# Existence Results for Particular Instances of the Vector Quasi-Equilibrium Problem on Hadamard Manifolds.

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**Abstract**

We show the validity of select existence results for a vector optimization problem, and a variational inequality. More generally, we consider generalized vector quasi-variational inequalities, as well as, fixed point problems on genuine Hadamard manifolds.

**Keywords:** Variational inequality · Hadamard manifold · Quasi-convex Functions

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# 1 Introduction

Zhou and Huang [1], Colao et al. [2], and Li and Hang [3] considered, respectively, the existence of solutions to vector optimization problems, equilibrium problems, and more generally, generalized vector quasi-equilibrium problems, in the Riemannian context. To the best of our knowledge, Colao et al. [2] and Zhou and Huang [4] were the first to consider the existence of solutions to equilibrium problems in the Riemannian context by generalizing the Knaster-Kuratowski-Mazurkiewicz (KKM) lemma to an Hadamard manifold. In [2] Colao et al. presented some applications such as existence results for a variational inequality and a fixed point problem. The existence results in [1, 3] are also based on the KKM lemma.

In the aforementioned papers, the authors used the convexity of a certain function that as noted by Cruz Neto et al. in [5] and Kristály et al. in [6], reduces the corresponding results and consequences of these papers to nothing more than previously well-known facts in Euclidean space. In this paper, we show that [3] as well as others papers listed in [5, 6] remain valid, in part or in full, on genuine Hadamard manifolds.

The organization of this paper is as follows. In Section 2, notation and basic results are established. In Section 3, the main result is stated and proved. Sections 4, 5, and 6 present applications of our main result. Conclusions are made in Section 7.

# 2 Riemannian Geometry

In this section, we recall some basic concepts and properties that can be found in any introductory book on Riemannian geometry; see, for instance, do Carmo [7] and Sakay [8].

Let $M$ be an $n$-dimensional connected manifold. Let $T_xM$ denote the $n$-dimensional *tangent space* of $M$ at $x$, $TM = \cup_{x\in M} T_xM$ *denote the tangent bundle* of $M$ and $\mathcal{X}(M)$ denote the space of smooth vector fields over $M$. When $M$ is endowed with a Riemannian metric $\langle \cdot\, , \cdot \rangle$ and corresponding norm denoted by $\| \cdot \|$, $M$

is a Riemannian manifold. Recall that the metric can be used to define the length of a piecewise smooth curve $\gamma : [a, b] \to M$ joining $x$ to $y$, i.e., $\gamma(a) = x$ and $\gamma(b) = y$, by $l(\gamma) := \int_a^b \|\gamma'(t)\| dt$. By minimizing this length functional over the set of all such curves, we obtain the Riemannian distance $d(x, y)$ inducing the original topology on $M$.

From the completeness of the Riemannian manifold $M$, the *exponential map* $\exp_x : T_x M \to M$ is defined by, $\exp_x v = \gamma_v(1, x)$, for each $x \in M$. A complete simply-connected Riemannian manifold of nonpositive sectional curvature is called an Hadamard manifold. If $M$ is an Hadamard manifold, then $M$ has the same topology and differential structure as the Euclidean space $\mathbb{R}^n$. Throughout the remainder of this paper, we assume that $M$ is an n-dimensional Hadamard manifold. The following result is well-known and essential for our work.

**Theorem 2.1.** *Let $M$ be an Hadamard manifold and $x \in M$. Then $exp_x : T_x M \to M$ is a diffeomorphism.*

*Proof.* See [7, Theorem 3.1, page 149]. □

## 3 Quasi-convexity and Main Result

A set $\Omega \subset M$ is said to be *convex* iff any geodesic segment with end points in $\Omega$ is contained in $\Omega$; that is, if $\gamma : [0, 1] \to M$ is a geodesic such that $x = \gamma(0) \in \Omega$ and $y = \gamma(1) \in \Omega$, then $\gamma(t) \in \Omega$ for all $t \in [0, 1]$. A real-valued function $f$ defined on $M$ is said to be convex on a convex set $\Omega$ iff

$$f(\gamma(t)) \leq (1 - t) f(x) + t f(y), \qquad t \in [0, 1], \qquad x, y \in \Omega.$$

The function $f$ is said to be concave iff $-f$ is convex. If $g$ is both convex and concave, then $g$ is said to be linear affine. Furthermore, $f$ is quasi-convex on $\Omega$ iff

$$f(\gamma(t)) \leq \max\{f(x), f(y)\}, \qquad t \in [0, 1], \qquad x, y \in \Omega.$$

Given $\alpha \in \mathbb{R}$, let $L_f(\alpha)$ denote the level set $\alpha$ of $f$ defined as follows:

$$L_f(\alpha) = \{x \in M : f(x) \leq \alpha\}.$$

**Proposition 3.1.** *Let $f\colon M \to \mathbb{R}$ be a function. Then, $f$ is a quasi-convex function if only if $L_f(\alpha)$ is a convex set for all $\alpha \in \mathbb{R}$.*

*Proof.* Similar to the proof in the Euclidean context, the proof follows from the definitions of quasi-convexity of functions and convexity of a set. □

**Theorem 3.1.** *Let $\Omega \subset M$ be a convex set, $h : \Omega \to \mathbb{R}$ be a quasi-convex function, and $G$ be a diffeomorphism. Then $h \circ G^{-1}$ is a quasi-convex function on $G(\Omega)$.*

*Proof.* See [9, Theorem 10.9, page 101]. □

The following is our main result.

**Theorem 3.2.** *Let $x \in M$ and $u \in T_xM \setminus \{0\}$. Then, $g : M \to \mathbb{R}$ defined by*

$$g(y) = \langle u, \exp_x^{-1} y\rangle, \tag{1}$$

*is a quasi-convex function.*

*Proof.* Let $f : T_xM \to \mathbb{R}$ be defined by $f(w) = \langle u, w\rangle$. Note that $f$ is a convex function and, in particular, quasi-convex. Let $F : T_xM \to M$ be a function given by $F(w) = \exp_x w$. From Theorem 2.1, it follows that $F$ is a diffeomorphism. Since $g(y) = (f \circ F^{-1})(y)$, the desired result it follows immediately from Theorem 3.1 with $h = f$ and $G = F$. □

Recently, in [5, 6] the authors showed that the convexity of $g$ given in (1) only happens in Hadamard manifolds that are isometric to the usual Euclidean space.

**Theorem 3.3.** *If $M$ is an Hadamard manifold with null sectional curvature, $x \in M$ and $u \in T_xM \setminus \{0\}$, then the function $g$ defined in* (1) *is an affine linear function.*

*Proof.* See [5, 6]. □

**Theorem 3.4.** *If $M$ is an Hadamard manifold with negative sectional curvature, $y \in M$ and $u \in T_yM \setminus \{0\}$, then the function $g : M \to \mathbb{R}$ defined in* (1) *is not convex.*

*Proof.* See [5, 6]. □

A list of recent papers was presented in [5, 6]; although they were written in the context of general Hadamard manifolds, their use of the convexity of $g$ reduces their corresponding results and consequences to nothing more than previously well-known facts in Euclidean space. In the next sections, we show that Theorem 3.2 ensures that some of the papers listed in [5, 6] remain, in part or in full, valid for genuine Hadamard manifolds.

# 4 The Multicriteria Problem

Consider the following vector optimization problem on Hadamard manifold:

$$\begin{cases} \min_{\mathbb{R}^p_+} f(x), \\ s.t. \; x \in C, \end{cases} \tag{2}$$

where $C \subset M$ is a nonempty set and $f : M \to \mathbb{R}^p$ is a vector function. Recall that a point $x^* \in C$ is a weak minimum point of $F$ iff there is no $x \in C$ with $f(x) \prec f(x^*)$ or, equivalently, iff

$$f(x) - f(x^*) \notin -\mathbb{R}^p_{++}, \qquad x \in C.$$

Zhou and Huang [1] presented an existence result for solutions to (2) by showing its equivalence with the variational inequality problem on an Hadamard manifold:

$$\text{Find } x^* \in C : \quad \langle A(x^*), \exp_{x^*}^{-1} y\rangle \notin -\mathbb{R}^p_{++}, \qquad y \in C, \tag{3}$$

where $C$ is a convex set, $f$ is a differentiable convex vector function, and $A$ is the Riemannian Jacobian of $f$. Assuming that $f$ is given by $f(x) := (f_1(x), \ldots, f_p(x))$, we denote the Riemannian Jacobian of $f$ by

$$Jf(x) := (\operatorname{grad} f_1(x), \ldots, \operatorname{grad} f_p(x)), \qquad x \in M.$$

Németh [10] studied the case when $p = 1$ in (3). From Theorems 3.3 and 3.4, is possible to note that the existence result for solutions to (2) presented in [1, Theorem 3.2] is restricted to Hadamard manifolds with null sectional curvature. Next, we show that Theorem 3.2 implies that [1, Theorem 3.2] really holds in genuine Hadamard manifolds. In fact, it guarantees the convexity of the set

$$\{y \in C : \langle \nabla f(x), exp_x^{-1} y\rangle \in -\,int\,\mathbb{R}^p_+\} = \{y \in C : Jf(x)exp_x^{-1} y \prec 0\}. \tag{4}$$

**Definition 4.1.** *A function $H\colon M \to \mathbb{R}^p$ is said to be $\mathbb{R}^p_+$-quasi-convex iff for each $x, y \in M$ and each geodesic segment $\gamma : [0, 1] \to M$ joining $x$ to $y$, the following holds:*

$$H(\gamma(t)) \preceq \max\{H(x), H(y)\},$$

*where the maximum is considered coordinate by coordinate.*

**Remark 4.1.** *This definition appeared in [11]. Note that, $H := (h_1, \ldots, h_p)$ is a quasi-convex function if, only if, $h_i$ is a quasi-convex function for all $i = 1, \ldots, p$.*

**Definition 4.2.** *Let $\alpha \in \mathbb{R}$ and $H\colon M \to \mathbb{R}^p$ be a vectorial function. The level set $\alpha$ of $H$ is defined by*

$$L_H(\alpha) = \{x \in M : H(x) \preceq \alpha\, e_p\},$$

*where $e_p = (1, \cdots, 1) \in \mathbb{R}^p$.*

**Proposition 4.1.** *The function $H\colon M \to \mathbb{R}^p$ is quasi-convex if, and only if, $L_H(\alpha)$ is a convex set for all $\alpha \in \mathbb{R}$.*

*Proof.* First, note that $L_H(\alpha) = \cap_1^m L_{h_i}(\alpha)$. The remainder of the proof follows from Remark 4.1 since $h_i$ is a quasi-convex function. In this case, $L_{h_i}(\alpha)$ is a convex set for each $i = 1, \dots, p$. □

From Theorem 3.2, when $u = \operatorname{grad} f_i$, it follows that $M \ni y \mapsto \langle \operatorname{grad} f_i(p), \exp_x^{-1} y\rangle$ is a quasi-convex function for each $i = 1, \dots, p$; hence, Remark 4.1 implies that $M \ni y \mapsto Jf(x) \exp_x^{-1} y$ is a quasi-convex vectorial function. Combining Proposition 4.1 with $H(y) = Jf(x) \exp_x^{-1} y$ and $\alpha = 0$, we conclude that (4) is a convex set. Therefore, [1, Theorem 3.2] indeed holds in genuine Hadamard manifolds.

# 5 Equilibrium Problem

Given $K \subset M$ a nonempty, closed and convex set and $F : K \times K \to \mathbb{R}$ a bifunction satisfying the property $F(x, x) \geq 0$, for all $x \in K$, the *equilibrium problem* (EP) in the Riemannian context is as follows:

$$\text{Find } x^* \in \Omega : \quad F(x^*, y) \geq 0, \qquad y \in K. \tag{5}$$

In this case, the bifunction $F$ is called an *equilibrium bifunction.* To the best of our knowledge, this problem was first considered by Colao et al. in [2], where the authors pointed out important problems, which are retrieved from (5); in particular, given $A \in \mathcal{X}(M)$, if

$$F(x, y) = \langle A(x), \exp_x^{-1} y\rangle, \qquad x, y \in K, \tag{6}$$

(5) reduces to the variational inequality problem on an Hadamard manifold. The authors also presented an existence result for the (EP) and applications to variational inequality, fixed point, and Nash equilibrium problems. Moreover, they presented the convergence of a proximal point-type algorithm based on the Picard iteration for firmly nonexpansive mappings. Application to the variational inequality and fixed point

problems, such as the convergence analysis of the iterative process, is based on the convexity of the function $g$ given in (1) which, as noted in [5, 6], makes the results in [2, Theorem 3.5, 3.10 and 4.9] to be restricted to Hadamard manifolds with null sectional curvature. For now, Theorem 3.2 allows us to ensure that the applications to variational inequality (in particular when $f = 0$) and fixed point problems remain valid for genuine Hadamard manifolds. To be precise, the quasi- convexity of $g$ in (1) ensures the convexity of the following sets:

$$\{y \in K : \langle A(x), \exp_x^{-1} y\rangle < 0\}, \qquad \text{and} \qquad \{y \in K : F(x, y) < 0\},$$

where $A \in \mathcal{X}(M)$ and $F(x, y) := \max\{-\langle \exp_x^{-1} z, \exp_x^{-1} y\rangle : z \in T(x)\}$, with $T : K \rightrightarrows K$ being a set-valued mapping.

## 6 Generalized Vector Quasi-Variational Inequalities

Given $K \subset M$ a nonempty and closed set, a set-valued vector field $V_i : K \rightrightarrows TM$ (i=1,. . . , p), and $S : K \rightrightarrows K$ a set-valued mapping, the generalized vector quasi-variational inequality (GVQVI) on Hadamard manifolds, is as follows:

$$\text{Find } x^* \in K,\ v^* \in V(x^*) :\ x^* \in S(x^*) \quad \text{and} \quad \langle A(x^*), \exp_{x^*}^{-1} y\rangle \notin -\mathbb{R}^p_{++}, \qquad y \in S(x^*), \tag{7}$$

where $V := V_1 \times \ldots \times V_p$. The GVQVI in (7) was considered by Li and Hang in [3] as a particular case of the generalized vector quasi-equilibrium problem (GVQEP), which was introduced and studied by the authors on Hadamard manifolds, whose main result is an existence theorem of solutions; see [3, Theorem 3.1]. In the proof of the existence theorem for GVQVI in (7), namely, [3, Theorem 4.1], the authors used the convexity of $g$ in (1). From Theorems 3.3 and 3.4, is restricted to Hadamard manifolds with null sectional curvature. However, it easy to verify that Theorem 3.2 allows us conclude that [3, Theorem 4.1] really holds in genuine

Hadamard manifolds since Theorem 3.2 implies $\mathbb{R}^p_+$-quasi-convexity of

$$K \ni y \longmapsto \langle z, exp_x^{-1} y \rangle, \qquad x \in K, \quad z \in C := C_1 \times \ldots \times C_p, \quad p \geq 1, \tag{8}$$

where $C_i \in TM$ for $i = 1, \ldots, p$.

# 7 Conclusions

It is well-known that $g$ in (1) can be used to develop an algorithm to approximate equilibrium points; see [2]. The convergence analysis presented in the said paper utilized the convexity of $g$, which restricted your validity just on Hadamard manifolds that are isometric to the usual Euclidean space (this follows from Theorems 3.3 and 3.4). In the future, we plan to extend the convergence result in [2] to genuine Hadamard manifolds.

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