	u_2	u_3
$\overline{V}$	0	0	1	0	0	0	0	$\overline{V}$	0	0	1	0	0	0	0
$\overline{V}+V$	$\overline{V}$	0	1	1	0	0	0	$\underline{C}+V$	$\underline{C}$	0	1	1	0	0	0
$\underline{C}+2V$	$\underline{C}+V$	$\underline{C}$	1	1	1	0	0	$\underline{C}+V+\epsilon$	$\underline{C}+\epsilon$	0	1	1	0	0	0
$\underline{C}+2V+\epsilon$	$\underline{C}+V+\epsilon$	$\underline{C}+\epsilon$	1	1	1	0	0	$\overline{C}$	$\overline{C}-V$	$\overline{C}$	1	1	1	0	0
0	$\underline{C}$	$\underline{C}$	0	1	1	1	0	0	$\underline{C}$	$\underline{C}+V$	0	1	1	1	0
0	$\underline{C}+\epsilon$	$\underline{C}$	0	1	1	1	0	0	$\underline{C}+\epsilon$	$\underline{C}+V+\epsilon$	0	1	1	1	0
0	0	$\underline{C}$	0	0	1	0	1	0	0	$\overline{V}$	0	0	1	0	1

First of all, it is clear that $\sum_{s=1}^{6} \lambda_s = 1$ and $\lambda_s \geq 0$ for $\forall s = 1, \cdots, 6$ due to (3a) - (3c) and (14).

Next, it is also obvious that $\bar{y}_i = y_i(z) = \sum_{s=1}^{6} \lambda_s y_i(z^s)$ for $i = 1, 2, 3$ and $\bar{u}_i = u_i(z) = \sum_{s=1}^{6} \lambda_s u_i(z^s)$ for $i = 2, 3$. In the following, we decide the values of $\hat{x}_i$ for $i = 1, \cdots, 9$ and show $\bar{x}_i = x_i(z) = \sum_{s=1}^{6} \lambda_s x_i(z^s)$ for $i = 1, 2, 3$, i.e., $\bar{x}_1 = \lambda_1 \hat{x}_1 + \lambda_2 \hat{x}_2 + \lambda_3 \hat{x}_4$, $\bar{x}_2 = \lambda_2 \hat{x}_3 + \lambda_3 \hat{x}_5 + \lambda_4 \hat{x}_7$, and $\bar{x}_3 = \lambda_3 \hat{x}_6 + \lambda_4 \hat{x}_8 + \lambda_5 \hat{x}_9$. Note that y and u are given in $z^1, \cdots, z^6$, the corresponding feasible region for $(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9)$ can be described as set $A = \{(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^9 : \underline{C} \leq \hat{x}_1 \leq \overline{V},\ \underline{C} \leq \hat{x}_2 \leq \overline{V} + V,\ \underline{C} \leq \hat{x}_3 \leq \overline{V},\ -V \leq \hat{x}_3 - \hat{x}_2 \leq \overline{V} - \underline{C},\ \underline{C} \leq \hat{x}_4 \leq \overline{C},\ \underline{C} \leq \hat{x}_5 \leq \overline{C},\ \underline{C} \leq \hat{x}_6 \leq \overline{C},\ -V \leq \hat{x}_5 - \hat{x}_4 \leq V,\ -V \leq \hat{x}_6 - \hat{x}_5 \leq V,\ \underline{C} \leq \hat{x}_7 \leq \overline{V},\ \underline{C} \leq \hat{x}_8 \leq \overline{V} + V,\ \underline{C} - \overline{V} \leq \hat{x}_8 - \hat{x}_7 \leq V,\ \underline{C} \leq \hat{x}_9 \leq \overline{V}\}$. To show $\bar{x}_i = \sum_{s=1}^{6} \lambda_s x_i(z^s)$ for $i = 1, 2, 3$, equivalently we prove that fixing $(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{u}_2, \bar{u}_3) \in B = \{(y_1, y_2, y_3, u_2, u_3) \in [0, 1]^5 : \text{(3a)} - \text{(3c)}, \text{(14)}\}$, for $\forall (\bar{x}_1, \bar{x}_2, \bar{x}_3)$ belonging the set

$$C = \Big\{ (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \mathbb{R}^3 : \quad \bar{x}_1 \geq \underline{C}\bar{y}_1,\ \bar{x}_2 \geq \underline{C}\bar{y}_2,\ \bar{x}_3 \geq \underline{C}\bar{y}_3, \tag{15a}$$

$$\bar{x}_1 \leq \overline{V}\bar{y}_1 + V(\bar{y}_2 - \bar{u}_2) + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2), \tag{15b}$$

$$\bar{x}_2 \leq \overline{V}\bar{y}_2 + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2), \tag{15c}$$

$$\bar{x}_3 \leq \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \overline{V} - V)\bar{u}_2, \tag{15d}$$

$$\bar{x}_2 - \bar{x}_1 \leq \overline{V}\bar{y}_2 - \underline{C}\bar{y}_1 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2), \tag{15e}$$

$$\bar{x}_3 - \bar{x}_2 \leq (\underline{C} + V)\bar{y}_3 - \underline{C}\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_3, \tag{15f}$$

$$\bar{x}_1 - \bar{x}_2 \leq \overline{V}\bar{y}_1 - (\overline{V} - V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2, \tag{15g}$$

$$\bar{x}_2 - \bar{x}_3 \leq \overline{V}\bar{y}_2 - \underline{C}\bar{y}_3 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2), \tag{15h}$$

$$\bar{x}_3 - \bar{x}_1 \leq (\underline{C} + 2V)\bar{y}_3 - \underline{C}\bar{y}_1 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2, \tag{15i}$$

$$\left.\begin{aligned} \bar{x}_1 - \bar{x}_3 &\le \overline{V}\bar{y}_1 - \underline{C}\bar{y}_3 + V(\bar{y}_2 - \bar{u}_2) + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2), \quad (15j)\\ \bar{x}_1 - \bar{x}_2 + \bar{x}_3 &\le \overline{V}\bar{y}_1 - (\overline{V} - V)\bar{y}_2 + \overline{V}\bar{y}_3 + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \end{aligned}\right\}, \quad (15k)$$

there exists $(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9) \in A$ such that

$$\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4,\ \bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7,\ \bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9, \qquad (16)$$

i.e., the linear transformation $F : A \to C$ is surjective, where

$$F = \begin{pmatrix} \bar{y}_1 - \bar{y}_2 + \bar{u}_2 & \bar{y}_2 - \bar{y}_3 + \bar{u}_3 & 0 & \bar{y}_3 - \bar{u}_2 - \bar{u}_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{y}_2 - \bar{y}_3 + \bar{u}_3 & 0 & \bar{y}_3 - \bar{u}_2 - \bar{u}_3 & 0 & \bar{u}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{y}_3 - \bar{u}_2 - \bar{u}_3 & 0 & \bar{u}_2 & \bar{u}_3 \end{pmatrix}.$$

Since C is a closed and bounded polytope, any point can be expressed as a convex combination of the extreme points in C. Accordingly, we only need to show that for any extreme point $w^i \in C$ ($i = 1, \cdots, M$), there exists a point $p^i \in A$ such that $Fp^i = w^i$, where M represents the number of extreme points in C (because for an arbitrary point $w \in C$, which can be rewritten as $w = \sum_{i=1}^{M} \mu_i w^i$ and $\sum_{i=1}^{M} \mu_i = 1$, there exists $p = \sum_{i=1}^{M} \mu_i p_i \in A$ such that $Fp = w$ due to the linearity of F and the convexity of A). Since it is difficult to enumerate all the extreme points in C, in the following proof we show the conclusion holds for any point in the faces of C, i.e., satisfying one of (15a) - (15k) at equality, which implies the conclusion holds for extreme points.

Satisfying $\bar{x}_1 \ge \underline{C}\bar{y}_1$ at equality. For this case, substituting $\bar{x}_1 = \underline{C}\bar{y}_1$ into (15b) - (15k), we obtain the feasible region of $(\bar{x}_2, \bar{x}_3)$ as $C' = \{(\bar{x}_2, \bar{x}_3) \in \mathbb{R}^2 : \underline{C}\bar{y}_2 \le \bar{x}_2 \le \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2),\ \underline{C}\bar{y}_3 \le \bar{x}_3 \le (\underline{C} + 2V)\bar{y}_3 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2,\ \bar{x}_3 - \bar{x}_2 \le (\underline{C} + V)\bar{y}_3 - \underline{C}\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_3\}$.

First, by letting $\hat{x}_1 = \hat{x}_2 = \hat{x}_4 = \underline{C}$, it is easy to check that $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$, following (16). Note here that once $(\hat{x}_1, \hat{x}_2, \hat{x}_4)$ fixed, the corresponding feasible region for $(\hat{x}_3, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9)$ can be described as set $A' = \{(\hat{x}_3, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^6 : \underline{C} \le \hat{x}_3 \le \overline{V},\ \underline{C} \le \hat{x}_5 \le \underline{C} + V,\ \underline{C} \le \hat{x}_6 \le \overline{C},\ -V \le \hat{x}_6 - \hat{x}_5 \le V,\ \underline{C} \le \hat{x}_7 \le \overline{V},\ \underline{C} \le \hat{x}_8 \le \overline{V} + V,\ \underline{C} - \overline{V} \le \hat{x}_8 - \hat{x}_7 \le V,\ \underline{C} \le \hat{x}_9 \le \overline{V}\}$. In the following, we repeat the argument above to consider that one of inequalities in C' is satisfied at equality to obtain the values of $(\hat{x}_3, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9)$ from A'.

1) Satisfying $\bar{x}_2 \ge \underline{C}\bar{y}_2$ at equality. We obtain $\bar{x}_3 \in C'' = \{\bar{x}_3 \in \mathbb{R} : \underline{C}\bar{y}_3 \le \bar{x}_3 \le (\underline{C} + V)\bar{y}_3 - (\underline{C} + V - \overline{V})\bar{u}_3\}$ through substituting $\bar{x}_2 = \underline{C}\bar{y}_2$ into C'. By letting $\hat{x}_3 = \hat{x}_5 = \hat{x}_7 = \underline{C}$, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$, following (16). Thus, the corresponding feasible region for

$(\hat{x}_6, \hat{x}_8, \hat{x}_9)$ can be described as set $A'' = \{(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^3 : \underline{C} \le \hat{x}_6 \le \underline{C} + V,\ \underline{C} \le \hat{x}_8 \le \underline{C} + V,\ \underline{C} \le \hat{x}_9 \le \overline{V}\}$. If $\bar{x}_3 \ge \underline{C}\bar{y}_3$ is satisfied at equality, we let $\hat{x}_6 = \hat{x}_8 = \hat{x}_9 = \underline{C}$; if $\bar{x}_3 \le (\underline{C} + V)\bar{y}_3 - (\underline{C} + V - \overline{V})\bar{u}_3$ is satisfied equality, we let $\hat{x}_6 = \hat{x}_8 = \underline{C} + V$ and $\hat{x}_9 = \overline{V}$. It is easy to check that $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$.

2) Satisfying $\bar{x}_2 \le \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ at equality. We obtain $\bar{x}_3 \in C'' = \{\bar{x}_3 \in \mathbb{R} : \underline{C}\bar{y}_3 \le \bar{x}_3 \le (\underline{C} + 2V)\bar{y}_3 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2\}$. By letting $\hat{x}_3 = \hat{x}_7 = \overline{V}$ and $\hat{x}_5 = \underline{C} + V$, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$, following (16). Thus, the corresponding feasible region for $(\hat{x}_6, \hat{x}_8, \hat{x}_9)$ can be described as set $A'' = \{(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^3 : \underline{C} \le \hat{x}_6 \le \underline{C} + 2V,\ \underline{C} \le \hat{x}_8 \le \overline{V} + V,\ \underline{C} \le \hat{x}_9 \le \overline{V}\}$. If $\bar{x}_3 \ge \underline{C}\bar{y}_3$ is satisfied at equality, we let $\hat{x}_6 = \hat{x}_8 = \hat{x}_9 = \underline{C}$; if $\bar{x}_3 \le (\underline{C} + 2V)\bar{y}_3 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2$ is satisfied equality, we let $\hat{x}_6 = \underline{C} + 2V$, $\hat{x}_8 = \overline{V} + V$ and $\hat{x}_9 = \overline{V}$. In this way, we have $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$.

3) Satisfying $\bar{x}_3 \ge \underline{C}\bar{y}_3$ at equality. We obtain $\bar{x}_2 \in C'' = \{\bar{x}_2 \in \mathbb{R} : \underline{C}\bar{y}_2 \le \bar{x}_2 \le \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)\}$. By letting $\hat{x}_6 = \hat{x}_8 = \hat{x}_9 = \underline{C}$, we have $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$, following (16). Thus, the corresponding feasible region for $(\hat{x}_3, \hat{x}_5, \hat{x}_7)$ can be described as set $A'' = \{(\hat{x}_3, \hat{x}_5, \hat{x}_7) \in \mathbb{R}^3 : \underline{C} \le \hat{x}_3 \le \overline{V},\ \underline{C} \le \hat{x}_5 \le \underline{C} + V,\ \underline{C} \le \hat{x}_7 \le \overline{V}\}$. If $\bar{x}_2 \ge \underline{C}\bar{y}_2$ is satisfied at equality, we let $\hat{x}_3 = \hat{x}_5 = \hat{x}_7 = \underline{C}$; if $\bar{x}_2 \le \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_3 = \hat{x}_7 = \overline{V}$ and $\hat{x}_5 = \underline{C} + V$.

4) Satisfying $\bar{x}_3 \le (\underline{C} + 2V)\bar{y}_3 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2$ at equality. We obtain $\bar{x}_2 \in C'' = \{\bar{x}_2 \in \mathbb{R} : \underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) + (\overline{V} - \underline{C})\bar{u}_2 \le \bar{x}_2 \le \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)\}$. By letting $\hat{x}_6 = \underline{C} + 2V$, $\hat{x}_8 = \overline{V} + V$, and $\hat{x}_9 = \overline{V}$, we have $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$, following (16). Thus, the corresponding feasible region for $(\hat{x}_3, \hat{x}_5, \hat{x}_7)$ can be described as set $A'' = \{(\hat{x}_3, \hat{x}_5, \hat{x}_7) \in \mathbb{R}^3 : \underline{C} \le \hat{x}_3 \le \overline{V}, \hat{x}_5 = \underline{C} + V, \hat{x}_7 = \overline{V}\}$. If $\bar{x}_2 \ge \underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) + (\overline{V} - \underline{C})\bar{u}_2$ is satisfied at equality, we let $\bar{x}_3 = \underline{C}$; if $\bar{x}_2 \le \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ is satisfied at equality, we let $\bar{x}_3 = \overline{V}$.

5) Satisfying $\bar{x}_3 - \bar{x}_2 \le (\underline{C} + V)\bar{y}_3 - \underline{C}\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_3$ at equality. We obtain $\bar{x}_2 \in C'' = \{\bar{x}_2 \in \mathbb{R} : \underline{C}\bar{y}_2 \le \bar{x}_2 \le \underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3) - (\underline{C} + V - \overline{V})\bar{u}_2\}$ through substituting $\bar{x}_3 = \bar{x}_2 + (\underline{C} + V)\bar{y}_3 - \underline{C}\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_3$ into set C'. By letting $\hat{x}_3 = \underline{C}$, $\hat{x}_9 = \overline{V}$, and $\hat{x}_6 - \hat{x}_5 = \hat{x}_8 - \hat{x}_7 = V$, we have $\bar{x}_3 - \bar{x}_2 = (\lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9) - (\lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7)$. If $\bar{x}_2 = \underline{C}\bar{y}_2$, we let $\hat{x}_3 = \hat{x}_5 = \hat{x}_7 = \underline{C}$;

if $\bar{x}_2 = \underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3) - (\underline{C} + V - \overline{V})\bar{u}_2$, we let $\hat{x}_3 = \underline{C}$, $\hat{x}_5 = \underline{C} + V$, and $\hat{x}_7 = \overline{V}$. For both cases, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$ and thus $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$.

Similar analyses hold for $\bar{x}_2 \geq \underline{C}\bar{y}_2$ and $\bar{x}_3 \geq \underline{C}\bar{y}_3$ due to the similar structure between $\bar{x}_1 \geq \underline{C}\bar{y}_1$, $\bar{x}_2 \geq \underline{C}\bar{y}_2$, and $\bar{x}_3 \geq \underline{C}\bar{y}_3$ and thus are omitted here.

Satisfying (15b) **at equality**. For this case, substituting $\bar{x}_1 = \overline{V}\bar{y}_1 + V(\bar{y}_2 - \bar{u}_2) + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ into (15e) - (15k), we obtain the feasible region of $(\bar{x}_2, \bar{x}_3)$ as $C' = \{(\bar{x}_2, \bar{x}_3) \in \mathbb{R}^2 : \overline{V}\bar{y}_2 - (\overline{C} - \underline{C} - V)\bar{u}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3) \leq \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2),\ \underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \leq \bar{x}_3 \leq \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \overline{V} - V)\bar{u}_2,\ \bar{x}_3 - \bar{x}_2 \leq (\overline{V} + V)\bar{y}_3 - \overline{V}\bar{y}_2 - V\bar{u}_3,\ \bar{x}_2 - \bar{x}_3 \leq \overline{V}\bar{y}_2 - \underline{C}\bar{y}_3 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)\}$.

First, by letting $\hat{x}_1 = \overline{V}$, $\hat{x}_2 = \overline{V} + V$, and $\hat{x}_4 = \overline{C}$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$. Then the corresponding feasible region for $(\hat{x}_3, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9)$ can be described as set $A' = \{(\hat{x}_3, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^6 : \hat{x}_3 = \overline{V},\ \overline{C} - V \leq \hat{x}_5 \leq \overline{C},\ \underline{C} \leq \hat{x}_6 \leq \overline{C},\ -V \leq \hat{x}_6 - \hat{x}_5 \leq V,\ \underline{C} \leq \hat{x}_7 \leq \overline{V},\ \underline{C} \leq \hat{x}_8 \leq \overline{V} + V,\ \underline{C} - \overline{V} \leq \hat{x}_8 - \hat{x}_7 \leq V,\ \underline{C} \leq \hat{x}_9 \leq \overline{V}\}$. We consider that one of inequalities in C' is satisfied at equality to obtain the values of $(\hat{x}_3, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9)$ from A' as follows.

1) Satisfying $\bar{x}_2 \geq \overline{V}\bar{y}_2 - (\overline{C} - \underline{C} - V)\bar{u}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3)$ at equality. We obtain $\underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \leq \bar{x}_3 \leq \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \underline{C} - V)\bar{u}_2$. By letting $\hat{x}_3 = \overline{V}$, $\hat{x}_5 = \overline{C} - V$, and $\hat{x}_7 = \underline{C}$, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$. As a result, we have $(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in A'' = \{(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^3 : \overline{C} - 2V \leq \hat{x}_6 \leq \overline{C},\ \underline{C} \leq \hat{x}_8 \leq \underline{C} + V,\ \underline{C} \leq \hat{x}_9 \leq \overline{V}\}$. If $\bar{x}_3 = \underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_6 = \overline{C} - 2V$ and $\hat{x}_8 = \hat{x}_9 = \underline{C}$; if $\bar{x}_3 = \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \underline{C} - V)\bar{u}_2$, we let $\hat{x}_6 = \overline{C}$, $\hat{x}_8 = \underline{C} + V$, and $\hat{x}_9 = \overline{V}$.

2) Satisfying $\bar{x}_2 \leq \overline{V}\bar{y}_2 + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ at equality. We obtain $\underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \leq \bar{x}_3 \leq \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \overline{V} - V)\bar{u}_2$. By letting $\hat{x}_3 = \hat{x}_7 = \overline{V}$ and $\hat{x}_5 = \overline{C}$, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$. As a result, we have $(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in A'' = \{(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^3 : \overline{C} - V \leq \hat{x}_6 \leq \overline{C},\ \underline{C} \leq \hat{x}_8 \leq \overline{V} + V,\ \underline{C} \leq \hat{x}_9 \leq \overline{V}\}$. If $\bar{x}_3 = \underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_6 = \overline{C} - V$ and $\hat{x}_8 = \hat{x}_9 = \underline{C}$; if $\bar{x}_3 = \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \overline{V} - V)\bar{u}_2$, we let $\hat{x}_6 = \overline{C}$, $\hat{x}_8 = \overline{V} + V$, and $\hat{x}_9 = \overline{V}$.

3) Satisfying $\bar{x}_3 \geq \underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ at equality. We obtain $\overline{V}\bar{y}_2 - (\overline{C} - \underline{C} - V)\bar{u}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3) \leq \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_6 = \overline{C} - 2V$ and $\hat{x}_8 = \hat{x}_9 = \underline{C}$,

we have $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$. As a result, we have $(\hat{x}_3, \hat{x}_5, \hat{x}_7) \in A'' = \{(\hat{x}_3, \hat{x}_5, \hat{x}_7) \in \mathbb{R}^3 : \hat{x}_3 = \overline{V},\ \hat{x}_5 = \overline{C} - V,\ \underline{C} \le \hat{x}_7 \le \overline{V}\}$. If $\bar{x}_2 = \overline{V}\bar{y}_2 - (\overline{C} - \underline{C} - V)\bar{u}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3)$, we let $\hat{x}_7 = \underline{C}$; if $\bar{x}_2 \le \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_7 = \overline{V}$.

4) Satisfying $\bar{x}_3 \le \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \overline{V} - V)\bar{u}_2$ at equality. We obtain $\overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \le \bar{x}_2 \le \overline{V}\bar{y}_2 + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_6 = \overline{C}$, $\hat{x}_8 = \overline{V} + V$, and $\hat{x}_9 = \overline{V}$, we have $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$. As a result, we have $(\hat{x}_3, \hat{x}_5, \hat{x}_7) \in A'' = \{(\hat{x}_3, \hat{x}_5, \hat{x}_7) \in \mathbb{R}^3 : \hat{x}_3 = \overline{V},\ \overline{C} - V \le \hat{x}_5 \le \overline{C},\ \hat{x}_7 = \overline{V}\}$. If $\bar{x}_2 = \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_5 = \overline{C} - V$; if $\bar{x}_2 = \overline{V}\bar{y}_2 + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_5 = \overline{C}$.

5) Satisfying $\bar{x}_3 - \bar{x}_2 \le (\overline{V} + V)\bar{y}_3 - \overline{V}\bar{y}_2 - V\bar{u}_3$ at equality. We obtain $\overline{V}\bar{y}_2 - (\overline{C} - \underline{C} - V)\bar{u}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3) \le \bar{x}_2 \le \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ through substituting $\bar{x}_3 = \bar{x}_2 + (\overline{V} + V)\bar{y}_3 - \overline{V}\bar{y}_2 - V\bar{u}_3$ into set C'. By letting $\hat{x}_3 = \hat{x}_9 = \overline{V}$ and $\hat{x}_6 - \hat{x}_5 = \hat{x}_8 - \hat{x}_7 = V$, we have $\bar{x}_3 - \bar{x}_2 = (\lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9) - (\lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7)$. If $\bar{x}_2 = \overline{V}\bar{y}_2 - (\overline{C} - \underline{C} - V)\bar{u}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3)$, we let $\hat{x}_5 = \overline{C} - V$ and $\hat{x}_7 = \underline{C}$; if $\bar{x}_2 \le \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_5 = \overline{C} - V$ and $\hat{x}_7 = \overline{V}$. For both cases, we have $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$ and thus $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$.

6) Satisfying $\bar{x}_2 - \bar{x}_3 \le \overline{V}\bar{y}_2 - \underline{C}\bar{y}_3 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ at equality. We obtain $\underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \le \bar{x}_3 \le \underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_3 = \overline{V}$, $\hat{x}_5 - \hat{x}_6 = V$, $\hat{x}_8 - \hat{x}_7 = \underline{C} - \overline{V}$, and $\hat{x}_9 = \underline{C}$, we have $\bar{x}_2 - \bar{x}_3 = (\lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7) - (\lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9)$. If $\bar{x}_3 = \underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_6 = \overline{C} - 2V$ and $\hat{x}_8 = \hat{x}_9 = \underline{C}$; if $\bar{x}_3 = \underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_6 = \overline{C} - V$ and $\hat{x}_8 = \hat{x}_9 = \underline{C}$.

Similar analyses hold for (15c) and (15d) due to the similar structure between (15b), (15c), and (15d) and thus are omitted here.

Satisfying (15e) **at equality**. For this case, substituting $\bar{x}_2 = \bar{x}_1 + \overline{V}\bar{y}_2 - \underline{C}\bar{y}_1 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ into (15a) - (15k), we obtain the feasible region of $(\bar{x}_1, \bar{x}_3)$ as $C' = \{(\bar{x}_1, \bar{x}_3) \in \mathbb{R}^2 : \underline{C}\bar{y}_1 \le \bar{x}_1 \le \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2),\ \underline{C}\bar{y}_3 \le \bar{x}_3 \le \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \overline{V} - V)\bar{u}_2,\ \bar{x}_3 - \bar{x}_1 \le (\underline{C} + 2V)\bar{y}_3 - \underline{C}\bar{y}_1 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2,\ \bar{x}_1 - \bar{x}_3 \le \underline{C}\bar{y}_1 - \underline{C}\bar{y}_3\}$.

First, by letting $\hat{x}_1 = \underline{C}$, $\hat{x}_3 - \hat{x}_2 = \overline{V} - \underline{C}$, $\hat{x}_5 - \hat{x}_4 = V$, and $\hat{x}_7 = \overline{V}$, we have $\bar{x}_2 - \bar{x}_1 = (\lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7) - (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4)$. Since $\underline{C} \le \hat{x}_3 \le \overline{V}$, it follows that $\hat{x}_2 = \underline{C}$ and $\hat{x}_3 = \overline{V}$. Then the corresponding feasible region for $(\hat{x}_4, \hat{x}_6, \hat{x}_8, \hat{x}_9)$ can be described as set $A' = \{(\hat{x}_4, \hat{x}_6, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^6 : \underline{C} \le \hat{x}_4 \le \overline{C} - V,\ \underline{C} \le \hat{x}_6 \le \overline{C},\ 0 \le \hat{x}_6 - \hat{x}_4 \le 2V,\ \underline{C} \le \hat{x}_8 \le \overline{V} + V,\ \underline{C} \le \hat{x}_9 \le \overline{V}\}$. Next, we only need to show $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$ and $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$. We consider that one of inequalities in C' is satisfied at equality to obtain the values of $(\hat{x}_4, \hat{x}_6, \hat{x}_8, \hat{x}_9)$ from A' as follows.

1) Satisfying $\bar{x}_1 \ge \underline{C}\bar{y}_1$ at equality. We obtain $\underline{C}\bar{y}_3 \le \bar{x}_3 \le (\underline{C} + 2V)\bar{y}_3 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2$. By letting $\hat{x}_4 = \underline{C}$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$. As a result, we have $(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in A'' = \{(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^3 : \underline{C} \le \hat{x}_6 \le \underline{C} + 2V,\ \underline{C} \le \hat{x}_8 \le \overline{V} + V,\ \underline{C} \le \hat{x}_9 \le \overline{V}\}$. If $\bar{x}_3 = \underline{C}\bar{y}_3$, we let $\hat{x}_6 = \hat{x}_8 = \hat{x}_9 = \underline{C}$; if $\bar{x}_3 = (\underline{C} + 2V)\bar{y}_3 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2$, we let $\hat{x}_6 = \underline{C} + 2V$, $\hat{x}_8 = \overline{V} + V$, and $\hat{x}_9 = \overline{V}$.

2) Satisfying $\bar{x}_1 \le \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ at equality. We obtain $\underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \le \bar{x}_3 \le \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \overline{V} - V)\bar{u}_2$. By letting $\hat{x}_4 = \overline{C} - V$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$. As a result, we have $(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in A'' = \{(\hat{x}_6, \hat{x}_8, \hat{x}_9) \in \mathbb{R}^3 : \overline{C} - V \le \hat{x}_6 \le \overline{C},\ \underline{C} \le \hat{x}_8 \le \overline{V} + V,\ \underline{C} \le \hat{x}_9 \le \overline{V}\}$. If $\bar{x}_3 = \underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_6 = \overline{C} - V$ and $\hat{x}_8 = \hat{x}_9 = \underline{C}$; if $\bar{x}_3 = \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \overline{V} - V)\bar{u}_2$, we let $\hat{x}_6 = \overline{C}$, $\hat{x}_8 = \overline{V} + V$, and $\hat{x}_9 = \overline{V}$.

3) Satisfying $\bar{x}_3 \ge \underline{C}\bar{y}_3$ at equality. We obtain $\bar{x}_1 = \underline{C}\bar{y}_1$ since $\bar{x}_1 - \bar{x}_3 \le \underline{C}\bar{y}_1 - \underline{C}\bar{y}_3$. By letting $\hat{x}_4 = \hat{x}_6 = \hat{x}_8 = \hat{x}_9 = \underline{C}$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$ and $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$.

4) Satisfying $\bar{x}_3 \le \overline{C}\bar{y}_3 - (\overline{C} - \overline{V})\bar{u}_3 - (\overline{C} - \overline{V} - V)\bar{u}_2$ at equality. We obtain $\underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \le \bar{x}_1 \le \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_6 = \overline{C}$, $\hat{x}_8 = \overline{V} + V$, and $\hat{x}_9 = \overline{V}$, we have $\bar{x}_3 = \lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9$. Thus it follows that $\overline{C} - 2V \le \hat{x}_4 \le \overline{C} - V$. If $\bar{x}_1 = \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_4 = \overline{C} - 2V$; if $\bar{x}_1 = \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_4 = \overline{C} - V$.

5) Satisfying $\bar{x}_3 - \bar{x}_1 \le (\underline{C} + 2V)\bar{y}_3 - \underline{C}\bar{y}_1 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2$ at equality. We obtain $\underline{C}\bar{y}_1 \le \bar{x}_1 \le \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_6 - \hat{x}_4 = 2V$, $\hat{x}_8 = \overline{V} + V$, and $\hat{x}_9 = \overline{V}$, we have $\bar{x}_3 - \bar{x}_1 = (\lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9) - (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4)$. If $\bar{x}_1 = \underline{C}\bar{y}_1$, we let $\hat{x}_4 = \underline{C}$ and thus $\hat{x}_6 = \underline{C} + 2V$; if $\bar{x}_1 \le \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_4 = \overline{C} - 2V$.

6) Satisfying $\bar{x}_1 - \bar{x}_3 \le \underline{C}\bar{y}_1 - \underline{C}\bar{y}_3$ at equality. We obtain $\underline{C}\bar{y}_3 \le \bar{x}_3 \le \underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$.

By letting $\hat{x}_4 = \hat{x}_6$, $\hat{x}_8 = \hat{x}_9 = \underline{C}$, we have $\bar{x}_1 - \bar{x}_3 = (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4) - (\lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9)$. If $\bar{x}_3 = \underline{C}\bar{y}_3$, we let $\hat{x}_6 = \underline{C}$; if $\bar{x}_3 \leq \underline{C}\bar{y}_3 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_6 = \overline{C} - V$.

Similar analyses hold for (15f) - (15h) due to the similar structure between (15e) and (15f) - (15h) and thus are omitted here.

Satisfying (15i) **at equality**. For this case, substituting $\bar{x}_3 = \bar{x}_1 + (\underline{C} + 2V)\bar{y}_3 - \underline{C}\bar{y}_1 - (\underline{C} + 2V - \overline{V})\bar{u}_3 - (\underline{C} + V - \overline{V})\bar{u}_2$ into (15a) - (15k), we obtain the feasible region of $(\bar{x}_1, \bar{x}_2)$ as $C' = \{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 : \underline{C}\bar{y}_1 \leq \bar{x}_1 \leq \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2),\ \bar{x}_2 - \bar{x}_1 \leq \overline{V}\bar{y}_2 - \underline{C}\bar{y}_1 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2),\ \bar{x}_1 - \bar{x}_2 \leq \underline{C}\bar{y}_1 - \underline{C}\bar{y}_2 - V\bar{y}_3 + V\bar{u}_3 + (\underline{C} + V - \overline{V})\bar{u}_2\}$.

First, by letting $\hat{x}_1 = \hat{x}_2 = \underline{C}$, $\hat{x}_6 - \hat{x}_4 = 2V$, $\hat{x}_8 = \overline{V} + V$, and $\hat{x}_9 = \overline{V}$, we have $\bar{x}_3 - \bar{x}_1 = (\lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9) - (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4)$. Since $\underline{C} \leq \hat{x}_7 \leq \overline{V}$ and $\underline{C} - \overline{V} \leq \hat{x}_8 - \hat{x}_7 \leq V$, we have $\hat{x}_7 = \overline{V}$. Then the corresponding feasible region for $(\hat{x}_3, \hat{x}_4, \hat{x}_5)$ can be described as set $A' = \{(\hat{x}_3, \hat{x}_4, \hat{x}_5) \in \mathbb{R}^3 : \underline{C} \leq \hat{x}_3 \leq \overline{V},\ \underline{C} \leq \hat{x}_4 \leq \overline{C} - 2V,\ \underline{C} \leq \hat{x}_5 \leq \overline{C} - V, \hat{x}_5 - \hat{x}_4 = V\}$. Next, we only need to show $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$ and $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$. We consider that one of inequalities in C' is satisfied at equality to obtain the values of $(\hat{x}_3, \hat{x}_4, \hat{x}_5)$ from A' as follows.

1) Satisfying $\bar{x}_1 \geq \underline{C}\bar{y}_1$ at equality. We obtain $\underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3) - (\underline{C} + V - \overline{V})\bar{u}_2 \leq \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_4 = \underline{C}$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$. As a result, we have $\underline{C} \leq \hat{x}_3 \leq \overline{V}$ and $\hat{x}_5 = \underline{C} + V$. If $\bar{x}_2 = \underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3) - (\underline{C} + V - \overline{V})\bar{u}_2$, we let $\hat{x}_3 = \underline{C}$; if $\bar{x}_2 = \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_3 = \overline{V}$.

2) Satisfying $\bar{x}_1 \leq \underline{C}\bar{y}_1 + (\overline{C} - \underline{C} - 2V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ at equality. We obtain $\underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\bar{u}_2 \leq \hat{x}_2 \leq \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_4 = \overline{C} - 2V$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$. As a result, we have $\underline{C} \leq \hat{x}_3 \leq \overline{V}$ and $\hat{x}_5 = \overline{C} - V$. If $\bar{x}_2 = \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\bar{u}_2$, we let $\hat{x}_3 = \underline{C}$; if $\hat{x}_2 = \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_3 = \overline{V}$.

3) Satisfying $\bar{x}_2 - \bar{x}_1 \leq \overline{V}\bar{y}_2 - \underline{C}\bar{y}_1 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ at equality. We obtain $\overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \leq \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_3 = \overline{V}$, we have $\bar{x}_2 - \bar{x}_1 = (\lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7) - (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4)$. As result, we have $\underline{C} + V \leq \hat{x}_5 \leq \overline{C} - V$. If $\hat{x}_2 = \overline{V}\bar{y}_2 + (\underline{C} + V - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_5 = \underline{C} + V$; if $\bar{x}_2 = \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_5 = \overline{C} - V$.

4) Satisfying $\bar{x}_1 - \bar{x}_2 \leq \underline{C}\bar{y}_1 - \underline{C}\bar{y}_2 - V\bar{y}_3 + V\bar{u}_3 + (\underline{C} + V - \overline{V})\bar{u}_2$ at equality. We obtain $\underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3) - (\underline{C} + V - \overline{V})\bar{u}_2 \leq \bar{x}_2 \leq \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) + (\overline{V} - \underline{C})\bar{u}_2$. By letting $\hat{x}_3 = \underline{C}$, we have $\bar{x}_1 - \bar{x}_2 = (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4) - (\lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7)$. As result, we have $\underline{C} + V \leq \hat{x}_5 \leq \overline{C} - V$. If $\bar{x}_2 = \underline{C}\bar{y}_2 + V(\bar{y}_3 - \bar{u}_3) - (\underline{C} + V - \overline{V})\bar{u}_2$, we let $\hat{x}_5 = \underline{C} + V$; if $\bar{x}_2 = \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) + (\overline{V} - \underline{C})\bar{u}_2$, we let $\hat{x}_5 = \overline{C} - V$.

Similar analyses hold for (15j) due to the similar structure between (15i) and (15j) and thus are omitted here.

Satisfying (15k) **at equality**. For this case, substituting $\bar{x}_3 = \bar{x}_2 - \bar{x}_1 + \overline{V}\bar{y}_1 - (\overline{V} - V)\bar{y}_2 + \overline{V}\bar{y}_3 + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ into (15a) - (15k), we obtain the feasible region of $(\bar{x}_1, \bar{x}_2)$ as $C' = \{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 : \overline{V}\bar{y}_1 + (\underline{C} + V - \overline{V})(\bar{y}_2 - \bar{y}_3 + \bar{u}_3) + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \leq \bar{x}_1 \leq \overline{V}\bar{y}_1 + V(\bar{y}_2 - \bar{u}_2) + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2),\ \bar{x}_2 - \bar{x}_1 \leq (\overline{V} - V)\bar{y}_2 - \overline{V}\bar{y}_1 + V\bar{u}_2,\ \bar{x}_1 - \bar{x}_2 \leq \overline{V}\bar{y}_1 - (\overline{V} - V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2\}$.

First, by letting $\hat{x}_1 = \overline{V}$, $\hat{x}_2 - \hat{x}_3 = V$, $\hat{x}_4 = \hat{x}_6 = \overline{C}$, $\hat{x}_5 = \overline{C} - V$, $\hat{x}_8 - \hat{x}_7 = V$, and $\hat{x}_9 = \overline{V}$, we have $\bar{x}_1 - \bar{x}_2 + \bar{x}_3 = (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4) - (\lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7) + (\lambda_3\hat{x}_6 + \lambda_4\hat{x}_8 + \lambda_5\hat{x}_9)$. Then the corresponding feasible region for $(\hat{x}_3, \hat{x}_7)$ can be described as set $A' = \{(\hat{x}_3, \hat{x}_7) \in \mathbb{R}^2 : \underline{C} \leq \hat{x}_3 \leq \overline{V},\ \underline{C} \leq \hat{x}_7 \leq \overline{V}\}$. Next, we only need to show $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$ and $\bar{x}_2 = \lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7$. We consider that one of inequalities in C' is satisfied at equality to obtain the values of $(\hat{x}_3, \hat{x}_7)$ from A' as follows.

1) Satisfying $\bar{x}_1 \geq \overline{V}\bar{y}_1 + (\underline{C} + V - \overline{V})(\bar{y}_2 - \bar{y}_3 + \bar{u}_3) + (\overline{C} - \overline{V})(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ at equality. We obtain $\underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \leq \bar{x}_2 \leq \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\bar{u}_2$. By letting $\hat{x}_3 = \underline{C}$ and thus $\hat{x}_2 = \underline{C} + V$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$. If $\bar{x}_2 = \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_7 = \underline{C}$ and thus $\hat{x}_8 = \underline{C} + V$; if $\bar{x}_2 = \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\bar{u}_2$, we let $\hat{x}_7 = \overline{V}$ and thus $\hat{x}_8 = \overline{V} + V$.

2) Satisfying $\bar{x}_1 \leq \overline{V}\bar{y}_1 + V(\bar{y}_2 - \bar{u}_2) + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$ at equality. We obtain $\overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \underline{C} - V)\bar{u}_2 \leq \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_3 = \overline{V}$ and thus $\hat{x}_2 = \overline{V} + V$, we have $\bar{x}_1 = \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4$. If $\bar{x}_2 = \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \underline{C} - V)\bar{u}_2$, we let $\hat{x}_7 = \underline{C}$ and thus $\hat{x}_8 = \underline{C} + V$; if $\bar{x}_2 = \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_7 = \overline{V}$ and thus $\hat{x}_8 = \overline{V} + V$.

3) Satisfying $\bar{x}_2 - \bar{x}_1 \leq (\overline{V} - V)\bar{y}_2 - \overline{V}\bar{y}_1 + V\bar{u}_2$ at equality. We obtain $\underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 -$

$\bar{u}_3) - (\overline{C} - \overline{V} - V)\bar{u}_2 \leq \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$. By letting $\hat{x}_7 = \overline{V}$ and thus $\hat{x}_8 = \overline{V} + V$, we have $\bar{x}_2 - \bar{x}_1 = (\lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7) - (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4)$. If $\bar{x}_2 = \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \overline{V} - V)\bar{u}_2$, we let $\hat{x}_3 = \underline{C}$ and thus $\hat{x}_2 = \underline{C} + V$; if $\bar{x}_2 = \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_3 = \overline{V}$ and thus $\hat{x}_2 = \overline{V} + V$.

4) Satisfying $\bar{x}_1 - \bar{x}_2 \leq \overline{V}\bar{y}_1 - (\overline{V} - V)\bar{y}_2 - (\underline{C} + V - \overline{V})\bar{u}_2$ at equality. We obtain $\underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2) \leq \bar{x}_2 \leq \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \underline{C} - V)\bar{u}_2$. By letting $\hat{x}_7 = \underline{C}$ and thus $\hat{x}_8 = \underline{C} + V$, we have $\bar{x}_1 - \bar{x}_2 = (\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 + \lambda_3\hat{x}_4) - (\lambda_2\hat{x}_3 + \lambda_3\hat{x}_5 + \lambda_4\hat{x}_7)$. If $\bar{x}_2 = \underline{C}\bar{y}_2 + (\overline{C} - \underline{C} - V)(\bar{y}_3 - \bar{u}_3 - \bar{u}_2)$, we let $\hat{x}_3 = \underline{C}$ and thus $\hat{x}_2 = \underline{C} + V$; if $\bar{x}_2 = \overline{V}\bar{y}_2 + (\overline{C} - \overline{V} - V)(\bar{y}_3 - \bar{u}_3) - (\overline{C} - \underline{C} - V)\bar{u}_2$, we let $\hat{x}_3 = \overline{V}$ and thus $\hat{x}_2 = \overline{V} + V$.

This completes the proof. ■

Theorem 2 $Q_3^2 =$ *conv*(P_3^2).

Proof: First, we have both P_3^2 and Q_3^2 bounded from their formulation representations. Since all the inequalities in Q_3^2 are valid and facet-defining for conv(P_3^2) based on Propositions 1 and 3, we have $Q_3^2 \supseteq \text{conv}(P_3^2)$. Meanwhile, we have that any extreme point in Q_3^2 is integral in y and u based on Proposition 4. Thus $Q_3^2 = \text{conv}(P_3^2)$. ■

For the case $L = \ell = 2$ and $\overline{C} - \underline{C} - 2V < 0$, we can obtain the similar convex hull representation of the original polytope (i.e., $\hat{P}_3^2$) described as follows:

Theorem 3 $\hat{Q}_3^2 =$ *conv*$(\hat{P}_3^2) = \{(x, y, u) \in \mathbb{R}^8 : (3a) - (3d), (4) - (10), (13) - (14)\}$.

Proof: The proofs are similar with those for Theorem 2 and thus omitted here. ■

Remark 2 *Since the start-up decision is not considered in the first-time period in Q_3^2, the strong valid inequalities in Q_3^2 (e.g., (4) - (13)) can be applied to any three consecutive time periods.*

Remark 3 *Besides the case in which $L = \ell = 2$, the convex hull results for the cases in which $L = \ell = 1$, $L = 1$ and $\ell = 2$, and $L = 2$ and $\ell = 1$ under the condition of either $\overline{C} - \underline{C} - 2V \geq 0$ or $\overline{C} - \underline{C} - 2V < 0$ can be obtained similarly. Descriptions are omitted here for brevity.*

3 Strengthening Multi-period Formulations

First of all, the inequalities we derived in the previous sections can be applied to solve the general multi-period problems, because the start-up decision is not considered for the first-time period. These inequalities are polynomial in the order of $\mathcal{O}(T)$. In this section, we further strengthen the formulation for the general polytope P by exploring the inequalities covering multiple periods. For notation brevity, we let $\sum_{t=a}^{b} x_t = \sum_{t=a}^{b} y_t = \sum_{t=a}^{b} u_t = 0$ if $b < a$.

Proposition 5 *For $1 \leq k \leq \min\{L, \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor + 1\}$, $t \in [k+1, T]_{\mathbb{Z}}$, the inequality*

$$x_t \leq \overline{C} y_t - \sum_{s=0}^{k-1} (\overline{C} - \overline{V} - sV) u_{t-s} \tag{17}$$

is valid for conv(P). Furthermore, it is facet-defining for conv(P) when $t = T$ and $k = \min\{L, \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor + 1\}$.

Proof: **(Validity)** We discuss the following two cases in terms of the value of y_t:

1) If $y_t = 0$, we have $x_t = 0$ due to constraints (1e) and $u_{t-s} = 0$ for all $s \in [0, k-1]_{\mathbb{Z}}$ due to constraints (1a) since $k \leq L$. Thus, (17) holds.

2) If $y_t = 1$, we have $\sum_{s=0}^{k-1} u_{t-s} \leq 1$ due to constraints (1a) since $k \leq L$. We discuss the following two cases:

   - If $u_{t-s} = 0$ for all $s \in [0, k-1]_{\mathbb{Z}}$, (17) converts to $x_t \leq \overline{C}$, which is valid because of (1e).
   - If $u_{t-s} = 0$ for some $s \in [0, k-1]_{\mathbb{Z}}$, (17) converts to $x_t \leq \overline{V} + sV$, which is valid because of ramp-up constraints (1f).

(Facet-defining) We generate $3T - 1$ affinely independent points in conv(P) that satisfy (17) at equality. Since $0 \in \text{conv}(P)$, we generate another $3T - 2$ linearly independent points in conv(P) in the following groups. In the following proofs, we use the superscript of (x, y, u), e.g., r in (x^r, y^r, u^r), to indicate the index of different points in conv(P).

First, we create T linearly independent points $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ ($r \in [1, T]_{\mathbb{Z}}$) such that $\bar{y}_s^r = 1$ for each $s \in [1, r]_{\mathbb{Z}}$ and $\bar{y}_s^r = 0$ otherwise. Thus we have $\bar{u}_s^r = 0$ for all $s \in [2, T]_{\mathbb{Z}}$. For the value of $\bar{x}^r$, we consider the following cases: 1) for each $r \in [1, T-1]_{\mathbb{Z}}$, we have $\bar{x}_s^r = \underline{C}$ for each $s \in [1, r]_{\mathbb{Z}}$ and $\bar{x}_s^r = 0$ otherwise; 2) for each $r = T$, we have $\bar{x}_s^r = \overline{C}$ for each $s \in [1, T]_{\mathbb{Z}}$.

Second, we create $T-1$ linearly independent points $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ $(r \in [1, T-1]_{\mathbb{Z}})$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C} + \epsilon, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \hat{u}_s^r = 0, \\ \forall s \end{array}.$$

Third, we create k linearly independent points $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ $(r \in [T-k+1, T]_{\mathbb{Z}})$ such that

$$\acute{x}_s^r = \begin{cases} \overline{V} + (s-r)V, & s \in [r, T]_{\mathbb{Z}} \\ 0, & s \in [1, r-1]_{\mathbb{Z}} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & s \in [r, T]_{\mathbb{Z}} \\ 0, & s \in [1, r-1]_{\mathbb{Z}} \end{cases}, \text{ and } \acute{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

Fourth, for the remaining $T-k-1$ points, we consider $k = L$ and $\lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor + 1$ respectively, since the condition requires $k = \min\{L, \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor + 1\}$.

1) If $k = L$, we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ for each $r \in [2, T-k]_{\mathbb{Z}}$, where

$$\acute{x}_s^r = \begin{cases} \underline{C}, & s \in [r, T-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & s \in [r, T-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \acute{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

2) If $k = \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor + 1$, we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ for each $r \in [2, T-k]_{\mathbb{Z}}$, where

$$\acute{x}_s^r = \begin{cases} \overline{V} + (s-r)V, & s \in [r, r+k-1]_{\mathbb{Z}} \\ \overline{C}, & s \in [r+k, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & s \in [r, T-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \acute{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

Finally, it is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^T$ and $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=2}^T$ are linearly independent because they construct a lower-diagonal matrix. In addition, $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=1}^{T-1}$ are also linearly independent with them after Gaussian elimination between $(\bar{x}, \bar{y}, \bar{u})$ and $(\hat{x}, \hat{y}, \hat{u})$. Therefore the statement is proved.

■

Proposition 6 *For $1 \le k \le \min\{L, \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor + 2\}$, $t \in [k, T-1]_{\mathbb{Z}}$, the inequality*

$$x_t \le \overline{V} y_t + (\overline{C} - \overline{V})(y_{t+1} - u_{t+1}) - \sum_{s=1}^{k-1} (\overline{C} - \overline{V} - (s-1)V) u_{t-s+1} \tag{18}$$

is valid for conv(P). Furthermore, it is facet-defining for conv(P) when one of the following conditions is satisfied: (1) $L \le 3$, $k = \min\{L, \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor + 2\}$ for all $t \in [k, T-1]_{\mathbb{Z}}$; (2) $L \ge 4$, $k = \min\{L, \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor + 2\}$ for $t = T-1$.

Proof: **(Validity)** We discuss the following four cases in terms of the values of y_t and y_{t+1}:

1) If $y_t = y_{t+1} = 1$, we have $u_{t+1} = 0$ due to constraints (1b) and $\sum_{s=1}^{k-1} u_{t-s+1} \leq 1$ due to constraints (1a) since $k \leq L$. We further discuss the following two cases.

   - If $u_{t-s+1} = 0$ for all $s \in [1, k-1]_{\mathbb{Z}}$, then (18) converts to $x_t \leq \overline{C}$, which is valid because of constraints (1e).
   - If $u_{t-s+1} = 1$ for some $s \in [1, k-1]_{\mathbb{Z}}$, then (18) converts to $x_t \leq \overline{V} + (s-1)V$, which is valid because of ramp-up constraints (1f).

2) If $y_t = 1$ and $y_{t+1} = 0$, then $u_{t-s+1} = 0$ for all $s \in [0, k-1]_{\mathbb{Z}}$ due to constraints (1a) since $k \leq L$. It follows that (18) converts to $x_t \leq \overline{V}$, which is valid because of ramp-down constraints (1g).

3) If $y_t = 0$ and $y_{t+1} = 1$, we have $u_{t+1} = 1$ due to constraints (1c) and $u_{t-s+1} = 0$ for all $s \in [1, k-1]_{\mathbb{Z}}$ due to constraints (1a) since $k \leq L$. It follows (18) is valid.

4) If $y_t = y_{t+1} = 0$, (18) is clearly valid.

(Facet-defining) We provide the facet-defining proof for condition (1), as the proof for condition (2) is similar with that for Proposition 5 and thus omitted here.

We generate $3T - 2$ linearly independent points in $\text{conv}(P)$ that satisfy (18) at equality in the following groups.

1) For each $r \in [1, t-1]_{\mathbb{Z}}$ (totally $t-1$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ such that

$$\acute{x}_s^r = \begin{cases} \underline{C}, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \text{and} \quad \begin{array}{l} \acute{u}_s^r = 0, \\ \forall s \end{array}.$$

2) For each $r \in [1, t-1]_{\mathbb{Z}}$ (totally $t-1$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C} + \epsilon, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \text{and} \quad \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

3) For $r = t$ (totally one points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{V}, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \text{and} \quad \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

4) For each $r \in [t+1, T-1]_{\mathbb{Z}}$ (totally $T-t-1$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}_s^r = 1$ for each $s \in [t-k+2, r]_{\mathbb{Z}}$ and $\bar{y}_s^r = 0$ otherwise. Thus $\bar{u}_s^r = 1$ for $s = t-k+2$ and $\bar{u}_s^r = 0$ otherwise. Moreover, we let $\bar{x}_s^r = \max\{\underline{C}, \overline{V} + (s - (t-k+2))V\}$ for each $s \in [t-k+2, t]_{\mathbb{Z}}$, $\bar{x}_s^r = \max\{\underline{C}, \overline{V} + (k-3)V\}\}$ for each $s \in [t+1, r]_{\mathbb{Z}}$, and $\bar{x}_s^r = 0$ otherwise.

5) For $r = T$ (totally one points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}^r_s = 1$ for each $s \in [1, T]_{\mathbb{Z}}$, $\bar{u}^r_s = 0$ for each $s \in [2, T]_{\mathbb{Z}}$, and $\bar{x}^r_s = \overline{C}$ for each $s \in [1, T]_{\mathbb{Z}}$.

6) For each $r \in [2, t-k+1]_{\mathbb{Z}}$ (totally $t-k$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}^r_s = \begin{cases} \overline{V}, & s \in [r, t]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}^r_s = \begin{cases} 1, & s \in [r, t]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}^r_s = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

7) For each $r \in [t-k+2, t]_{\mathbb{Z}}$ (totally $k-1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}^r_s = \begin{cases} \overline{V} + (s-r)V, & s \in [r, t]_{\mathbb{Z}} \\ \max\{\underline{C}, \overline{V} + (t-r-1)V\}, & s \in [t+1, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}^r_s = \begin{cases} 1, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}^r_s = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

8) For each $r \in [t+1, T]_{\mathbb{Z}}$ (totally $T-t$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}^r_s = \begin{cases} \underline{C}, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}^r_s = \begin{cases} 1, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}^r_s = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

9) For each $r \in [t+1, T]_{\mathbb{Z}}$ (totally $T-t$ points), we create $(\grave{x}^r, \grave{y}^r, \grave{u}^r) \in \text{conv}(P)$ such that

$$\grave{x}^r_s = \begin{cases} \underline{C} + \epsilon, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \grave{y}^r_s = \begin{cases} 1, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \grave{u}^r_s = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

Finally, it is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^T$ and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=2}^T$ are linearly independent because they can construct a lower-diagonal matrix. In addition, $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1}^{t-1}$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t+1}^T$ are also linearly independent with them after Gaussian eliminations between $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1}^{t-1}$ and $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^{t-1}$, and between $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=t+1}^T$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t+1}^T$. ■

Proposition 7 *For $k = \min\{L-1, \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor\}$, $t \in [k+3, T]_{\mathbb{Z}}$, the inequality*

$$x_{t-1} \leq (\overline{C} - kV)y_{t-1} + kV(y_t - u_t) - \sum_{s=0}^{k} (\overline{C} - \overline{V} - sV)u_{t-s-1} \tag{19}$$

is valid for conv(P). Furthermore, it is facet-defining for conv(P) when one of the following conditions is satisfied: (1) $L \leq 3$, $L-1 \leq \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor$ for all $t \in [k+3, T]_{\mathbb{Z}}$; (2) $L \geq 4$, $L-1 \leq \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor$ for $t = T$.

Proof: **(Validity)** We discuss the following four cases in terms of the values of y_{t-1} and y_t:

1) If $y_{t-1} = y_t = 1$, we have $u_t = 0$ due to constraints (1b) and $\sum_{s=0}^{k} u_{t-s-1} \leq 1$ due to constraints (1a) since $k \leq L-1$. We further discuss the following two cases.

- If $u_{t-s-1} = 0$ for all $s \in [0, k]_{\mathbb{Z}}$, then (19) converts to $x_t \leq \overline{C}$, which is valid because of constraints (1e).
- If $u_{t-s+1} = 1$ for some $s \in [0, k]_{\mathbb{Z}}$, then (19) converts to $x_t \leq \overline{V} + sV$, which is valid because of ramp-up constraints (1f).

2) If $y_{t-1} = 1$ and $y_t = 0$, then $u_{t-s-1} = 0$ for all $s \in [0, L-2]_{\mathbb{Z}}$ and $\sum_{s=0}^{k} u_{t-s-1} \leq 1$ due to constraints (1a) since $k \leq L - 1$. We further discuss the following two cases.

   - If $u_{t-s-1} = 0$ for all $s \in [0, k]_{\mathbb{Z}}$, then (19) converts to $x_t \leq \overline{C} - kV$, which is valid since $x_t \leq \overline{V}$ due to ramp-down constraints (1g) and $k \leq \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor$.
   - If $k = L - 1$ and $u_{t-k-1} = 1$, then (19) converts to $x_t \leq \overline{V}$, which is valid because of ramp-down constraints (1g).

3) If $y_{t-1} = 0$ and $y_t = 1$, we have $u_t = 1$ due to constraints (1c) and $u_{t-s-1} = 0$ for all $s \in [0, k]_{\mathbb{Z}}$ due to constraints (1a) since $k \leq L - 1$. It follows (19) is valid.

4) If $y_t = y_{t+1} = 0$, (19) is clearly valid.

(Facet-defining) We provide the facet-defining proof for condition (1), as the proof for condition (2) is similar with that for Proposition 5 and thus omitted here. Since $L - 1 \leq \lfloor \frac{\overline{C}-\overline{V}}{V} \rfloor$, we have $k = L - 1$.

We generate $3T - 2$ linearly independent points in $\text{conv}(P)$ that satisfy (19) at equality in the following groups.

1) For each $r \in [1, t-2]_{\mathbb{Z}}$ (totally $t - 2$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ such that

$$\acute{x}_s^r = \begin{cases} \underline{C}, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \acute{u}_s^r = 0, \\ \forall s \end{array}.$$

2) For each $r \in [1, t-2]_{\mathbb{Z}}$ (totally $t - 2$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C} + \epsilon, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

3) For $r = t - 1$ (totally one points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{V}, & s \in [t-k-1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s \in [t-k-1, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \bar{u}_s^r = \begin{cases} 1, & s = t-k-1 \\ 0, & \text{o.w.} \end{cases}.$$

4) For each $r \in [t, T-1]_{\mathbb{Z}}$ (totally $T-t$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}^r_s = 1$ for each $s \in [t-k, r]_{\mathbb{Z}}$ and $\bar{y}^r_s = 0$ otherwise. Thus $\bar{u}^r_s = 1$ for $s = t-k$ and $\bar{u}^r_s = 0$ otherwise. Moreover, we let $\bar{x}^r_s = \overline{V} + (s-(t-k))V$ for each $s \in [t-k, t-1]_{\mathbb{Z}}$, $\bar{x}^r_s = \max\{\underline{C}, \overline{V} + (k-2)V\}\}$ for each $s \in [t, r]_{\mathbb{Z}}$, and $\bar{x}^r_s = 0$ otherwise.

5) For $r = T$ (totally one points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}^r_s = 1$ for each $s \in [1, T]_{\mathbb{Z}}$, $\bar{u}^r_s = 0$ for each $s \in [2, T]_{\mathbb{Z}}$, and $\bar{x}^r_s = \overline{C}$ for each $s \in [1, T]_{\mathbb{Z}}$.

6) For each $r \in [2, t-k-2]_{\mathbb{Z}}$ (totally $t-k-3$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}^r_s = \begin{cases} \underline{C}, & s \in [r, t-2]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \ \hat{y}^r_s = \begin{cases} 1, & s \in [r, t-2]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}^r_s = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

7) For each $r \in [t-k-1, t-1]_{\mathbb{Z}}$ (totally $k+1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}^r_s = \begin{cases} \overline{V} + (s-r)V, & s \in [r, t]_{\mathbb{Z}} \\ \overline{V} + (t-r)V, & s \in [t+1, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \ \hat{y}^r_s = \begin{cases} 1, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}^r_s = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

8) For each $r \in [t, T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}^r_s = \begin{cases} \underline{C}, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \ \hat{y}^r_s = \begin{cases} 1, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}^r_s = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

9) For each $r \in [t, T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we create $(\grave{x}^r, \grave{y}^r, \grave{u}^r) \in \text{conv}(P)$ such that

$$\grave{x}^r_s = \begin{cases} \underline{C} + \epsilon, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \ \grave{y}^r_s = \begin{cases} 1, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \grave{u}^r_s = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

Finally, it is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^T$ and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=2}^T$ are linearly independent because they can construct a lower-diagonal matrix. In addition, $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1}^{t-2}$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t}^T$ are also linearly independent with them after Gaussian eliminations between $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1}^{t-2}$ and $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^{t-2}$, and between $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=t}^T$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t}^T$. ∎

Proposition 8 *For each $k \in \{[2, T-2]_{\mathbb{Z}} : \overline{C} - \overline{V} - (k-1)V > 0\}$, the inequality*

$$x_{t-k} \leq \overline{V} y_{t-k} + V \sum_{s=1}^{k-1} (y_{t-s} - \sum_{i=s}^{\min\{k, s+L-1\}} u_{t-i}) + (\overline{C} - \overline{V} - (k-1)V)(y_t - \sum_{s=0}^{\min\{k, L-1\}} u_{t-s}) \quad (20)$$

is valid for conv(P) for each $t \in [\max\{\min\{k, k+L-2\}+2, \min\{k, L-1\}+2\}, T]_{\mathbb{Z}}$. Furthermore, it is facet-defining for conv(P) when one of the following conditions is satisfied: (1) $L \leq 3$ and $t = T$; (2) $L \leq 3$ and $k = \lfloor \frac{\overline{C} - \overline{V}}{V} \rfloor + 1$ for all $t \in [\max\{\min\{k, k+L-2\}+2, \min\{k, L-1\}+2\}, T]_{\mathbb{Z}}$.

Proof: **(Validity)** We discuss the following possible two cases in terms of the value of y_{t-k}:

1) If $y_{t-k} = 0$, $x_{t-k} = 0$ due to constraints (1e). It follows that inequality (20) is valid since $y_{t-s} - \sum_{i=s}^{\min\{k,s+L-1\}} u_{t-i} \geq 0$ for all $s \in [1, k-1]_{\mathbb{Z}}$ and $y_t - \sum_{s=0}^{\min\{k,L-1\}} u_{t-s} \geq 0$ due to minimum-up time constraints (1a).

2) If $y_{t-k} = 1$, then we consider the following two cases in terms of the value of u_{t-k}:

   (1) If $u_{t-k} = 1$, then we have $x_{t-k} \leq \overline{V}$ due to ramp-up constraints (1f). It follows that inequality (20) is valid since $y_{t-s} - \sum_{i=s}^{\min\{k,s+L-1\}} u_{t-i} \geq 0$ for all $s \in [1, k-1]_{\mathbb{Z}}$ and $y_t - \sum_{s=0}^{\min\{k,L-1\}} u_{t-s} \geq 0$ due to minimum-up time constraints (1a).

   (2) If $u_{t-k} = 0$, it means that the generator starts up at a time period prior to time $t-k$. To show inequality (20) is valid, we consider the following two cases based on when this generator shuts down as follows.

   - If the generator shuts down at $t - \bar{s}$ for some $\bar{s} \in [1, k-1]_{\mathbb{Z}}$, i.e., $y_{t-\bar{s}} = 0$, then $u_{t-s} = 0$ for all $s \in [\bar{s}, \min\{k, k+L-2\}]_{\mathbb{Z}}$. It follows that inequality (20) converts to $x_{t-k} \leq \overline{V} + (k - \bar{s} - 1)V + V\sum_{s=1}^{\bar{s}-1}(y_{t-s} - \sum_{i=s}^{\min\{k,s+L-1\}} u_{t-i}) + (\overline{C} - \overline{V} - (k-1)V)(y_t - \sum_{s=0}^{\min\{k,L-1\}} u_{t-s})$, which is valid since $x_{t-k} \leq \overline{V} + (k - \bar{s} - 1)V$ due to ramp-down constraints (1g), $y_{t-s} - \sum_{i=s}^{\min\{k,s+L-1\}} u_{t-i} \geq 0$ for all $s \in [1, \bar{s}-1]_{\mathbb{Z}}$, and $y_t - \sum_{s=0}^{\min\{k,L-1\}} u_{t-s} \geq 0$.
   - If the generator shuts down at $\bar{t}$ such that $\bar{t} \geq t$, then inequality (20) converts to $x_{t-k} \leq \overline{C}$, which is clearly valid due to constraints (1e).

(Facet-defining) We provide the facet-defining proof for condition (2), as the proof for condition (1) is similar with that for Proposition 5 and thus omitted here.

We have $\overline{C} \leq \overline{V} + kV$ from condition (2) and generate $3T-2$ linearly independent points in conv(P) that satisfy (20) at equality in the following groups.

1) For each $r \in [1, t-k-1]_{\mathbb{Z}}$ (totally $t-k-1$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ such that
$$\acute{x}_s^r = \begin{cases} \underline{C}, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \acute{u}_s^r = 0, \\ \forall s \end{array}.$$

2) For each $r \in [1, t-k-1]_{\mathbb{Z}}$ (totally $t-k-1$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that
$$\bar{x}_s^r = \begin{cases} \underline{C} + \epsilon, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

3) For each $r \in [t-k, t-1]_{\mathbb{Z}}$ (totally k points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{V} + (r-(t-k))V, \ s \in [1, t-k-1]_{\mathbb{Z}} \\ \overline{V} + (r-s)V, \ s \in [t-k, r]_{\mathbb{Z}} \\ 0, \ s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \ \bar{y}_s^r = \begin{cases} 1, \ s \in [1, r]_{\mathbb{Z}} \\ 0, \ s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

4) For each $r \in [t, T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{C}, \ s \in [1, t-k]_{\mathbb{Z}} \\ \overline{V} + (t-s)V, \ s \in [t-k+1, t-1]_{\mathbb{Z}} \\ \overline{V}, \ s \in [t, r]_{\mathbb{Z}} \\ 0, \ s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \ \bar{y}_s^r = \begin{cases} 1, \ s \in [1, r]_{\mathbb{Z}} \\ 0, \ s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

5) For each $r \in [2, t-k]_{\mathbb{Z}}$ (totally $t-k-1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that $\hat{y}_s^r = 1$ for each $s \in [r, r+L-1]_{\mathbb{Z}}$ and $\hat{y}_s^r = 0$ otherwise. Thus $\hat{u}_s^r = 1$ for each $s = r$ and $\hat{u}_s^r = 0$ otherwise. Meanwhile, we let $\hat{x}_s^r = \overline{V}$ for each $s \in [r, r+L-1]_{\mathbb{Z}} \setminus \{t-k\}$ and $\hat{x}_s^r = 0$ for each $s \in [1, r-1]_{\mathbb{Z}} \cup [r+L, T]_{\mathbb{Z}}$. In addition, for the value of $\hat{x}_{t-k}^r$: 1) If $\hat{y}_{t-k}^r = 1$, we let $\hat{x}_{t-k}^r = \overline{V}$ if $\hat{y}_{t-k+1}^r = 0$ and $\hat{x}_{t-k}^r = \overline{V} + V$ otherwise; 2) If $\hat{y}_{t-k}^r = 0$, we let $\hat{x}_{t-k}^r = 0$.

6) For each $r \in [t-k+1, T]_{\mathbb{Z}}$ (totally $T-t+k$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, \ s \in [r, \min\{r+L-1, T\}]_{\mathbb{Z}} \\ 0, \ \text{o.w.} \end{cases}, \ \hat{y}_s^r = \begin{cases} 1, \ s \in [r, \min\{r+L-1, T\}]_{\mathbb{Z}} \\ 0, \ \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, \ s = r \\ 0, \ \text{o.w.} \end{cases}.$$

7) For each $r \in [t-k+1, T]_{\mathbb{Z}}$ (totally $T-t+k$ points), we create $(\grave{x}^r, \grave{y}^r, \grave{u}^r) \in \text{conv}(P)$ such that

$$\grave{x}_s^r = \begin{cases} \underline{C}, \ s \in [r, \min\{r+L-1, T\}]_{\mathbb{Z}} \\ 0, \ \text{o.w.} \end{cases}, \ \grave{y}_s^r = \begin{cases} 1, \ s \in [r, \min\{r+L-1, T\}]_{\mathbb{Z}} \\ 0, \ \text{o.w.} \end{cases}, \text{ and } \grave{u}_s^r = \begin{cases} 1, \ s = r \\ 0, \ \text{o.w.} \end{cases}.$$

Finally, it is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^T$ and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=2}^T$ are linearly independent because they can construct a lower-diagonal matrix. In addition, $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1}^{t-k-1}$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t-k+1}^T$ are also linearly independent with them after Gaussian eliminations between $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1}^{t-k-1}$ and $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^{t-k-1}$, and between $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=t-k+1}^T$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t-k+1}^T$. ■

Proposition 9 *For each $k \in \{[2, T-1]_{\mathbb{Z}} : \overline{C} - \overline{V} - (k-1)V > 0\}$, the inequality*

$$x_1 \leq \overline{V} y_1 + V \sum_{s=2}^{k} (y_s - \sum_{i=\max\{2, s-L+1\}}^{s} u_i) + (\overline{C} - \overline{V} - (k-1)V)(y_{k+1} - \sum_{i=\max\{2, k-L+2\}}^{k+1} u_i) \quad (21)$$

is valid and facet-defining for conv(P) for each $k \in [2, T-1]_{\mathbb{Z}}$.

Proof: The proofs are similar with that for Proposition 8 and thus omitted here. ■

From Propositions 5 - 9, we can observe that these derived inequalities contain a single continuous variable and the total number of inequalities is in the order of up to $\mathcal{O}(T^2)$.

Proposition 10 *For each $k \in [1, T-1]_{\mathbb{Z}}$ such that $\overline{C} - \underline{C} - kV > 0$, $t \in [k+1, T]_{\mathbb{Z}}$, the inequality*

$$x_t - x_{t-k} \leq (\underline{C} + kV)y_t - \underline{C}y_{t-k} - \sum_{s=0}^{\min\{k-1,L-1\}} (\underline{C} + (k-s)V - \overline{V})u_{t-s} \tag{22}$$

is valid for conv(P). Furthermore, it is facet-defining for conv(P) when $t = T$.

Proof: **(Validity)** We discuss the following four cases in terms of the values of y_{t-k} and y_t:

1) If $y_{t-k} = y_t = 1$, then $\sum_{s=0}^{\min\{k-1,L-1\}} u_{t-s} \leq 1$ due to constraints (1a). We further discuss the following two cases.

   - If $u_{t-s} = 0$ for all $s \in [0, \min\{k-1, L-1\}]_{\mathbb{Z}}$, (22) converts to $x_t - x_{t-k} \leq kV$, which is valid due to ramp-up constraints (1f).

   - If $u_{t-s} = 1$ for some $s \in [0, \min\{k-1, L-1\}]_{\mathbb{Z}}$, (22) converts to $x_t - x_{t-k} \leq sV - \underline{C}$, which is valid since $x_t \leq sV$ due to ramp-up constraints (1f) and $x_{t-k} \geq \underline{C}$ due to constraints (1d).

2) If $y_{t-k} = 1$ and $y_t = 0$, then $\sum_{s=0}^{\min\{k-1,L-1\}} u_{t-s} = 0$ due to constraints (1a). (22) converts to $x_{t-k} \geq \underline{C}$, which is valid due to constraints (1d).

3) If $y_{t-k} = 0$ and $y_t = 1$, then the generator should start up at time period $\bar{t} \in [t-k+1, t]_{\mathbb{Z}}$. Meanwhile, we have $\sum_{s=0}^{\min\{k-1,L-1\}} u_{t-s} \leq 1$ due to constraints (1a). We further discuss the following two cases.

   - If $u_{t-s} = 0$ for all $s \in [0, \min\{k-1, L-1\}]_{\mathbb{Z}}$, it follows $\bar{t} \in [t-k+1, t-\min\{k-1, L-1\}-1]_{\mathbb{Z}}$, i.e., $t - \bar{t} \in [\min\{k-1, L-1\}+1, k-1]_{\mathbb{Z}}$. Meanwhile, (22) converts to $x_t \leq \underline{C} + kV$, which is valid since $x_t \leq \overline{V} + (t-\bar{t})V \leq \overline{V} + (k-1)V$ due to ramp-up constraints (1f) and $\overline{V} < \underline{C} + V$.

   - If $u_{t-s} = 1$ for some $s \in [0, \min\{k-1, L-1\}]_{\mathbb{Z}}$, (22) converts to $x_t \leq sV$, which is valid since $x_t \leq sV$ due to ramp-up constraints (1f).

4) If $y_{t-k} = y_t = 0$, then (22) is clearly valid.

(Facet-defining) The proof is similar with that for Proposition 5 and thus omitted here. ■

Proposition 11 *For each* $k \in \{[1, T-2]_{\mathbb{Z}} : \overline{C} - \underline{C} - kV > 0\}$*,* $t \in [k+2, T]_{\mathbb{Z}}$*, the inequality*

$$x_{t-1} - x_{t-k-1} \leq \overline{V} y_{t-1} - \underline{C} y_{t-k-1} + (\underline{C} + kV - \overline{V})(y_t - u_t) - \sum_{s=1}^{\min\{k,L-1\}} (\underline{C} + (k-s+1)V - \overline{V}) u_{t-s} \quad (23)$$

is valid for conv(P). Furthermore, it is facet-defining for conv(P) when one of the following conditions is satisfied: (1) $t = T$*; (2)* $\min\{k, L-1\} \leq 2$ *for all* $t \in [k+2, T]_{\mathbb{Z}}$*.*

Proof: **(Validity)** We discuss the following two cases in terms of the value of y_t:

1) If $y_t = 0$, then $u_{t-s} = 0$ for all $s \in [0, \min\{k, L-1\}]_{\mathbb{Z}}$ due to constraints (1a). Inequality (23) converts to $x_{t-1} - x_{t-k-1} \leq \overline{V} y_{t-1} - \underline{C} y_{t-k-1}$ since we have $x_{t-1} \leq \overline{V} y_{t-1}$ due to constraints (1g) and $x_{t-k-1} \geq \underline{C} y_{t-k-1}$ due to constraints (1d).

2) If $y_t = 1$, then $\sum_{s=0}^{\min\{k,L-1\}} u_{t-s} \leq 1$ due to constraints (1a). We further discuss the following three cases.

   (1) If $u_{t-s} = 0$ for all $s \in [0, \min\{k, L-1\}]_{\mathbb{Z}}$, then $y_{t-1} = 1$ due to constraints (1c). Thus (23) converts to $x_{t-1} - x_{t-k-1} \leq \underline{C} + kV - \underline{C} y_{t-k-1}$. We further discuss the following two case in terms of the value of y_{t-k-1}.

   - If $y_{t-k-1} = 1$, then (23) converts to $x_{t-1} - x_{t-k-1} \leq kV$, which is valid due to ramp-up constraints (1f).
   - If $y_{t-k-1} = 0$, then it follows the generator starts up at time $\bar{t} \in [t-k, \min\{k, L-1\}-1]_{\mathbb{Z}}$. Meanwhile, (23) converts to $x_{t-1} \leq \underline{C} + kV$, which is valid since $x_{t-1} \leq \overline{V} + (t-1-\bar{t})V < \underline{C} + V + (k-1)V = \underline{C} + kV$, where the first inequality is due to ramp-up constraints (1f) and the second inequality is due to $\overline{V} < \underline{C} + V$.

   (2) If $u_t = 1$, then $y_{t-1} = 0$ due to constraints (1b). It follows that inequality (23) converts to $x_{t-k-1} \geq \underline{C} y_{t-k-1}$, which is valid due to constraints (1d).

   (3) If $u_{t-s} = 1$ for some $s \in [1, \min\{k, L-1\}]_{\mathbb{Z}}$, then inequality (23) converts to $x_{t-1} - x_{t-k-1} \leq \overline{V} + (s-1)V - \underline{C} y_{t-k-1}$, which is valid since $x_{t-1} \leq \overline{V} + (s-1)V$ due to ramp-up constraints (1f) and $x_{t-k-1} \geq \underline{C} y_{t-k-1}$ due to constraints (1d).

(Facet-defining) We provide the facet-defining proof for condition (2), as the proof for condition (1) is similar with that for Proposition 5 and thus omitted here.

We let $\kappa = \min\{k, L-1\}$ and generate $3T-2$ linearly independent points in $\mathrm{conv}(P)$ that satisfy (23) at equality in the following groups.

1) For each $r \in [1, t-2]_{\mathbb{Z}}$ (totally $t-2$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \mathrm{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C}, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

2) For $r = t-1$ (totally one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \mathrm{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \underline{C}, & s \in [1, r-1]_{\mathbb{Z}} \\ \overline{V}, & s = r \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \bar{u}_s^r = 0, \\ \forall s \end{array}.$$

3) For each $r \in [t, T-1]_{\mathbb{Z}}$ (totally $T-t$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \mathrm{conv}(P)$ such that $\bar{y}_s^r = 1$ for each $s \in [t-\kappa, r]_{\mathbb{Z}}$ and $\bar{y}_s^r = 0$ otherwise. Thus $\bar{u}_s^r = 1$ for $s = t-\kappa$ and $\bar{u}_s^r = 0$ otherwise. Moreover, we let $\bar{x}_s^r = \overline{V} + (s-(t-\kappa))V$ for each $s \in [t-\kappa, t-1]_{\mathbb{Z}}$, $\bar{x}_s^r = \max\{\underline{C}, \overline{V} + (\kappa-2)V\}$ for each $s \in [t, r]_{\mathbb{Z}}$, and $\bar{x}_s^r = 0$ otherwise.

4) For $r = T$ (totally one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \mathrm{conv}(P)$ such that $\bar{y}_s^r = 1$ for each $s \in [1, T]_{\mathbb{Z}}$, $\bar{u}_s^r = 0$ for each $s \in [2, T]_{\mathbb{Z}}$, and $\bar{x}_s^r = \underline{C}$ for each $s \in [1, t-k-1]_{\mathbb{Z}}$, $\bar{x}_s^r = \underline{C} + (s-(t-k-1))V$ for each $s \in [1, t-1]_{\mathbb{Z}}$, and $\bar{x}_s^r = \underline{C} + kV$ for each $s \in [t, T]_{\mathbb{Z}}$.

5) We create $(\dot{x}, \dot{y}, \dot{u}) \in \mathrm{conv}(P)$ (totally one point) such that $\dot{y}_s = 1$ for each $s \in [1, T]_{\mathbb{Z}}$, $\bar{u}_s = 0$ for each $s \in [2, T]_{\mathbb{Z}}$, and $\dot{x}_s = \underline{C} + \epsilon$ for each $s \in [1, t-k-1]_{\mathbb{Z}}$, $\bar{x}_s = \underline{C} + (s-(t-k-1))V + \epsilon$ for each $s \in [1, t-1]_{\mathbb{Z}}$, and $\bar{x}_s = \underline{C} + kV + \epsilon$ for each $s \in [t, T]_{\mathbb{Z}}$.

6) For each $r \in [1, t-2]_{\mathbb{Z}} \setminus \{t-k-1\}$ (totally $t-3$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \mathrm{conv}(P)$ such that

$$\acute{x}_s^r = \begin{cases} \underline{C} + \epsilon, & s \in [1, r]_{\mathbb{Z}} \setminus \{t-k-1\} \\ \underline{C}, & s \in [1, r]_{\mathbb{Z}} \cap \{t-k-1\} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \acute{u}_s^r = 0, \\ \forall s \end{array}.$$

7) For each $r \in [t-\kappa, t-1]_{\mathbb{Z}}$ (totally κ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \mathrm{conv}(P)$ such that

$$\hat{x}_s^r = \begin{cases} \overline{V} + (s-r)V, & s \in [r, t-1]_{\mathbb{Z}} \\ \overline{V} + (t-1-r)V, & s \in [t, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [r, T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

8) For each $r \in [t,T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & s \in [r,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [r,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

9) For each $r \in [t,T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we create $(\grave{x}^r, \grave{y}^r, \grave{u}^r) \in \text{conv}(P)$ such that

$$\grave{x}_s^r = \begin{cases} \underline{C} + \epsilon, & s \in [r,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \grave{y}_s^r = \begin{cases} 1, & s \in [r,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \grave{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

For the remaining $\kappa - t - 2$ points, we consider $\kappa = L-1$ and k respectively since $\kappa = \min\{k, L-1\}$.

- If $\kappa = L-1$, we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ for each $r \in [2, T-\kappa-1]_{\mathbb{Z}}$, where

$$\hat{x}_s^r = \begin{cases} \underline{C}, & s \in [r,t-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [r,t-1]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

- If $\kappa = k$, we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ for each $r \in [2, T-\kappa-1]_{\mathbb{Z}}$, where

$$\hat{x}_s^r = \begin{cases} \underline{C}, & s \in [r, t-\kappa-1]_{\mathbb{Z}} \\ \underline{C} + (s-(t-\kappa-1))V, & \\ \quad s \in [t-\kappa, t-1]_{\mathbb{Z}} & \\ \underline{C} + \kappa V, & s \in [t,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [r,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

Finally, it is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^T$ and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=2}^T$ are linearly independent because they can construct a lower-diagonal matrix. In addition, $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1, r\neq t-k-1}^{t-2}$, $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t}^T$, and $(\dot{x}, \dot{y}, \dot{u})$ are also linearly independent with them after Gaussian eliminations between $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1, r\neq t-k-1}^{t-2}$, $(\dot{x}, \dot{y}, \dot{u})$, and $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^{t-2}$, and between $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=t}^T$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t}^T$. ■

Proposition 12 *For each $k \in \{[2, T-1]_{\mathbb{Z}} : \overline{C} - \underline{C} - kV > 0\}$, $t \in [k + \min\{k, L-1\} + 1, T]_{\mathbb{Z}}$, the inequality*

$$x_{t-k} - x_t \leq \overline{V} y_{t-k} - \underline{C} y_t + (\underline{C} + kV - \overline{V})(y_{t-k+1} - u_{t-k+1}) - \sum_{s=1}^{\min\{k,L-1\}} (\underline{C} + (k-s+1)V - \overline{V}) u_{t-k-s+1} \tag{24}$$

is valid for conv(P). Furthermore, it is facet-defining for conv(P) when $\min\{k, L-1\} \leq 2$ *for all* $t \in [k + \min\{k, L-1\} + 1, T]_{\mathbb{Z}}$.

Proof: As a symmetry of (23), inequality (24) can be proved to be valid and facet-defining similarly and thus the proofs are omitted here. ■

Proposition 13 *For each* $k \in \{[1, T-1]_{\mathbb{Z}} : \overline{C} - \overline{V} - (k-1)V > 0\}$*, the inequality*

$$x_{t-k} - x_t \leq \overline{V} y_{t-k} - \underline{C} y_t + V \sum_{s=1}^{k-1} (y_{t-s} - \sum_{i=s}^{\min\{k, s+L-1\}} u_{t-i}) + (\underline{C} + V - \overline{V})(y_t - \sum_{s=0}^{\min\{k, L-1\}} u_{t-s}) \quad (25)$$

is valid for conv(P) for each $t \in [\max\{\min\{k, k+L-2\}+2, \min\{k, L-1\}+2\}, T]_{\mathbb{Z}}$*. Furthermore, it is facet-defining for conv(P) for each* $t \in [\max\{\min\{k, k+L-2\}+2, \min\{k, L-1\}+2\}, T]_{\mathbb{Z}}$ *when* $L \leq 3$*.*

Proof: **(Validity)** We discuss the following two cases in terms of the values of y_{t-k}:

1) If $y_{t-k} = 0$, then inequality (25) is valid since $x_t \geq \underline{C} y_t$ due to constraints (1d), $y_{t-s} - \sum_{i=s}^{\min\{k,s+L-1\}} u_{t-i} \geq 0$ for all $s \in [1, k-1]_{\mathbb{Z}}$ and $y_t - \sum_{s=0}^{\min\{k,L-1\}} u_{t-s} \geq 0$ due to constraints (1a), and $\underline{C} + V - \overline{V} > 0$.

2) If $y_{t-k} = 1$, then we only consider the case that $u_{t-k} = 0$ since we can easily verify that (25) is valid when $u_{t-k} = 1$ (following $x_{t-k} \leq \overline{V}$ and the case 1) above). We further discuss the following cases in terms of the time period when the generator shuts down.

   (1) If the generator shuts down at $\bar{t}$ such that $\bar{t} \geq t$, then inequality (25) converts to $x_{t-k} - x_t \leq kV$, which is valid due to ramp-down constraints (1g).

   (2) If the generator shuts down at $t - \bar{s}$ such that $\bar{s} \in [1, k-1]_{\mathbb{Z}}$, i.e., $y_{t-\bar{s}} = 0$, then inequality (25) converts to $x_{t-k} - x_t \leq \overline{V} + (k-1-\bar{s})V - \underline{C} y_t + V \sum_{s=1}^{\bar{s}-1} (y_{t-s} - \sum_{i=s}^{\min\{k,s+L-1\}} u_{t-i}) + (\underline{C} + V - \overline{V})(y_t - \sum_{s=0}^{\min\{k,L-1\}} u_{t-s})$, which is clearly valid since $x_{t-k} \leq \overline{V} + (k-1-\bar{s})V$ due to ramp-down constraints (1g), $x_t \geq \underline{C} y_t$ due to constraints (1d), $y_{t-s} - \sum_{i=s}^{\min\{k,s+L-1\}} u_{t-i} \geq 0$ for all $s \in [1, \bar{s}-1]_{\mathbb{Z}}$ and $y_t - \sum_{s=0}^{\min\{k,L-1\}} u_{t-s} \geq 0$ due to constraints (1a), and $\underline{C} + V - \overline{V} > 0$.

(Facet-defining) The proof is similar with that for Proposition 8 and thus omitted here. ∎

From Propositions 10 - 13, we can observe that these derived inequalities contain two continuous variables and the total number of inequalities is in other order of $\mathcal{O}(T^2)$.

Proposition 14 *For each* $t \in [\max\{L+2, 4\}, T]_{\mathbb{Z}}$*, the inequality*

$$x_{t-3} - x_{t-2} + x_{t-1} \leq \overline{V} y_{t-3} - (\overline{V} - V) y_{t-2} + \overline{V} y_{t-1} + (\underline{C} + V - \overline{V})(y_t - u_t - y_{t-1})$$

$$+(\overline{C}-\overline{V})(y_{t-1}-u_{t-1}-u_{t-2})-\sum_{s=0}^{L-3}(\overline{C}-\overline{V}-sV)u_{t-s-3} \quad (26)$$

is valid for conv(P) when $L \geq 2$*. Furthermore, it is facet-defining for conv(P) for each* $t \in [\max\{L+2,4\},T]_{\mathbb{Z}}$ *when* $L \leq 3$*.*

Proof: **(Validity)** We discuss the following two cases in terms of the value of y_t:

1) If $y_{t-1} = 0$, then $u_{t-s-1} = 0$ for all $s \in [0, L-1]_{\mathbb{Z}}$ due to constraints (1a) and $y_t = u_t$ due to constraints (1c) and (1a). It follows that inequality (26) converts to $x_{t-3} - x_{t-2} \leq \overline{V}y_{t-3} - (\overline{V} - V)y_{t-2}$, which can be easily verified to be valid through consider all the three possible cases, i.e., (1) $y_{t-3} = y_{t-2} = 1$, (2) $y_{t-3} = 1$ and $y_{t-2} = 0$, and (3) $y_{t-3} = y_{t-2} = 0$.

2) If $y_{t-1} = 1$, then $\sum_{s=0}^{L-1} u_{t-s-1} \leq 1$ due to constraints (1a). We further discuss the following four possible cases.

   (1) If $u_{t-s-1} = 0$ for all $s \in [0, L-1]_{\mathbb{Z}}$, then $y_{t-2} = 1$ due to constraints (1c) and $L \geq 2$. It follows that inequality (26) converts to $x_{t-3}-x_{t-2}+x_{t-1} \leq \overline{V}y_{t-3}-(\overline{V}-V)+\overline{C}+(\underline{C}+V-\overline{V})(y_t-1)$, which can be easily verified to be valid through consider all the four possible cases, i.e., (1) $y_{t-3} = y_t = 1$, (2) $y_{t-3} = 1$ and $y_t = 0$, (3) $y_{t-3} = 0$ and $y_t = 1$, and (4) $y_{t-3} = y_t = 0$.

   (2) If $u_{t-1} = 1$, then $u_{t-s-1} = 0$ for all $s \in [1, L-1]_{\mathbb{Z}}$. Meanwhile, we have $y_t = 1$ and $u_t = 0$ due to $L \geq 2$ and $y_{t-2} = 0$ due to constraints (1g). It follows that inequality (26) converts to $x_{t-3}+x_{t-1} \leq \overline{V}y_{t-3}+\overline{V}$, which is valid since $x_{t-3} \leq \overline{V}y_{t-3}$ and $x_{t-1} \leq \overline{V}$ due to constraints (1f) and (1g).

   (3) If $u_{t-2} = 1$, then $u_{t-s-1} = 0$ for all $s \in [2, L-1]_{\mathbb{Z}}$ and $u_{t-1} = 0$. Meanwhile, we have $y_{t-3} = 0$ due to (1g) and $y_{t-1} = 1$ and $u_t = 0$ due to $L \geq 2$. It follows that inequality (26) converts to $x_{t-1} - x_{t-2} \leq V + (\underline{C} + V - \overline{V})(y_t - 1)$, which can be easily verified to be valid either $y_t = 1$ or $y_t = 0$.

   (4) If $u_{t-s-3} = 1$ for some $s \in [0, L-3]_{\mathbb{Z}}$ when $L \geq 3$, then $y_{t-3} = y_{t-2} = y_{t-1} = 1$ due to minimum-up time constraints (1a). It follows that inequality (26) converts to $x_{t-3} - x_{t-2} + x_{t-1} \leq \overline{V} + sV + V + (\underline{C} + V - \overline{V})(y_t - 1)$, which can be easily verified to be valid either $y_t = 1$ or $y_t = 0$ since $x_{t-3} \leq \overline{V} + sV$ and $x_{t-1} - x_{t-2} \leq V + (\underline{C} + V - \overline{V})(y_t - 1)$.

(Facet-defining) We only provide the facet-defining proof for the case when $L = 3$ since the case when $L = 2$ can be proved similarly and thus omitted here.

We generate generate $3T-2$ linearly independent points in $\text{conv}(P)$ that satisfy (26) at equality in the following groups.

1) For each $r \in [1, t-4]_{\mathbb{Z}}$ (totally $t-4$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ such that

$$\acute{x}_s^r = \begin{cases} \underline{C}, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \acute{u}_s^r = 0, \\ \forall s \end{array}.$$

2) For $r = t-2$ (totally one point), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ such that

$$\acute{x}_s^r = \begin{cases} \underline{C} + V, & s \in [1, r-1]_{\mathbb{Z}} \\ \underline{C}, & s = r \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & s \in [1, r]_{\mathbb{Z}} \\ 0, & s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \acute{u}_s^r = 0, \\ \forall s \end{array}.$$

3) For each $r \in [1, t-2]_{\mathbb{Z}}$ (totally $t-2$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}_s^r = 1$ for each $s \in [1, r]_{\mathbb{Z}}$ and $\bar{y}_s^r = 0$ otherwise. Thus $\bar{u}_s^r = 0$ for $\forall s \in [2, T]_{\mathbb{Z}}$. For the value of $\bar{x}^r$: (1) for each $r \in [1, t-4]_{\mathbb{Z}}$, we let $\bar{x}_s^r = \underline{C} + \epsilon$ for each $s \in [1, r]_{\mathbb{Z}}$; (2) for $r = t-3$, we let $\bar{x}_s^r = \overline{V}$ for each $s \in [1, r]_{\mathbb{Z}}$; (3) for $r = t-2$, we let $\bar{x}_s^r = \underline{C} + V + \epsilon$ for each $s \in [1, r-1]_{\mathbb{Z}}$ and $\bar{x}_s^r = \underline{C} + \epsilon$ for $s = r$.

4) For $r = t-1$ (totally one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}_s^r = 1$ for $\forall s \in [1, T]_{\mathbb{Z}}$ and thus $\bar{u}_s^r = 0$ for $\forall s$. For the value of $\bar{x}^r$, we let $\bar{x}_s^r = \overline{C} - V$ for $s = t-2$ and $\bar{x}_s^r = \overline{C}$ otherwise.

5) For each $r \in [t, T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that

$$\bar{x}_s^r = \begin{cases} \overline{V}, & s = t-3 \\ \underline{C} + V, & s = t-1 \\ \underline{C}, & s \in [r, T]_{\mathbb{Z}} \cup \{t-2\} \\ 0, & \text{o.w.} \end{cases}, \quad \bar{y}_s^r = \begin{cases} 1, & s \in [t-3, r]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \bar{u}_s^r = \begin{cases} 1, & s = t-3 \\ 0, & \text{o.w.} \end{cases}.$$

6) For each $r \in [2, t-1]_{\mathbb{Z}}$ (totally $t-2$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that $\hat{y}_s^r = 1$ for each $s \in [r, r+L-1]_{\mathbb{Z}}$ (i.e., $s \in [r, r+2]_{\mathbb{Z}}$) and $\hat{y}_s^r = 0$ otherwise. Thus $\hat{u}_s^r = 1$ for $s = r$. For the value of $\hat{x}^r$: (1) for each $r \in [2, t-4]_{\mathbb{Z}} \cup \{t-2\}$, we let $\hat{x}_s^r = \underline{C}$ for each $s \in [r, r+2]_{\mathbb{Z}} \setminus \{t-3\}$ and $\hat{x}_s^r = \underline{C} + V$ for each $s \in [r, r+2]_{\mathbb{Z}} \cap \{t-3\}$; (2) for $r = t-3$, we let $\hat{x}_s^r = \overline{V}$ for each $s \in \{t-3, t-1\}$ and $\hat{x}_s^r = \underline{C}$ for each $s = t-2$; (3) for $r = t-1$, we let $\hat{x}_s^r = \overline{V}$ for each $s \in [r, r+L-1]_{\mathbb{Z}}$.

7) For each $r \in [t,T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that

$$\hat{x}_s^r = \begin{cases} \underline{C}, & s \in [r,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \hat{y}_s^r = \begin{cases} 1, & s \in [r,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

8) For each $r \in [t,T]_{\mathbb{Z}}$ (totally $T-t+1$ points), we create $(\grave{x}^r, \grave{y}^r, \grave{u}^r) \in \text{conv}(P)$ such that

$$\grave{x}_s^r = \begin{cases} \underline{C}+\epsilon, & s \in [r,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \quad \grave{y}_s^r = \begin{cases} 1, & s \in [r,T]_{\mathbb{Z}} \\ 0, & \text{o.w.} \end{cases}, \text{ and } \grave{u}_s^r = \begin{cases} 1, & s = r \\ 0, & \text{o.w.} \end{cases}.$$

9) We create $(\dot{x}, \dot{y}, \dot{u}) \in \text{conv}(P)$ such that $\dot{y}_s = 1$ for each $s \in \{t-2, t-1, t\}$ and $\dot{y}_s = 0$ otherwise. Thus we have $\dot{u}_s = 1$ for $s = t-2$. Meanwhile, we let $\dot{x}_{t-2} = \dot{x}_t = \underline{C} + \epsilon$ and $\dot{x}_{t-1} = \underline{C} + V + \epsilon$.

Finally, it is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^T$ and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=2}^T$ are linearly independent because they can construct a lower-diagonal matrix. In addition, $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1, r\neq t-3}^{t-2}$, $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t}^T$, and $(\dot{x}, \dot{y}, \dot{u})$ are also linearly independent with them after Gaussian eliminations between $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1, r\neq t-3}^{t-2}$, $(\dot{x}, \dot{y}, \dot{u})$, and $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^{t-2}$, and between $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=t}^T$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t}^T$. ■

Proposition 15 *For each $k \in \{[0, T-4]_{\mathbb{Z}} : \overline{C} - \overline{V} - kV > 0\}$, $t \in [\max\{1, L-2\}, T-k-3]_{\mathbb{Z}}$, the inequality*

$$\begin{aligned} x_t - x_{t+1} + x_{t+2} \le \overline{V} y_t - (\overline{V} - V) y_{t+1} + \overline{V} y_{t+2} - \phi \\ + V \sum_{s=1}^{k} (y_{t+s+2} - \sum_{i=0}^{L-1} u_{t+s-i+2}) + (\overline{C} - \overline{V} - kV)(y_{t+k+3} - \sum_{j=0}^{L-1} u_{t+k-j+3}) \end{aligned} \quad (27)$$

is valid and facet-defining for conv(P) when $L \ge 2$, where $\phi = 0$ if $L \ge 4$ or $t = 1$, and $\phi = (\underline{C} + V - \overline{V}) u_t$ otherwise.

Proof: **(Validity)** We prove the validity for the case that $\phi = (\underline{C} + V - \overline{V}) u_t$, i.e., $L \le 3$ and $t \ge 2$, while other cases can be proved similarly. We discuss the following two cases in terms of the value of y_{t+2}:

1) If $y_{t+2} = 0$, to show inequality (27) is valid, we show $x_t - x_{t+1} \le \overline{V} y_t - (\overline{V} - V) y_{t+1} - (\underline{C} + V - \overline{V}) u_t$. Then inequality (27) is valid since $y_{t+s+2} - \sum_{i=0}^{L-1} u_{t+s-i+2} \ge 0$ for all $s \in [1,k]_{\mathbb{Z}}$ and $y_{t+k+3} - \sum_{j=0}^{L-1} u_{t+k-j+3} \ge 0$ due to minimum-up constraints (1a). We discuss the following three possible cases.

   (1) If $y_t = y_{t+1} = 1$ and $u_t = 0$, then (27) converts to $x_t - x_{t+1} \le V$, which is valid due to ramp-down constraints (1g).

(2) If $y_t = 1$ and $y_{t+1} = u_t = 0$, then (27) converts to $x_t \leq \overline{V}$, which is valid due to ramp-down constraints (1g).

(3) If $y_t = y_{t+1} = u_t = 1$, then (27) converts to $x_t - x_{t+1} \leq \overline{V} - \underline{C}$, which is valid since $x_t \leq \overline{V}$ due to ramp-up constraints (1f) and $x_{t+1} \geq \underline{C}$ due to constraints (1d).

2) If $y_{t+2} = 1$, we discuss the following two cases in terms of the value of u_{t+2}:

(1) If $u_{t+2} = 1$, then we have $y_{t+1} = u_t = 0$ due to constraints (1a) - (1c). Thus, we have $x_t \leq \overline{V} y_t$ due to ramp-down constraints (1g) and $x_{t+2} \leq \overline{V}$ due to ramp-up constraints (1f). It follows that inequality (27) is valid since $y_{t+s+2} - \sum_{i=0}^{L-1} u_{t+s-i+2} \geq 0$ for all $s \in [1,k]_{\mathbb{Z}}$ and $y_{t+k+3} - \sum_{j=0}^{L-1} u_{t+k-j+3} \geq 0$ due to minimum-up constraints (1a).

(2) If $u_{t+2} = 0$, we discuss the following three possible cases.

- If $u_{t+1} = 1$, then $y_t = u_t = 0$ due to constraints (1b) - (1c) and $y_{t+1} = y_{t+2} = 1$ due to $L \geq 2$. Then (27) is clearly valid since $x_{t+2} - x_{t+1} \leq V$ due to ramp-up constrains (1f).
- If $u_t = 1$, then $y_{\bar{s}} = 1$ for all $\bar{s} \in [t, t+L-1]_{\mathbb{Z}}$ and $u_{\hat{s}} = 0$ for all $\hat{s} \in [t-L+4, t+L]_{\mathbb{Z}}$ since we consider $L \leq 3$. Inequality (27) converts to $x_t - x_{t+1} + x_{t+2} \leq \overline{V} - \underline{C} + \overline{V} + V\sum_{s=1}^{k}(y_{t+s+2} - \sum_{i=0}^{L-1} u_{t+s-i+2}) + (\overline{C} - \overline{V} - kV)(y_{t+k+3} - \sum_{j=0}^{L-1} u_{t+k-j+3})$, which is valid since $x_t \leq \overline{V}$, $x_{t+1} \geq \underline{C}$, and $x_{t+2} \leq \overline{V} + V\sum_{s=1}^{k}(y_{t+s+2} - \sum_{i=0}^{L-1} u_{t+s-i+2}) + (\overline{C} - \overline{V} - kV)(y_{t+k+3} - \sum_{j=0}^{L-1} u_{t+k-j+3})$ due to inequality (20).
- If $u_{\bar{t}} = 1$ for some $\bar{t} \leq t-1$, then (27) converts to $x_t - x_{t+1} + x_{t+2} \leq V + \overline{V} + V\sum_{s=1}^{k}(y_{t+s+2} - \sum_{i=0}^{L-1} u_{t+s-i+2}) + (\overline{C} - \overline{V} - kV)(y_{t+k+3} - \sum_{j=0}^{L-1} u_{t+k-j+3})$, which is valid since $x_t - x_{t+1} \leq V$, and $x_{t+2} \leq \overline{V} + V\sum_{s=1}^{k}(y_{t+s+2} - \sum_{i=0}^{L-1} u_{t+s-i+2}) + (\overline{C} - \overline{V} - kV)(y_{t+k+3} - \sum_{j=0}^{L-1} u_{t+k-j+3})$ due to inequality (20).

(Facet-defining) We only provide the facet-defining proof for the case when $L \geq 4$ since other cases can be proved similarly and thus omitted here.

We generate generate $3T-2$ linearly independent points in conv(P) that satisfy (27) at equality in the following groups.

1) For each $r \in [1, t-1]_{\mathbb{Z}}$ (totally $t-1$ points), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ such that

$$\acute{x}_s^r = \begin{cases} \underline{C}, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \quad \acute{y}_s^r = \begin{cases} 1, & s \in [1,r]_{\mathbb{Z}} \\ 0, & s \in [r+1,T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \acute{u}_s^r = 0, \\ \forall s \end{array}.$$

2) For $r = t+1$ (totally one point), we create $(\acute{x}^r, \acute{y}^r, \acute{u}^r) \in \text{conv}(P)$ such that

$$\acute{x}^r_s = \begin{cases} \underline{C} + V, \ s \in [1, r-1]_{\mathbb{Z}} \\ \underline{C}, \ s = r \\ 0, \ s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \ \acute{y}^r_s = \begin{cases} 1, \ s \in [1, r]_{\mathbb{Z}} \\ 0, \ s \in [r+1, T]_{\mathbb{Z}} \end{cases}, \text{ and } \begin{array}{l} \acute{u}^r_s = 0, \\ \forall s \end{array}.$$

3) For each $r \in [1, t+k+2]_{\mathbb{Z}}$ (totally $t+k+2$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}^r_s = 1$ for each $s \in [1, r]_{\mathbb{Z}}$ and $\bar{y}^r_s = 0$ otherwise. Thus $\bar{u}^r_s = 0$ for $\forall s \in [2, T]_{\mathbb{Z}}$. For the value of $\bar{x}^r$: (1) for each $r \in [1, t-1]_{\mathbb{Z}}$, we let $\bar{x}^r_s = \underline{C} + \epsilon$ for each $s \in [1, r]_{\mathbb{Z}}$; (2) for $r = t$, we let $\bar{x}^r_s = \overline{V}$ for each $s \in [1, r]_{\mathbb{Z}}$; (3) for $r = t+1$, we let $\bar{x}^r_s = \underline{C} + V + \epsilon$ for each $s \in [1, r-1]_{\mathbb{Z}}$ and $\bar{x}^r_s = \underline{C} + \epsilon$ for $s = r$; (4) for $r = t+2$, we let $\bar{x}^r_s = \underline{C} + V$ for each $s \in [1, t]_{\mathbb{Z}}$, $\bar{x}^r_s = \underline{C}$ for $s = t+1$, and $\bar{x}^r_s = \overline{V}$ for $s = t+2$; (5) for each $r \in [t+3, t+k+2]_{\mathbb{Z}}$, we let $\bar{x}^r_s = \overline{V} + (r-s)V$ for each $s \in [t+2, r]_{\mathbb{Z}}$, $\bar{x}^r_s = \overline{V} + (r-t-3)V$ for $s = t+1$, and $\bar{x}^r_s = \overline{V} + (r-t-2)V$ for each $s \in [1, t]_{\mathbb{Z}}$.

4) For $r = t+k+3$ (totally one point), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}^r_s = 1$ for $\forall s \in [1, T]_{\mathbb{Z}}$ and thus $\bar{u}^r_s = 0$ for $\forall s$. For the value of $\bar{x}^r$, we let $\bar{x}^r_s = \overline{C} - V$ for $s = t+1$ and $\bar{x}^r_s = \overline{C}$ otherwise.

5) For each $r \in [t+k+4, T]_{\mathbb{Z}}$ (totally $T-t-k-3$ points), we create $(\bar{x}^r, \bar{y}^r, \bar{u}^r) \in \text{conv}(P)$ such that $\bar{y}^r_s = 1$ for each $s \in [t+k-L+4, r]_{\mathbb{Z}}$ and $\bar{y}^r_s = 0$ otherwise. Thus $\bar{u}^r_s = 1$ for $s = t+k-L+4$. For the value of $\bar{x}^r$, we consider the following cases:

- If $t+k-L+4 \geq t+3$, we let $\bar{x}^r_s = \underline{C}$ for each $s \in [t+k-L+4, r]_{\mathbb{Z}}$;
- If $t+k-L+4 = t+2$, we let $\bar{x}^r_s = \overline{V}$ for each $s \in [t+k-L+4, r]_{\mathbb{Z}}$;
- If $t+k-L+4 = t+1$, we let $\bar{x}^r_s = \underline{C} + V$ for $s = t+2$ and $\bar{x}^r_s = \underline{C}$ for each $s \in [t+k-L+4, r]_{\mathbb{Z}} \setminus \{t+2\}$;
- If $t+k-L+4 \leq t$, we let $\bar{x}^r_s = \overline{V}$ for each $s \in [t+k-L+4, t]_{\mathbb{Z}}$, $\bar{x}^r_s = \underline{C} + V$ for $s = t+2$, and $\bar{x}^r_s = \underline{C}$ for each $s \in [t+k-L+4, r]_{\mathbb{Z}} \setminus \{t+2\}$;

6) For each $r \in [2, T]_{\mathbb{Z}}$ (totally $T-1$ points), we create $(\hat{x}^r, \hat{y}^r, \hat{u}^r) \in \text{conv}(P)$ such that $\hat{u}^r_s = 1$ for $s = r$ and $\hat{u}^r_s = 0$ otherwise. For the values of $\hat{x}^r$ and $\hat{y}^r$: (1) for each $r \in [2, t-L+3]_{\mathbb{Z}}$, we let $\hat{y}^r_s = 1$ for $s \in [r, t+2]_{\mathbb{Z}}$ and $\hat{y}^r_s = 0$ otherwise; we let $\hat{x}^r_s = \underline{C} + V$ for each $s \in [r, t]_{\mathbb{Z}}$, $\hat{x}^r_s = \underline{C}$ for $s = t+1$, and $\hat{x}^r_s = \overline{V}$ for $s = t+2$; (2) for each $r \in [t-L+4, T]_{\mathbb{Z}}$, we let $\hat{y}^r_s = 1$

for $s \in [r, r+L-1]_{\mathbb{Z}}$ and $\hat{y}^r_s = 0$ otherwise; the value of $\hat{x}^r$ can be assigned similarly as above and thus omitted here.

7) For each $r \in [t+1, T]_{\mathbb{Z}} \setminus \{t+2\}$ (totally $T-t-1$ points), we create $(\grave{x}^r, \grave{y}^r, \grave{u}^r) \in \mathrm{conv}(P)$ such that $\grave{y}^r_s = 1$ for $s \in [r, r+L-1]_{\mathbb{Z}}$ and $\grave{y}^r_s = 0$ otherwise. Thus $\grave{u}^r_s = 1$ for $s = r$. We assign the value of $\grave{x}^r$ to make $(\grave{x}^r, \grave{y}^r, \grave{u}^r)$ and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)$ linearly independent for each $r \in [t+1, T]_{\mathbb{Z}} \setminus \{t+2\}$. It can be easily assigned following the similar rule above and thus omitted here.

Finally, it is clear that $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1}^T$ and $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=2}^T$ are linearly independent because they can construct a lower-diagonal matrix. In addition, $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1, r\neq t}^{t+1}$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t+1, r\neq t+2}^T$ are also linearly independent with them after Gaussian eliminations between $(\acute{x}^r, \acute{y}^r, \acute{u}^r)_{r=1, r\neq t}^{t+1}$ and $(\bar{x}^r, \bar{y}^r, \bar{u}^r)_{r=1, r\neq t}^{t+1}$, and between $(\hat{x}^r, \hat{y}^r, \hat{u}^r)_{r=t+1, r\neq t+2}^T$ and $(\grave{x}^r, \grave{y}^r, \grave{u}^r)_{r=t+1, r\neq t+2}^T$. ■

We can observe that the inequalities derived in Propositions 14 and 15 contain three continuous variables and are polynomial in the order of $\mathcal{O}(T)$.

To summarize, all the derived strong valid inequalities in this section covering multiple periods are polynomial in the order of up to $\mathcal{O}(T^2)$. Therefore, we do not need to perform a separation approach.

4 Computational Experiments

In this section, we show the effectiveness of our proposed strong valid inequalities on solving both the network-constrained unit commitment (used by ISOs) and self-scheduling unit commitment (used by market participants) problems. The experiments were performed on a computer node with two AMD Opteron 2378 Quad Core Processors at 2.4GHz. The addressable memory is 4GB and the time limit was set at one hour per run. CPLEX 12.3 with default settings were used to solve the problems.

4.1 Network-Constrained Unit Commitment Problem

For the network-constrained unit commitment problem, we first provide the mathematical formulation and then report the computational results for the power system data based on [3] and [17], and a modified IEEE 118-bus system based on the one given online at `http://motor.ece.iit.edu/data/SCUC_118/`, respectively.

For the mathematical formulation, we set the operational time interval to be 24 hours (i.e., $T = 24$) and let $\mathcal{G}$ and $\mathcal{B}$ represent the set of generators and buses respectively, with $|\mathcal{G}| = G$ and $|\mathcal{B}| = B$. Besides, we let $\mathcal{E}$ represent the set of transmission lines linking two buses. With superscripts g and b representing generator and bus index respectively, we introduce the notation for the whole system, with a part of them similar to those defined in Section 1. For each generator g, we let L^g (ℓ^g) be its minimum-up (-down) time limit, $\overline{C}^g$ ($\underline{C}^g$) be its generation upper (lower) bound, $\overline{V}^g$ be its start-up/shut-down ramp rate, V^g be its ramp-up/-down rate in stable generation, SU^g (SD^g) represent its start-up (shut-down) cost of generator g, (x_t^g, y_t^g, u_t^g) represent its status at each time period t for $t \in [1, T]_{\mathbb{Z}}$, and $f^g(x_t^g)$ represent its generation cost of generator g when its generation amount is x_t^g at t. In addition, we let d_t^b represent the load (demand) at bus b at time period t and r_t represent the system reserve factor at t. For each transmission line $(j, h) \in \mathcal{E}$, we let C_{jh} represent its capacity, and K_{jh}^b represent the line flow distribution factor for the flow on the transmission line (j, h) contributed by the net injection at bus b. Meanwhile, for notation convenience, we let $\mathcal{G}_b \subseteq \mathcal{G}$ represent the set of generators at bus b (e.g., $\mathcal{G}_i \cap \mathcal{G}_j = \emptyset$ for $i, j \in \mathcal{B}$ and $i \neq j$, $\bigcup_{b=1}^{B} \mathcal{G}_b = \mathcal{G}$) and $G_b = |\mathcal{G}_b|$. Accordingly, the network-constrained unit commitment problem can be described as follows:

$$
\begin{aligned}
\min_{x,y,u} \quad & \sum_{g=1}^{G}\Big(\sum_{t=2}^{T}\big(\mathrm{SU}^g u_t^g + \mathrm{SD}^g(y_{t-1}^g - y_t^g + u_t^g)\big) + \sum_{t=1}^{T} f^g(x_t^g)\Big) && \text{(28a)}\\
\text{s.t.} \quad & \sum_{i=t-L^g+1}^{t} u_i^g \leq y_t^g, \quad \forall t \in [L^g+1, T]_{\mathbb{Z}}, \forall g \in [1, G]_{\mathbb{Z}}, && \text{(28b)}\\
& \sum_{i=t-\ell^g+1}^{t} u_i^g \leq 1 - y_{t-\ell^g}^g, \forall t \in [\ell^g+1, T]_{\mathbb{Z}}, \forall g \in [1, G]_{\mathbb{Z}}, && \text{(28c)}\\
& -y_{t-1}^g + y_t^g - u_t^g \leq 0, \quad \forall t \in [2, T]_{\mathbb{Z}}, \forall g \in [1, G]_{\mathbb{Z}}, && \text{(28d)}\\
& \underline{C}^g y_t^g \leq x_t^g \leq \overline{C}^g y_t^g, \quad \forall t \in [1, T]_{\mathbb{Z}}, \forall g \in [1, G]_{\mathbb{Z}}, && \text{(28e)}\\
& x_t^g - x_{t-1}^g \leq V^g y_{t-1}^g + \overline{V}^g(1 - y_{t-1}^g), \quad \forall t \in [2, T]_{\mathbb{Z}}, \forall g \in [1, G]_{\mathbb{Z}}, && \text{(28f)}\\
& x_{t-1}^g - x_t^g \leq V^g y_t^g + \overline{V}^g(1 - y_t^g), \quad \forall t \in [2, T]_{\mathbb{Z}}, \forall g \in [1, G]_{\mathbb{Z}}, && \text{(28g)}\\
& \sum_{g=1}^{G} x_t^g = \sum_{b=1}^{B} d_t^b, \quad \forall t \in [1, T]_{\mathbb{Z}}, && \text{(28h)}\\
& \sum_{g=1}^{G} \overline{C}_g y_t^g \geq r_t \sum_{b=1}^{B} d_t^b, \quad \forall t \in [1, T]_{\mathbb{Z}}, && \text{(28i)}
\end{aligned}
$$

$$-C_{jh} \leq \sum_{b=1}^{B} K_{jh}^{b} \Big(\sum_{g=1}^{G_b} x_t^g - d_t^b \Big) \leq C_{jh}, \quad \forall t \in [1,T]_{\mathbb{Z}}, \forall (j,h) \in \mathcal{E}, \tag{28j}$$

$$y_t^g \in \{0,1\}, \ \forall t \in [1,T]_{\mathbb{Z}}; \ u_t^g \in \{0,1\}, \ \forall t \in [2,T]_{\mathbb{Z}}, \forall g \in [1,G]_{\mathbb{Z}}, \tag{28k}$$

where the objective is to minimize the total cost, including start-up cost, shut-down cost, and the generation cost that is represented by $f^g(x_t^g)$, which is typically a nondecreasing quadratic function, i.e., $f^g(x_t^g) = a^g (x_t^g)^2 + b^g x_t^g + c^g$. Constraints (28b) (resp. (28c)) describe the minimum-up (resp. minimum-down) time restrictions and constraints (28d) describe the relationship between y and u. Constraints (28e) describe the generation upper and lower bound for generator g if it is online at time period t. Constraints (28f) (resp. (28g)) describe the maximum generation increment (resp. decrement) between two consecutive time periods (i.e., ramp-rates restrictions). Constraints (28h) enforce the load balance at each time period t. Constraints (28i) describe the system reserve requirements. Finally, constraints (28j) represent the capacity limit of each transmission line (j,h) (see, e.g., [23]). Note here that the generation cost function $f^g(\cdot)$ can be approximated by a piecewise linear function [3]. With this approximation, the formulation above can be reformulated as an MILP formulation.

4.1.1 Power System Data Based on [3] and [17]

In this experiment, there are eight types of generators (see Table 7), and twenty instances with each containing different combinations of each type of generators (see Table 8). The system load setting is reported in Table 9. Constraints (28i) and (28j) are not included in this experiment since the system reserve and transmission data are not provided in [3] and [17].

Table 7: Generator Data

Generators	$\underline{C}$ (MW)	$\overline{C}$ (MW)	L/ℓ (h)	V (MW/h)	$\overline{V}$ (MW/h)	SU (\$/h)	a (\$/MW^{2}h)	b (\$/MWh)	c (\$/h)
1	150	455	8	91	180	2000	0.00048	16.19	1000
2	150	455	8	91	180	2000	0.00031	17.26	970
3	20	130	5	26	35	500	0.002	16.6	700
4	20	130	5	26	35	500	0.00211	16.5	680
5	25	162	6	32.4	40	700	0.00398	19.7	450
6	20	80	3	16	28	150	0.00712	22.26	370
7	25	85	3	17	33	200	0.00079	27.74	480
8	10	55	1	11	15	60	0.00413	25.92	660

For each instance, we compare four formulations (i.e., “MILP”, “Strong”, “Strong-1”, and

Table 8: Problem Instances [17]

Instances	Generators								# of Generators
	1	2	3	4	5	6	7	8	
1	12	11	0	0	1	4	0	0	28
2	13	15	2	0	4	0	0	1	35
3	15	13	2	6	3	1	1	3	44
4	15	11	0	1	4	5	6	3	45
5	15	13	3	7	5	3	2	1	49
6	10	10	2	5	7	5	6	5	50
7	17	16	1	3	1	7	2	4	51
8	17	10	6	5	2	1	3	7	51
9	12	17	4	7	5	2	0	5	52
10	13	12	5	7	2	5	4	6	54
11	46	45	8	0	5	0	12	16	132
12	40	54	14	8	3	15	9	13	156
13	50	41	19	11	4	4	12	15	156
14	51	58	17	19	16	1	2	1	165
15	43	46	17	15	13	15	6	12	167
16	50	59	8	15	1	18	4	17	172
17	53	50	17	15	16	5	14	12	182
18	45	57	19	7	19	19	5	11	182
19	58	50	15	7	16	18	7	12	183
20	55	48	18	5	18	17	15	11	187

Table 9: System Load (% of Total Capacity) [17]

Time	1	2	3	4	5	6	7	8	9	10	11	12
Load	71%	65%	62%	60%	58%	58%	60%	64%	73%	80%	82%	83%
Time	13	14	15	16	17	18	19	20	21	22	23	24
Load	82%	80%	79%	79%	83%	91%	90%	88%	85%	84%	79%	74%

"Strong-2") and report the results in Table 10, where "MILP" represents the original MILP formulation given in (28), "Strong" represents the original MILP formulation plus our proposed strong valid inequalities in Sections 1 - 3 (i.e., (2d) - (2g), (4) - (13), and (17) - (27)) as constraints in the formulation, "Strong-1" represents the original MILP formulation plus inequalities (2d) - (2g) as constraints and inequalities in Sections 2 and 3 (i.e., (4) - (13) and (17) - (27)) as user cuts, and "Strong-2" represents the original MILP formulation plus all the strong valid inequalities added as user cuts.

In Table 10, the column labelled "Integer OBJ. ($)" provides the best objective value corresponding to the best integer solution obtained from all four different formulations, i.e., "MILP", "Strong", "Strong-1", and "Strong-2", within the time limit. The column labelled "IGap (%)"

Table 10: Computational Performance for the Data Based on [3] and [17]

Inst	Integer OBJ. ($)	IGap (%)		Percentage (%)	CPU Time(s) (TGap (10^{-4}))				# of Nodes				# of User	
		MILP	Strong		MILP	Strong	Strong-1	Strong-2	MILP	Strong	Strong-1	Strong-2	Strong-1	Strong-2
1	3794100	0.76	0.12	84.94	*** (6.97)	2132.38	*** (1.18)	*** (3.22)	204521	61127	361175	258106	28	315
2	4770702	0.78	0.14	82.56	*** (14.7)	*** (2.54)	*** (4.96)	*** (5.42)	131085	64282	188069	135795	171	654
3	5080033	0.82	0.06	92.62	*** (1.93)	*** (1.15)	1060.64	*** (1.49)	155292	74126	56715	231671	262	884
4	4755459	0.78	0.05	93.05	1921.24	1640.69	510.39	723.99	90231	83913	64685	63963	90	756
5	5354093	0.91	0.04	95.63	*** (1.39)	*** (1.28)	2107.45	*** (1.34)	161350	61669	110622	424892	295	1539
6	4383414	1.09	0.04	95.94	*** (1.4)	*** (1.11)	*** (1.32)	1361.15	546367	149716	944040	324041	209	1581
7	5784804	0.75	0.08	88.7	*** (3.33)	*** (1.82)	*** (2.52)	*** (2.64)	145248	127640	499276	139909	165	1391
8	5136903	0.96	0.04	95.76	*** (1.3)	707.3	590.98	436.67	167748	19879	41011	21481	164	1251
9	5584115	0.91	0.05	95.01	*** (2.2)	*** (1.85)	*** (1.36)	669.01	174427	49344	166697	29233	313	1800
10	5046209	1.15	0.06	94.52	*** (1.93)	*** (1.5)	*** (1.69)	*** (1.48)	140382	119148	383328	544832	252	2129
11	15681132	0.72	0.07	89.92	*** (9.99)	*** (2.25)	*** (3.53)	*** (4.9)	28211	4368	38212	25296	646	3066
12	17079158	0.78	0.04	95.17	*** (8.17)	*** (1.14)	2315.29	*** (1.72)	33515	10457	12621	22343	447	3768
13	16758002	0.85	0.03	96.07	*** (6.57)	*** (1.08)	*** (1.11)	*** (1.73)	41118	10847	26170	27864	660	3656
14	19976963	0.8	0.04	95.01	*** (6.76)	*** (1.33)	*** (1.33)	*** (1.95)	42719	2537	13296	15090	1262	3983
15	17242043	0.93	0.03	97.29	*** (1.9)	1652.43	*** (1.07)	870.97	29106	1654	33767	5577	820	5692
16	19342401	0.74	0.04	94.03	*** (8.15)	3356.66	2084.96	*** (1.1)	60224	5235	11787	22108	867	4038
17	19534390	0.87	0.02	97.35	*** (2.24)	1445.89	2482.16	908.61	13924	769	11988	2152	981	5399
18	19455610	0.85	0.03	96.74	*** (2.24)	*** (1.08)	1958.65	2224.8	16340	3452	12177	12809	661	4829
19	19963596	0.81	0.03	96.3	*** (5.08)	*** (1.13)	*** (1.25)	*** (1.49)	24223	5595	14013	17027	814	4501
20	19571381	0.86	0.03	96.86	*** (1.93)	3595.03	2248.15	666.37	16862	5406	11336	1447	483	6314

provides the root-node integrality gaps of "MILP" and "Strong", respectively. The integrality gap is defined as $(Z_{\text{MILP}} - Z_{\text{LP}})/Z_{\text{MILP}}$, where Z_{LP} is the objective value of the LP relaxation and Z_{MILP} is the objective value of the best integer solution, i.e., the value in the column labelled "Integer OBJ. ($)". We can observe that, our proposed strong valid inequalities tighten the LP relaxation dramatically, with the integrality gap reduction (from "MILP" to "Strong") reported in the column labelled "Percentage (%)". In the column labelled "CPU Time(s) (TGap (10^{-4}))", we report the computational time that CPLEX takes to solve the problem for each approach. For the cases in which CPLEX cannot solve the problem to optimality (i.e., reach the default 0.01% optimality gap) within one hour time limit, we provide the label "***" and accordingly report the terminating gap labelled "TGap (10^{-4})", which indicates the relative gap between the objective value corresponding to the best integer solution and the best lower bound when the time limit is reached. We can observe that all "Strong", "Strong-1", and "Strong-2" approaches perform much better than the original model "MILP". Almost all instances cannot be solved to optimality by "MILP" within one hour limit (except instance 4), while most instances can be solved by at least one of "Strong",

"Strong-1", and "Strong-2" approaches with our proposed strong valid inequalities added. The number of explored branch-and-bound nodes is reported in the column labelled "# of Nodes". The final column labelled "# of User" reports the number of user cuts added to solve the problem for "Strong-1" and "Strong-2".

4.1.2 Modified IEEE 118-Bus System

For this experiment, there are 54 generators, 118 buses, 186 transmission lines, and 91 load buses in the modified IEEE 118-bus system. We generate 15 instances, each with different load profile. Corresponding to each nominal load d_t^n given in the IEEE 118-bus system, we randomly generate a load $\bar{d}_t^n \in [1.8d_t^n, 2.2d_t^n]$. This random generation process is conducted for fifteen times corresponding to each (n, t) to generate the 15 instances. In this experiment, both constraints (28i) and (28j) are included with the system reserve factor r_t set at 5% for each time period $t \in [1, T]_{\mathbb{Z}}$.

Table 11: Computational Performance for the IEEE 118-Bus System

Inst	Integer OBJ. ($)	IGap (%)		Percent -age (%)	CPU Time(s) (TGap (10^{-4}))		# of Nodes		# of User
		MILP	Strong-1		MILP	Strong-1	MILP	Strong-1	
1	3358217	1.54	0.09	94.42	*** (1.39)	1432.02	180121	85936	100
2	3356847	1.37	0.05	96.65	*** (1.43)	2371.36	229259	342774	222
3	3367104	1.61	0.06	96.29	*** (3)	*** (1.8)	159795	136426	340
4	3362632	1.64	0.06	96.26	*** (1.96)	*** (1.37)	272480	238904	225
5	3349280	1.47	0.09	93.97	*** (2.23)	*** (1.47)	150695	373875	299
6	3364177	1.45	0.07	95.28	*** (1.28)	848.11	152427	69191	257
7	3353272	1.58	0.08	95.19	*** (2.29)	*** (1.51)	180557	594986	182
8	3348885	1.27	0.04	97.12	758.44	289.94	54354	28080	215
9	3354399	1.5	0.06	96.02	*** (3.27)	*** (1.9)	127050	102107	199
10	3352652	1.53	0.06	96.21	*** (1.91)	*** (1.38)	191125	187788	280
11	3357921	1.54	0.06	95.85	*** (1.31)	665.88	166568	58687	249
12	3359379	1.55	0.05	96.57	1074.87	405.07	94365	29781	262
13	3359624	1.57	0.07	95.78	*** (1.23)	1162.33	166052	66590	236
14	3362072	1.57	0.06	96.07	671.6	480.58	36746	19262	271
15	3351562	1.51	0.1	93.61	*** (2.12)	2615.75	142626	98899	294

For each instance, we compare the computational performance between "MILP" and "Strong-1", as defined in Section 4.1.1 and shown in Table 11. The labels in Table 11 are similar to those in Table 10. For "Strong-1", in the column labelled "IGap (%)", we report the integrality gap when all the strong valid inequalities are added as constraints. We continue to observe that the strong valid inequalities tighten the LP relaxation significantly, with about 95% reduction between the integrality gaps of the original MILP model and the model with the strong valid inequalities added. "Strong-1" also performs much better in terms of the computational time and terminating gap reported in the column labelled "CPU Time(s) (TGap (10^{-4}))". The number of explored branch-and-bound nodes is also reduced for most instances as indicated in the column labelled "# of Nodes". The final column reports the number of user cuts added in the formulation "Strong-1".

4.2 Self-Scheduling Unit Commitment Problem

For the self-scheduling unit commitment problem in which a single generator is considered, we first provide the mathematical formulation and then report the computational results for the eight single generators described in Table 7.

For the mathematical formulation, besides the notation defined in Section 1, we let p_t represent the electricity price at time period t, $f(x_t)$ represent the generation cost corresponding to the generation amount of x_t at t, and SU (SD) represent the start-up (shut-down) cost. Accordingly, the self-scheduling unit commitment problem can be described as follows:

$$
\begin{aligned}
\max_{x,y,u} \quad & \sum_{t=1}^{T}\big(p_t x_t - f(x_t)\big) - \sum_{t=2}^{T}\big(\mathrm{SU} u_t + \mathrm{SD}(y_{t-1} - y_t + u_t)\big) && \text{(29a)} \\
\text{s.t.} \quad & \sum_{i=t-L+1}^{t} u_i \leq y_t, \quad \forall t \in [L+1, T]_{\mathbb{Z}}, && \text{(29b)} \\
& \sum_{i=t-\ell+1}^{t} u_i \leq 1 - y_{t-\ell}, \ \forall t \in [\ell+1, T]_{\mathbb{Z}}, && \text{(29c)} \\
& -y_{t-1} + y_t - u_t \leq 0, \quad \forall t \in [2, T]_{\mathbb{Z}}, && \text{(29d)} \\
& \underline{C} y_t \leq x_t \leq \overline{C} y_t, \quad \forall t \in [1, T]_{\mathbb{Z}}, && \text{(29e)} \\
& x_t - x_{t-1} \leq V y_{t-1} + \overline{V}(1 - y_{t-1}), \quad \forall t \in [2, T]_{\mathbb{Z}}, && \text{(29f)} \\
& x_{t-1} - x_t \leq V y_t + \overline{V}(1 - y_t), \quad \forall t \in [2, T]_{\mathbb{Z}}, && \text{(29g)} \\
& y_t \in \{0, 1\}, \ \forall t \in [1, T]_{\mathbb{Z}}; \ u_t \in \{0, 1\}, \ \forall t \in [2, T]_{\mathbb{Z}}, && \text{(29h)}
\end{aligned}
$$

where the objective is to maximize the total profit, i.e., the total revenue from selling electricity minus the total cost from producing electricity. The generation cost function $f(x_t) = a(x_t)^2 + bx_t + c$ can be approximated by a piecewise linear function and accordingly the above formulation can be reformulated as an MILP formulation. Constraints (29b) (resp. (29c)) represent the minimum-up (resp. minimum-down) time restrictions, constraints (29d) represent the relationship between y

and u, constraints (29e) represent the generation upper and lower bound, and constraints (29f) (resp. (29g)) represent the ramp-up (resp. ramp-down) rate limits.

For each generator in Table 7, we test three instances with the price p_t, $\forall t \in [1, T]_{\mathbb{Z}}$ with $T = 10000$, randomly generated and report the average result over these three instances. For generators 1 and 2, we randomly generate $p_t \in [0, 35]$; for generators 3 and 4, we randomly generate $p_t \in [0, 41]$; for generator 5, we randomly generate $p_t \in [0, 44]$; for generator 6, we randomly generate $p_t \in [0, 48]$; for generator 7, we randomly generate $p_t \in [0, 60]$; for generator 8, we randomly generate $p_t \in [0, 67]$. These price ranges are selected based on the generator data in Table 7. We compare two formulations for each generator: "MILP" and "Strong" that are similarly defined in Section 4.1.1, i.e, "MILP" represents the original MILP formulation described in (29), "Strong" represents the original MILP formulation plus our proposed strong valid inequalities in Sections 1 - 3 (i.e., (2d) - (2g), (4) - (13), and (17) - (27)) as constraints.

Table 12: Computational Performance for Eight Single Generators

Generator	IGap (%)		Percent -age (%)	CPU Time(s) (TGap (%))		# of Nodes	
	MILP	Strong		MILP	Strong	MILP	Strong
1	30.96	0.07	99.76	1612.55	84.4	11396	0
2	39.07	0.09	99.78	1557.55	66.11	11716	0
3	56.77	0.16	99.71	*** (0.91) [3]	82.16	49493	0
4	53.31	0.15	99.71	*** (0.82) [3]	104.69	44752	0
5	32.2	0.26	99.2	*** (0.1) [3]	201.97	47635	55
6	57.69	0.58	98.99	*** (0.15) [3]	109.45	58441	0
7	50.18	0.19	99.62	1387.44	88.96	11640	0
8	81.63	6.54	91.99	*** (3.49) [3]	403.65	36612	591

We report the computational results in Table 12. The integrality gaps of two formulations are reported in the column labelled "IGap (%)", in which the integrality gap is defined as $(Z_{\text{LP}} - Z_{\text{MILP}})/Z_{\text{LP}}$, where Z_{LP} is the objective value of the LP relaxation and Z_{MILP} is the objective value corresponding to the best integer solution we obtained from these two formulations within the time limit. Since the self-scheduling unit commitment problem is a maximization problem, the integrality gap definition is different that defined for the network-constrained unit commitment problem in Section 4.1. We can observe that the strong valid inequalities tighten the LP relaxation dramatically, as the gap reduction between these two formulations is reported in the column labelled "Percentage (%)". In the column labelled "CPU Time(s) (TGap (%))", we report the computational time that CPLEX needs to solve each instance. For the case in which CPLEX cannot solve it to optimality (i.e., 0.01%) within one hour time limit, we use "***" to indicate it and report the terminating gap ("TGap (%)"). The number in the square bracket indicates the number of instances not solved to default optimality when the one hour time limit is reached. The column labelled "# of Nodes" reports explored branch-and-bound nodes for each formulation. From the table, we can observe significant advantages of applying our derived strong valid inequalities as cutting planes. For most cases, the "Strong" formulation can be solved at the root node without getting into the branch-and-bound procedure.

5 Conclusions

In this paper, we performed the polyhedral study of the integrated minimum-up/-down time and ramping polytope for the unit commitment problem. We derived strong valid inequalities to strengthen the original MILP formulation. In particular, our derived valid inequalities are strong enough to provide the convex hull description for the polytope up to three time periods with different minimum-up/-down time limits. To the best of our knowledge, this is the first study that provides the convex hull description for the three-period cases. In addition, our derived strong valid inequalities for the general multi-period case cover one, two, and three continuous variables, respectively. They are facet-defining under certain conditions. Furthermore, these inequalities are in polynomial size in the order of $\mathcal{O}(T^2)$. Finally, the computational results showed the effectiveness of our proposed strong valid inequalities by solving both the network-constrained unit commitment and self-scheduling unit commitment problems under various data settings. Therefore, Our derived strong valid inequalities can be adopted not only by an ISO to solve a system-level network-constrained unit commitment problem in order to minimize the total cost for the whole system, but also by a generation company (GENCO) to self-schedule its generators in order to maximize its total profit.

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